Semi-Lipschitz Functions, Best Approximation, and Fuzzy Quasi-Metric Hyperspaces.

Memoria presentada por
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Certifico: que la presente memoria "Semi-Lipschitz Functions, Best Approximation, and Fuzzy Quasi-Metric Hyperspaces," ha sido realizada bajo mi dirección por D. José Manuel Sánchez Álvarez, en el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas.

Y para que así conste, presento la referida tesis, firmando el presente certificado.

Valencia, 1 de abril de 2009

Fdo. Salvador Romaguera Bonilla

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Quiero señalar desde estas líneas a aquellas personas que de algún modo han hecho posible la realización de este trabajo.

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Los dioses nos han revelado todas las cosas desde el principio. Pero el hombre busca y con el tiempo encuentra. Supongamos que esas cosas son como si fueran verdades. Porque, seguramente, ningún hombre conoce ni conocerá jamás la verdad sobre los dioses y sobre todo aquello de lo que hablo. Pues aun si da la casualidad de que dice la verdad perfecta no la conoce, sino que la apariencia todo lo envuelve.

(JENÓFANES (570-478 a. C.))

Funciones Semi-Lipschitz, Mejor Aproximación e Hiperespacios Casi-Métricos Fuzzy

José Manuel Sánchez Álvarez

En los últimos años se ha desarrollado una teoría matemática que permite generalizar algunas teorías matemáticas clásicas: hiperespacios, espacios de funciones, topología algebraica, etc. Este hecho viene motivado, en parte, por ciertos problemas de análisis funcional, concentración de medidas, sistemas dinámicos, teoría de las ciencias de la computación, matemática económica, etc.

Esta tesis doctoral está dedicada al estudio de algunas de estas generalizaciones desde un punto de vista no simetrico. En la primera parte, estudiamos el conjunto de funciones semi-Lipschitz; mostramos que este conjunto admite una estructura de cono normado. Estudiaremos diversos tipos de completitud (bicompletitud, right k-completitud, D-completitud, etc), y también analizaremos cuando la casi-distancia correspondiente es balancea-
da. Además presentamos un modelo adecuado para el computo de la complejidad de ciertos algoritmos mediante el uso de normas relativas. Esto se consigue seleccionando un espacio de funciones semi-Lipschitz apropiado. Por otra parte, mostraremos que estos espacios proporcionan un contexto adecuado en el que caracterizar los puntos de mejor aproximación en espacios casi-métricos.

El hecho de que varias hipertopologías hayan sido aplicadas con éxito en diversas áreas de Ciencias de la Computación ha contribuido a un considerable aumento del interés en el estudio de los hiperespacios desde un punto de vista no simétrico. Así, en la segunda parte de la tesis, estudiamos algunas condiciones de mejor aproximación en el contexto de hiperespacios

casi-métricos. Por otro lado, caracterizamos la completitud de un espacio uniforme usando la completitud de Sieber-Pervin, la de Smyth y la D-completitud de su casi-uniformidad superior de Hausdorff-Bourbaki, definida en los subconjuntos compactos no vacíos.

Finalmente, introducimos dos nociones de hiperespacio casi-métrico fuzzy que generalizan las correspondientes nociones de espacio métrico fuzzy de Kramosil y Michalek, y de George y Veeramani respectivamente al contexto de hiperespacios casi-métricos. Presentamos diversas nociones básicas de completitud, precompacidad y compacidad. Aplicamos esta teoría a varios ejemplos y ponemos de manifiesto las ventajas del uso de casi-métricas fuzzy en lugar de las métricas y casi-métricas clásicas.

Funcions semi-Lipschitz, Millor Aproximaciò i Hiperespai Quasi-Mètrics Fuzzy

José Manuel Sánchez Álvarez

En els darrers anys s'ha desenvolupat una teoria matemàtica que permet generalitzar algunes teories matemàtiques clàssiques: hiperespais, espais de funcions, topologia algebraica, etc. Aquest fet ve motivat,en part, per certs problemes de anàlisi funcional, concentració de mesures, sistemes dinàmic, teoria de les Ciències de la Computació, matemàtica econòmica, etc.

Aquesta tesi doctoral està dedicada a l'estudi d'algunes d'aquestes generalitzacions des d'un punt de vista no simètric. A la primera part, estudiem el conjunt de funcions semi-Lipschitz; mostrem que aquest conjunt admet una estructura de con normand. Estudiarem diversos tipus de completesa (bicompletesa, right k-completesa, D-completesa, etc), i també analitzarem quan la quasi-distància correspondent és balanceada. A més presentem un model adequat per al comput de la complexitat de certs algorismes mitjançant l'ùs de normes relatives. Això és aconsegueix seleccionant un espai de funcions semi-Lipschitz apropiat. D'altra banda, mostrarem que aquests espais proporcionen un context adequat en que caracteritzar els punts de millor aproximaciò en espais quasi-mètrics.

El fet de que varies hipertopologies aixen segut aplicades amb exit a diferents arees de Ciència de la Computació, ha contribuit a un considerable aument del interés en l'estudi d'estos hiperespais des de un punt de vista no simètric. Així, a la segona part de la tesi, estudiem algunes condicions de millor aproximaciò en el context de hiperespai quasi-mètrics. D'altra banda, caracteritzem la completesa d'un espai uniforme usant la completesa de Sieber-Pervin, la d'Smyth o la D-completesa de la seva quasi-uniformitat superior d' Hausdorff-Bourbaki definida en els subconjunts compactes no

buits.

Finalment introduïm dues nocions de hiperespai quasi-mètric fuzzy que generalitzen les corresponents nocions d'espai mètric fuzzy de Kramosil i Michalek, i de George i Veeramani respectivament al context de hiperespais quasi-mètrics. Presentem diverses nocions bàsics de completesa, precompacitat i compatacitat. Apliquem aquesta teoria a alguns exemples i posem de manifest els avantatges de l'ús de quasi-mètriques fuzzy en lloc de les mètriques i quasi-mètriques clàsiques.

Semi-Lipschitz Functions, Best Approximation, and Fuzzy Quasi-Metric Hyperspaces

José Manuel Sánchez Álvarez

Motivated, in part, by some problems from functional analysis, concentration of measures, dynamical systems, theoretical computer science, mathematical economics, etc, in the last years a mathematical theory has been developed in order to generalize classical mathematical theories: hyperspaces, function spaces, topological algebra, etc.

This doctoral thesis is devoted to study some of these generalizations from a nonsymmetric point of view. In the first part, we study the set of semi-Lipschitz functions, we show that this set can be endowed with the structure of normed cone. We also study different types of completeness, (bi-completeness, right k-completeness, D-completeness), and we explore when the corresponding quasi-distance is balanced. Using relativized norms we present a model for computing the complexity of certain algorithms, which is done with the help of a suitable space of semi-Lipschitz functions. On the other hand, we show that our approach provides an appropriate setting to characterize the points of best approximation of quasi-metric spaces.

The fact that some hypertopologies have been successfully applied to several areas of Computer Science has contributed to increase the interest of a nonsymmetric study of hypertopologies. Thus, in a second part, we study some conditions on best approximation in the realm of quasi-metric hyperspaces. By other hand, we characterize completeness of a uniform space using Sieber-Pervin completeness, Smyth completeness and D-completeness of its upper Hausdorff-Bourbaki quasi-uniformity, on the collection of its nonempty compact subsets.

Finally we introduce two notions of fuzzy quasi-metric hyperspace that generalize the corresponding notions of fuzzy metric space by Kramosil and Michalek, and by George and Veeramani respectively, to the quasi-metric hyperspace context. Several basic properties of completeness, precompactness, and compactness of these spaces are obtained. We apply this theory to some examples and we point out some advantages of the use of fuzzy quasi-metrics instead of classical metrics and quasi-metrics.

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

We begin with a short summary of the history of some concepts and we give some references. The concept of a metric space was defined by Fréchet in [36]. Sierpinski's book ([125]) and Kuratowski's book ([68]) contain general results on these spaces. On the other hand, asymmetric distance functions had already been considered by Hausdorff in the beginning of the last century when in his classical book on set-theory ([49]) he discussed the Hausdorff metric of a metric space. Although the notion of quasi-metric was formally introduced in 1931 by Niemytzki ([85]) and Wilson ([138]), Niemytzki explored the interplay of the various assumptions in the usual axiomatization of a metric space. Kelly in [56] noticed that given a quasi-pseudo-metric d we can associate its conjugate quasi-pseudo-metric d^{-1}.

The notion of quasi-norm that we are using was introduced in ([28]). By other hand Alegre, Ferrer and Gregori introduced this concept to the study of functional analysis from a non symmetric point of view ([3, 4]).

The theory of uniformities was introduced by Weil. He wanted to develop a tool which, in contrast to metrics, could be applied to spaces not necessarily satisfying the axioms of countability. Bourbaki ([14]) developed the theory of uniform spaces presenting an axiomatic theory parallel to the theory of topological spaces.

The beginning of quasi-uniformities is due to Nachbin ([84]) in 1948, motivated by the study of the uniform preordered spaces. He called the studied nonsymmetric structures semi-uniformities. The term quasi-uniformity was lather suggested by Császár in [20]. Krishnan ([58]) showed that every topological space is quasi-uniformizable; subsequently, a proof of this result were obtained by Császár ([20]) in terms of syntopogenous structures, and by Pervin ([87]), in a direct fashion.

Hausdorff started the study of topologies defined over a collection of subsets of a topological space. In [49] he defined a metric on the space of all nonempty closed subsets of a bounded metric space, called the Hausdorff metric. If we extend this metric to every nonempty set we obtain the so-called Hausdorff pseudo-metric. Later on, Bourbaki introduced in [13] the so-called Bourbaki or Hausdorff uniformity, which generates a T_1 topology over the collection of all nonempty closed subsets of a uniform space. Other authors (see [55, 135]) also studied these topologies which started to be called hypertopologies and whose corresponding spaces were called hyperspaces.

On other hand, the best studied hypertopology from a nonsymmetric point of view is the Hausdorff quasi-uniform topology. Levine and Stager ([69]) as well as Berthiaume ([12]) noticed that it can be defined, in the set of all nonempty subsets of a set in the same way as it is done with the Hausdorff uniformity. The study of the Hausdorff quasi-uniformity was continued by Cao, Künzi, Ryser, Reilly, Romaguera, etc ([16, 17, 64, 66, 67]).

Recently, the study of nonsymmetric structures has received a new drive as consequence of their applications to Computer Science. This theory began with Smyth (see [128, 129]). He tried to find a convenient category for computation. The two main spaces used in semantics are Scott domains ([119]) and metric spaces. He wanted to unify these spaces and obtain their advantages. He asserted that the more suitable structures are the quasi-metric spaces and the quasi-uniform spaces.

Schellekens introduced in [116] a quasi-pseudo-metric on function spaces which is suitable for the study of complexity analysis of programs, which is called the complexity distance. He showed that the complexity distance is weightable. Using this theory, he also proved that each Divide & Conquer algorithm induces a contraction map on a complexity space (the sequential Smyth completion of the complexity spaces). The complexity of such an algorithm then is represented via the fixed point of the map obtained by the Banach Fixed Point Theorem. As an application of this theory, he gave a new proof showing that the mergesort program has optimal asymptotic average time.

This work is continued by Romaguera and Schellekens ([108]). They introduced the dual complexity space which is isometric to the complexity space and put their interest in this space rather than the complexity space as the dual space admits a structure of quasi-normed semilinear space, so the presentations of the proofs becomes somewhat more elegant. They proved that the dual complexity space is a Smyth-complete Baire quasi-metric space and that the complexity subspaces having lower bound are totally bounded. Furthermore, the dual complexity space has the advantage that it respects the interpretation usually given to the minimum in semantic domains.

Investigation in this topic is being developed by García-Raffi, Romaguera, Sánchez-Pérez and Schellekens ([38, 106, 109]).

In 1965, the concept of fuzzy set was introduced by Zadeh ([139]). Many authors have introduced and discussed several notions of fuzzy metric space from different points of view ([32, 42, 43, 45, 47]). Kramosil and Michalek introduced and studied in [57] an interesting notion of fuzzy metric space which makes use of the concept of continuous t-norm ([118]); this notion of fuzzy metric space is closely related to a class of probabilistic metric spaces, the so-called (generalized) Menger spaces. Later on, George and Veeramani started, in [42] (see also [43]), the study of a stronger form of metric fuzziness. In [47] Gregori and Romaguera introduced two notions of fuzzy quasi-metric space that generalize the corresponding notions of fuzzy metric space by Kramosil and Michalek, and by George and Veeramani, to the quasi-metric context, and in [92], a suitable definition for the Hausdorff fuzzy metric of a fuzzy metric space in the sense of George and Veeramani is introduced.

This doctoral thesis is devoted to study some generalizations, from a nonsymmetric point of view, of classical mathematical theories. We summarize the whole investigation:

In **Chapter 1**, we comment on the origins of the asymmetric topology and we briefly present the main results which will be developed in the thesis. Notions and concepts which are used, will be defined in section 1.2 below.

In [103], S. Romaguera and M. Sanchis discussed several properties of real valued semi-Lipschitz functions. In **Chapter 2** we extend their study to the space of semi-Lipschitz functions that are valued in a quasi-normed linear space. Our approach is motivated, in part, by the fact that this structure can be applied to study some processes in the theory of complexity spaces.

We show that the set of semi-Lipschitz functions can be endowed whit structure of a normed cone we also study different types of completeness, (bicompleteness, right k-completeness, D-completeness)

We show that if (X, d) is a T_1 quasi-metric space and (Y, q) is a quasi-normed linear space, then the normed cone of semi-Lipschitz functions from (X, d) to (Y, q) that vanish at a point $x_0 \in X$, is balanced. Moreover, it is complete in the sense of D. Doitchinov whenever (Y, q) is a biBanach space.

Let T be the recurrence equation on $\mathbb{N}$ associated to a given algorithm. If we denote by f the complexity function which is the solution of such a recurrence equation, then f constitutes a total mapping defined recursively that is, at the same time, the limit of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of partial mappings also defined recursively. In **Chapter 3** we present a model to compute the complexity represented by f, by means of the values that take a certain relativized norm on the sequence $\{f_n\}_{n\in\mathbb{N}}$ and its initial segments. This is done with the help of a suitable space of semi-Lipschitz functions which is constructed here.

In **Chapter 4** we show that the set of semi-Lipschitz functions, defined on a quasi-metric space (X, d) with values in a quasi-normed linear space (Y, q), that vanish at a fixed point $x_0 \in X$ with the structure of a quasi-normed semilinear space provides an appropriate setting in which to characterize the points of best approximation of quasi-metric spaces. And apply our methods to obtain some conditions on best approximation in the realm of quasi-metric hyperspaces.

A celebrated theorem due to Morita states that a separated uniform space $(X,\mathcal{U})$ is complete if and only if its Hausdorff-Bourbaki uniformity is complete on the collection of its nonempty compact subsets. In **Chapter 5**, we show that completeness of $(X,\mathcal{U})$ can be also characterized, among others, by Sieber-Pervin completeness, Smyth completeness or D-completeness of its upper Bourbaki quasi-uniformity on the collection of its nonempty compact subsets.

Finally in **Chapter 6**, we construct and discuss a notion of Hausdorff fuzzy quasi-metric, based on the notion of fuzzy (quasi-)metric of Kramosil and Michalek. We show that this new concept has several nice properties of completeness, precompactness and compactness. We also consider a notion of Hausdorff fuzzy quasi-metric, based on the notion of fuzzy (quasi-)metric in the sense of George and Veeramani. Finally we apply the developed theory to the domain of words and we point out some advantages of the use of fuzzy quasi-metrics instead of classical metrics and quasi-metrics.

1.2 Preliminaries

Next we recall some pertinent concepts. In what follows X will always denote a nonempty set.

A *quasi-pseudo-metric* on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$:

(i) $d(x, x) = 0$

(ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If, in addition, d satisfies the condition:

(iii) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,

then d is said to be a *quasi-metric* on X.

By other hand, if d can take the value ∞ then it is called an *extended quasi* (*-pseudo*)*-metric* on X. In the case that d is an extended quasi-metric we also refer to it as a *quasi-distance*.

A(n *extended*) *quasi-*(*pseudo-*) *metric space* is a pair (X, d) such that d is a (n *extended*) *quasi-*(*pseudo-*)*metric* on X.

If d is a(n extended) quasi-(pseudo-)metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a(n extended) quasi-(pseudo-)metric on X, called the *conjugate* (*extended*) *quasi-*(*pseudo-*)*metric* of d, and the function d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ for all $x, y \in X$, is a(n extended) (pseudo-)metric on X.

The following is a simple but paradigmatic example of a quasi-metric space.

Example 1.1. Let ℓ be the real-valued function defined on $\mathbb{R}\times\mathbb{R}$ by $\ell(x, y) = \max\{x - y, 0\}$. Then ℓ is a quasi-metric on $\mathbb{R}$ such that ℓ^s is the Euclidean metric on $\mathbb{R}$.

Each extended quasi-pseudo-metric d on X generates a topology T_d on X which has as a base the family of balls of the form $B_d(x, r) = \{y \in X : d(x, y) < r\}$, where $x \in X$ and $r > 0$. Note that if d is a quasi-metric, then T_d is a T_0 topology on X. Moreover, if condition (iii) above is replaced by

(iii') $d(x, y) = 0 \Leftrightarrow x = y$,

then T_d is a T_1 topology.

A topological space (X, T) is said to be quasi-(pseudo-)metrizable if there is a quasi-(pseudo-)metric d on X such that $T = T(d)$.

A subset A of a quasi-(pseudo-)metric space (X, d) is called bounded if A is bounded in the (pseudo-)metric space (X, d^s).

A quasi-metric d is said to be bicomplete if d^s is a complete metric.

An extended quasi-(pseudo-)metric d is said to be bicomplete if d^s is a complete extended (pseudo-)metric.

If d is an extended quasi-pseudo-metric on a set X, then the relation $\leq_d$ on X given by $x \leq_d y$ if and only if $d(x, y) = 0$, is a preorder on X (i.e., $\leq_d$ is reflexive and transitive).

It is clear that d is an extended quasi-metric on a set X if and only if $\leq_d$ is a (partial) order on X (i.e., the preorder $\leq_d$ is antisymmetric, which means that $x \leq_d y$ and $y \leq_d x$, implies $x = y$). In this case, $\leq_d$ is called the specialization order.

Note that in Example 1.1 the specialization order coincides with the usual order on $\mathbb{R}$.

Remark 1.1. The natural connection between asymmetric distances and order, described above, provides some advantages in certain settings, if one works with quasi-metrics instead of metrics. Thus, in modeling a computational process on a collection X of elements (for example, chains of information, words of an alphabet in a programming language, complexity functions in analysis of algorithms, etc.) we can define an order $\leq$ on X given by $x \leq y$ if and only if the element y contains all the information provided by the element x, and then it is possible, in many cases, to construct a suitable (extended) quasi-metric d on X such that the order $\leq$ is exactly the specialization order of d, transmitting in this way the information provided by $\leq$ to the quasi-metric space (X, d) (see, for instance, [76, 116]).

For more information about quasi-metric spaces see [35] and [62].

Let us recall that a quasi-uniformity on X is a filter $\mathcal{U}$ on $X \times X$ such that:

(i) for each $U \in \mathcal{U}$, $\Delta \subseteq U$, where $\Delta = \{(x, x) : x \in X\}$;

(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V^2 \subseteq U$, where $V^2 = \{(x, y) \in X \times X : \text{there is } z \in X \text{ with } (x, z) \in V \text{ and } (z, y) \in V\}$.

By a quasi-uniform space we mean a pair $(X, \mathcal{U})$ such that $\mathcal{U}$ is a quasi-uniformity on X. The members of $\mathcal{U}$ are called *entourages*.

Each quasi-uniformity $\mathcal{U}$ on X generates a topology $T_{\mathcal{U}}$ on X such that a neighborhood base for each point $x \in X$ is given by $\{U(x) : U \in \mathcal{U}\}$, where $U(x) = \{y \in X : (x, y) \in \mathcal{U}\}$.

If (X, T) is a topological space, and $\mathcal{U}$ is a quasi-uniformity (resp. d is a quasi-pseudo-metric) on X such that $T_{\mathcal{U}} = T$ (resp. $T_d = T$) , we say that $\mathcal{U}$ is a quasi-uniformity (resp. d is a quasi-pseudo-metric) compatible with T.

The filter $\mathcal{U}^{-1}$, formed by all sets of the form $U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$ where $U \in \mathcal{U}$, is a quasi-uniformity on X called the *conjugate quasi-uniformity* of $\mathcal{U}$.

If $\mathcal{U}$ is a quasi-uniformity on X, then the family $\{U^s = U \cap U^{-1} : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}^s$ (in fact, it is a uniformity), which is the coarsest uniformity containing both $\mathcal{U}$ and $\mathcal{U}^{-1}$. Hence this uniformity is the supremum of the quasi-uniformities $\mathcal{U}$ and $\mathcal{U}^{-1}$, i.e. $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$.

Of course, a quasi-uniformity $\mathcal{U}$ on X is a uniformity on X if and only if $\mathcal{U} = \mathcal{U}^{-1}$.

Each (extended) quasi-pseudo-metric d on X induces a quasi-uniformity $\mathcal{U}_d$ on X which has as a base the countable family

$$\big\{\{(x, y) \in X \times X : d(x, y) < 2^{-n}\} : n \in \mathbb{N}\big\}.$$

In connection with this fact we have the following useful result which can be found, for instance, in ([35, Theorem 1.5]).

Proposition 1.1. *Let $(X, \mathcal{U})$ be a quasi-uniform space. $\mathcal{U}$ has a countable base if and only if there exists a quasi-pseudo-metric d on X such that $\mathcal{U}_d = \mathcal{U}$.*

On the other hand, one problem which arises when working with nonsymmetric structures is that there are several theories about completion. Since Reilly, Subrahmanyam and Vamanamurthy [89] began a systematized study of several definitions of Cauchy sequence in quasi-pseudo-metric spaces, various authors have investigated different notions of completeness for these spaces. We recall the notions of left K-Cauchy and right K-Cauchy sequence respectively.

According to [89] a sequence $\{x_n\}_{n\in\mathbb{N}}$ in a quasi-pseudo-metric space (X,d) is called *right K-Cauchy* if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for $n \geq m \geq k$, and $\{x_n\}_{n\in\mathbb{N}}$ is called *left K-Cauchy* if for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for $n \geq m \geq k$. (X,d) is said to be *left K-sequentially complete* (resp. *right K-sequentially complete*) if each left K-Cauchy (resp. right K-Cauchy) sequence is T_d-convergent.

The corresponding notions for filters were studied by Romaguera [98].

If (X,T) is a topological space, we denote by $\mathcal{P}_0(X)$, $\mathcal{C}_0(X)$ and $\mathcal{K}_0(X)$, the collection of all nonempty subsets of X, the collection of all nonempty closed subsets of X and the collection of all nonempty compact subsets of X.

It is well known ([12, 67, 69]) that, similarly to the theory of uniform spaces ([78]), given a quasi-uniform space $(X,\mathcal{U})$ we can construct three quasi-uniformities $\mathcal{U}_H^+$, $\mathcal{U}_H^-$ and $\mathcal{U}_H$ on $\mathcal{P}_0(X)$, as follows:

For each $U \in \mathcal{U}$, put

$$U_H^+ = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\},$$

and

$$U_H^- = \left\{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\right\}.$$

Then the collection $\{U_H^+ : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_H^+$ on $\mathcal{P}_0(X)$, called the upper Bourbaki (or Hausdorff) quasi-uniformity of $(X,\mathcal{U})$ on $\mathcal{P}_0(X)$, and the collection $\{U_H^- : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\mathcal{U}_H^-$ on $\mathcal{P}_0(X)$, called the lower Bourbaki (or Hausdorff) quasi-uniformity of $(X,\mathcal{U})$ on $\mathcal{P}_0(X)$.

The quasi-uniformity $\mathcal{U}_H = \mathcal{U}_H^+ \vee \mathcal{U}_H^-$ is said to be the Bourbaki (or Hausdorff) quasi-uniformity of $(X,\mathcal{U})$ on $\mathcal{P}_0(X)$.

Obviously, if $(X,\mathcal{U})$ is a uniform space then $\mathcal{U}_H$ is exactly the Bourbaki (or Hausdorff) uniformity on $\mathcal{P}_0(X)$ of $(X,\mathcal{U})$. It is interesting to note that in this case, both $\mathcal{U}_H^+$ and $\mathcal{U}_H^-$ are quasi-uniformities but not uniformities on $\mathcal{P}_0(X)$ unless $|X| = 1$ ([69]).

Remark 1.2. Observe that for a uniform space $(X,\mathcal{U})$, we have $(\mathcal{U}_H^+)^{-1} = \mathcal{U}_H^-$ on $\mathcal{P}_0(X)$. Therefore $(\mathcal{U}_H^+)^s = (\mathcal{U}_H^-)^s = \mathcal{U}_H$ on $\mathcal{P}_0(X)$.

Given a uniform space $(X,\mathcal{U})$, the restrictions of $\mathcal{U}_H^+$, $\mathcal{U}_H^-$ and $\mathcal{U}_H$ to $\mathcal{K}_0(X) \times \mathcal{K}_0(X)$, will be also denoted by $\mathcal{U}_H^+$, $\mathcal{U}_H^-$ and $\mathcal{U}_H$, respectively.

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of real numbers, the set of nonnegative real numbers, the set of positive integers numbers and the set of nonnegative integers, respectively.

The notion of a cone will be also useful.

As usual by a *monoid* we mean a semigroup $(X,+)$ with neutral element. Similarly to [54], by a cone we mean a triple $(X,+,\cdot)$ such that $(X,+)$ is an Abelian monoid, and $\cdot$ is a function from $\mathbb{R}^+ \times X$ to X such that for all $x,y \in X$ and $r,s \in \mathbb{R}^+$:

(i) $r \cdot (s \cdot x) = (rs) \cdot x$;

(ii) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$;

(iii) $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;

(iv) $1 \cdot x = x$.

A *quasi-norm* on a cone $(X, +, \cdot)$ [102, 103] is a function $q : X \to \mathbb{R}^+$ such that for all $x, y \in X$ and $r \in \mathbb{R}^+$:

(i) $x = 0$ if and only if there is $-x \in X$ and $q(x) = 0 = q(-x)$,

(ii) $q(r \cdot x) = rq(x)$,

(iii) $q(x + y) \leq q(x) + q(y)$.

If the quasi-norm q satisfies:

(i') $q(x) = 0$ if and only if $x = 0$,

then q is called a *norm* on the cone $(X, +, \cdot)$.

A (*quasi-*)*normed cone* is a pair (X, q) such that X is a cone and q is a (quasi-)norm on X.

If $(X, +, \cdot)$ is a linear space and q is a quasi-norm on X, then the pair (X, q) is called a *quasi-normed* (*linear*) *space* (*asymmetric normed linear space* in [38]). Note that in this case, the function $q^{-1} : X \to \mathbb{R}^+$ given by $q^{-1}(x) = q(-x)$ is also a quasi-norm on X and the function $q^s : X \to \mathbb{R}^+$ given by $q^s(x) = q(x) \vee q^{-1}(x)$ is a norm on X. As in [39], we say that (X, q) is a *biBanach space* if (X, q^s) is a Banach space.

A simple but crucial example of a biBanach space is the pair $(\mathbb{R}, u)$, where u is the quasi-norm on $\mathbb{R}$ given by $u(x) = \max\{x, 0\}$ for all $x \in \mathbb{R}$. Note that $u^s(x) = |x|$ for all $x \in \mathbb{R}$, so $(\mathbb{R}, u)$ is a biBanach space.

It is well known that each quasi-norm q on a linear space X induces a quasi-metric d_q on X defined as:

$$d_q(x,y) = q(x-y) \qquad \text{for all } x,y \in X.$$

The *complexity* (*quasi-metric*) *space* was introduced by M. Schellekens ([116]) to obtain a topological foundation to the complexity analysis of programs and algorithms. This space consists of the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \Big\{ f : \omega \longrightarrow (0,\infty] : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \Big\},$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f,g) = \sum_{n=0}^{\infty} 2^{-n} \Big[\big(\frac{1}{g(n)} - \frac{1}{f(n)} \big) \vee 0 \Big],$$

for all $f, g \in \mathcal{C}$, where we adopt the convention that $1/\infty = 0$. The elements of $\mathcal{C}$ are called *complexity functions.*

It is well known that $\mathcal{C}$ is a cone when it is equipped with the pointwise operations.

Following [116] the intuition behind the *complexity distance* $d_{\mathcal{C}}(f,g)$ is that it measures relative progress made in lowering the complexity by replacing the complexity function f by the complexity function g. Thus $d_{\mathcal{C}}(f,g) = 0$ if and only if $f \leq g$, and, thus, condition $d_{\mathcal{C}}(f,g) = 0$, with $f \neq g$, can be interpreted as f is more efficient than g on all inputs. Observe that the above information is not provided by using the metric $(d_{\mathcal{C}})^s$ because from the value $(d_{\mathcal{C}})^s(f,g)$ is not possible to determine which complexity function would be more efficient.

If (X,d) is a quasi-pseudo-metric space given A is a subset of X we denote by $\overline{A}^T$ the closure of A with respect to T, we define

$$\mathcal{C}_{\cap}(X) = \big\{ \overline{A}^{T_d} \cap \overline{A}^{T_{d^{-1}}} : A \in \mathcal{P}_0(X) \big\}.$$

Remark 1.3. In a quasi-pseudo-metric space (X,d) the following inclusions are obvious: $\mathcal{C}_0(X) \subseteq \mathcal{C}_\cap(X) \subseteq \mathcal{P}_0(X)$. Moreover, if (X,d) is a metric space, then $\mathcal{K}_0(X) \subseteq \mathcal{C}_0(X)$ and $\mathcal{C}_0(X) = \mathcal{C}_\cap(X)$. It is well known (see Example 1.2 below) that the situation is quite different for quasi-metric spaces).

Now, for each $A, B \in \mathcal{P}_0(X)$ let

$$H_d^-(A,B) = \sup_{a\in A} d(a,B), \quad H_d^+(A,B) = \sup_{b\in B} d(A,b),$$

and

$$H_d(A,B) = \max\{H_d^-(A,B), H_d^+(A,B)\}.$$

Then H_d^-, H_d^+ and H_d are extended quasi-pseudo-metrics on $\mathcal{P}_0(X)$ (see, for instance, [12, 67, 93, 94, etc]). Moreover H_d is an extended quasi-metric on $C_\cap(X)$ (compare [67, Lemma 2]), and it is a quasi-metric on the set of all bounded subsets of X that are in $C_\cap(X)$. In this case we say that H_d is the Hausdorff quasi-metric of d. Note that if (X,d) is a metric space, then H_d is the extended Hausdorff metric of d (on $\mathcal{C}_0(X)$).

Example 1.2. Let $X = \{a,b,c\}$ and let d be the quasi-metric on X given by $d(b,a) = d(c,b) = d(c,a) = 1$, and $d(x,y) = 0$ otherwise. Let $F = \{a,c\}$. Then $\overline{F}^{T_d} \cap \overline{F}^{T_{d^{-1}}} = X$, because $d(b,c) = d(a,b) = 0$. Therefore $F \notin \mathcal{C}_\cap(X)$, and hence $\mathcal{K}_0(X) \nsubseteq \mathcal{C}_\cap(X)$.

Chapter 2

Semi-Lipschitz Functions that are Valued in a Quasi-Normed Linear Space

2.1 Introduction

Motivated, in part, by some problems from computer science and their applications (see for instance [38, 40, 101, 106, 116, 129]), the theory of completeness has received a certain attention in the recent years (see, among other contributions, [24, 26, 34, 61, 97, 127, 132]). These advances have also permitted the development of generalizations, to the nonsymmetric case, of classical mathematical theories: hyperspaces, function spaces, topological algebra, etc.

Let us recall that in [102] it was shown that the set of real-valued semi-Lipschitz functions defined on a quasi-metric space (X, d) that vanish at a point $x_0 \in X$ can be structured as a normed cone. Applications of semi-Lipschitz functions to questions on best approximation, global attractors on dynamical systems, and concentration of measure can be found in [83, 102], [101] and [130], respectively.

In [112], semi-Lipschitz functions that are valued in a quasi-normed linear space have been discussed. This study was motivated, in great part, by the fact that quasi-normed linear spaces provide suitable mathematical models in the theory of computational complexity (see [38, 40, 106]).

The complexity quasi-metric space was introduced in [116] to study complexity analysis of programs and algorithms. Later on, it was introduced in [108] the dual complexity space. Several quasi-metric properties of the complexity space were obtained via the analysis of the dual complexity space. In [106] Romaguera and Schellekens show that the structure of a quasi-normed semilinear space provides a suitable setting to carry out an analysis of the dual complexity space.

This chapter is a contribution to the study of the set of semi-Lipschitz functions from a nonsymmetric point of view. We show that this set can be endowed with structure of a quasi-normed linear space when they are defined on a quasi-metric space taking values in a quasi-normed space. We show that this space is bicomplete and we also study other types of completeness.

Moreover we prove the somewhat surprising fact that when the quasi-metric space is T_1 this space is balanced in the sense of Doitchinov ([26]). We also prove that it is complete in the sense of Doitchinov whenever (Y, q) is a biBanach space. As an application of these results to asymmetric functional analysis, we deduce that the dual space of a T_1 quasi-normed linear space is balanced and Doitchinov complete (see [99]). It is interesting to recall that the study of balanced quasi-metric spaces from a fuzzy point of view has been recently discussed in [46, 105], and that, on the other hand, some applications of balanced (extended) quasi-metrics to theoretical computer science have been given in [96, 100].

2.2 The Structure of the Set of Semi-Lipschitz Functions

In this section we show a new concept of semi-Lipschitz functions, we are going to define a quasi-distance and a quasi-norm on this set. Some internal and external composition operations are defined, this operations give the structure of normed cone. It will be interesting to know when this space has the structure of normed cone it will be showed in some results in this chapter.

Definition 2.1. Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normed space, respectively. A function $f : X \longrightarrow Y$ is said to be a *semi-Lipschitz function* if there exists $k \geq 0$ such that

$$q(f(x) - f(y)) \leq kd(x, y)$$

for all $x, y \in X$. The number k is called a semi-Lipschitz constant for f.

Another interesting set could be the following:

Definition 2.2. A function f on a quasi-metric space (X, d) with values in a quasi-normed linear space (Y, q) is called $\leq_{(d,q)}$-*increasing* if

$$q(f(x) - f(y)) = 0 \text{ whenever } d(x, y) = 0.$$

By $Y^X_{(d,q)}$ we shall denote the set of all $\leq_{(d,q)}$-increasing functions from (X, d) to (Y, q).

It is clear that if (X, d) is a T_1 quasi-metric space, then every function from X to Y is $\leq_{(d,q)}$ -increasing.

If for each $f, g \in Y^X_{(d,q)}$ and $a \in \mathbb{R}^+$ we define $f+g$ and af in the usual way; then it is a routine to show that $(Y^X_{(d,q)}, +, \cdot)$ is a cone.

This example shows that the set of $\leq_{(d,q)}$-increasing functions is not necessarily a linear space:

Example 2.1. Let $X = \mathbb{Z}_3$, where, as usual, by $\mathbb{Z}_3$ we denote the quotient field between the set $\mathbb{Z}$ integer numbers and $3\mathbb{Z}$ multiples of 3. Let d be the quasi-metric on X given by

$$d(x,y) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \leq y. \end{cases}$$

Let $Y = \mathbb{R}$, $q(x) = x \vee 0$ and take f such that $f(0) = 0, f(1) = 1$ and $f(-1) = -2$. It is easy to see that $f \in Y^X_{(d,q)}$ but $-f \notin Y^X_{(d,q)}$. Thus $Y^X_{(d,q)}$ is not a linear space.

A simple but interesting example of a semi-Lipschitz function is the following:

Example 2.2. Let $(\mathbb{N}, d)$ be a quasi-metric space where:

$$d(x,y) = \begin{cases} 1 & \text{if } y > x, \\ 0 & \text{if } y \leq x. \end{cases}$$

Then, the dual complexity space, is the quasi-normed space $(\mathbb{B}^*, q)$, with

$$\mathbb{B}^* = \{f : \omega \to \mathbb{R} : \sum_{n=0}^{\infty} 2^{-n}(f(n) \vee 0) < \infty\}$$

and

$$q(f) = \sum_{n=0}^{\infty} 2^{-n}(f(n) \vee 0).$$

Let now $F : (\mathbb{N}, d) \to (\mathbb{B}^*, q)$ be the function defined by $F(0) = \mathbf{0}$ and $F(n) = f_n$ for all $n > 0$ (where $\{f_n\}_{n \in \mathbb{N}}$ is any strictly decreasing sequence of functions in $(\mathbb{B}^*, q)$, for the usual order, i.e, $g > h \iff g(x) > h(x)$for all $x \in \omega$).

It is easy to check that F is a semi-Lipschitz function, let see it:

Case 1: If $x \geq y$ then $d(x, y) = 0$ and we have that $q(F(x) - F(y)) = q(f_x - f_y) = 0$ because F is a decreasing function, thus

$$q(F(x) - F(y)) \leq kd(x, y).$$

Case 2: If $x < y$, then $d(x, y) = 1$ and we have that

$$q(F(x) - F(y)) = q(f_x - f_y) = \sum_{n=0}^{\infty} 2^{-n}(f_x(n) - f_y(n) \vee 0) \leq$$

$$\sum_{n=0}^{\infty} 2^{-n}(f_x(n) \vee 0) \leq \sum_{n=0}^{\infty} 2^{-n}(f_1(n) \vee 0).$$

Since $f \in \mathbb{B}^*$, then

$$\sum_{n=0}^{\infty} 2^{-n}(f_1(n) \vee 0) = k < \infty$$

so that

$$q(F(x) - F(y)) \leq q(F(1)) < k = kd(x, y).$$

Given a quasi-metric space (X, d) and a quasi-normed space (Y, q), fix $x_0 \in X$ and put

$$\mathcal{SL}_0(d,q) = \Big\{ f \in Y^X_{(d,q)} : \sup_{d(x,y)\neq 0} \frac{q(f(x)-f(y))}{d(x,y)} < \infty \ , f(x_0) = 0 \Big\}.$$

Then $\mathcal{SL}_0(d,q)$ is exactly the set of all semi-Lipschitz functions that vanishes at x_0, and it is clear that $(\mathcal{SL}_0(d,q), +, \cdot)$ is a subcone of $(Y^X_{(d,q)}, +, \cdot)$.

Now let $\rho_{(d,q)} : \mathcal{SL}_0(d,q) \times \mathcal{SL}_0(d,q) \longrightarrow [0, \infty]$ defined by

$$\rho_{(d,q)}(f,g) = \sup_{d(x,y)\neq 0} \frac{q((f-g)(x) - (f-g)(y))}{d(x,y)}$$

for all $f, g \in \mathcal{SL}_0(d,q)$. Then $\rho_{(d,q)}$ is a quasi-distance on $\mathcal{SL}_0(d,q)$. However $\rho_{(d,q)}$ is not a quasi-metric in general, as Example 2.3 below shows.

Furthermore, it is clear that for each $f, g, h \in \mathcal{SL}_0(d,q)$ and each $r > 0$,

$$\rho_{(d,q)}(f+h, g+h) = \rho_{(d,q)}(f,g)$$

and

$$\rho_{(d,q)}(rf, rg) = r\rho_{(d,q)}(f,g)$$

i.e., $\rho_{(d,q)}$ is an invariant quasi-distance. Moreover, it is easy to check that $\rho_{(d,q)}(f, \mathbf{0}) = 0$ if and only if $f = \mathbf{0}$, where by $\mathbf{0}$ we denote the function that vanishes at every $x \in X$.

Consequently, the nonnegative function $\| \cdot \|_{(d,q)}$ defined on $\mathcal{SL}_0(d,q)$ by

$$\|f\|_{(d,q)} = \rho_{(d,q)}(f, \mathbf{0})$$

is a norm on $\mathcal{SL}_0(d,q)$. Therefore $(\mathcal{SL}_0(d,q), \| \cdot \|_{(d,q)})$ is a normed cone.

Example 2.3. ([103]) Let d be the T_1 quasi-metric defined on $\mathbb{R}$ by $d(x,y) = x-y$ if $x \geq y$ and $d(x,y) = 1$ otherwise. Then T_d is the Sorgenfrey topology on $\mathbb{R}$. Now, let q be the quasi-norm defined on $\mathbb{R}$ by $q(x) = x \vee 0$. Let $x_0 = 0$. If we denote the identity function on $\mathbb{R}$ by id and $\mathbf{0}$ denotes the function on $\mathbb{R}$ that vanishes at every $x \in \mathbb{R}$, then $\rho_{(d,q)}(\mathbf{0}, id) = \infty$.

Note that the above example shows that there exists $f \in \mathcal{SL}_0(d,q)$ such that $\rho_{(d,q)}(\mathbf{0}, f) = 0$ but $f \neq \mathbf{0}$.

This example also provides an instance of a T_1 quasi-metric space (X,d) such that $(\mathcal{SL}_0(d,q),+)$ is not a group for some $x_0 \in X$, which suggests the question of characterizing when $(\mathcal{SL}_0(d,q),+)$ is a group. In order to give an answer to this question note that if x_0 is a point in the quasi-metric space (X,d), then the set

$$\mathcal{SL}_0(d^{-1},q) = \left\{ f \in \ Y^X_{(d^{-1},q)} : \sup_{d(y,x)\neq 0} \frac{q(f(x)-f(y))}{d(y,x)} < \infty \ , f(x_0) = 0 \right\}$$

has also a structure of a cone and $(\mathcal{SL}_0(d^{-1},q), \|\cdot\|_{(d^{-1},q)})$ is a normed cone, where $\|f\|_{(d^{-1},q)} = \rho_{(d^{-1},q)}(f, \mathbf{0})$, i.e.,

$$\|f\|_{(d^{-1},q)} = \sup_{d(y,x)\neq 0} \frac{q(f(x)-f(y))}{d(y,x)}$$

for all $f \in \mathcal{SL}_0(d^{-1},q)$.

Proposition 2.1. *Let (X,d) and (Y,q) be a quasi-metric space and a quasi-normed space, respectively. Then*

$$f \in \mathcal{SL}_0(d,q) \Longleftrightarrow -f \in \mathcal{SL}_0(d^{-1},q).$$

Proof. Let $f \in \mathcal{SL}_0(d,q)$ then there exists $k \in \mathbb{R}^+$ such that

$$q(f(x)-f(y)) \leq \ kd(x,y)$$

for all $x, y \in X$. We change x by y and hence

$$q(f(y) - f(x)) \leq \; kd(y, x)$$

and

$$q(-f(x) - (-f(y))) \leq \; kd^{-1}(x, y)$$

then $-f \in \mathcal{SL}_0(d^{-1}, q)$. The converse is analogous. $\square$

This is the reason why we can say that the biggest set with group structure in $(\mathcal{SL}_0(d, q), +)$ is the intersection set.

Corollary 2.1. *Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normed space, respectively.Then $(\mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q), +, \cdot)$ is a linear space.*

Proof. It follows from Proposition 2.1 that $f \in \mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q)$ if and only if $-f \in \mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q)$. $\square$

Remark 2.1. Note that for each $f \in \mathcal{SL}_0(d, q)$, $\|f\|_{(d,q)} = \|-f\|_{(d^{-1},q)}$. Thus the normed cones $(\mathcal{SL}_0(d, q), \|\cdot\|_{(d,q)})$ and $(\mathcal{SL}_0(d^{-1}, q), \|\cdot\|_{(d^{-1},q)})$ are isometrically isomorphic by the bijective map

$$\phi : \mathcal{SL}_0(d, q) \longrightarrow \mathcal{SL}_0(d^{-1}, q)$$

defined by $\phi(f) = -f$.

Furthermore, we have

$$\mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q) =$$

$$\big\{f \in Y^X_{(d,q)} \cap Y^X_{(d^{-1},q)} : \sup_{d(x,y)\neq 0} \frac{q(f(x)-f(y)) \vee q(f(y)-f(x))}{d(x,y)} < \infty, f(x_0) = 0\big\}.$$

Hence $(\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$ is a normed linear space, where $\|\cdot\|_B$ is the norm defined by

$$\|f\|_B = \sup_{d(x,y)\neq 0} \frac{q(f(x)-f(y)) \vee q(f(y)-f(x))}{d(x,y)},$$

for all $f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$.

Observe that $\|\cdot\|_B = \|\cdot\|_{(d,q)} \vee \|\cdot\|_{(d^{-1},q)}$ on $\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$.

The next result, whose proof is very easy, provides a characterization that will be useful, later on.

Proposition 2.2. *Let (X,d) and (Y,q) be a quasi-metric space and a quasi-normed space, respectively. Then $f \in \mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q)$ if and only if $f(x_0) = 0$ and there exists $k \geq 0$ such that $q^s(f(x)-f(y)) \leq kd(x,y)$.*

Remark 2.2. It is straightforward to see that $f : (X,d) \longrightarrow (Y,q)$ belongs to $Y^X_{(d,q)} \cap Y^X_{(d^{-1},q)}$ if and only if $f(x) = f(y)$ whenever $d(x,y) = 0$.

The next example shows that the intersection set is not "always interesting".

Example 2.4. Let (X,d) and (Y,q) be a quasi-metric and a quasi-normed space, respectively, such that there is $x_0 \in X$ satisfying $d(x,x_0) \wedge d(x_0,x) = 0$ for all $x \in X$. Then $\mathcal{SL}_0(d,q) \cap \mathcal{SL}_0(d^{-1},q) = \{\mathbf{0}\}$.

Now we see another example using a classical quasi-metric:

Example 2.5. Let $X = [0, 1]$ and let d be the quasi-metric on X given by $d(x, y) = y - x$ if $x \leq y$ and $d(x, y) = 1$ otherwise. Clearly T_d is the restriction of the Sorgenfrey topology to $[0, 1]$. Let (Y, q) be a quasi-normed space and put $x_0 = 0$. Then, a function $f : X \longrightarrow Y$ satisfies $f \in \mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q)$ if and only if there is $k \geq 0$ such that

$$q(f(x) - f(y)) \vee q(f(y) - f(x)) \leq k(d(x, y) \wedge d(y, x))$$

for all $x, y \in X$.

The next result characterizes when the semi-Lipschitz function space with the internal composition law defined is a group.

Theorem 2.1. *Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normed space, respectively. Then the following assertions are equivalent:*

(1) $\mathcal{SL}_0(d, q) = \mathcal{SL}_0(d^{-1}, q)$.

(2) $\mathcal{SL}_0(d, q)$ *is a group.*

(3) $\mathcal{SL}_0(d^{-1}, q)$ *is a group.*

(4) $\mathcal{SL}_0(d, q) \subset \mathcal{SL}_0(d^{-1}, q)$.

(5) $\mathcal{SL}_0(d^{-1}, q) \subset \mathcal{SL}_0(d, q)$.

Proof. (1) $\Rightarrow$ (2) By Corollary 2.1 $(\mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q), +, \cdot)$ is a linear space. If $\mathcal{SL}_0(d, q) = \mathcal{SL}_0(d^{-1}, q)$ then $(\mathcal{SL}_0(d, q), +)$ is a group.

(2) $\Rightarrow$ (3) Let $f \in \mathcal{SL}_0(d^{-1}, q)$. By Proposition 2.1 $-f \in \mathcal{SL}_0(d, q)$, since $\mathcal{SL}_0(d, q)$ is a group, $f \in \mathcal{SL}_0(d, q)$, by Proposition 2.1 $-f \in \mathcal{SL}_0(d^{-1}, q)$.

(3) $\Rightarrow$ (4) The proof is similar to the proof of (2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (5) Let $f \in \mathcal{SL}_0(d^{-1}, q)$. Then $-f \in \mathcal{SL}_0(d, q) \subset \mathcal{SL}_0(d^{-1}, q)$ hence $-f \in \mathcal{SL}_0(d^{-1}, q)$.Thus $f \in \mathcal{SL}_0(d, q)$.

(5) $\Rightarrow$ (1) Is the same that (4) $\Rightarrow$ (5). □

Let see that all this does not depend on the x_0 chosen:

Proposition 2.3. *Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normed space, respectively. If there exists $x_0 \in X$ such that $\mathcal{SL}_0(d, q) = \mathcal{SL}_0(d^{-1}, q)$, then $\mathcal{SL}_1(d, q) = \mathcal{SL}_1(d^{-1}, q)$ for each $x_1 \in X$.*

Proof. Let $f \in \mathcal{SL}_1(d, q)$. Define a function g on X by $g(x) = f(x) - f(x_0)$ for all $x \in X$. It easy to check that $g \in \mathcal{SL}_0(d, q)$. Thus, $g \in \mathcal{SL}_0(d^{-1}, q)$. Since $g(x) - g(y) = f(x) - f(y)$ for all $x, y \in X$ we obtain that $f \in \mathcal{SL}_1(d^{-1}, q)$. □

2.3 Completeness Properties

In this section, we discuss the completeness properties of the semi-Lipschitz function space.

Theorem 2.2. *Let (X, d) and (Y, q) be a quasi-metric space and a biBanach space, respectively. Consider the following conditions:*

(1) (X, d) *is a metric space.*

(2) $\mathcal{SL}_0(d, q) = \mathcal{SL}_0(d^{-1}, q)$ *and* $\|\cdot\|_{(d,q)} = \|\cdot\|_{(d^{-1},q)}$.

(3) $(\mathcal{SL}_0(d, q), \|\cdot\|_{(d,q)})$ *is a Banach space.*

Then: $(1) \Rightarrow (2)$, *and* $(2) \Leftrightarrow (3)$.

Proof. $(1) \Rightarrow (2)$ Obvious.

$(2) \Rightarrow (3)$ From (2) $\mathcal{SL}_0(d, q) = \mathcal{SL}_0(d^{-1}, q)$, thus $\mathcal{SL}_0(d, q) \cap \mathcal{SL}_0(d^{-1}, q) = \mathcal{SL}_0(d, q)$ is a linear space, by other hand $\|\cdot\|_{(d,q)} = \|\cdot\|_{(d^{-1},q)}$ therefore $\|\cdot\|_{(d,q)} = \|\cdot\|_B$ from Theorem 2.3 $(\mathcal{SL}_0(d, q), \|\cdot\|_{(d,q)})$ is a Banach space.

$(3) \Rightarrow (2)$ Suppose that $(\mathcal{SL}_0(d, q), \|\cdot\|_{(d,q)})$ is a Banach space. Then $\mathcal{SL}_0(d, q)$ is a group, so $\mathcal{SL}_0(d, q) = \mathcal{SL}_0(d^{-1}, q)$. Moreover $\|\cdot\|_{(d,q)}$ is a norm on $\mathcal{SL}_0(d, q)$, so that $\|f\|_{(d,q)} = \|-f\|_{(d,q)}$ for all $f \in \mathcal{SL}_0(d, q)$. Since $-f \in \mathcal{SL}_0(d, q)$ it follows that $\|-f\|_{(d,q)} = \|f\|_{(d^{-1},q)}$. We conclude that $\|\cdot\|_{(d,q)} = \|\cdot\|_{(d^{-1},q)}$ on $\mathcal{SL}_0(d, q)$. $\square$

To see that in general $(3) \Rightarrow (1)$ is not true we take $Y = \mathbf{0}$ for any quasi-metric space (X, d) that is not a metric space.

The following result allows us to prove that if (Y, q) is a biBanach space then $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$ is a bicomplete space:

Theorem 2.3. *Let (X, d) and (Y, q) be a quasi-metric space and a biBanach space, respectively. Then $\rho_{(d,q)}$ is a bicomplete quasi-distance on $\mathcal{SL}_0(d,q)$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$. Then, given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\sup_{d(x,y)\neq 0} \frac{q((f_n - f_m)(x) - (f_n - f_m)(y))}{d(x,y)} < \varepsilon \qquad (*)$$

for all $n, m \geq n_0$.

If $x = x_0$ then $f_n(x) = 0$ for all $n \in \mathbb{N}$.

Let $x \neq x_0$.We consider the following cases.

Case 1. $d(x, x_0) \neq 0$. Then, we deduce from $(*)$ that given $\varepsilon > 0$ there exists $n_0' \in \mathbb{N}$ such that if $n, m \geq n_0'$ then $q(f_n(x) - f_m(x)) < \varepsilon$ and if we change n and m, then $q(f_m(x) - f_n(x)) < \varepsilon$. Therefore, $\{f_n(x)\}$ is a Cauchy sequence in (Y, q^s).

Case 2. $d(x, x_0) = 0$. Then $d(x_0, x) \neq 0$ so $q(f_m(x) - f_n(x)) < \varepsilon$ and $q(f_n(x) - f_m(x)) < \varepsilon$.

Consequently, $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in (Y, q^s), thus $\{f_n(x)\}_{n\in\mathbb{N}}$ converges in (Y, q^s) and we define f as the function pointwise limit of $\{f_n(x)\}_{n\in\mathbb{N}}$ in (Y, q) for all $x \in X$. We shall prove that $\{f_n\}_{n\in\mathbb{N}}$ converges to f in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$.

Indeed, given $\varepsilon > 0$, since $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ in (Y, q), for all

$x \in X$, then for each x, y there exists n' such that if $m' \geq n'$ then

$$\frac{q^s(f(x) - f_{m'}(x) - (f(y) - f_{m'}(y)))}{d(x,y)} < \frac{\varepsilon}{2}$$

and since $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, we can also find n_0 such that if $m', n \geq n_0$ then

$$\frac{q^s(f_{m'}(x) - f_n(x) - (f_{m'}(y) - f_n(y)))}{d(x,y)} < \frac{\varepsilon}{2}$$

for all $x, y \in X$.

Thus we have

$$\frac{\varepsilon}{2} > \frac{q^s(f(x) - f_{m'}(x) - (f(y) - f_{m'}(y)))}{d(x,y)}$$

$$\geq \frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} - \frac{q^s(f'_m(x) - f_n(x) - (f'_m(y) - f_n(y)))}{d(x,y)}$$

and hence

$$\frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since n_0 is independent from x and y, we obtain

$$\sup_{d(x,y)\neq 0} \frac{q^s(f(x) - f_n(x) - (f(y) - f_n(y)))}{d(x,y)} < \varepsilon,$$

for all $n \geq n_0$. Consequently $\rho_{(d,q)}$ is a bicomplete quasi-distance on $(\mathcal{SL}_0(d, q)$.

$\square$

Corollary 2.2. *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a Cauchy sequence in* $(\mathcal{SL}_0(d,q), \|\cdot\|_{(d,q)})$ *then there exists a convergent sequence* $\{k_n\}_{n\in\mathbb{N}}$ *in* $(\mathbb{R}, T_E)$ *such that* k_n *is a semi-Lipschitz constant for* f_n.

The following result allows us to prove that if (Y,q) is a biBanach space then $(\mathcal{SL}_0(d,q)\cap\mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$ is a Banach space.

Corollary 2.3. *Let* (X,d) *and* (Y,q) *be a quasi-metric space and a biBanach space, respectively. Then* $(\mathcal{SL}_0(d,q)\cap\mathcal{SL}_0(d^{-1},q), \|\cdot\|_B)$ *is a Banach space.*

Proof. Given $\{f_n\}_{n\in\mathbb{N}} \in \mathcal{SL}_0(d,q)\cap\mathcal{SL}_0(d^{-1},q)$ a convergent sequence, let f be the limit of $\{f_n\}_{n\in\mathbb{N}}$ respect to $\|\cdot\|_B$. Given $\varepsilon > 0$ there exists n_0 such that if $n \geq n_0$:

$$\sup_{d(x,y)\neq 0} \frac{q^s(f(x)-f_n(x)-(f(y)-f_n(y)))}{d(x,y)} < \varepsilon,$$

then let see that $f \in \mathcal{SL}_0(d,q)\cap\mathcal{SL}_0(d^{-1},q)$:

$$\sup_{d(x,y)\neq 0} \frac{q^s(f(x)-f(y))}{d(x,y)}$$

$$\leq \sup_{d(x,y)\neq 0} \frac{q^s(f(x)-f_n(x)-(f(y)-f_n(y)))}{d(x,y)} + \sup_{d(x,y)\neq 0} \frac{q^s(f_n(x)-f_n(y))}{d(x,y)}$$

$$< \varepsilon + \|f\|_B < \infty.$$

Thus $(\mathcal{SL}_0(d,q)\cap\mathcal{SL}_0(d^{-1},q), \|\cdot\|)$ is a closed subset of $(\mathcal{SL}_0(d,q), \rho^s_{(d,q)})$, by Theorem 2.3 is complete. □

2.4 Other Completeness Properties

In this section, we discuss other completeness properties of the semi-Lipschitz function space.

Let us recall that right K-completeness and left K-completeness constitute very useful extensions of the notion of completeness to the nonsymmetric context.

In fact, they have been successfully applied to different fields like hyperspaces and function spaces, topological algebra and theoretical computer science.

Definition 2.3. Let (X, d) be a quasi-pseudo-metric space. A net $\{x_\delta\}_{\delta\in\Lambda}$ in X, is called *left K-Cauchy* provided that for each $\varepsilon > 0$ there is δ_0 such that $d(x_{\delta_1}, x_{\delta_2}) < \varepsilon$ for all $\delta_2 \geq \delta_1 \geq \delta_0$, and $\{x_\delta\}_{\delta\in\Lambda}$ in X is called *right K-Cauchy* provided that for each $\varepsilon > 0$ there exists δ_0 such that $d(x_{\delta_2}, x_{\delta_1}) < \varepsilon$ for all $\delta_2 \geq \delta_1 \geq \delta_0$.

Definition 2.4. A quasi-pseudo-metric space (X, d) is called *left K-complete (resp. right K-complete) space* if each left K-Cauchy net (resp. right K-Cauchy net) is T_d-convergent. (Compare with [65, Section 2]).

Remark 2.3. ([5, Remark 2]) It is clear that each right K-complete quasi-metric space is right K-sequentially complete. However, the converse does not hold even for quasi-metric spaces. In fact, in [131, Example 2.4], Stoltenberg presented an example of a right K-sequentially complete quasi-metric space that is not right K-complete. The space of this example fails to be regular.

The following theorem allows us to prove that if (Y, q) is a biBanach finite dimensional space then $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$ is right K-complete. We will need an auxiliary lemma.

Lemma 2.1. *Let (X, d) a quasi-metric space. If $\{x_\delta\}_{\delta\in\Lambda}$ is a right k-Cauchy net with a cluster point x in (X, d^s), then $\{x_\delta\}_{\delta\in\Lambda}$ converges to x in (X, d^s).*

Theorem 2.4. *Let (X, d) and (Y, q) be a quasi-metric space and a biBanach finite dimensional space, respectively. Then $\rho_{(d,q)}$ is right K-complete on $\mathcal{SL}_0(d,q)$.*

Proof. Let $\{f_\delta\}_{\delta\in\Lambda}$ be a right K-Cauchy net in $(\mathcal{SL}_0(d,q), \rho_{(d,q)})$. Then, given $\varepsilon > 0$ there is $\delta_0 \in \Lambda$ such that

$$\sup_{d(x,y)\neq 0} \frac{q((f_{\delta_2} - f_{\delta_1})(x) - (f_{\delta_2} - f_{\delta_1})(y))}{d(x,y)} < \varepsilon \qquad (**)$$

for all $\delta_2 \geq \delta_1 \geq \delta_0$.

Let $x \neq x_0$. We consider the following cases.

Case 1. $d(x, x_0) \neq 0$ and $d(x_0, x) \neq 0$. Then, we deduce from $(**)$ that given $\varepsilon > 0$ there exists δ_0' such that if $\delta_2 \geq \delta_1 \geq \delta_0'$ then $q^s(f_{\delta_2}(x) - f_{\delta_1}(x)) < \varepsilon$. Therefore$\{f_\delta(x)\}_{\delta\in\Lambda}$ is a Cauchy net in (Y, q^s).

Case 2. $d(x_0, x) = 0$ and $d(x, x_0) \neq 0$. Then

$$q(f_{\delta_2}(x)) - q(f_{\delta_1}(x)) \leq q(f_{\delta_2}(x) - f_{\delta_1}(x)) < \varepsilon d(x, x_0)$$

for all $\delta_2 \geq \delta_1 \geq \delta_0$. Since $q(-f_\delta(x)) = 0$ for all $\delta \in \Lambda$, it follows that $q^s(f_{\delta_2}(x)) < \varepsilon d(x, x_0) + q(f_{\delta_0}(x))$ for all $\delta_2 \geq \delta_0$. Thus $\{f_\delta(x)\}_{\delta \geq \delta_0}$ is a bounded net in the finite dimensional space (Y, q^s), so it has a cluster point in (Y, q^s). Hence, by Lemma 2.1, $\{f_\delta(x)\}_{\delta \in \Lambda}$ converges in (Y, q^s).

Case 3. $d(x_0, x) \neq 0$ and $d(x, x_0) = 0$. A slight modification of Case 2 shows that $\{f_\delta(x)\}_{\delta \in \Lambda}$ converges in (Y, q^s).

We define f such that $\{f_\delta(x)\}_{\delta \in \Lambda}$ converges to $f(x)$ for each $x \in X$. We first note that f is $\leq_{(d,q)}$-increasing.

Next we see that the net

$$\Big\{\frac{q(f_\delta(x) - f_\delta(y))}{d(x, y)}\Big\}_{\delta \in \Lambda}$$

converges to

$$\frac{q(f(x) - f(y))}{d(x, y)}$$

for all $x, y \in X$ such that $d(x, y) \neq 0$.

For $\varepsilon > 0$, for $\delta_1 \geq \delta_0$ and $x, y \in X$ such that $d(x, y) \neq 0$ we take $\delta_2 \in \Lambda$, with $\delta_2 \geq \delta_1$ such that

$$\frac{q^s(f(x) - f_{\delta_2}(x) - (f(y) - f_{\delta_2}(y)))}{d(x, y)} < \frac{\varepsilon}{2}.$$

so that

$$\frac{q(f(x) - f_{\delta_1}(x) - (f(y) - f_{\delta_1}(y)))}{d(x, y)}$$

$$= \frac{q(f(x) - f_{\delta_1}(x) - f_{\delta_2}(x) + f_{\delta_2}(x) - (f(y) - f_{\delta_1}(y) - f_{\delta_2}(y) + f_{\delta_2}(y)))}{d(x, y)}$$

$$\leq \frac{q(f(x) - f_{\delta_2}(x) - (f(y) - f_{\delta_2}(y)))}{d(x,y)}$$
$$+\frac{q(f_{\delta_2}(x) - f_{\delta_1}(x) - (f_{\delta_2}(y) - f_{\delta_1}(y)))}{d(x,y)} < \varepsilon.$$

We conclude that

$$\sup_{d(x,y)\neq 0} \frac{q(f(x) - f_{\delta'}(x) - (f(y) - f_{\delta'}(y)))}{d(x,y)} < \varepsilon.$$

The proof is finished. □

Corollary 2.4. *Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normed space respectively. Let $\{f_\delta\}_{\delta\in\Lambda}$ be a right K-Cauchy net in $(\mathcal{SL}_0(d, q), \rho_{(d,q)})$. If for each $x \in X$ $\{f_\delta(x)\}_{\delta\in\Lambda}$ converges to $f(x)$ in (Y, q^s) then $\{f_\delta\}_{\delta\in\Lambda}$ converges to f in $(\mathcal{SL}_0(d, q), \rho_{(d,q)})$.*

Theorem 2.5. *Let (X, d) and (Y, q) be a T_1 quasi-metric space and a biBanach space respectively. Then $\rho_{(d,q)}$ is right K-complete.*

Proof. Let $\{f_\delta\}_{\delta\in\Lambda}$ be a right K-Cauchy net in $(\mathcal{SL}_0(d, q), \rho_{(d,q)})$. Then, given $\varepsilon \geq 0$ there is δ_0 such that

$$\sup_{d(x,y)\neq 0} \frac{q((f_{\delta_2} - f_{\delta_1})(x) - (f_{\delta_2} - f_{\delta_1})(y)))}{d(x,y)} < \varepsilon$$

for all $\delta_2 \geq \delta_1 \geq \delta_0$.

Let $x \neq x_0$.

Since (X,d) is T_1, $d(x,x_0)\neq 0$ and $d(x_0,x)\neq 0$ then, we deduce from $(**)$ that given $\frac{\varepsilon}{d(x,x_0)}$ and $\frac{\varepsilon}{d(x_0,x)}$ there exists δ_0' such that if $\delta_2\geq\delta_1\geq\delta_0'$ then $q^s(f_{\delta_2}(x)-f_{\delta_1}(x))<\varepsilon$. Therefore $\{f_\delta(x)\}_{\delta\in\Lambda}$ is a Cauchy net in (Y,q) for all $x\in X$. Thus $\{f_\delta(x)\}_{\delta\in\Lambda}$ is a convergent net in (Y,q^s) and we define f such that $\{f_\delta(x)\}_{\delta\in\Lambda}$ converges to $f(x)$ for each $x\in X$. Let $\{f_\delta\}_{\delta\in\Lambda}$ a right K-Cauchy net in $(\mathcal{SL}_0(d,q),\rho_{(d,q)})$.

Let us see that the net

$$\{\frac{q(f_\delta(x)-f_\delta(y))}{d(x,y)}\}_{\delta\in\Lambda}$$

converges to

$$\frac{q(f(x)-f(y))}{d(x,y)}$$

for all $x,y\in X$ such that $d(x,y)\neq 0$.

Since $\{f_\delta\}_{\delta\in\Lambda}$ is a right K-Cauchy net, given $\varepsilon>0$ there exists δ_0 such that if $\delta_2\geq\delta_1\geq\delta_0$ then

$$\sup_{d(x,y)\neq 0}\frac{q((f_{\delta_2}-f_{\delta_1})(x)-(f_{\delta_2}-f_{\delta_1})(y))}{d(x,y)}<\frac{\varepsilon}{2}.$$

Since $\{f_\delta(x)\}_{\delta\in\Lambda}$ converges to $f(x)$ for all $x\in X$, then for each $x,y\in X$ there exists δ_0' such that if $\delta_1'\geq\delta_0'$ then

$$\frac{q^s(f(x)-f_{\delta_1'}(x)-(f(y)-f_{\delta_1'}(y)))}{d(x,y)}<\frac{\varepsilon}{2}.$$

Thus given $\varepsilon>0$, for all $\delta'\geq\delta_0$ and for each $x,y\in X$ such that $d(x,y)\neq 0$ and we take $\delta_1\geq\delta'$ and $\delta_1\geq\delta_0'$ then

$$\frac{q(f(x)-f_{\delta'}(x)-(f(y)-f_{\delta'}(y)))}{d(x,y)}$$

$$= \frac{q(f(x) - f_{\delta'}(x) - f_{\delta_1}(x) + f_{\delta_1}(x) - (f(y) - f_{\delta'}(y) - f_{\delta_1}(y) + f_{\delta_1}(y)))}{d(x,y)}$$

$$\leq \frac{q(f(x) - f_{\delta_1}(x) - (f(y) + f_{\delta_1}(y)))}{d(x,y)} + \frac{q(f_{\delta_1}(x) - f_{\delta'}(x) - (f_{\delta'}(y) + f_{\delta_1}(y)))}{d(x,y)} < \varepsilon$$

for all $x, y \in X$ such that $d(x,y) \neq 0$

$$\sup_{d(x,y)\neq 0} \frac{q(f(x) - f_{\delta'}(x) - (f(y) - f_{\delta'}(y)))}{d(x,y)} < \varepsilon,$$

for all $\delta'_n \geq \delta_0$. □

2.5 On Balancedness and D-Completeness of the Space of Semi-Lipschitz Functions

In this section we suppose that all quasi-metrics are T_1 quasi-metrics.

In [26] Doitchinov introduced an important property of symmetry in quasi-metric spaces, namely *balancedness,* in order to develop a satisfactory theory of completion. He observed that paradigmatic examples of quasi-metric spaces, as the Sorgenfrey line, the Kofner plane and the Pixley-Roy spaces are balanced, and proved that every balanced quasi-metric generates a Hausdorff and completely regular topology.

Recall that a(n extended) quasi-metric space (X, d) is *balanced* if for each pair of sequences $\{y_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\lim_{n,m\to\infty} d(y_m, x_n) = 0$, and each $x, y \in X$ and $r_1, r_2 \in \mathbb{R}^+$ satisfying $d(x, x_n) \leq r_1$ and $d(y_n, y) \leq r_2$ for all $n \in \mathbb{N}$, it follows that $d(x, y) \leq r_1 + r_2$. In this case, d is called a *balanced* quasi-metric.

We say that the normed cone $(SL_0(d, q), \|\cdot\|_{d,q})$ is a *balanced normed cone* if the extended quasi-metric $\rho_{d,q}$ is balanced on $SL_0(d, q)$.

According to [26], by a *Cauchy* sequence in the sense of Doitchinov in a quasi-metric space (X, d) we mean a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X satisfying $\lim_{n,m\to\infty} d(y_m, x_n) = 0$. The quasi-metric space (X, d) is said to be *complete* in the sense of Doitchinov if every Cauchy sequence is convergent with respect to $T(d)$.

Later on, Doitchinov proved that each balanced quasi-metric space (X, d) is isometrically isomorphic to a $T(d)$ and $T(d^{-1})$-dense subspace of a balanced complete quasi-metric space.

Following the modern terminology of [62], Cauchy sequences in the sense of Doitchinov will be called, in the sequel, *D-Cauchy* sequences and complete quasi-metric spaces will be called *D-(sequentially) complete* quasi-metric spaces.

Theorem 2.6. *Let (X, d) be a quasi-metric space, (Y, q) be a quasi-normed linear space and $x_0 \in X$. Then $(\mathcal{SL}_0(d, q), \|\cdot\|_{d,q})$ is a balanced normed cone.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$, $\{g_n\}_{n\in\mathbb{N}}$ be sequences in $\mathcal{SL}_0(d, q)$ with $\lim_{n,m\to\infty} \rho_{d,q}(g_m, f_n) = 0$, and let $f, g \in \mathcal{SL}_0(d, q)$ and $r_1, r_2 \in \mathbb{R}^+$ such that $\rho_{d,q}(f, f_n) \leq r_1$ and $\rho_{d,q}(g_n, g) \leq r_2$ for all $n \in \mathbb{N}$. Choose $x, y \in X$ with $x \neq y$.

Then
$$q((f - f_n)(x) - (f - f_n)(y)) \leq r_1 d(x, y),$$

and
$$q((g_n - g)(x) - (g_n - g)(y)) \leq r_2 d(x, y),$$

for all $n \in \mathbb{N}$. Moreover, for an arbitrary $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that
$$q((g_n - f_n)(y) - (g_n - f_n)(x)) < \varepsilon d(y, x),$$

for all $n \geq n_0$. Consequently
$$\begin{aligned} &q((f - g)(x) - (f - g)(y)) \\ \leq\ &q((f - f_{n_0})(x) - (f - f_{n_0})(y)) + q((f_{n_0} - g_{n_0})(x) - (f_{n_0} - g_{n_0})(y)) \\ &+q((g_{n_0} - g)(x) - (g_{n_0} - g)(y)) \\ <\ &r_1 d(x, y) + \varepsilon d(y, x) + r_2 d(x, y). \end{aligned}$$

Since ε is arbitrary, it follows that

$$q((f-g)(x)-(f-g)(y)) \leq r_1 d(x,y) + r_2 d(x,y).$$

Therefore

$$\rho_{d,q}(f,g) \leq r_1 + r_2.$$

We conclude that $(\mathcal{SL}_0(d,q), \|\cdot\|_{d,q})$ is balanced. □

Corollary 2.5. *Let (X,d) be a quasi-metric space, (Y,q) be a quasi-normed linear space and $x_0 \in X$. Then $(\mathcal{SL}_0(d,q), T(\rho_{d,q}))$ is a Hausdorff and completely regular topological space.*

Theorem 2.7. *Let (X,d) be a quasi-metric space, (Y,q) be a biBanach space and $x_0 \in X$. Then $(\mathcal{SL}_0(d,q), \|\cdot\|_{d,q})$ is D-complete.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a D-Cauchy sequence in $\mathcal{SL}_0(d,q)$. Then, there is a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{SL}_0(d,q)$ such that

$$\lim_{n,m\to\infty} \rho_{d,q}(g_m, f_n) = 0.$$

Thus, given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $\rho_{d,q}(g_m, f_n) < \varepsilon$ for all $n,m \geq n_0$.

Now fix $x \in X$. Then

$$q((g_m - f_n)(x)) < \varepsilon d(x,x_0) \quad \text{and} \quad q((f_n - g_m)(x)) < \varepsilon d(x_0,x),$$

so,

$$q^s((g_m - f_n)(x)) < \varepsilon d^s(x,x_0) \quad \text{for all} \quad n,m \geq n_0. \tag{$*$}$$

Therefore, for each $n, m \geq n_0$,

$$q^s((f_n - f_m)(x)) \leq q^s((f_n - g_{n_0})(x)) + q^s((g_{n_0} - f_m)(x)) < 2\varepsilon d^s(x, x_0),$$

and, since (Y, q) is a biBanach space, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is convergent in (Y, q^s). Then, we can construct a function $f : X \to Y$ such that $\{f_n\}_{n\in\mathbb{N}}$ is pointwise convergent to f with respect to the norm q^s. Observe that, by condition $(*)$, the sequence $\{g_n\}_{n\in\mathbb{N}}$ is also pointwise convergent to f with respect to q^s.

We shall prove that $f \in \mathcal{SL}_0(d)$ and that $\lim_{n\to\infty} \rho_{d,q}(f, f_n) = 0$. Indeed, we first note that $f(x_0) = \mathbf{0}$ because $f_n(x_0) = \mathbf{0}$ for all $n \in \mathbb{N}$. Now, for the given $\varepsilon > 0$, for $n \geq n_0$ and for $x, y \in X$ with $x \neq y$, there exists $m \geq n$ such that

$$q^s((f - g_m)(x)) < \varepsilon d(x, y) \quad \text{and} \quad q^s((f - g_m)(y)) < \varepsilon d(x, y).$$

Hence

$$\frac{q((f - f_n)(x) - (f - f_n)(y))}{d(x, y)}$$

$$\leq \frac{q((f - g_m)(x) - (f - g_m)(y))}{d(x, y)} + \frac{q((g_m - f_n)(x)) - (g_m - f_n)(y))}{d(x, y)}$$

$$< \frac{q^s((f - g_m)(x)) + q^s((f - g_m)(y))}{d(x, y)} + \varepsilon < 3\varepsilon.$$

It then follows that

$$\sup_{x\neq y} \frac{q(f(x) - f(y))}{d(x, y)} \leq 3\varepsilon + \sup_{x\neq y} \frac{q(f_{n_0}(x) - f_{n_0}(y))}{d(x, y)}.$$

Thus, we have shown that $f \in \mathcal{SL}_0(d, q)$ and $\rho_{d,q}(f, f_n) \leq 3\varepsilon$ for all $n \geq n_0$. Consequently $(\mathcal{SL}_0(d, q), \|\cdot\|_{d,q})$ is D-complete. $\square$

As an application of the previous results we next show that if (X, p) is a T_1 quasi-normed linear space (i.e. the quasi-pseudo-metric d_p induced by the quasi-norm p is actually a quasi-metric), then the dual space (X^*, p^*) of (X, p) is balanced and D-complete in the natural sense that we explain in the following.

Let us recall ([2, 39]) that if (X, p) is a quasi-normed linear space then the so-called *algebraic dual* of (X, p) is the cone X^* consisting of all linear real-valued functions on X that are upper semicontinuous on $(X, T((d_p)^{-1}))$. Equivalently, X^* consists of all linear real-valued functions on X that are lower semicontinuous on $(X, T(d_p))$ ([103, p. 58]). It immediately follows ([83, 103]) that $X^* = L(X) \cap \mathcal{SL}_0(d_p, u^{-1})$, where $L(X)$ denotes the space of all linear real-valued functions on X and $\mathcal{SL}_0(d_p, u^{-1})$ denotes the space of all semi-Lipschitz functions from (X, d_p) to the quasi-normed linear space $(\mathbb{R}, u^{-1})$ that vanish at $\mathbf{0}$. Note that in this case we have

$$\rho_{d_p,u^{-1}}(f, g) = \sup_{d(x,y)\neq 0} \frac{((f-g)(x) - (f-g)(y)) \vee 0}{p(x-y)},$$

for all $f, g \in \mathcal{SL}_0(d_p, u^{-1})$.

Let us also recall that p^* is the function from X^* to $\mathbb{R}^+$ defined by $p^*(f) = \sup\{f(x) : p(x) \leq 1\}$ for all $f \in X^*$ ([2, 39]), and thus (X^*, p^*) is a normed cone which is said to be the *dual space* of (X, p).

It is routine to see that X^* is a closed subspace of the metrizable space $(\mathcal{SL}_0(d_p, u^{-1}), (\rho_{d_p,u^{-1}})^s)$.

From the preceding facts and Theorems 2.6 and 2.7 we immediately deduce the following.

Theorem 2.8. *Let (X, p) be a T_1 quasi-normed linear space. Then (X^*, d_{p^*}) is a balanced D-complete extended quasi-metric space.*

In the light of Theorem 2.8 it seems interesting to recall that there exist T_1 (actually Hausdorff) quasi-normed nonnormable linear spaces in abundance. See for instance:

Example 2.6. ([37, Example 1]) Consider the linear lattice $(\mathrm{E}_0, \leq)$ defined by all the sequences of real numbers that are different from zero only in a finite set of indexes endowed with its natural order, and the function $q_0 : \mathrm{E}_0 \longrightarrow \mathbb{R}^+$ defined by:

$$q_0(x) = \|x \vee 0\|_1 + \|x \wedge 0\|_2.$$

We prove that this function is in fact an asymmetric norm. Since $x = x \vee 0 + x \wedge 0$ for all $x \in E_0$, we have that $q(x) = 0$ if and only if $x = 0$. Furthermore, it is obviously positively homogeneous. We just need to show that it satisfies the triangle inequality.

First note that for a pair of elements $x, y \in E_0$, $(x+y) \vee 0 \leq x \vee 0 + y \vee 0$. For each $1 \leq p < \infty$, the norm properties related to the order operations of the normed lattices $(E_0, \|\cdot\|_p, \leq)$ (see Chapter I, Vol.II, in [70]) leads to the inequality

$$\|(x+y) \vee 0\|_p \leq \|x \vee 0\|_p + \|y \vee 0\|_p.$$

The following equalities are also satisfied for every $x \in E_0$ and for each $1 \leq p < \infty$ (Chapter I, Vol. II, in [70]),

$$\|x \wedge 0\|_p = \| - (x \wedge 0\|_p = \| - (-((-x) \vee 0))\|_p = \| - x \vee 0\|_p.$$

Then

$$q_0(x+y) = \|(x+y) \vee 0\|_1 + \|(x+y) \wedge 0\|_2$$
$$= \|(x+y) \vee 0\|_1 + \|(-x-y) \vee 0\|_2$$
$$\leq \|x \vee 0\|_1 + \|y \vee 0\|_1 + \| - x \vee 0\|_2 + \| - y \vee 0\|_2 = q_0(x) + q_0(y).$$

Clearly (E_0, q_0) is and asymmetric normed linear space.

Note that (E_0, q_0) is a Hausdorff space since for every $x \in E_0$, the norm w, given by $w(x) := \|x \vee 0\|_2 + \|x \wedge 0\|_2$ is equivalent to $\|\cdot\|_2$, and $w(x) \leq q_0(x)$. This means that the open balls defined by q_0 are contained in the open balls defined by the norm w on E_0, and then (E_0, q_0) is a Hausdorff space. The proof of the fact that (E_0, q_0) is not isomorphic to any normed space is a consequence of Theorem 7, [37]. Straightforward calculations show that in this case

$$\begin{aligned} \|x\|_{q_0} &= \inf_{x_1 \in E_0} \{q_0(x_1 - x)\} \\ &= \|x \vee 0\|_2 + \|x \wedge 0\|_2, \end{aligned}$$

$\|\cdot\|_{q_0}$ is equivalent to $\|\cdot\|_2$, and $q_0^s(x)$ is exactly $\|\cdot\|_1$. The condition for (E_0, q_0) to be isomorphic to a normed space given in Theorem 7 of [37] would imply that q^s and $\|\cdot\|_q$ are equivalent. But this is not true, since $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent in E_0. Note that the construction provides more examples of the same situation just by replacing the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ by $\|\cdot\|_r$ and $\|\cdot\|_s$ respectively for any $1 \leq r < \infty$ and $1 \leq s < \infty$, $r \neq s$.

Chapter 3

Norms on Semi-Lipschitz Functions: An Approach to Computing Complexity by Partial Functions

3.1 Introduction

As we indicated in Chapter 2 the notion of a semi-Lipschitz function was introduced and discussed in [102]. In particular, it was proved that the set of semi-Lipschitz functions, defined on a quasi-metric space (X, d), that vanish at a point $x_0 \in X$ can be structured as a normed cone. From then, semi-Lipschitz functions have been successfully applied to some questions in asymmetric functional analysis [81, 82, 103], concentration of measure [130], global attractors on dynamical systems, and theoretical computer science

[101].

In this chapter we show that semi-Lipschitz functions also provide an efficient tool to compute the complexity of certain algorithms in the following sense: If T is the recurrence equation on $\mathbb{N}$ associated to a given algorithm (with $T(n) > 0$ for all $n \in \mathbb{N}$) and denote by f the complexity function which is the solution of this recurrence equation, then the complexity of this algorithm is represented via f. Moreover, f constitutes a total mapping defined recursively that is, at the same time, the limit of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of partial mappings also defined recursively.

Here, we present a model for computing the complexity represented by f by means of the values that take a certain relativized norm and its induced quasi-metric on the sequence $\{f_n\}_{n\in\mathbb{N}}$ and its initial segments. This is done with the help of a suitable space of semi-Lipschitz functions which is constructed here; in particular, the induced quasi-metric will permit us to easily measure progress made in lowering complexity when the sequence $\{f_n\}_{n\in\mathbb{N}}$ is replaced by any initial segment of it.

3.2 Computing Complexity by Semi-Lipschitz Functions and Partial Functions

Let (X, d) and (Y, d') be two quasi-metric spaces and let $f : (X, d) \to (Y, d')$; then f is said to be a semi-Lipschitz function ([102]) if there exists $k \geq 0$ such that $d'(f(x), f(y)) \leq kd(x, y)$ for all $x, y \in X$. The number k is called a semi-Lipschitz constant for f.

A function $f : (X, d) \to (Y, d')$ between two quasi-metrics spaces is called $\leq_{(d,d')}$-increasing (compare ([103, 102])) if $d'(f(x), f(y)) = 0$ whenever $d(x, y) = 0$. By $Y^X_{(d,d')}$ we shall denote the set of all $\leq_{(d,d')}$-increasing functions from (X, d) to (Y, d').

It is clear that if T_d is a T_1 topology, then every function from X to Y is $\leq_{(d,d')}$ -increasing.

If we denote by $\mathcal{SL}(d, d')$ the set of all semi-Lipschitz functions from the quasi-metric space (X, d) to the quasi-metric space (X, d') then it is clear that

$$\mathcal{SL}(d, d') = \{f \in Y^X_{(d,d')} : \sup_{d(x,y)\neq 0} \frac{d'(f(x), f(y))}{d(x, y)} < \infty\}.$$

In the following, we shall apply the structure of the space $\mathcal{SL}(d, d')$ of semi-Lipschitz functions to the case that (Y, d') is the complexity quasi-metric space as defined above. To this end, we shall consider the quasi-metric space (ω, d_α) where $\alpha > 0$, d_α is the quasi-metric on ω, given by $d_\alpha(n, m) = 0$ if $n \geq m$, and $d_\alpha(n, m) = \alpha$ if $n < m$.

Thus

$$\mathcal{SL}(d_\alpha, d_{\mathcal{C}}) = \{F \in \mathcal{C}^\omega_{(d_\alpha,d_{\mathcal{C}})} : \frac{1}{\alpha} \sup_{n<m} d_{\mathcal{C}}(F(n), F(m)) < \infty\}.$$

Next we observe that $\mathcal{SL}(d_\alpha, d_\mathcal{C})$ can be structured as a relativized normed cone, where by a relativized normed cone we mean a pair $(C, \|.\|)$ such that $C(= (C, +, \cdot))$ is a cone and $\|.\|$ is a nonnegative real-valued function on C satisfying the following conditions for all $x, y \in C$ and $r > 0$:

(i) $\|r \cdot x\| = \|x\| / r,$

(ii) $\|x + y\| \leq \|x\| + \|y\| .$

In this case, we say that $\|.\|$ is a relativized norm on C.

Indeed, given $F, G \in \mathcal{SL}(d_\alpha, d_\mathcal{C})$ and $r > 0$, define $(F + G)(n) = F(n) + G(n)$ for all $n \in \omega$, and $(r \cdot F)(n) = r \cdot F(n)$ for all $n \in \omega$, where $F(n) + G(n)$ and $r \cdot F(n)$ are pointwise defined.

We also define $\| \cdot \|_{(d_\alpha, d_\mathcal{C})} : \mathcal{SL}(d_\alpha, d_\mathcal{C}) \to \mathbb{R}^+$ by

$$\|F\|_{(d_\alpha, d_\mathcal{C})} = \frac{1}{\alpha} \sup_{n<m} d_\mathcal{C}(F(n), F(m)),$$

for all $F \in \mathcal{SL}(d_\alpha, d_\mathcal{C})$. Then we have the following result.

Theorem 3.1. $(\mathcal{SL}(d_\alpha, d_\mathcal{C}), \|.\|_{(d_\alpha, d_\mathcal{C})})$ *is a relativized normed cone.*

Furthermore, it is immediate to see that the nonnegative real valued function D defined on $\mathcal{SL}(d_\alpha, d_\mathcal{C}) \times \mathcal{SL}(d_\alpha, d_\mathcal{C})$ by

$$D(F, G) = ((\|G\|_{(d_\alpha, d_\mathcal{C})} - \|F\|_{(d_\alpha, d_\mathcal{C})}) \vee 0),$$

is a quasi-pseudo-metric on $\mathcal{SL}(d_\alpha, d_\mathcal{C})$. It is called the quasi-pseudo-metric induced by $\|.\|_{(d_\alpha, d_\mathcal{C})}$.

In the rest of the chapter we shall apply the relativized norm $\|.\|_{(d_\alpha,d_\mathcal{C})}$ and the quasi-pseudo-metric D to analyze the complexity of algorithms having an associated recurrence equation .

Let T be the recurrence equation on $\mathbb{N}$ associated to a given algorithm and denote by f the complexity function which is the solution of this recurrence equation (we assume that $f(0) = \infty$). Then f constitutes a total mapping defined recursively.

Of course, the function f can be approximated by a sequence $\{p_n\}_{n\in\mathbb{N}}$ of partial functions, where each $p_n : \{0, 1, ..., n\} \to (0, \infty]$ is defined by $p_n(0) = \infty$, $p_n(k) = T(k)$ for $k = 1, ..., n$.

It is clear that each function p_n can be identified with a complexity function f_n defined as follows:

$$f_n(0) = \infty, f_n(k) = T(k) \text{ for } k = 1, ..., n, \text{ and } f_n(k) = \infty \text{ for } k > n. \qquad (*)$$

Then we have that $f \leq f_n$ and $f_{n+1} \leq f_n$ for all $n \in \mathbb{N}$.

First we show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f both with respect to the quasi-metric $d_\mathcal{C}$ and its conjugate, which agrees with the computational interpretation of the partial functions p_n.

Indeed, since $f \leq f_n$ it immediately follows that $d_\mathcal{C}(f, f_n) = 0$ for all $n \in \mathbb{N}$. On the other hand, since $f_n(k) = f(k)$ for $k = 0, 1, ..., n$, we obtain

$$d_\mathcal{C}(f_n, f) = \sum_{k=n+1}^{\infty} 2^{-k} \frac{1}{f(k)} \leq \frac{1}{T(1)} \sum_{k=n+1}^{\infty} 2^{-k} = \frac{2^{-n}}{T(1)}.$$

Consequently $d_\mathcal{C}(f_n, f) \to 0$.

Next we show that the complexity represented by f can be derived by computing the relativized norm $\|.\|_{(d_\alpha,d_\mathcal{C})}$ on certain semi-Lipschitz functions that take values on the sequence $\{f_n\}_{n\in\mathbb{N}}$ and its initial segments; thus we obtain the advantage to calculate on finite sums (those that they come represented by the initial segments of $\{f_n\}_{n\in\mathbb{N}}$), avoiding to have to calculate with infinite series.

Indeed, let $F:\omega\to\mathcal{C}$ given by $F(n)=f_n$ for all $n\in\mathbb{N}$ and $F(0)=f_\infty$ where $f_\infty(n)=\infty$ for all $n\in\omega$.

Note that in certain sense F can be identified with $\{f_n\}_{n\in\mathbb{N}}$ and thus with f. On the other hand, $F\in\mathcal{SL}(d_\alpha,d_\mathcal{C})$ because $d_\mathcal{C}(F(n),F(m))=0$ whenever $d_\alpha(n,m)=0$, and

$$\frac{1}{\alpha}\sup_{n<m} d_\mathcal{C}(F(n),F(m))=\frac{1}{\alpha}\lim_{m\to\infty} d_\mathcal{C}(f_\infty,f_m)=\frac{1}{\alpha}\sum_{k=1}^{\infty}2^{-k}\frac{1}{f(k)}<\infty.$$

We also obtain that

$$\|F\|_{(d_\alpha,d_\mathcal{C})}=\frac{1}{\alpha}\sum_{k=1}^{\infty}2^{-k}\frac{1}{f(k)}.$$

Now construct a sequence $\{F_j\}_{j\in\mathbb{N}}$ of functions from ω to $\mathcal{C}$ as follows: $F_j(0)=f_\infty$, $F_j(n)=f_n$ for $0<n<j$, and $F_j(n)=f_j$ otherwise. Observe that $F\leq F_j$ and $F_{j+1}\leq F_j$ for all $j\in\mathbb{N}$. Moreover, it is easy to see that $F_j\in\mathcal{SL}(d_\alpha,d_\mathcal{C})$. In particular

$$\|F_j\|_{(d_\alpha,d_\mathcal{C})}=\frac{1}{\alpha}\sum_{k=1}^{j}2^{-k}\frac{1}{f_j(k)},$$

and consequently

$$\lim_{j\to\infty}\|F_j\|_{(d_\alpha,d_\mathcal{C})}=\|F\|_{(d_\alpha,d_\mathcal{C})}.$$

More exactly, we have

$$\|F\|_{(d_\alpha,d_\mathcal{C})}-\|F_j\|_{(d_\alpha,d_\mathcal{C})}$$

$$= \frac{1}{\alpha}\sum_{k=j+1}^{\infty} 2^{-k}\frac{1}{f(k)} + \frac{1}{\alpha}\sum_{k=1}^{j} 2^{-k}(\frac{1}{f(k)} - \frac{1}{f_j(k)})$$

$$= \frac{1}{\alpha}\sum_{k=j+1}^{\infty} 2^{-k}\frac{1}{f(k)},$$

and thus

$$\|F\|_{(d_\alpha,d_{\mathcal{C}})} - \|F_j\|_{(d_\alpha,d_{\mathcal{C}})} = \frac{1}{\alpha}\, d_{\mathcal{C}}(f_j, f).$$

Hence

$$\|F\|_{(d_\alpha,d_{\mathcal{C}})} - \|F_j\|_{(d_\alpha,d_{\mathcal{C}})} \leq \frac{1}{\alpha}\frac{2^{-j}}{T(1)}.$$

Finally, note that the use of the induced quasi-pseudo-metric D also provides a satisfactory interpretation in this context because the relations

$$D(F, F_j) = 0 \text{ and } D(F_{j+1}, F_j) = 0$$

agree with the facts that F is more efficient than F_j and that F_{j+1} is more efficient than F_j for all $j \in \mathbb{N}$.

We conclude the chapter with and example which illustrates the method developed above.

Example 3.1. Consider the average case of Quicksort as discussed in ([59]), where the following recurrence equation for this algorithm is obtained: $T(1) = 1$ and

$$T(n) = \frac{n+2}{n+1}T(n-1) + \frac{2n}{n+1}$$

for all $n \geq 2$.

It well know that there is $f \in \mathcal{C}$ which the unique solution for this recurrence equation.

Let $\{f_n\}_{n\in\mathbb{N}}$ defined as in (*).

Hence in this case we obtain for the three first terms F_1, F_2 and F_3 :

$$\|F_1\| = 0, \quad \|F_2\| = \frac{3}{32\alpha} \quad \text{and} \quad \|F_3\| = \frac{111}{928}.$$

Then, the relativized norm $\|\cdot\|_{(d_\alpha,d_\mathcal{C})}$ yields us a quantification of the nearness of the algorithm to the solution in each step. Thus given an error level we can obtain the number of steps that we must give to come near to the real solution as much as we wish.

Therefore if we consider an error level $\varepsilon = 0.003$ we can obtain the number of steps:

$$\frac{2^{-j}}{\alpha} \leq 0.003$$

and then for $\alpha = 1$ we obtain that $j \geq 9$.

Chapter 4

Semi-Lipschitz Functions and Best Approximation in Quasi-Metric Spaces

4.1 Introduction

This chapter is a contribution to the study of semi-Lipschitz functions and best approximation from a nonsymmetric point of view. We show how the quasi-normed space structure provides an appropriate setting to characterize the points of best approximation. In this way our results generalize the quasi-metric theory of semi-Lipschitz functions and best approximation [102].

4.2 Best Approximation in Quasi-Metric Spaces

Let (X, d) be a quasi-metric space and let $a \in X$. We shall denote by $cl_d\{a\}$ the closure of the subset $\{a\}$ in the topology T_d, i. e., $cl_d\{a\} = \{x \in X : d(x, a) = 0\}$. As usual, if $A \subset X$, by $d(p, A)$ we shall denote the $inf\{d(p, a) : a \in A\}$ for each $p \in X$.

Definition 4.1. Let (X, d) be a quasi-metric space. Let $A \subset X$ and $p \in X$. An element $a_0 \in A$ such that $d(p, A) = d(p, a_0)$ is said to be an element of best approximation to p by elements of A, it it exists.

Note that if $d(p, a_0) = 0$ for some $a_0 \in A$ then a_0 is obviously an element of best approximation to p by elements of A. Therefore we focus our attention on those points $p \notin \bigcup\{cl_d\{a\} : a \in A\}$.

Proposition 4.1. *Let (X, d) be a quasi-metric space. Let $A \subset X$, $x_0 \in A$ and $p \notin \bigcup\{cl_d\{a\} : a \in A\}$. Then $a_0 \in A$ is an element of best approximation to p by elements of A if and only if for any quasi-normal space (Y, q) such that there exists a $e \in Y$ with $q(e) = 1$ and $q(-e) = 0$, then there is $f \in \mathcal{SL}_0(d, q)$ such that:*

(1) $\|f\|_{(d,q)} = 1$

(2) $f|_A = 0$

(3) $d(p, a_0) = q(f(p) - f(a_0))$

Proof. Suppose first that $a_0 \in A$ is an element of best approximation to p by elements of A. Let (Y, q) be a quasi-normed space and let $e \in Y$ such that

$q(e) = 1$ and $q(-e) = 0$. Define $f : (X, d) \longrightarrow (Y, q)$ by $f(x) = d(x, A)e$ for all $x \in X$.

Let see that $f \in \mathcal{SL}_0(d, q)$:

If $x_0 \in A$ then $f(x_0) = 0$.

Now, given two points $x, y \in X$ with $d(x, y) = 0$ the triangle inequality say that $d(x, a) \leq d(y, a)$ for each $a \in A$, that is $d(x, A) \leq d(y, A)$. Thus

$$q(f(x) - f(y)) = q(d(x, A)e - d(y, A)e) = q((d(x, A) - d(y, A))e)$$

Then

$$q((d(x, A) - d(y, A))e) = (d(y, A) - d(x, A))q(-e) = 0.$$

Given $x, y \in X$ such that $d(x, y) \neq 0$; we have:

- If $d(x, A) \leq d(y, A)$, then

$$\frac{q(f(x) - f(y))}{d(x, y)} = \frac{(d(y, A) - d(x, A))q(-e)}{d(x, y)} = 0.$$

- If $d(y, A) \leq d(x, A)$, then

$$\frac{q(f(x) - f(y))}{d(x, y)} = \frac{(d(x, A) - d(y, A))q(e)}{d(x, y)}$$

  but $d(x, A) - d(y, A) \leq d(x, y)$.

Thus

$$\frac{q(f(x) - f(y))}{d(x, y)} \leq 1$$

for all $x, y \in X$ such that $d(x, y) \neq 0$ then

$$\|f\|_{(d,q)} = \sup_{d(x,y)\neq 0} \frac{q(f(x) - f(y))}{d(x, y)} \leq 1.$$

(1) Let see that $\|f\|_{(d,q)} = 1$. We know that $\|f\|_{(d,q)} \leq 1$. On the other hand $d(p, a_0) \neq 0$. Then

$$\|f\|_{(d,q)} \geq \frac{q(f(p) - f(a_0))}{d(p, a_0)} = \frac{d(p, A)q(e)}{d(p, a_0)} = 1.$$

(2) By definition, $f(a) = d(a, A)e$, so $f|_A = 0$.

(3) We know that

$$\frac{q(f(p) - f(a_0))}{d(p, a_0)} = 1$$

Then $q(f(p) - f(a_0)) = d(p, a_0)$.

Conversely, for each $a \in A$, $d(p, a) \neq 0$, and hence

$$\begin{aligned} d(p, a) &= \|f\|_{(d,q)} d(p, a) \\ &\geq \frac{q(f(p) - f(a))}{d(p, a)} d(p, a) = q(f(p) - f(a)) \\ &= q(f(p)) = q(f(p) - f(a_0)) = d(p, a_0). \end{aligned}$$

Therefore, for each $a \in A$, $d(p, a) \geq d(p, a_0)$. Hence $d(p, A) = d(p, a_0)$, which proves that a_0 is an element of best approximation to p by elements of A. □

Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normal space respectively, let $A \subset X$ and $x_0 \in A$. Put

$$A_0 = \{f : X \longrightarrow Y : f \in \mathcal{SL}_0(d, q) \text{ and } f|_A = 0\},$$

and let us define for each $x, y \in X$ such that $d(x, y) \neq 0$,

$$d_{A_0}(x, y) = \sup\{\frac{q(f(x) - f(y))}{\|f\|_{(d,q)}} : f \in A_0 \text{ and } \|f\|_{(d,q)} \neq 0\}.$$

Given $x, y \in X$, for all $f \in \mathcal{SL}_0(d, q)$, if $d(x, y) \neq 0$ then

$$q(f(x) - f(y)) \leq \|f\|_{(d,q)} d(x, y).$$

Thus

$$d_{A_0}(x, y) \leq \sup\{\frac{q(f(x) - f(y))}{\|f\|_{(d,q)}} : f \in \mathcal{SL}_0(d, q) \text{ and } \|f\|_{(d,q)} \neq 0\} \leq d(x, y).$$

We now have the following result.

Proposition 4.2. *Let (X, d) and (Y, q) be a quasi-metric space and a quasi-normed space, respectively. Let $A \subset X$, $a_0 \in A$ and $p \notin \bigcup\{cl_d\{a\} : a \in A\}$. Then a_0, is an element of best approximation to p by elements of A if and only if $d_{A_0}(p, a_0) = d(p, a_0)$.*

Proof. Suppose that a_0 is an element of best approximation to p by elements of A. By Proposition 4.1 there is $f \in A_0$ such that $\|f\|_{(d,q)} = 1$ and $d(p, a_0) = q(f(p) - f(a_0))$. Therefore

$$d_{A_0}(p, a_0) = \sup\{\frac{q(h(p) - h(a_0))}{\|h\|_{(d,q)}} : f \in A_0 \text{ and } \|h\|_{(d,q)} \neq 0\}$$

$$\leq \frac{q(f(p) - f(a_0))}{\|f\|_{(d,q)}} = d(p, a_0).$$

Since $d_{A_0}(p, a_0) \leq d(p, a_0)$, we denote that $d_{A_0}(p, a_0) = d(p, a_0)$.

Conversely, for all $a \in A$, we have:

$$d(p,a_0) = d_{A_0}(p,a_0) = \sup\{\frac{q(f(x)-f(y))}{\|f\|_{(d,q)}} : f \in A_0 \text{ and } \|f\|_{(d,q)} \neq 0\},$$

$$= \sup\{\frac{q(f(p)-f(a)}{\|f\|_{(d,q)}} : f \in A_0, \|f\|_{(d,q)} \neq 0\} = d_{A_0}(p,a) \leq d(p,a),$$

so that a_0 is an element of best approximation to p of A. □

We shall denote by $P_A(p)$ the set of all point of best approximation to p by elements of A, where $A \subset X$ satisfies $\bigcap\{cl_d\{a\} : a \in A\} \neq \emptyset$. Then A is said to be semi-Chebyschev if card$P_A(p) \leq 1$ for each $p \notin \bigcap\{cl_A\{a\} : a \in A\}$.

Proposition 4.3. *Let (X,d) and (Y,q) be a quasi-metric space and a quasi-normed space, respectively. Let $A, M \subset X|\emptyset$. Then $M \subset P_A(p)$ if and only if there is a $f \in \mathcal{SL}_0(d,q)$ such that*

(1) $\|f\|_{(d,q)} = 1$

(2) $f|_A = 0$

(3) $d(p,a) = q(f(p)-f(a))$ *for all* $a \in M$.

Proof. Suppose $M \subset P_A(p)$. Fix $a_0 \in M$. By Proposition 4.1 there exists $f \in \mathcal{SL}_0(d,q)$ satisfying (1) and (2) and $d(p,a_0) = q(f(p)-f(a_0))$. Let $a \in M$. Then

$$d(p,a) = d(p,A) = d(p,a_0),$$

so

$$d(p,y) = q(f(p)-f(a_0)).$$

Since $f|_A = 0$ we obtain that $d(p, a) = q(f(p)) = q(f(p) - f(y))$.

Conversely, suppose that there exists $f \in \mathcal{SL}_0(d, q)$ satisfying (1), (2) and (3) and let $a_0 \in M$. By Proposition 4.1, $a_0 \in P_A(p)$. Hence $M \subset P_A(p)$. □

From the Proposition 4.3 we obtain the next corollary:

Corollary 4.1. *Let (X, d) and (Y, q) be a quasi-metric space and quasi-normal space, respectively. Let $A \subset X$ and $a_0 \in A$. Then A is semi-Chebyshev if and only if there does not exist $f \in \mathcal{SL}_0(d, q)$, $p_1 \in X$ and $a_1, a_2 \in A$, $a_1 \neq a_2$, such that*

(1) $\|f\|_{(d,q)} = 1$

(2) $f|_A = 0$

(3) $q(f(p_1)) = d(p_1, a_1) = d(p_1, a_2)$.

4.3 Best Approximation in Quasi-Metric Hyperspaces

In this section we apply our methods to obtain some conditions on best approximation in the realm of quasi-metric hyperspaces.

Proposition 4.4. *Let (X,d) a quasi-metric space, given $A_n, A \in \mathcal{P}_0(X)$ if there exist a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X and a in $\overline{A}^d$, such that $\{a_n\}_{n\in\mathbb{N}}$ converges to a in (X,d^{-1}), the sequence $\{A_n\}_{n\in\mathbb{N}}$ converges to A with respect to H_d^- and $d(p,a_n) = d(p,A_n)$ then $d(p,a) = d(p,A)$.*

Proof. We first going to see that $d(p,A) \leq d(p,a)$. If $a \in \overline{A}^d$ we obtain that $d(a,A) = 0$ and $d(p,A) - d(p,a) \leq d(a,A) = 0$ thus $d(p,A) \leq d(p,a)$.

By other hand, by the convergence of $\{a_n\}_{n\in\mathbb{N}}$ to a in (X,d^{-1}) we obtain that for each $\varepsilon > 0$ there exist $n_0 \in \mathcal{N}$ such that if $n \geq n_0$ then

$$d(p,a) - d(p,a_n) < \frac{\varepsilon}{2}.$$

If $\{A_n\}_{n\in\mathbb{N}}$ converges to A with the H_d^- quasi-metric and $d(p,a_n) = d(p,A_n)$ for each $\varepsilon > 0$ we can take the same n_0 without loss of generality, and for each $n \geq n_0$:

$$d(p,a_n) - d(p,A) = d(p,A_n) - d(p,A)$$

$$= H_d^-(p,A_n) - H_d^-(p,A) \leq H^-(A,A_n) < \frac{\varepsilon}{2}.$$

Thus

$$d(p,a) - d(p,A)$$

$$\leq d(p,a) - d(p,a_n) + d(p,a_n) - d(p,A)$$

$$\leq d(a_n, a) + H^-(A, A_n) = \varepsilon,$$

then $d(p, a) \leq d(p, A)$. □

Corollary 4.2. *If (X, d) a metric space, given $A_n, A \in \mathcal{P}_0(X)$ if there exist a sequence $\{a_n\}_{n\in\mathbb{N}}$ in X and a in $\overline{A}$, such that $\{a_n\}_{n\in\mathbb{N}}$ converges to a, the sequence $\{A_n\}_{n\in\mathbb{N}}$ converges to A with respect to H_d^- and $d(p, a_n) = d(p, A_n)$ then $d(p, a) = d(p, A)$.*

Chapter 5

Completeness of the Upper Bourbaki Quasi-Uniformity of a Uniform Space

5.1 Introduction.

In [52, p. 31], Isbell gave an example of a complete uniform space whose Bourbaki uniformity is not complete on $\mathcal{P}_0(X)$. Answering a question posed by Császár ([22, p. 199]), Burdick characterized in [15] those complete uniform spaces $(X,\mathcal{U})$ for which the Bourbaki uniformity is complete on $\mathcal{P}_0(X)$, and Künzi and Ryser obtained in [67, Proposition 6] a nice quasi-uniform version of Burdick's theorem in terms of right K-completeness.

On the other hand, Morita proved in [79] that a separated uniform space $(X,\mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X),\mathcal{U}_H)$ is complete. Attempts of obtaining a satisfactory quasi-uniform extension of Morita's theorem may be found in [17, 64, 113], where partial positive results were obtained for different notions of quasi-uniform completeness.

Here we show that a careful examination of the proof of Morita's theorem which is based on the analysis of the properties of the filter associated to a Cauchy net in $(\mathcal{K}_0(X), \mathcal{U}_H)$, allows us to characterize any separated complete uniform space $(X, \mathcal{U})$ in terms of many kinds of completeness of the quasi-uniform space $(\mathcal{K}_0(X), \mathcal{U}_H^+)$, like convergence completeness, Sieber-Pervin completeness, Smyth completeness, left K-completeness, D-completeness and half-completeness. In this way, we obtain an apparently unexpected link between completeness of a (separated) uniform space $(X, \mathcal{U})$ and several types of quasi-uniform completeness of $\mathcal{U}_H^+$. We also give an example of a complete uniform space $(X, \mathcal{U})$ for which $(\mathcal{K}_0(X), \mathcal{U}_H^+)$ is not right K-complete.

5.2 The Results

In order to help the reader, we start this section by giving some pertinent concepts.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\mathcal{F}$ be a filter on X. Then $\mathcal{F}$ is said to be:

(i) SP-Cauchy (Cauchy in [35, 124]) if for each $U \in \mathcal{U}$ there is $x \in X$ such that $U(x) \in \mathcal{F}$.

(ii) left K-Cauchy ([98]) if for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$.

(iii) D-Cauchy ([27]) if there is a filter $\mathcal{G}$ on X such that for each $U \in \mathcal{U}$ there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ with $G \times F \subseteq U$.

Following [62] (see also [98]), a quasi-uniform space $(X,\mathcal{U})$ is called convergence complete (respectively, left K-complete, D-complete) if every SP-Cauchy (respectively, left K-Cauchy, D-Cauchy) filter is convergent in $(X, T(\mathcal{U}))$. $(X,\mathcal{U})$ is called Sieber-Pervin complete if every SP-Cauchy filter has a cluster point in $(X, T(\mathcal{U}))$; it is called half-complete (respectively, bicomplete) if every Cauchy filter on $(X,\mathcal{U}^s)$ is convergent in $(X, T(\mathcal{U}))$ (respectively, in $(X, T(\mathcal{U}^s))$, and it is Smyth complete provided that every left K-Cauchy filter on $(X,\mathcal{U})$ converges (equivalently, has a cluster point) in $(X, T(\mathcal{U}^s))$ (see, for instance, [62, 98]).

Convergence complete quasi-uniform spaces are called complete in [124] and SP-complete in [19]. Sieber-Pervin complete quasi-uniform spaces are called complete in [80] and MN-complete in [24], and bicomplete quasi-uniform spaces are called doubly complete in [21], pair complete in [71] and L-complete in ([24]).

Remark 5.1. It is well known, and easy to see, that the above notions of quasi-uniform completeness are related as follows:

(i) convergence complete ⇒ Sieber-Pervin complete ⇒ left K-complete ⇒ half-complete;

(ii) convergence complete ⇒ D-complete ⇒ half-complete;

(iii) Smyth complete ⇒ left K-complete;

(iv) Smyth complete ⇒ bicomplete ⇒ half-complete.

Moreover, none of the other possible implications are true (see, for instance, [98, Remark 3] for some counterexamples).

Since in the realm of the Bourbaki (quasi-)uniformity is convenient, in many cases, working with nets instead of filters, we recall the notions of Cauchy kind corresponding to nets.

Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a net in X. Then $\{x_\alpha\}_{\alpha\in\Lambda}$ is said to be:

(i) SP-Cauchy if for each $U \in \mathcal{U}$ there are $x \in X$ and $\alpha_U \in \Lambda$ such that $x_\alpha \in U(x)$ whenever $\alpha \geq \alpha_U$.

(ii) left K-Cauchy if for each $U \in \mathcal{U}$ there is $\alpha_U \in \Lambda$ such that $(x_\alpha, x_\beta) \in U$ whenever $\beta \geq \alpha \geq \alpha_U$.

(iii) D-Cauchy if there is a net $\{y_\beta\}_{\beta\in\Delta}$ such that for each $U \in \mathcal{U}$ there are $\beta_U \in \Delta$ and $\alpha_U \in \Lambda$ with $(y_\beta, x_\alpha) \in U$ whenever $\beta \geq \beta_U$ and $\alpha \geq \alpha_U$.

Then, the following facts are well known, and easy to shown (see, for instance, [61, p. 325] for the left K- case).

A quasi-uniform space $(X,\mathcal{U})$ is convergence complete (respectively, left K-complete, D-complete, half-complete) if and only if every PS-Cauchy (respectively, left K-Cauchy, D-Cauchy, Cauchy in $(X,\mathcal{U}^s)$) net is convergent in $(X,T(\mathcal{U}))$. Furthermore $(X,\mathcal{U})$ is Sieber-Pervin complete if and only if every SP-Cauchy net has a cluster point in $(X,T(\mathcal{U}))$, and it is Smyth complete if and only if every left K-Cauchy net converges (equivalently, has a cluster point) in $(X,T(\mathcal{U})^s)$.

Note that in a first attempt to characterize completeness of a separated uniform space $(X,\mathcal{U})$ in terms of completeness of $(\mathcal{K}_0(X),\mathcal{U}_H^+)$, a direct application of Morita's theorem provides the following restatement of it.

Proposition 5.1. *A separated uniform space $(X,\mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is bicomplete.*

Proof. By Remark 1.2, $(\mathcal{U}_H^+)^s = \mathcal{U}_H$, so, we obviously have that $(\mathcal{K}_0(X),\mathcal{U}_H)$ is complete if and only $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is bicomplete. Then, the result follows from Morita's theorem. □

From the above Proposition we deduce that $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is half-complete whenever $(X,\mathcal{U})$ is a separated complete uniform space.

Clearly, and contrarily to Proposition 5.1, it is not possible to deduce in a straightforward way convergence completeness of $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ from completeness of $(X,\mathcal{U})$ via Morita's theorem. In fact, it is easy to give examples of SP-Cauchy (actually, convergent) nets in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ which are not Cauchy in $(\mathcal{K}_0(X),\mathcal{U}_H)$, as we show below.

Example 5.1. Denote by $\mathcal{U}$ the uniformity induced on $\mathbb{R}$ by the Euclidean metric. Of course $(\mathbb{R},\mathcal{U})$ is a separated complete uniform space. Let $\{A_n\}_{n\in\mathbb{N}}$ be the sequence in $\mathcal{K}_0(\mathbb{R})$ defined by

$$A_{2n-1} = [0, 1/n] \quad \textit{and} \quad A_{2n} = [2, 2 + 1/n],$$

for all $n \in \mathbb{N}$. Put $B = [0, 3]$. Since $A_n \subset B$ for all $n \in \mathbb{N}$, then $\{A_n\}_{n\in\mathbb{N}}$ converges to B in $(\mathcal{K}_0(\mathbb{R}), T(\mathcal{U}_H^+))$, so, as a net, it is PS-Cauchy in $(\mathcal{K}_0(\mathbb{R}),\mathcal{U}_H^+)$. However $\{A_n\}_{n\in\mathbb{N}}$ does not converges in $(\mathcal{K}_0(\mathbb{R}), T(\mathcal{U}_H^-))$, because for each $n \in \mathbb{N}$, we have $U(A_{2n}) \cap U(A_{2n-1}) = \emptyset$, where

$$U = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x - y| < 1/2\}.$$

Therefore $\{A_n\}_{n\in\mathbb{N}}$ does not converges in $(\mathcal{K}_0(\mathbb{R}), T(\mathcal{U}_H))$, and hence, as a net, it is not Cauchy in $(\mathcal{K}_0(\mathbb{R}),\mathcal{U}_H)$.

Given a (nonempty) set X and a net $\{A_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{P}_0(X)$, then the filter on X that has as a base the family $\{F_\alpha : \alpha \in \Lambda\}$, where for each $\alpha \in \Lambda$, $F_\alpha = \bigcup_{\beta\geq\alpha} A_\beta$, is called the filter associated to $\{A_\alpha\}_{\alpha\in\Lambda}$.

Recall ([22, 52]) that a filter on a uniform space $(X,\mathcal{U})$ is semi-Cauchy provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) \in \mathcal{F}$.

Then, the proof of Morita's theorem as given in [8] (see also [17, Theorem 3.5]) is obtained by showing the following facts:

Fact 1. If $(X,\mathcal{U})$ is a uniform space and $\{A_\alpha\}_{\alpha\in\Lambda}$ is a Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H)$, then the filter associated to $\{A_\alpha\}_{\alpha\in\Lambda}$ is semi-Cauchy on $(X,\mathcal{U})$.

Fact 2. If, in addition, $(X,\mathcal{U})$ is complete, then $C := \bigcap_{F\in\mathcal{F}} \overline{F}$ belongs to $\mathcal{K}_0(X)$.

Fact 3. If $(X,\mathcal{U})$ is separated and complete, then $\{A_\alpha\}_{\alpha\in\Lambda}$ converges to C in $(\mathcal{K}_0(X), T(\mathcal{U}_H))$.

Our main result will be deduced from the following lemmas which essentially constitute the upper Bourbaki quasi-uniform counterparts of the preceding statements.

Lemma 5.1. *Let $(X,\mathcal{U})$ be a uniform space. Then, for each PS-Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$, its associated filter is semi-Cauchy on $(X,\mathcal{U})$.*

Proof. Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$. Given $U\in\mathcal{U}$ choose $V\in\mathcal{U}$ such that $V^2\subseteq U$. Then, there exist $B_V\in\mathcal{K}_0(X)$ and $\alpha_V\in\Lambda$ such that $A_\alpha\subseteq V(B_V)$, for all $\alpha\geq\alpha_V$. Since B_V is compact there exists a finite subset B_V' of B_V such that $B_V\subseteq V(B_V')$. Hence

$$\bigcup_{\alpha\geq\alpha_V} A_\alpha \subseteq V(B_V)\subseteq V^2(B_V')\subseteq U(B_V').$$

Therefore $U(B_V')$ belongs to the filter associated to the net $\{A_\alpha\}_{\alpha\in\Lambda}$, and consequently it is a semi-Cauchy filter on $(X,\mathcal{U})$. □

In the proof of Fact 2 above, the quasi-uniformity $\mathcal{U}_H^-$ does not play a relevant role. Thus we are able to prove the following result.

Lemma 5.2. *Let $(X,\mathcal{U})$ be a complete uniform space and let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a PS-Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$. If we denote by $\mathcal{F}$ the filter associated to $\{A_\alpha\}_{\alpha\in\Lambda}$, then $C:=\bigcap_{F\in\mathcal{F}}\overline{F}$ belongs to $\mathcal{K}_0(X)$.*

Proof. By Lemma 5.1, $\mathcal{F}$ is a semi-Cauchy filter on $(X,\mathcal{U})$. Let $\mathcal{F}'$ be an ultrafilter on X containing $\mathcal{F}$. Then $\mathcal{F}'$ is clearly a Cauchy (ultra)filter on $(X,\mathcal{U})$, and thus it converges. Therefore $\mathcal{F}$ has a cluster point in $(X,T(\mathcal{U}))$, so that $C \neq \emptyset$.

Next we show that C is compact in $(X,T(\mathcal{U}))$. Indeed, let $\mathcal{G}$ be a filter on C and let

$$\mathcal{H} = \text{fil}\left\{U(G) \cap F : G \in \mathcal{G}, U \in \mathcal{U}, F \in \mathcal{F}\right\}.$$

Clearly $\mathcal{H}$ is a filter on X because for each $G \in \mathcal{G}$, $G \subseteq C$ and thus $U(G) \cap F \neq \emptyset$ whenever $U \in \mathcal{U}$ and $F \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathcal{H}$, it clearly follows that $\mathcal{H}$ is a semi-Cauchy filter on $(X,\mathcal{U})$. Let $\mathcal{H}'$ be an ultrafilter on X containing $\mathcal{H}$. Then $\mathcal{H}'$ is a Cauchy (ultra)filter on $(X,\mathcal{U})$, and thus $\mathcal{H}$ has a cluster point $x_0 \in X$. Obviously x_0 is also a cluster point of $\mathcal{F}$, and thus $x_0 \in C$. In order to see that $\mathcal{G}$ clusters to x_0, choose $U \in \mathcal{U}$ and $G \in \mathcal{G}$. Let $V \in \mathcal{U}$ symmetric, with $V^2 \subseteq U$. Since $V(G) \in \mathcal{H}$, then there exists $y \in V(x_0) \cap V(G)$,so that $V^2(x_0) \cap G \neq \emptyset$, i.e., $U(x_0) \cap G \neq \emptyset$. Therefore x_0 is a cluster point of $\mathcal{G}$. We conclude that $C \in \mathcal{K}_0(X)$. □

Lemma 5.3. *Under the conditions of Lemma 5.2 and if, in addition,* $(X,\mathcal{U})$ *is separated, then the net* $\{A_\alpha\}_{\alpha\in\Lambda}$ *converges to* C *in* $(\mathcal{K}_0(X), T(\mathcal{U}_H^+))$.

Proof. Assume the contrary. Then there exists $W \in \mathcal{U}$ such that for each $\alpha \in \Lambda$ we find $\gamma(\alpha) \in \Lambda$ with $\gamma(\alpha) \geq \alpha$ satisfying $A_{\gamma(\alpha)} \backslash W(C) \neq \emptyset$. For each $\alpha \in \Lambda$ define

$$E_\alpha = A_{\gamma(\alpha)} \backslash \text{int} W(C).$$

Then $E_\alpha \neq \emptyset$. Moreover E_α is compact because it is a closed subset of the compact (hence, closed) set $A_{\gamma(\alpha)}$. Thus $E_\alpha \in \mathcal{K}_0(X)$ for all $\alpha \in \Lambda$. Denote by $\mathcal{L}$ the filter on X associated to the net $\{E_\alpha\}_{\alpha\in\Lambda}$. Since for each $\beta \in \Lambda$, $E_\beta \subset A_{\gamma(\beta)}$ and $\gamma(\beta) \geq \beta$, it follows that

$$\bigcup_{\beta\geq\alpha} E_\beta \subseteq \bigcup_{\beta\geq\alpha} A_\beta,$$

for all $\alpha \in \Lambda$, so $\mathcal{F} \subseteq \mathcal{L}$. Let $\mathcal{L}'$ be an ultrafilter on X containing $\mathcal{L}$. Thus $\mathcal{L}'$ is a Cauchy (ultra)filter on $(X,\mathcal{U})$ by Lemma 5.1. Hence $\mathcal{L}$ has a cluster point $x_0 \in X$. Since x_0 is also a cluster point of $\mathcal{F}$, we deduce that $x_0 \in C$. Therefore

$$E_\alpha \subseteq A_{\gamma(\alpha)} \backslash \mathrm{int} W(x_0),$$

so that $E_\alpha \cap \mathrm{int} W(x_0) = \emptyset$ for all $\alpha \in \Lambda$. This contradicts the fact that x_0 is a cluster point of $\mathcal{L}$. We conclude that $\{A_\alpha\}_{\alpha\in\Lambda}$ converges to C in $(\mathcal{K}_0(X), T(\mathcal{U}_H^+))$. □

Theorem 5.1. *For a separated uniform space $(X,\mathcal{U})$ the following statements are equivalent:*

(1) $(X,\mathcal{U})$ is complete.

(2) $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is convergence complete.

(3) $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is Sieber-Pervin complete.

(4) $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is Smyth complete.

(5) $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is left K-complete.

(6) $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is D-complete.

(7) $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is half-complete.

Proof. (1) $\Rightarrow$ (2). It follows from Lemmas 5.2 and 5.3.

(1) $\Rightarrow$ (4). Let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a left K-Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$. Then, it is PS-Cauchy in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$, and hence it converges to $C := \bigcap_{F\in\mathcal{F}} \overline{F}$ in $(\mathcal{K}_0(X),T(\mathcal{U}_H^+))$, by Lemmas 5.2 and 5.3. We shall show that $\{A_\alpha\}_{\alpha\in\Lambda}$ clusters to C in $(\mathcal{K}_0(X),T(\mathcal{U}_H^-))$. Assume the contrary. Then, there exist $U \in \mathcal{U}$, $\alpha_0 \in \Lambda$ and a net $\{c_\alpha\}_{\alpha\in\Lambda}$ in C such that $c_\alpha \in X\backslash U(A_\alpha)$ for all $\alpha \geq \alpha_0$. Let $c \in C$ be a cluster point of $\{c_\alpha\}_{\alpha\in\Lambda}$. Choose $V \in \mathcal{U}$ symmetric, with $V^3 \subseteq U$. Since $\{A_\alpha\}_{\alpha\in\Lambda}$ is left K-Cauchy in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$, there is $\alpha_V \geq \alpha_0$ such that $A_\beta \subseteq V(A_\alpha)$ whenever $\beta \geq \alpha \geq \alpha_V$. Let $\alpha \geq \alpha_V$ such that $c_\alpha \in V(c)$. Then, there is $\beta \geq \alpha$ such that $V(c) \cap A_\beta \neq \emptyset$. Thus $V(c) \cap V(A_\alpha) \neq \emptyset$, and hence $c_\alpha \in V^3(A_\alpha)$, which contradicts the fact that $c_\alpha \in X\backslash U(A_\alpha)$ for all $\alpha \geq \alpha_0$.

By Remark 1.2, we conclude that the net $\{A_\alpha\}_{\alpha\in\Lambda}$ clusters to C in $(\mathcal{K}_0(X),T(\mathcal{U}_H^+)^s))$. Consequently $(\mathcal{K}_0(X),\mathcal{U}_H^+)$ is Smyth complete.

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (7), (2) $\Rightarrow$ (6) $\Rightarrow$ (7), and (4) $\Rightarrow$ (5) $\Rightarrow$ (7) are obvious (see Remark 5.1).

(7) $\Rightarrow$ (1). Let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a Cauchy net in $(X,\mathcal{U})$. Then, for each $U \in \mathcal{U}$ there is $\alpha_U \in \Lambda$ such that $x_\alpha \in U(x_\beta)$ for all $\alpha, \beta \geq \alpha_U$. Hence $\{\{x_\alpha\}\}_{\alpha\in\Lambda}$ is a Cauchy net in the uniform space $(\mathcal{K}_0(X),\mathcal{U}_H)$. Let $K \in \mathcal{K}_0(X)$ be a limit point of $\{\{x_\alpha\}\}_{\alpha\in\Lambda}$ in $(\mathcal{K}_0(X),T(\mathcal{U}_H^+))$. Thus, for each $U \in \mathcal{U}$ there is $\alpha_U \in \Lambda$ such that

$$\{x_\alpha : \alpha \geq \alpha_U\} \subseteq U(K).$$

Therefore, for each $U \in \mathcal{U}$ and each $\alpha \geq \alpha_U$ there exists $y_{U,\alpha} \in K$ with $x_\alpha \in U(y_{U,\alpha})$.

Now, define a directed relation $\preceq$ on $\mathcal{U} \times \Lambda$ by $(U,\alpha) \preceq (V,\beta)$ if (i)

$(U,\alpha) = (V,\beta)$, or (ii) $V \subseteq U$, $\alpha_U \leq \alpha$, $\alpha_V \leq \beta$, and $\alpha \leq \beta$.

Let $y \in K$ be a cluster point of the net $\{y_{U,\alpha}\}_{(U,\alpha)\in(\mathcal{U}\times\Lambda)}$. We shall show that y is a cluster point of the net $\{x_\alpha\}_{\alpha\in\Lambda}$. Indeed, given $U \in \mathcal{U}$ and $\alpha \in \Lambda$, choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$, and $\beta \in \Lambda$ with $\beta \geq \alpha$ and $\beta \geq \alpha_V$. Then $x_\beta \in V(y_{V,\beta})$. Let $(W,\gamma) \succeq (V,\beta)$ such that $y_{W,\gamma} \in V(y)$. Then $\gamma \geq \alpha$ and

$$x_\gamma \in W(y_{W,\gamma}) \subseteq V(y_{W,\gamma}) \subseteq V^2(y) \subseteq U(y).$$

We conclude that y is a cluster point of the net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $(X, T(\mathcal{U}))$, so $(X,\mathcal{U})$ is complete. □

Remark 5.2. Morita's theorem is now a consequence of Theorem 5.1 (4), because every Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H)$ is obviously a left K-Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H^+)$.

Remark 5.3. In the light of Theorem 5.1, it seems natural to ask for characterizations of completeness of $(X,\mathcal{U})$ in terms of completeness of $(\mathcal{K}_0(X),\mathcal{U}_H^-)$. In this direction, first note that by Remark 1.2 and Proposition 5.1 we have that a separated uniform space $(X,\mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X),\mathcal{U}_H^-)$ is bicomplete. Hence, if $(X,\mathcal{U})$ is complete, then $(\mathcal{K}_0(X),\mathcal{U}_H^-)$ is half-complete. Now suppose that $(\mathcal{K}_0(X),\mathcal{U}_H^-)$ is half-complete and let $\{x_\alpha\}_{\alpha\in\Lambda}$ be a Cauchy net in $(X,\mathcal{U})$. Then $\{\{x_\alpha\}\}_{\alpha\in\Lambda}$ is a Cauchy net in $(\mathcal{K}_0(X),\mathcal{U}_H)$, i.e., in $((\mathcal{K}_0(X),(\mathcal{U}_H^-)^s)$, and thus $\{\{x_\alpha\}\}_{\alpha\in\Lambda}$ converges to some C in $(\mathcal{K}_0(X), T(\mathcal{U}_H^-))$. Let $c \in C$; then it is obvious that $\{x_\alpha\}_{\alpha\in\Lambda}$ converges to c in $(X, T(\mathcal{U}))$. Thus, we have shown that a separated uniform space $(X,\mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X),\mathcal{U}_H^-)$ is half-complete.

Nevertheless, the situation is quite different for Smyth completeness as the following example shows.

Example 5.2. As in Example 5.1, let $\mathcal{U}$ be the uniformity on $\mathbb{R}$ induced by the Euclidean metric. For each $n \in \mathbb{N}$ set $A_n = [-n, n]$. Since $A_n \subset A_{n+1}$, it follows that $\{A_n\}_{n\in\mathbb{N}}$ is, as a net, left K-Cauchy in $(\mathcal{K}_0(\mathbb{R}), \mathcal{U}_H^-)$. Clearly $\{A_n\}_{n\in\mathbb{N}}$ does not converges in $(\mathcal{K}_0(\mathbb{R}), T(\mathcal{U}_H^+))$. Since $\mathcal{U}_H = (\mathcal{U}_H^-)^s$, we conclude that $(\mathcal{K}_0(\mathbb{R}), \mathcal{U}_H^-)$ is not Smyth complete.

As mentioned in Section 5.1, right K-completeness constitutes a suitable notion of quasi-uniform completeness to generalize Burdick's theorem to the quasi-uniform setting. We conclude the chapter with a brief discussion of this kind of completeness in our context.

Recall ([98]) that a filter on a quasi-uniform space $(X, \mathcal{U})$ is right K-Cauchy if it is a left K-Cauchy filter in $(X, \mathcal{U}^{-1})$. Similarly, a net in $(X, \mathcal{U})$ is said to be right K-Cauchy if it is left K-Cauchy in $(X, \mathcal{U}^{-1})$. A quasi-uniform space $(X, \mathcal{U})$ is called right K-complete if every right K-Cauchy filter (equivalently, net) converges in $(X, T(\mathcal{U}))$ (see ([65, 98])). We have the following.

Proposition 5.2. *A separated uniform space $(X, \mathcal{U})$ is complete if and only if $(\mathcal{K}_0(X), \mathcal{U}_H^-)$ is right K-complete.*

Proof. Suppose that $(X, \mathcal{U})$ is complete and let $\{A_\alpha\}_{\alpha\in\Lambda}$ be a right K-Cauchy net in $(\mathcal{K}_0(X), \mathcal{U}_H^-)$. Since, by Remark 1.2, $(\mathcal{U}_H^-)^{-1} = \mathcal{U}_H^+$, it follows that $\{A_\alpha\}_{\alpha\in\Lambda}$ is left K-Cauchy in $(X, \mathcal{U}_H^+)$, so, by Theorem 5.1 (4), it converges in $(\mathcal{K}_0(X), T(\mathcal{U}_H))$, and hence in $(\mathcal{K}_0(X), T(\mathcal{U}_H^-))$. Therefore $(\mathcal{K}_0(X), \mathcal{U}_H^-)$ is right K-complete. The converse follows from Remark 5.3 because every right K-complete quasi-uniform space is clearly half-complete. □

Finally, observe that completeness of $(X, \mathcal{U})$ cannot be characterized by right K-completeness of $(\mathcal{K}_0(X), \mathcal{U}_H^+)$ as Example 5.2 shows (indeed, it suffices to note that $\{A_n\}_{n\in\mathbb{N}}$ is right K-Cauchy in $(\mathcal{K}_0(X), \mathcal{U}_H^+)$).

Chapter 6

The Hausdorff Fuzzy Quasi-Metric

6.1 Introduction.

It is well known that the Hausdorff distance has an undoubted importance not only in general topology but also in other areas of Mathematics and Computer Science, such as convex analysis and optimization [11, 74, 90], dynamical systems [33, 86, 111, 136], mathematical morphology [123] , fractals [7, 29], image processing [51, 73, 122, 140] , programming language and semantics [9, 10], computational biology [48, 126], etc.

In [30], Egbert extended the classical construction of the Hausdorff distance of a metric space to Menger spaces. Later on, Tardiff [133] (see also [117, 121]), generalized Egbert's construction to probabilistic metric spaces, obtaining in this way a suitable notion of a Hausdorff probabilistic distance.

Since fuzzy metric spaces, in the sense of Kramosil and Michalek, are closely related to Menger spaces [57], one can easily define, from Egbert-Tardiff's construction, a Hausdorff fuzzy distance for a given fuzzy metric space. In connection with these constructions, a notion of Hausdorff fuzzy metric for fuzzy metric spaces in the sense of George and Veeramani [42, 43] was discussed in [92].

On the other hand, it is well known that several structures of asymmetric topology like quasi-uniformities and (fuzzy) quasi-metrics, constitute efficient tools to formulate and solve problems in hyperspaces, function spaces, topological algebra, asymmetric functional analysis, point-free geometry, complexity of algorithms, theoretical computer science, etc. (see, for instance, Chapters 11 and 12 of [62], Section 3 of [60], and also [1, 6, 25, 41, 44, 63, 75, 91, 96, 104, 110, 114, 120, 137, etc] for recent contributions).

In this chapter we introduce and study notions of Hausdorff fuzzy quasi-metric (in the senses of Kramosil and Michalek, and George and Veeramani, respectively) that generalize to the asymmetric setting the corresponding notions of Hausdorff fuzzy metric. In this way, we partially reconcile the theory of fuzzy metric hyperspaces with the theory of asymmetric topology. Furthermore, we apply our approach to the domain of words a paradigmatic example of a space that naturally appear in the theory of computation.

6.2 Basic Notions and Preliminary Results

In this section we recall the concepts and results on fuzzy quasi-metric spaces which we will need in the rest of the chapter. They are taken from [47] (see also [18]). Moreover, we shall observe that the attractive relationship between quasi-metrics and order, recalled in Remark 1.1, is preserved in this framework.

According to [117], a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is associative and commutative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for every $a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

It is well known and easy to see that for each continuous t-norm $*$ one has $* \leq \wedge$, where $\wedge$ is the continuous t-norm given by $a \wedge b = \min\{a, b\}$ for all $a, b \in [0,1]$.

By a KM-fuzzy quasi-pseudo-metric on a set X we mean a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:

(i) $M(x, y, 0) = 0$;

(ii) $M(x, x, t) = 1$ for all $t > 0$;

(iii) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;

(iv) $M(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous.

A KM-fuzzy quasi-metric on X is a KM-fuzzy quasi-pseudo-metric $(M, *)$ on X which satisfies the following condition: (ii') $x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$.

A KM-fuzzy (pseudo-)metric on X is a KM-fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for each $x, y \in X$: (v) $M(x, y, t) = M(y, x, t)$ for all $t > 0$.

A KM-fuzzy quasi-(pseudo-)metric space is a triple $(X, M, *)$ such that X is a (nonempty) set and $(M, *)$ is a KM-fuzzy quasi-(pseudo-)metric on X. The notion of a KM-fuzzy (pseudo-)metric space is defined in the obvious manner. Note that the KM-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of Kramosil and Michalek ([57]).

Each KM-fuzzy quasi-pseudo-metric $(M, *)$ on X generates a topology T_M on X which has as a base the family of open balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$. Observe that if $(M, *)$ is a quasi-metric on X, then T_M is a T_0 topology on X.

It is obvious from the definition of T_M that a sequence $\{x_n\}_{n \in \mathbb{N}}$ in a KM-fuzzy quasi-pseudo-metric $(X, M, *)$ converges to a point $x \in X$ with respect to T_M if and only if $\lim_n M(x, x_n, t) = 1$ for all $t > 0$.

If $(M, *)$ is a KM-fuzzy quasi-(pseudo-)metric on a set X, then $(M^{-1}, *)$ is also a KM-fuzzy quasi-(pseudo-)metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by $M^{-1}(x, y, t) = M(y, x, t)$. Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0, \infty)$ given by $M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\}$, then $(M^i, *)$ is, clearly, a KM-fuzzy (pseudo-)metric on X.

It is well known ([47, Proposition 1]) that if$(X, M, *)$ is a KM-fuzzy quasi-pseudo-metric space, then, for each $x, y \in X$ the function $M(x, y, \_)$ is non-decreasing.

In the rest of the chapter, KM-fuzzy quasi-(pseudo-)metrics and KM-fuzzy quasi-(pseudo-)metric spaces will be simply called fuzzy quasi-(pseudo-)metrics and fuzzy quasi-(pseudo-)metric spaces, respectively.

Remark 6.1. Notice that if $(M, *)$ is a fuzzy quasi-pseudo-metric on a set X, then the relation $\leq_M$ on X given by

$$x \leq_M y \Longleftrightarrow M(x, y, t) = 1 \text{ for all } t > 0,$$

is a preorder on X. Moreover, it is a (partial) order on X if and only if $(M, *)$ is a fuzzy quasi-metric on X. As in the quasi-metric case, $\leq_M$ is called the specialization order of $(M, *)$.

Example 6.1. (compare [47, Example 2.16]). Let d be a (n extended) quasi-(pseudo-)metric on a set X and let M_d be the function defined on $X \times X \times [0, \infty)$ by $M(x, y, 0) = 0$ and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

for all $t > 0$. Then, for each continuous t-norm $*$, $(X, M_d, *)$ is a fuzzy quasi-(pseudo-)metric space called standard fuzzy quasi-(pseudo-)metric space and $(M_d, *)$ is the standard fuzzy quasi-(pseudo-)metric of (X, d). Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^i = M_{d^s}$, and that the topology T_d, generated by d, coincides with the topology T_{M_d} generated by $(M_d, *)$.

We say that a topological space (X, T) admits a compatible fuzzy quasi-(pseudo-)metric if there is a fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that $T = T_M$.

It follows from Example 6.1 that every quasi-(pseudo-)metrizable topological space admits a compatible fuzzy quasi-(pseudo-)metric.

Conversely, we have:

Proposition 6.1. *([47, 105]). Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. Then $\{U_n : n \in \mathbb{N}\}$ is a base for a quasi-uniformity $\mathcal{U}_M$ on X such that $T_{\mathcal{U}_M} = T_M$, where $U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}$ for all $n \in \mathbb{N}$. Moreover the conjugate quasi-uniformity $(\mathcal{U}_M)^{-1}$ coincides with $\mathcal{U}_{M^{-1}}$ and $T_{(\mathcal{U}_M)^{-1}} = T_{M^{-1}}$.*

From Propositions 1.1 and 6.1 we deduce the following:

Corollary 6.1. *Let $(X, M, *)$ be a fuzzy quasi-(pseudo-)metric space. Then there is a quasi-(pseudo-)metric d on X such that $\mathcal{U}_d = \mathcal{U}_M$.*

6.3 Construction of the Hausdorff Fuzzy Quasi-Metric

We start this section by recalling the construction of the Hausdorff fuzzy metric of a fuzzy metric space $(X, M, *)$. In fact, it is a simple adaptation to the fuzzy setting of the definition of the Hausdorff probabilistic metric of a probabilistic metric space ([30, 117, 121, 133]).

Given $x \in X,\ A \in \mathcal{P}_0(X)$ and $t > 0,$ set $M(x, A, t) = \sup_{a \in A} M(x, a, t)$.

Now, for each $A, B \in \mathcal{P}_0(X)$ let

$$H_M^-(A, B, 0) = H_M^+(A, B, 0) = 0,$$

$$H_M^-(A, B, t) = \sup_{0<s<t} \inf_{a \in A} M(a, B, s), \quad H_M^+(A, B, t) = \sup_{0<s<t} \inf_{b \in B} M(A, b, s),$$

for all $t > 0$, and

$$H_M(A, B, t) = \min\{H_M^-(A, B, t), H_M^+(A, B, t)\},$$

for all $t \geq 0$.

Then H_M^-, H_M^+ are fuzzy quasi-pseudo-metrics on $\mathcal{P}_0(X)$ and H_M is a pseudo-metric on $\mathcal{P}_0(X)$. Furthermore H_M is a fuzzy metric on $\mathcal{C}_0(X)$, called the Hausdorff fuzzy pseudo-metric of $(X, M, *)$.

In the light of the above notions and of the construction of Liu and Li [72, p. 67] of a "Hausdorff fuzzy metric" in their recent study of coincidence point theorems for multivalued maps in complete fuzzy metric spaces, one can attempt to define the Hausdorff fuzzy metric in a more simplified way, as follows:

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\},$$

for all $A, B \in \mathcal{C}_0(X)$ and $t > 0$.

The following example shows that, for this alternative definition, H_M is not a fuzzy metric, in general.

Example 6.2. Let $X = \mathbb{N} \cup \{0\}$, and let consider the fuzzy set M in $X \times X \times [0, \infty)$ given by

$M(x, y, 0) = 0$ for all $x, y \in X$;

$M(x, x, t) = 1$ for all $x \in x$ and $t > 0$;

$M(x, y, t) = 0$ whenever $2^{-(x \wedge y)} - 2^{-(x \vee y)} \geq t$, with $x \neq y$ and $t > 0$;

and

$M(x, y, t) = 1$ otherwise.

We shall show that $(X, M, \wedge)$ is a fuzzy metric space.

Indeed, first note that if $M(x, y, t) = 1$ for all $t > 0$, then $x = y$, and that, clearly, $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$.

Let $x, y, z \in X$ and $t, s \geq 0$; if $M(x, y, t) = 0$ or $M(y, z, s) = 0$, then, obviously, $M(x, z, t + s) \geq M(x, y, t) \wedge M(y, z, s)$; otherwise, we have $M(x, y, t) = M(y, z, s) = 1$, and we assume the nontrivial case $x \neq z$, $x \neq y$ and $y \neq z$; then, we have

$$2^{-(x \wedge y)} - 2^{-(x \vee y)} < t \quad \text{and} \quad 2^{-(y \wedge z)} - 2^{-(y \vee z)} < s.$$

On the other hand, an easy computation shows that

$$2^{-(x \wedge z)} - 2^{-(x \vee z)} \leq 2^{-(x \wedge y)} - 2^{-(x \vee y)} + 2^{-(y \wedge z)} - 2^{-(y \vee z)},$$

and thus $2^{-(x\wedge z)} - 2^{-(x\vee z)} < t+s$, i.e., $M(x,z,t+s)=1$.

Now, let $x,y \in X$, $t>0$, and $\{t_k\}_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}^+$ such that $t_k \to t^-$. We assume the nontrivial case $x \neq y$. If $M(x,y,t)=0$, then $2^{-(x\wedge y)} - 2^{-(x\vee y)} \geq t$, since $t > t_k$ eventually, there is $k_0 \in \mathbb{N}$ such that $2^{-(x\wedge y)} - 2^{-(x\vee y)} > t_k$ for all $k \geq k_0$, i.e., $M(x,y,t_k)=0$ for all $k\geq k_0$. If $M(x,y,t)=1$, then $2^{-(x\wedge y)} - 2^{-(x\vee y)} < t$, so there is $k_0\in\mathbb{N}$ such that $2^{-(x\wedge y)} - 2^{-(x\vee y)} < t_k$ for all $k \geq k_0$, i.e., $M(x,y,t_k)=1$ for all $k\geq k_0$. We have shown that $M(x,y,\_)$ is left continuous on $(0,\infty)$. Consequently $(X,M,\wedge)$ (in fact $(X,M,*)$ for any continuous t-norm) is a fuzzy metric space.

Note also that T_M is the discrete topology on X because for each $x\in X$ and $\varepsilon\in(0,1)$, we have $B_M(x,\varepsilon,2^{-(x+1)}) = \{x\}$. Indeed, fix $x \in X$ and let $y\in X$ with $y\neq x$. If $y<x$, we have

$$2^{-(x\wedge y)} - 2^{-(x\vee y)} = 2^{-y} - 2^{-x} \geq 2^{-(x-1)} - 2^{-x} = 2^{-x},$$

and if $y > x$, we have

$$2^{-(x\wedge y)} - 2^{-(x\vee y)} = 2^{-x} - 2^{-y} \geq 2^{-x} - 2^{-(x+1)} = 2^{-(x+1)}.$$

Therefore $M(x,y,2^{-(x+1)})=0$. Hence $B_M(x,\varepsilon,2^{-(x+1)})=\{x\}$, so T_M is the discrete topology on X.

Finally, consider the elements A,B of $\mathcal{C}_0(X)$, where $A=\mathbb{N}$ and $B=\{0\}$. Note that $M(x,0,1)=1$ for all $x\in\mathbb{N}$, and that for each $t\in(0,1)$, there is $x_t\in\mathbb{N}$ such that $1-2^{-x_t}\geq t$, so $M(x_t,0,t)=0$. Hence

$$\min\{\inf_{a\in A} M(a,B,1), \inf_{b\in B} M(A,b,1)\} = \min\{\inf_{x\in\mathbb{N}} M(x,\{0\},1), M(\mathbb{N},0,1)\} = 1,$$

and, for each $t\in(0,1)$,

$$\min\{\inf_{a\in A} M(a,B,t), \inf_{b\in B} M(A,b,t)\} = \min\{\inf_{x\in\mathbb{N}} M(x,\{0\},t), M(\mathbb{N},0,t)\} = 0.$$

We conclude that $H_M(\mathbb{N}, \{0\}, \_)$ is not left continuous at $t = 1$, for the alternative definition of H_M,suggested above, so it is not a fuzzy metric on $\mathcal{C}_0(X)$.

Next we shall construct the Hausdorff fuzzy quasi-metric of a fuzzy quasi-metric space $(X, M, *)$.

Let $(X, M, *)$ be a fuzzy quasi-metric space. If A is a subset of X, the sets $\overline{A}^{T_M}$ and $\overline{A}^{T_{M^{-1}}}$ will be simply denoted by $\overline{A}^M$ and $\overline{A}^{M^{-1}}$, respectively.

The sets $\mathcal{P}_0(X)$, $\mathcal{C}_0(X)$, $\mathcal{K}_0(X)$ and $\mathcal{C}_\cap(X)$ are defined in the obvious manner, as in Section 6.1. In particular, we have

$$\mathcal{C}_\cap(X) = \{\overline{A}^M \cap \overline{A}^{M^{-1}} : A \in \mathcal{P}_0(X)\}.$$

Remark 6.2. It is straightforward to show (compare [91]) that if $A \in \mathcal{P}_0(X)$, then $A \in \mathcal{C}_\cap(X)$ if and only if $A = \overline{A}^M \cap \overline{A}^{M^{-1}}$.

As in the fuzzy metric case, given $x \in X$, $A \in \mathcal{P}_0(X)$ and $t > 0$, put $M(x, A, t) = \sup_{a \in A} M(x, a, t)$.

The following easy result will be useful later on.

Lemma 6.1. *Let $(X, M, *)$ be a fuzzy quasi-metric space. Then for each $x \in X$ and $A \in \mathcal{P}_0(X)$, the following hold:*

(1) $M(x, A, s) \leq M(x, A, t)$ whenever $0 \leq s < t$.

(2) $x \in \overline{A}^M \iff M(x, A, t) = 1$ for all $t > 0$.

Now, for a given fuzzy quasi-metric space $(X, M, *)$ and for each $A, B \in \mathcal{P}_0(X)$, define

$$H_M^-(A,B,0) = H_M^+(A,B,0) = 0,$$

and

$$H_M^-(A,B,t) = \sup_{0<s<t} \inf_{a\in A} M(a,B,s), \quad H_M^+(A,B,t) = \sup_{0<s<t} \inf_{b\in B} M(A,b,s),$$

for all $t > 0$. Then we obtain:

Lemma 6.2. *For each $A, B, C \in \mathcal{P}_0(X)$, the following hold:*

(1a) $A \subseteq \overline{B}^M \iff H_M^-(A,B,t) = 1$ *for all* $t > 0$.

(1b) $B \subseteq \overline{A}^{M^{-1}} \iff H_M^+(A,B,t) = 1$ *for all* $t > 0$.

(2a) $H_M^-(A,C,t+s) \geq H_M^-(A,B,t) * H_M^-(B,C,s)$ *for all* $t, s \geq 0$.

(2b) $H_M^+(A,C,t+s) \geq H_M^+(A,B,t) * H_M^+(B,C,s)$ *for all* $t, s \geq 0$.

(3a) $H_M^-(A,B,\_) : [0,\infty) \to [0,1]$ *is left continuous.*

(3b) $H_M^+(A,B,\_) : [0,\infty) \to [0,1]$ *is left continuous.*

Proof. (1a) Suppose that $A \subseteq \overline{B}^M$. Then for each $a \in A$ and $s > 0$, $M(a,B,s) = 1$ by Lemma 6.1 (2), so $\inf_{a\in A} M(a,B,s) = 1$. Choose any $t > 0$. Then

$$H_M^-(A,B,t) = \sup_{0<s<t} \inf_{a\in A} M(a,B,s) = 1.$$

Conversely, choose $t > 0$. Then by hypothesis, there is a sequence $\{s_k\}_{k\in\mathbb{N}}$, with $s_k \in (0,t)$ for all $k \in \mathbb{N}$, such that

$$\inf_{a\in A} M(a,B,s_k) > 1 - 1/k,$$

for all $k \in \mathbb{N}$. So, by Lemma 6.1 (1),

$$\inf_{a \in A} M(a, B, t) \geq \inf_{a \in A} M(a, B, s_k) > 1 - 1/k,$$

for all $k \in \mathbb{N}$. Consequently $M(a, B, t) = 1$ for all $a \in A$ and $t > 0$. So, by Lemma 6.1 (2), $A \subseteq \overline{B}^M$.

(1b) It follows similarly to the proof of (1a), so it is omitted.

(2a) Fix $t, s \geq 0$. Since the inequality is obvious if $t = 0$ or $s = 0$, we assume that $t, s > 0$. Let $r \in (0, t)$, and $r' \in (0, s)$. Then, for each $a \in A$, with $M(a, B, r) > 0$, and each $\varepsilon \in (0, M(a, B, r))$, there exists $b_a \in B$ such that $M(a, B, r) - \varepsilon \leq M(a, b_a, r)$. Hence

$$(M(a, B, r) - \varepsilon) * \inf_{b \in B} M(b, C, r') \leq M(a, b_a, r) * M(b_a, C, r') \leq M(a, C, r + r').$$

So, by the continuity of $*$, it follows

$$M(a, B, r) * \inf_{b \in B} M(b, C, r') \leq M(a, C, r + r'),$$

for each $a \in A$ with $M(a, B, r) > 0$ (Note that the preceding inequality obviously holds if $M(a, B, r) = 0$). Therefore

$$\inf_{a \in A} M(a, B, r) * \inf_{b \in B} M(b, C, r') \leq \inf_{a \in A} M(a, C, r + r')$$

Consequently, by definition of "sup" and by the continuity of $*$, it follows from standard arguments that

$$\sup_{0<r<t} \inf_{a \in A} M(a, B, r) * \sup_{0<r'<s} \inf_{b \in B} M(b, C, r')$$

$$\leq \sup_{\substack{0<r<t \\ 0<r'<s}} \inf_{a \in A} M(a, C, r + r').$$

Finally, since

$$\sup_{\substack{0<r<t\\0<r'<s}} \inf_{a\in A} M(a,C,r+r') = \sup_{0<r''<t+s} \inf_{a\in A} M(a,C,r''),$$

we conclude that

$$H_M^-(A,B,t) * H_M^-(B,C,s) \leq H_M^-(A,C,t+s).$$

$(2b)$ It follows similarly to the proof of $(2a)$, so it is omitted.

$(3a)$ Let $A, B \in \mathcal{P}_0(X)$, $t > 0$ and let $\{t_k\}_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{R}^+$ such that $t_k \to t$.

Since for each $k \in \mathbb{N}$, $t_k < t$, it immediately follows from Lemma 6.1 (1), that

$$H_M^-(A,B,t_k) \leq H_M^-(A,B,t),$$

for all $k \in \mathbb{N}$.

Now take an arbitrary $\varepsilon \in (0,1)$. Then there is $s_\varepsilon \in (0,t)$ such that

$$H_M^-(A,B,t) < \varepsilon + \inf_{a\in A} M(a,B,s_\varepsilon).$$

Let $k_0 \in \mathbb{N}$ such that $s_\varepsilon < t_k - 1/k$ for all $k \geq k_0$. Then $M(a,B,s_\varepsilon) \leq M(a,B,t_k - 1/k)$ for all $a \in A$ and $k \geq k_0$, so

$$H_M^-(A,B,t) < \varepsilon + \inf_{a\in A} M(a,B,t_k - \frac{1}{k}),$$

for all $k \geq k_0$. Therefore

$$H_M^-(A,B,t) < \varepsilon + \sup_{0<s<t_k} (\inf_{a\in A} M(a,B,s)),$$

for all $k \geq k_0$. We have proved that

$$H_M^-(A, B, t_k) \leq H_M^-(A, B, t) < \varepsilon + H_M^-(A, B, t_k),$$

for all $k \geq k_0$, and, consequently, $H_M^-(A, B, \_)$ is left continuous at $(0, \infty)$.

(3*b*) It follows similarly to the proof of (3*a*), so it is omitted. □

Now we define a fuzzy set H_M on $\mathcal{P}_0(X) \times \mathcal{P}_0(X) \times [0, \infty)$, by:

$$H_M(A, B, t) = \min\{H_M^-(A, B, t), H_M^+(A, B, t)\},$$

for all $A, B \in \mathcal{P}_0(X)$ and $t \geq 0$.

From the above lemma we obtain the following result.

Theorem 6.1. *For a fuzzy quasi-pseudo-metric space* $(X, M, *)$ *the following hold:*

(1) $(H_M^-, *)$, $(H_M^+, *)$ *and* $(H_M, *)$ *are fuzzy quasi-pseudo-metrics on* $\mathcal{P}_0(X)$.

(2) If $(X, M, *)$ *is a fuzzy quasi-metric space, then* $(H_M, *)$ *is a fuzzy quasi-metric on* $\mathcal{C}_\cap(X)$.

Proof. (1) From Lemma 6.2 (1a), (2a) and (3a), it follows that $(H_M^-, *)$ is a fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$. Moreover, from Lemma 6.2 (1b), (2b) and (3b), it follows that $(H_M^+, *)$ is a fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$. From these facts and the definition of H_M it immediately follows that $(H_M, *)$ is also a fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$.

(2) Since, by (1), $(H_M, *)$ is a fuzzy quasi-pseudo-metric on $\mathcal{C}_\cap(X)$, we only need to show that for $A, B \in \mathcal{C}_\cap(X)$, we have $A = B$ whenever $H_M(A, B, t) = H_M(B, A, t) = 1$ for all $t > 0$.

Indeed, suppose that $H_M(A, B, t) = H_M(B, A, t) = 1$ for all $t > 0$. Then, by Lemma 6.2 (1a), $A \subseteq \overline{B}^M$ and $B \subseteq \overline{A}^M$, and by Lemma 6.2 (1b), $B \subseteq \overline{A}^{M^{-1}}$ and $A \subseteq \overline{B}^{M^{-1}}$. Hence $A \subseteq \overline{B}^M \cap \overline{B}^{M^{-1}} = B$, and $B \subseteq \overline{A}^M \cap \overline{A}^{M^{-1}} = A$, so $A = B$. We conclude that $(H_M, *)$ is a fuzzy quasi-metric on $\mathcal{C}_\cap(X)$. □

The fuzzy quasi-metric $(H_M, *)$ of Theorem 6.1 is called the Hausdorff fuzzy quasi-metric of $(M, *)$ on $\mathcal{C}_\cap(X)$.

Example 6.3. Let (X, d) be a quasi-(pseudo-)metric space. Then $H_{M_d} = M_{H_d}$ on $\mathcal{P}_0(X)$, i.e., the Hausdorff fuzzy quasi-pseudo-metric of the standard fuzzy quasi-(pseudo-)metric $(M_d, *)$ coincides with the standard fuzzy quasi-pseudo-metric of the Hausdorff quasi-pseudo-metric of d on $\mathcal{P}_0(X)$.

Indeed, first note that $M_{H_d} = \min\{M_{H_d^-}, M_{H_d^+}\}$.

Now, given $A, B \in \mathcal{P}_0(X)$ and $s > 0$, an easy computation shows (compare [134, Result 2.6] or [92, Proposition 3]) that

$$M_d(a, B, s) = \frac{s}{s + d(a, B)},$$

and then

$$\inf_{a \in A} M_d(a, B, s) = \frac{s}{s + \sup_{a \in A} d(a, B)}.$$

Hence, for each $t > 0$,

$$H_{M_d}^-(A, B, t) = \sup_{0<s<t} \frac{s}{s + \sup_{a \in A} d(a, B)} = \frac{t}{t + H_d^-(A, B)} = M_{H_d^-}(A, B, t).$$

Similarly, we obtain that

$$H^+_{M_d}(A, B, t) = M_{H^+_d}(A, B, t),$$

and consequently

$$H_{M_d}(A, B, t) = \min\{M_{H^-_d}(A, B, t), M_{H^+_d}(A, B, t)\} = M_{H_d}(A, B, t).$$

We conclude that $H_{M_d} = M_{H_d}$ on $\mathcal{P}_0(X)$.

6.4 Some Properties of the Hausdorff Fuzzy Quasi-Metric

In this section we study properties of completeness, precompactness and compactness of the Hausdorff fuzzy quasi-metric.

In order to help the reader we first recall some pertinent concepts and results.

Let $(X, \mathcal{U})$ be a quasi-uniform space. For each $U \in \mathcal{U}$ put

$$H_U^- = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}$$

and

$$H_U^+ = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}.$$

Then $\{H_U^- : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $H_{\mathcal{U}}^-$ on $\mathcal{P}_0(X)$ and $\{H_U^+ : U \in \mathcal{U}\}$ is a base for a quasi-uniformity $H_{\mathcal{U}}^+$ on $\mathcal{P}_0(X)$ [12, 67]. The quasi-uniformity $H_{\mathcal{U}} = H_{\mathcal{U}}^- \vee H_{\mathcal{U}}^+$ is said to be the Hausdorff quasi-uniformity of $\mathcal{U}$ on $\mathcal{P}_0(X)$.

The following result was obtained by Berthiaume [12].

Theorem 6.2. *Let (X, d) be a quasi-pseudo-metric space. Then $\mathcal{U}_{H_d^-} = H_{\mathcal{U}_d}^-$, $\mathcal{U}_{H_d^+} = H_{\mathcal{U}_d}^+$ and $\mathcal{U}_{H_d} = H_{\mathcal{U}_d}$ on $\mathcal{P}_0(X)$.*

In our next result we present the analogue to this theorem for fuzzy quasi-pseudo-metric spaces. Furthermore, and similarly to the fuzzy metric setting (see [92, Theorem 2]), it will be the key to deduce in a direct way several properties of fuzzy quasi-pseudo-metric spaces from the corresponding well-known properties for quasi-pseudo-metric and quasi-uniform spaces.

Theorem 6.3. *Let* $(X, M, *)$ *be a fuzzy quasi-pseudo-metric space. Then* $\mathcal{U}_{H_M^-} = H_{\mathcal{U}_M}^-$, $\mathcal{U}_{H_M^+} = H_{\mathcal{U}_M}^+$ *and* $\mathcal{U}_{H_M} = H_{\mathcal{U}_M}$ *on* $\mathcal{P}_0(X)$.

Proof. Let $n \in \mathbb{N}$. If $A, B \in \mathcal{P}_0(X)$ verify $A \subseteq U_{n+1}^{-1}(B)$, then for each $a \in A$ there is $b_a \in B$ such that $M(a, b_a, 1/(n+1)) > 1 - 1/(n+1)$, Hence, for each $s \in (1/(n+1), 1/n)$, we have

$$M(a, B, s) \geq M(a, b_a, s) \geq M(a, b_a, 1/(n+1)) > 1 - 1/(n+1),$$

so

$$\inf_{a \in A} M(a, B, s) \geq 1 - 1/(n+1) > 1 - 1/n,$$

and consequently

$$H_M^-(A, B, 1/n) > 1 - 1/n.$$

Thus, we have shown that $H_{\mathcal{U}_M}^- \subseteq \mathcal{U}_{H_M^-}$ on $\mathcal{P}_0(X)$.

On the other hand, if $H_M^-(A, B, 1/n) > 1 - 1/n$, then there is $s \in (0, 1/n)$ such that $M(a, B, s) > 1 - 1/n$ for all $a \in A$, and hence $A \subseteq U_n^{-1}(B)$. Consequently $\mathcal{U}_{H_M^-} \subseteq H_{\mathcal{U}_M}^-$ on $\mathcal{P}_0(X)$.

Similarly we prove that $\mathcal{U}_{H_M^+} = H_{\mathcal{U}_M}^+$ on $\mathcal{P}_0(X)$. Hence $\mathcal{U}_{H_M} = H_{\mathcal{U}_M}$ on $\mathcal{P}_0(X)$. □

In the sequel we discuss the completeness of the Hausdorff fuzzy quasi-metric. We shall show that, in this context, right K-sequential completeness provides a satisfactory notion of (fuzzy) quasi-metric completeness. It is interesting to recall that right K-sequential completeness constitutes a suitable notion of quasi-metric completeness in the realm of spaces of functions and hyperspaces, respectively (see [62, Chapter 9]).

In the fuzzy setting we propose the following notions.

Definition 6.1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a fuzzy quasi-pseudo-metric space $(X, M, *)$ is called right K-Cauchy if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there is $n_0 \in \mathbb{N}$ such that $M(x_m, x_n, t) > 1 - \varepsilon$ whenever $n_0 \leq n \leq m$.

Definition 6.2. A fuzzy quasi-pseudo-metric space $(X, M, *)$ is called right K-sequentially complete if every right K-Cauchy sequence is convergent with respect to T_M.

Probabilistic quasi-pseudo-metric versions of the above concepts may be found in [18, p. 120].

The proof of the next result follows immediately from Proposition 6.1 and its Corollary, so it is omitted.

Proposition 6.2. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space and let d be a (n extended) quasi-pseudo-metric on X such that $\mathcal{U}_d = \mathcal{U}_M$. Then:*

(1) A sequence in X is right K-Cauchy in $(X, M, *)$ if and only if it is right K-Cauchy in (X, d).

(2) $(X, M, *)$ is right K-sequentially complete if and only if (X, d) is right K-sequentially complete.

Künzi and Ryser proved in [67] the following result (see also [113]).

Theorem 6.4. *Let* (X, d) *be quasi-pseudo-metric space. Then* $(\mathcal{P}_0(X), H_d)$ *is right K-sequentially complete if and only if* (X, d) *is right K-sequentially complete.*

The next result provides the fuzzy counterpart of the preceding theorem.

Theorem 6.5. *Let* $(X, M, *)$ *be a fuzzy quasi-pseudo-metric space. Then* $(\mathcal{P}_0(X), H_M, *)$ *is right K-sequentially complete if and only if* $(X, M, *)$ *is right K-sequentially complete.*

Proof. Let d be a quasi-pseudo-metric d on X such that $\mathcal{U}_d = \mathcal{U}_M$. Then $H_{\mathcal{U}_d} = H_{\mathcal{U}_M}$; so, by Theorems 6.2 and 6.3, $\mathcal{U}_{H_d} = \mathcal{U}_{H_M}$. It then follows from Proposition 6.2 (2) that $(\mathcal{P}_0(X), H_M, *)$ is right K-sequentially complete if and only if $(\mathcal{P}_0(X), H_d)$ is right K-sequentially complete. Now the conclusion follows from Theorem 6.4 and Proposition 6.2 (2). □

It is interesting to obtain a version of Theorem 6.5 for $\mathcal{C}_\cap(X)$, because in this case H_M is a fuzzy quasi-metric. Such a version is established in the next result.

Corollary 6.2. *Let* $(X, M, *)$ *be a fuzzy quasi-metric space. Then* $(\mathcal{C}_\cap(X), H_M, *)$ *is right K-sequentially complete if and only if* $(X, M, *)$ *is right K-sequentially complete.*

Proof. Suppose that $(\mathcal{C}_\cap(X), H_M, *)$ is right K-sequentially complete. Let $\{x_n\}_{n\in\mathbb{N}}$ be a right K-Cauchy sequence in $(X, M, *)$. Since $x_n = \overline{\{x_n\}}^M \cap \overline{\{x_n\}}^{M^{-1}}$ for all $n \in \mathbb{N}$, it follows that $\{\{x_n\}\}_{n\in\mathbb{N}}$ is a right K-Cauchy sequence in $(\mathcal{C}_\cap(X), H_M, *)$, so it converges to some $C \in \mathcal{C}_\cap(X)$ with respect to T_{H_M}. Then, it is immediate to check that each $c \in C$ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$ with respect to T_M. Since $\{x_n\}_{n\in\mathbb{N}}$ is right K-Cauchy we deduce that it converges to each $c \in C$ with respect to T_M. Therefore $(X, M, *)$ is right K-sequentially complete.

Conversely, let $\{A_n\}_{n\in\mathbb{N}}$ be a right K-Cauchy sequence in $(\mathcal{C}_\cap(X), H_M, *)$. By Theorem 5, $\{A_n\}_{n\in\mathbb{N}}$ converges to some $C \in \mathcal{P}_0(X)$ with respect to T_{H_M}. We shall show that $\{A_n\}_{n\in\mathbb{N}}$ converges to $\overline{C}^M$ with respect to T_{H_M}. First note that $\{A_n\}_{n\in\mathbb{N}}$ converges to $\overline{C}^M$ with respect to $T_{H_M^+}$ because $C \subseteq \overline{C}^M$. Now choose $x \in \overline{C}^M$ and $U \in \mathcal{U}_M$. Then there exist $c \in C$ and $n_0 \in \mathbb{N}$ such that $c \in U(x)$ and $C \subseteq U^{-1}(A_n)$ for all $n \geq n_0$. Hence $x \in U^{-2}(A_n)$ for all $n \geq n_0$. Thus $\overline{C}^M \subseteq U^{-2}(A_n)$ for all $n \geq n_0$, so $\{A_n\}_{n\in\mathbb{N}}$ converges to $\overline{C}^M$ with respect to $T_{H_M^-}$. Since $\overline{C}^M \in \mathcal{C}_\cap(X)$ we conclude that $(\mathcal{C}_\cap(X), H_M, *)$ is right K-sequentially complete. □

We finish this section by analyzing precompactness, total boundedness and compactness of the Hausdorff fuzzy quasi-metric.

Let us recall that a quasi-uniform space $(X, \mathcal{U})$ is precompact ([35, Chapter 3]) provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $X = \bigcup_{a\in A} U(a)$.

A quasi-uniform space $(X, \mathcal{U})$ is totally bounded provided that the uniform space $(X, \mathcal{U}^s)$ is totally bounded ([35, Chapter 3]).

It is well known that each totally bounded quasi-uniform space is precompact and that, contrarily to the uniform case, there exist precompact quasi-uniform spaces that are not totally bounded ([35, Chapter 3]).

In the fuzzy case, we have the following concepts (compare [105]):

A fuzzy quasi-pseudo-metric space $(X, M, *)$ is precompact (respectively, totally bounded) provided that the quasi-uniform space $(X,\mathcal{U}_M)$ is precompact (respectively, totally bounded).

Theorem 6.6. *[67]. Let $(X,\mathcal{U})$ be a quasi-uniform space. Then:*

(1) $(\mathcal{P}_0(X), H_\mathcal{U})$ is precompact if and only if $(X,\mathcal{U})$ is precompact.

(2) $(\mathcal{P}_0(X), H_\mathcal{U})$ is totally bounded if and only if $(X,\mathcal{U})$ is totally bounded.

(3) $(\mathcal{P}_0(X), (H_\mathcal{U})^s)$ is compact if and only if $(X,\mathcal{U}^s)$ is compact.

Related to the statement (3) of the above theorem, it is given in [67, Example 1] an example of a compact quasi-uniform space $(X,\mathcal{U})$ such that $(\mathcal{P}_0(X), H_\mathcal{U})$ is not compact.

In the fuzzy setting we have the following:

Theorem 6.7. *Let $(X, M, *)$ be a fuzzy quasi-pseudo-metric space. Then:*

(1) $(\mathcal{P}_0(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact.

(2) $(\mathcal{P}_0(X), H_M, *)$ is totally bounded if and only if $(X, M, *)$ is totally bounded.

(3) $(\mathcal{P}_0(X), (H_M)^i)$ is compact if and only if $(X, M^i, *)$ is compact.

Proof. We only show the statement (1), because (2) and (3) follow similarly. Indeed, by Theorem 6.7 (1) we have that $(\mathcal{P}_0(X), H_{\mathcal{U}_M}, *)$ is precompact if and only if $(X,\mathcal{U}_M)$ is precompact. Since, by Theorem 6.3, $\mathcal{U}_{H_M} = H_{\mathcal{U}_M}$ on $\mathcal{P}_0(X)$, we deduce that $(\mathcal{P}_0(X), H_M, *)$ is precompact if and only if $(X, M, *)$ is precompact. □

6.5 The Hausdorff GV-Fuzzy Quasi-Metric

Following [47], by a GV-fuzzy quasi-pseudo-metric on a set X we mean a pair $(M,*)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0,\infty)$ such that for all $x,y,z \in X$, $t,s > 0$:

(i) $M(x,y,t) > 0$;

(ii) $M(x,x,t) = 1$;

(iii) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$;

(iv) $M(x,y,\_) : (0,\infty) \to (0,1]$ is continuous.

A GV-fuzzy quasi-metric on X is a GV-fuzzy quasi-pseudo-metric $(M,*)$ on X which satisfies the following condition: (ii') $x = y \Leftrightarrow M(x,y,t) = M(y,x,t) = 1$ for some $t > 0$.

A GV-fuzzy (pseudo-)metric on X is a GV-fuzzy quasi-(pseudo-)metric $(M,*)$ on X such that for each $x,y \in X$: (v) $M(x,y,t) = M(y,x,t)$ for all $t > 0$.

The notion of a GV-fuzzy (pseudo-)metric space is defined in the obvious manner. Note that the GV-fuzzy metric spaces are exactly the fuzzy metric spaces in the sense of George and Veeramani [42].

If $(M,*)$ is a GV-fuzzy quasi-(pseudo-)metric on X, then the fuzzy sets in $X \times X \times (0,\infty)$, M^{-1} and M^i given by $M^{-1}(x,y,t) = M(y,x,t)$ and $M^i(x,y,t) = \min\{M(x,y,t), M^{-1}(x,y,t)\}$, are, as in the KM-case, a GV-fuzzy quasi-(pseudo-)metric and a GV-fuzzy (pseudo-)metric on X, respectively.

Obviously, each GV-fuzzy quasi-(pseudo-)metric $(M, *)$ can be considered as a KM-fuzzy quasi-(pseudo-)metric by defining $M(x, y, 0) = 0$ for all $x, y \in X$. Hence, each GV-fuzzy quasi-pseudo-metric space generates a topology T_M defined as in the KM-case.

Therefore, if $(X, M, *)$ is a GV-fuzzy quasi-pseudo-metric space, then $(H_M^-, *)$, $(H_M^+, *)$ and $(H_M, *)$ are fuzzy quasi-pseudo-metrics on $\mathcal{P}_0(X)$, and $(H_M, *)$ is a fuzzy quasi-metric on $\mathcal{C}_\cap(X)$ whenever $(X, M, *)$ is a GV-fuzzy quasi-metric space.

The next example, given in [92], shows that, however, $(H_M, *)$ is not a GV-fuzzy quasi-metric on $\mathcal{C}_\cap(X)$ even in the case that $(X, M, *)$ is a GV-fuzzy metric space.

Example 6.4. Denote by $*_L$ the Lukasiewicz t-norm, i.e., $a *_L b = \max\{, a + b - 1, 0\}$,for all $a, b \in [0, 1]$.

Now let $\{x_n\}_{n\geq 2}$ and $\{y_n\}_{n\geq 2}$ be two sequences of distinct points such that $A \cap B = \emptyset$, where $A = \{x_n : n \geq 2\}$ and $B = \{y_n : n \geq 2\}$.

Put $X = A \cup B$ and define a fuzzy set M in $X \times X \times (0, \infty)$ by:

$$M(x_n, x_m, t) = M(y_n, y_m, t) = 1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right],$$

$$M(x_n, y_m, t) = M(y_m, x_n, t) = \frac{1}{n} \wedge \frac{1}{m},$$

for all $n, m \geq 2$.

Then $(X, M, *_L)$ is a GV-fuzzy metric space. Since T_M is the discrete topology on X, it follows that $A, B \in \mathcal{C}_0(X)$. From the fact that for each $n \geq 2$ and each $s > 0$, $M(x_n, B, s) = M(A, y_n, s) = 1/n$, we deduce that

$H^-_M(A,B,t) = H^+_M(A,B,t) = 0$ and thus $H_M(A,B,t) = 0$ for all $t > 0$. Consequently $(H_M, *_L)$ is not a GV-fuzzy (quasi-)metric on $\mathcal{C}_0(X)$.

Despite the above example, it is shown in [92] that the formula given immediately before of Example 6.2 provides a suitable Hausdorff GV-fuzzy metric on $\mathcal{K}_0(X)$ for any GV-fuzzy metric space $(X, M, *)$.

In the rest of this section we discuss the corresponding situation to GV-fuzzy quasi-metric spaces.

We start this study with an example of a GV-fuzzy quasi-metric space $(X, M, *)$ whose induced topology is compact and metrizable, but for which $(H_M, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0(X)$.

Example 6.5. Let $X = \mathbb{N} \cup \{0\}$ and let d be the quasi-metric on X given by $d(x,x) = 0$ for all $x \in X$, $d(0,n) = 1/n$ for all $n \in \mathbb{N}$, and $d(n,x) = n$ for all $n \in \mathbb{N}$ and $x \in X \backslash \{n\}$. Clearly (X,d) is a quasi-metric space such that T_d is a compact and metrizable topology. Consider the standard fuzzy quasi-metric $(M_d, *)$ of (X,d) as given in Example 3, and denote also by $(M_d, *)$ its restriction to $X \times X \times (0,\infty)$. It is clear such a restriction is a GV-fuzzy quasi-metric on X.

Now put $A = X\backslash\{1\}$ and $B = \{1\}$. Then $A, B \in \mathcal{K}_0(X)$, and, by one of the formulas obtained in Example 5, we have that

$$H^-_{M_d}(A,B,t) = \frac{t}{t + H^-_d(A,B)},$$

for all $t > 0$. Therefore

$$H^-_{M_d}(A,B,t) = \frac{t}{t + \sup_{a\in A} d(a,\{1\})} = 0,$$

for all $t > 0$, so that $H_{M_d}(A,B,t) = 0$ for all $t > 0$. We conclude that $(H_{M_d}, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0(X) \cap \mathcal{C}_\cap(X)$.

The next is an example of a GV-fuzzy quasi-metric space $(X, M, *)$ for which $(H_M, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_D(X)$, where $\mathcal{K}_D(X)$ denotes the collection of all nonempty subsets of X that are T_M-compact and $T_{M^{-1}}$-compact.

Example 6.6. Let $X = \mathbb{N} \cup \{0\} \cup \{\infty\}$ and let d be the function defined on $X \times X$ by $d(x, x) = 0$ for all $x \in X$, $d(0, \infty) = 1$, $d(0, n) = n$ for all $n \in \mathbb{N}$, $d(x, 0) = 1$ for all $x \in X \backslash \{0\}$, $d(x, y) = 0$ whenever $x \in \mathbb{N}$ and $x \leq y$, and $d(x, y) = y$ otherwise (we assume that $\leq$ is the usual order on X).

It is a routine to show that d is a quasi-metric on X. As in Example 6.5 let $(M_d, *)$ be the GV-fuzzy quasi-metric on X obtained by restricting the standard fuzzy quasi-metric on (X, d) to $X \times X \times (0, \infty)$.

Now observe that $X \in \mathcal{K}_D(X)$. Indeed, since the only T_{M_d}-open set different from X that contains 1 is $\mathbb{N} \cup \{\infty\}$, we obtain that X is T_{M_d}-compact, and since the only $T_{M_{d^{-1}}}$-open set different from X that contains ∞ is $\mathbb{N} \cup \{\infty\}$, it follows that X is $T_{M_{d^{-1}}}$ -compact.

Finally, put $A = \{0\}$ and $B = X$. Then

$$H^+_{M_d}(A, B, t) = \frac{t}{t + \sup_{b \in B} d(\{0\}, b)} = \frac{t}{t + \sup_{n \in \mathbb{N}} n} = 0,$$

for all $t > 0$. So $H_{M_d}(A, B, t) = 0$ for all $t > 0$. We conclude that $(H_{M_d}, *)$ is not a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_D(X)$.

The rest of the section is devoted to prove that for a GV-fuzzy quasi-metric space $(X, M, *)$, $(H_M, *)$ is a GV-fuzzy quasi-pseudo-metric on the collection $\mathcal{K}_0^i$ of all nonempty subsets of X that are compact in the GV-fuzzy metric space $(X, M^i, *)$. In this way we extend the main result of [92] to the fuzzy quasi-metric framework.

To this end, we shall generalize the results of [92, Section 2] to GV-fuzzy quasi-metric spaces as follows (although the proofs are almost identical to the corresponding for GV-fuzzy metric spaces, we given them in order to help to the readers).

Proposition 6.3. *Let* $(X, M, *)$ *be a GV-fuzzy quasi-metric space. Then* M *is a continuous function on* $X \times X \times (0, \infty)$ *for the product topology* $T_{M^i} \times T_{M^i} \times T_E$, *where by* T_E *we denote the Euclidean topology on* $(0, \infty)$.

Proof. Let $x, y \in X$ and $t > 0$, and let $\{x'_n, y'_n, t'_n\}_{n' \in \mathbb{N}}$ be a sequence in $X \times X \times (0, \infty)$ that converges to (x, y, t) with respect to $T_{M^i} \times T_{M^i} \times T_E$.

Since $\{M(x'_n, y'_n, t'_n)\}_{n' \in \mathbb{N}}$ is a sequence in $(0, 1]$, there is a subsequence $\{x_n, y_n, t_n\}_{n \in \mathbb{N}}$ of $\{x'_n, y'_n, t'_n\}_{n \in \mathbb{N}}$ such that the sequence $\{M(x_n, y_n, t_n)\}_{n \in \mathbb{N}}$ converges to an element of $[0, 1]$.

Fix $\delta > 0$ such that $\delta < t/2$. Then, there is $n_0 \in \mathbb{N}$ such that $|t - t_n| < \delta$ for all $n \geq n_0$. Hence

$$M(x_n, y_n, t_n) \geq M(x_n, x, \delta/2) * M(x, y, t - 2\delta) * M(y, y_n, \delta/2),$$

and

$$M(x, y, t + 2\delta) \geq M(x, x_n, \delta/2) * M(x_n, y_n, t_n) * M(y_n, y, \delta/2),$$

for all $n \geq n_0$.

Since $\lim_n M^i(x, x_n, \delta/2) = \lim_n M^i(y, y_n, \delta/2) = 1$, we obtain, by taking limits when $n \to \infty$, that

$$\lim_n M(x_n, y_n, t_n) \geq 1 * M(x, y, t - 2\delta) * 1 = M(x, y, t - 2\delta),$$

and

$$M(x, y, t + 2\delta) \geq 1 * \lim_n M(x_n, y_n, t_n) * 1 = \lim_n M(x_n, y_n, t_n),$$

respectively.

So, by continuity of the function $t \longmapsto M(x, y, t)$, we immediately deduce that $M(x, y, t) = \lim_n M(x_n, y_n, t_n)$. Therefore M is continuous for $T_{M^i} \times T_{M^i} \times T_E$. $\square$

Lemma 6.3. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $a \in X$, $B \in \mathcal{K}_0^i(X)$ and $t > 0$, there is $b_0 \in B$ such that*

$$M(a, B, t) = M(a, b_0, t).$$

Proof. Let $a \in X$, $B \in \mathcal{K}_0^i(X)$ and $t > 0$. By Proposition 6.3, the function $y \longmapsto M(a, y, t)$ is continuous on X for T_{M^i}. Thus, by compactness of B, there exists $b_0 \in B$ such that $\sup_{b \in B} M(a, b, t) = M(a, b_0, t)$, i.e., $M(a, B, t) = M(a, b_0, t)$. $\square$

Lemma 6.4. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $a \in X$ and $B \in \mathcal{K}_0^i(X)$ the function*

$$t \longmapsto M(a, B, t)$$

is continuous on $(0, \infty)$.

Proof. Since $M(a, B, t) = \sup_{b \in B} M(a, b, t)$ and for each $b \in B$ the function $t \longmapsto M(a, b, t)$ is continuous on $(0, \infty)$, it follows that the function $t \longmapsto M(a, B, t)$ is lower semicontinuous on $(0, \infty)$.

We shall prove that $t \longmapsto M(a, B, t)$ is upper semicontinuous on $(0, \infty)$. To this end, let $t > 0$ and let $\{t_n\}_{n \in \mathbb{N}}$ be a sequence in $(0, \infty)$ that converges to t. By Lemma 6.3, for each $n \in \mathbb{N}$ there is $b_n \in B$ such that $M(a, B, t_n) =$

$M(a, b_n, t_n)$. Since $B \in \mathcal{K}_0^i(X)$, there exists a subsequence $\{b_{n_k}\}_{k\in\mathbb{N}}$ of $\{b_n\}_{n\in\mathbb{N}}$ and a point $b_0 \in B$ such that $b_{n_k} \to b_0$ in $(X, M^i, *)$. Hence $\lim_k M(a, b_{n_k}, t_{n_k}) = M(a, b_0, t)$, by Proposition 6.3, and thus

$$\lim_k M(a, B, t_{n_k}) = M(a, b_0, t) \leq M(a, B, t).$$

Consequently, the function $t \longmapsto M(a, B, t)$ is upper semicontinuous on $(0, \infty)$. This concludes the proof. □

Lemma 6.5. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $A \in \mathcal{K}_0^i(X)$, $B \in \mathcal{P}_0(X)$ and $t > 0$, there is $a_0 \in A$ such that*

$$\inf_{a\in A} M(a, B, t) = M(a_0, B, t).$$

Proof. Put $\alpha = \inf_{a\in A} M(a, B, t)$. Then, there is a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A such that $\alpha + 1/n > M(a_n, B, t)$ for all $n \in \mathbb{N}$. Since $A \in \mathcal{K}_0^i(X)$, there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$ and a point $a_0 \in A$ such that $a_{n_k} \to a_0$ in $(X, M^i, *)$.

Choose an arbitrary $b \in B$. By Proposition 6.3, $\lim_k M(a_{n_k}, b, t) = M(a_0, b, t)$. Since for each $k \in \mathbb{N}$, $\alpha + 1/n_k > M(a_{n_k}, b, t)$, it follows, taking limits when $k \to \infty$, that $\alpha \geq M(a_0, b, t)$. We conclude that $\alpha = M(a_0, B, t)$. □

From Lemmas 6.3 and 6.5 we immediately deduce the following.

Corollary 6.3. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space and let $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$. Then there exist $a_0 \in A$ and $b - 0 \in B$ such that*

$$\inf_{a\in A} M(a, B, t) = M(a_0, b_0, t).$$

Proposition 6.4. *Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then, for each $A, B \in \mathcal{K}_0^i(X)$ the function*

$$t \longmapsto \inf_{a \in A} M(a, B, t)$$

is continuous on $(0, \infty)$.

Proof. By Lemma 6.4, the function $t \longmapsto M(a, B, t)$ is continuous on $(0, \infty)$. Hence, the function $t \longmapsto \inf_{a\in A} M(a, B, t)$ is upper semicontinuous on $(0, \infty)$.

We shall prove that $t \longmapsto \inf_{a\in A} M(a, B, t)$ is lower semicontinuous on $(0, \infty)$. To this end, let $t > 0$ and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, \infty)$ that converges to t. By Lemma 6.5, for each $n \in \mathbb{N}$ there is $a_n \in A$ such that $M(a_n, B, t_n) = \inf_{a\in A} M(a, B, t_n)$. Since $A \in \mathcal{K}_0^i(X)$, there exists a subsequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ of $\{a_n\}_{n\in\mathbb{N}}$ and a point $a_0 \in A$ such that $a_{n_k} \to a_0$ in $(X, M^i, *)$. Then, by Lemma 6.3, there is $b_0 \in B$ such that $M(a_0, b_0, t) = M(a_0, B, t)$, and thus $\lim_k M(a_{n_k}, b_0, t_{n_k}) = M(a_0, b_0, t)$, by Proposition 6.3. Therefore, given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, $M(a_0, b_0, t) < \varepsilon + M(a_{n_k}, b_0, t_{n_k})$. So

$$\inf_{a\in A} M(a, B, t) \leq M(a_0, b_0, t) < \varepsilon + M(a_{n_k}, B, t_{n_k}) = \varepsilon + \inf_{a\in A} M(a, B, t_{n_k}),$$

for all $k \geq k_0$. Consequently, the function $t \longmapsto \inf_{a\in A} M(a, B, t)$ is lower semicontinuous on $(0, \infty)$. This concludes the proof. □

Remark 6.3. Note that Proposition 6.4 also shows that for $A, B \in \mathcal{K}_0^i(X)$ the function $t \longmapsto \inf_{b\in B} M(A, b, t)$ is continuous on $(0, \infty)$.

Now let $(X, M, *)$ be a GV-fuzzy quasi-metric and let $(H_M, *)$ be the Hausdorff fuzzy quasi-pseudo-metric on $\mathcal{P}_0(X)$ constructed in Section 6.3.

In order to prove that $(H_M, *)$ is actually a GV-fuzzy quasi-pseudo-metric on $\mathcal{K}_0^i(X)$, we first show that for each $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$, we have

$$H_M^-(A, B, t) = \inf_{a \in A} M(a, B, t).$$

Indeed, since

$$H_M^-(A, B, t) = \sup_{0<s<t} \inf_{a \in A} M(a, B, s),$$

there exists an increasing sequence $\{s_n\}_{n \in \mathbb{N}}$ convergent to t such that

$$H_M^-(A, B, t) = \lim_n \inf_{a \in A} M(a, B, s_n).$$

So, by Proposition 6.4, $H_M^-(A, B, t) = \inf_{a \in A} M(a, B, t)$.

Similarly, we obtain that

$$H_M^+(A, B, t) = \inf_{b \in B} M(A, b, t).$$

Thus

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\},$$

for all $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$.

Since $(M, *)$ is a GV-fuzzy quasi-metric, it follows from Lemma 6.5 that $H_M^-(A, B, t) > 0$ and $H_M^+(A, B, t) > 0$, so $H_M(A, B, t) > 0$ for all $A, B \in \mathcal{K}_0^i(X)$ and $t > 0$. Hence $(H_M, *)$ satisfies condition (i) of the definition of a GV-fuzzy quasi-pseudo-metric.

It is also clear that $H_M(A, A, t) = 1$ for all $A \in \mathcal{K}_0^i$ and all $t > 0$. Moreover, for each $A, B, C \in \mathcal{K}_0^i$ and $t, s \geq 0$, we have, by Lemma 6.2 (2a) and (2b) that $H_M(A, C, t+s) \geq H_M(A, B, t) * H_M(B, C, s)$.

Finally, given $A, B \in \mathcal{K}_0^i(X)$, continuity of $H_M(A, B, \_) : (0, \infty) \to (0, 1]$ is an immediate consequence of Proposition 6.4 and Remark 6.3.

Thus, we have shown the following:

Theorem 6.8. *Let* $(X, M, *)$ *be a GV-fuzzy quasi-metric space. Then* $(H_M, *)$ *is a GV-fuzzy quasi-pseudo-metric on* $\mathcal{K}_0^i(X)$. *Furthermore, we have*

$$H_M(A, B, t) = \min\{\inf_{a \in A} M(a, B, t), \inf_{b \in B} M(A, b, t)\},$$

for all $A, B \in \mathcal{K}_0^i(X)$ *and* $t > 0$.

6.6 A fuzzy approach to the domain of words

In this section we apply the results obtained in the preceding sections to modeling a typical example of theoretical computer science from a fuzzy quasi-metric point of view. This will be done with the help of the parameter t which provides a useful additional ingredient to construct such models.

Let us recall that the domain of words Σ^∞ ([61, 76, 88, 107, 115, 128, etc]) consists of all finite and infinite sequences ("words") over a nonempty set ("alphabet") Σ, ordered by the so-called information order $\sqsubseteq$ on Σ^∞ ([23, Example 1.6]), i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y, where we assume that the empty sequence ϕ is an element of Σ^∞. If $x \sqsubseteq y$ with $x \neq y$, we shall write $x \sqsubset y$.

For each $x, y \in \Sigma^\infty$ denote by $x \sqcap y$ the longest common prefix of x and y, and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x. Thus $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$, and $\ell(\phi) = 0$.

In theory of computation the fact that $x \sqsubseteq y$ is interpreted as the element y contains all the information provided by x, and thus the partially defined objects (finite words) customary represent stages of a computational process for which the totally defined objects (infinite words) contain exactly the amount of information provided by

Given a nonempty alphabet Σ, Smyth introduced in [128] a quasi-metric $d_\sqsubseteq$ on Σ^∞ given by $d_\sqsubseteq(x, y) = 0$ if $x \sqsubseteq y$, and $d(x, y) = 2^{-\ell(x \sqcap y)}$ otherwise (see also [61, 96, 104, etc]).

This quasi-metric has the advantage that its specialization order coincides with the order $\sqsubseteq$, and thus the quasi-metric space (Σ^∞, d) the information provided by $\sqsubseteq$ (compare Remark 1). Moreover, the metric $(d_\sqsubseteq)^s$ is given by $(d_\sqsubseteq)^s(x,y) = 0$ if $x = y$, and $(d_\sqsubseteq)^s(x,y) = 2^{-\ell(x \sqcap y)}$ otherwise; so that $(d_\sqsubseteq)^s$ is exactly the celebrated Baire metric on Σ^∞.

However, the quasi-metric $d_\sqsubseteq$ is unable to give us information on the degree of approximation to a word z from two different prefixes x, y of z. For instance, if we consider the totally defined object π and the partially defined ones $x = 3.14$ and $y = 3.141$, then it is clear that y contains more information on π than x, but $d_\sqsubseteq(x,\pi) = d_\sqsubseteq(y,\pi) = 0$, so $d_\sqsubseteq$ is not sensitive to this amount of information.

Motivated by this fact, we shall construct a fuzzy quasi-metric on Σ^∞ that preserves the advantages of $d_\sqsubseteq$ and that, in addition, permits us to measure, with the help of the parameter t, the degree of approximation to a given word of each of its prefix. Finally, we shall apply this construction to measure, in some representative cases, (fuzzy) distances between elements of $\mathcal{P}_0(\Sigma^\infty)$ via the Hausdorff fuzzy quasi-(pseudo-)metric.

Define a fuzzy set M in $\Sigma^\infty \times \Sigma^\infty \times [0,\infty)$ by

$M(x,y,0) = 0$ for all $x, y \in \Sigma^\infty$,

$M(x,x,t) = 1$ for all $x \in \Sigma^\infty$ and $t > 0$,

$M(x,y,t) = 1$ if $x \sqsubset y$ and $t > 2^{-\ell(x)}$,

$M(x,y,t) = 1 - 2^{\ell(x \sqcap y)}$ otherwise.

We wish to show that $(M,\wedge)$ is a fuzzy quasi-metric on Σ^∞. To this end we only prove that for each $x, y, z \in \Sigma^\infty$ and $t, s \geq 0$, one has $M(x,z,t+s) \geq M(x,y,t) \wedge M(y,z,s)$, because the rest of conditions in the definition of a fuzzy quasi-metric are obviously true.

Indeed, if $M(x,z,t+s)=1$, the conclusion is obvious. Assume now that $M(x,z,t+s)=1-2^{-\ell(x\sqcap z)}$. We distinguish two cases: (a) x is a prefix of z; (b) x is not a prefixes of z. In case (a) we have $M(x,z,t+s)=1-2^{-\ell(x)}$ and $t+s\leq 2^{-\ell(x)}$, and thus $M(x,y,t)=1-2^{-\ell(x\sqcap y)}$ because $t\leq 2^{-\ell(x)}$. Since $\ell(x)\geq \ell(x\sqcap y)$, it follows that $M(x,z,t+s)\geq M(x,y,t)$. In case (b) we have $M(x,z,t+s)=1-2^{-\ell(x\sqcap z)}$, and the conclusion follows immediately from the well-known facts that for each $x,y,z\in\Sigma^\infty$, one has: (i) $\ell(x\sqcap z)\geq \min\{\ell(x\sqcap y),\ell(y\sqcap z)\}$, and (ii) $\ell(x\sqcap z)=\ell(y\sqcap z)$ whenever x is a prefix of y but not a prefix of z.

We conclude that $(M,\wedge)$ is a fuzzy quasi-metric on Σ^∞.

Now, observe that if x,y are prefix of z, with $x\neq y$, and one obtains for some $t_0>0$, $M(x,z,t_0)>1$ and $M(y,z,t_0)=1$, then $2^{-\ell(y)}<t_0\leq 2^{-\ell(x)}$, so that $\ell(x)\leq\ell(y)$, i.e., $x\sqsubset y$; which shows that y is a better approximation to z than x.

Then, for each $z\in\Sigma^\infty\backslash\{\phi\}$, and each $x\sqsubset y$ we can define the degree of approximation of x to z, associated to $(M,\wedge)$, as the number $DA(x,z)=1/t_x$ where $t_x=\inf\{t>0: M(x,z,t)=1\}$. It is clear that $DA(x,z)=2^{\ell(x)}$.

In particular, for $x=3.14$ and $y=3.141$ as given above, one obtains for each $t\in]2^{-4},2^{-3}]$, $M(x,\pi,t)<1$ and $M(y,\pi,t)=1$, which agrees with the fact that y contains more information on π than x. Furthermore $DA(x,\pi)=2^3$ and $DA(y,\pi)=2^4$, which provides reasonable (and desirable) values on the degree of approximation of x and y to π, respectively.

Finally, we apply this approach to compute the distance between some interesting subsets of $\mathcal{P}_0(\Sigma^\infty)$ via the lower and upper Hausdorff fuzzy quasi-pseudo-metrics of $(\Sigma^\infty,M,\wedge)$.

Let $z\in\Sigma^\infty$ such that $\ell(z)=\infty$ and x be a prefix of z different from z, i.e., $x \sqsubset z$. Put $x_{\rightarrow} = \{y \in \Sigma^\infty : x \sqsubseteq y \sqsubset z\}$. Since $z \in \overline{x_{\rightarrow}}^{M} \cap \overline{x_{\rightarrow}}^{M^{-1}}$, it follows from Lemma 6.2 (1a), (1b), that for each $t > 0$,

$$H_M^-(\{z\}, x_{\rightarrow}, t) = H_M^+(x_{\rightarrow}, \{z\}, t) = 1.$$

Therefore (compare Remark 6.1), one has $\{z\} \leq_{H_M^-} x_{\rightarrow}$ and $x_{\rightarrow} \leq_{H_M^+} \{z\}$. The second relation is not a surprise because it can be computationally interpreted as that z contains at least the same amount of information of z than $x_{\rightarrow}$. However, the first relation seems certainly interesting because it can be computationally interpreted as that $x_{\rightarrow}$ contains at least the same amount of information of z than z, which is true because actually $x_{\rightarrow}$ has exactly the same amount of information of z than z.

Furthermore, it is easy to see that

$$H_M^-(x_{\rightarrow}, \{z\}, t) = 1 \Longleftrightarrow t > 2^{-\ell(x)},$$

so that $H_M(x_{\rightarrow}, \{z\}, t) = 1 \Longleftrightarrow t > 2^{-\ell(x)}$.

Bibliography

[1] C. Alegre, *Continuous operators of asymmetric normed spaces*, Acta Math. Hungar., in press.

[2] C. Alegre, J. Ferrer and V. Gregori, *On the Hahn-Banach theorem in certain linear quasi-uniform structures*, Acta Math. Hungar. **82** (1999), 325–330.

[3] C. Alegre, J. Ferrer and V. Gregori, *Quasi-uniformities on real vector spaces*, Indian J. Pure Appl., **28** (1997), 929–937.

[4] C. Alegre, J. Ferrer and V. Gregori, *Quasi-uniform structures in linear lattices*, Rocky Mountain J. Math., **23** (1993), 877–884.

[5] E. Alemany and S. Romaguera, *On right k-sequentially complete quasi-metric space*, Acta Math. Hungar. **75** (1997), 267–278.

[6] S. Andima, R. Kopperman and P. Nickolas, *An asymmetric Ellis theorem*, Topology Appl. **155** (2007), 146–160.

[7] J. Andres, J. Fišer, G. Gabor and K. Leśniak, *Multivalued fractals*, Chaos, Solitons and Fractals **24** (2005), 665–700.

[8] G. Artico and R. Moresco, *Notes on the topologies and uniformities of hyperspaces*, Rend. Seminar. Mat. Univ. Padova **63** (1980), 51–60.

[9] J.W. de Bakker and E.P. de Vink, *Control flow semantics*, The MIT Press, Cambridge, Massachusetts, 1996.

[10] J. W. de Bakker and E. P. de Vink. *Denotational models for programming languages: applications of Banach's fixed point theorem*, Topology Appl. **85** (1998), 35–52.

[11] G. Beer, *Topologies on Closed and Closed Convex sets*, Kluwer Academic Publishers, Dordrecht, 1993.

[12] G. Berthiaume, *On quasi-uniformities in hyperspaces*, Proc. Amer. Math. Soc. **66** (1977), 335–343.

[13] N. Bourbaki, *Topologie Generale*, Paris, Hermann, chaps. 1 and 2, 1940.

[14] N. Bourbaki, *General Topology*, Addison-Wesley, Reading, 1966.

[15] B.S. Burdick, *A note on completeness of hyperspaces*, in: S. Andima et al. (Eds.), General Topology and Application, 5th Northeast Conf., Marcel Dekker, New York, 1991, 19–24.

[16] J. Cao, *On hyperspace topologies of some quasi-uniform spaces*, J. Math. Research Exposition **11** (1991), 523–528.

[17] J. Cao, H.P.A. Künzi, I.L. Reilly and S. Romaguera, *Quasi-uniform hyperspaces of compact subsets*, Topology Appl. **87** (1998), 117–126.

[18] Y.J. Cho, M. Grabiec and V. Radu, *On nonsymmetric topological and probabilistic structures*, Nova Sci. Publ., New York, 2006.

[19] Á. Császár, *Complete extensions of quasi-uniform spaces, General Topology and its relations to modern analysis and algebra V* (Proc. Fifht Prague Topological Symp., 1981), Sigma Series in Pure Math. **3**, Heldermann, Berlin, 1983, 104–113.

[20] Á. Császár, *Fondements de la Topologie Générale*, Budapest-Paris, 1960.

[21] Á. Császár, *Foundations of General Topology*, Pergamon Press, Oxford, 1963.

[22] Á. Császár, *Strongly complete, supercomplete and ultracomplete spaces*, Mathematical Structures-Computational Mathematics-Mathematical Modelling, Papers dedicated to Prof. L. Iliev's 60th Anniversary, Sofia, 1975, 195–202.

[23] B.A. Davey and H.A. Priestley, *Introduction to lattices and order*, Cambridge Univ. Press 1990.

[24] J. Deák, *A bitopological view of quasi-uniform completeness*, I, Studia Sci. Math. Hungar. **30** (1995), 389–409; II **30** (1995), 411-431; **31** (1996), 385–404.

[25] A. Di Concilio, G. Gerla, *Quasi-metric spaces and point-free geometry*, Math. Struct. Comput. Sci. **16** (2006), 115–137.

[26] D. Doitchinov, *On completeness in quasi-metric spaces*, Topology Appl. **30** (1988), 127–148.

[27] D. Doitchinov, *On completeness of quasi-uniform spaces*, C.R. Acad. Bulg. Sci. (7) **41** (1988), 5–8.

[28] E.P. Dolzhenko and E.A. Sevast'yanov, *Sign-sensitive approsimations, the space of sign-sensitive weights. The rigidity and the freedom of a system*, Russian Acad. Sci. Dokl. Math., **48** (1994), 397–401.

[29] G.A. Edgar, *Measure*, Topology and Fractal Geometry, Springer, Berlin, New York, 1992.

[30] R.J. Egbert, *Products and quotients of probabilistic metric spaces*, Pacific J. Math. **24** (1968), 437–455.

[31] R. Engelking, *General Topology*, Polish. Sci. Publ., Warsaw, 1977.

[32] M. A. Erceg, *Metric spaces and fuzzy set theory*, J. Math. Anal. Appl. **69** (1979), 205–230.

[33] A. Fedeli, *On chaotic set-valued discrete dynamical systems*, Chaos, Solitons and Fractals, **23** (2005), 1381–1384.

[34] P. Flecher and W. Hunsaker, *Completeness using pairs of filters*, Topology Appl. **44** (1992), 149–155.

[35] P. Fletcher and W. F. Lindgren, *On quasi-uniform spaces*, Marcel Dekker, New York, (1982).

[36] M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rend. Circolo Mat. Palermo **22** (1906), 1–74.

[37] L.M. García-Raffi, S. Romaguera and E.A. Sánchez-Pérez, *On Hausdorff asymmetric normed linear spaces*, Houston J. Math. **29** (2003), 717–728.

[38] L. M. García-Raffi, S. Romaguera and E. A. Sánchez-Pérez, *Sequence spaces and asymmetric norms in the theory of computational complexity*, Math. Comput. Model. **36** (2002), 1–11.

[39] L.M. García-Raffi, S. Romaguera and E.A. Sánchez-Pérez, *The dual space of an asymmetric normed linear space*, Quaestiones Math. **26** (2003), 83–96.

[40] L. M. García-Raffi, S. Romaguera and E. A. Sánchez-Pérez, *The supremum asymmetric norm on sequence algebras: a*

general framework to measure complexity distances, Electronic Notes in Theoretical Computer Science **74** (2003). URL:Http://www.elsevier.nl/locate/entcs/volume74.html, 12 pages.

[41] L.M. García-Raffi, S. Romaguera and M. Schellekens, *Applications of the complexity space to the General Probabilistic Divide and Conquer Algorithms*, J. Math. Anal. Appl. **348** (2008), 346–355.

[42] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst. **64** (1994), 395–399.

[43] A. George and P. Veeramani, *On some results of analysis of fuzzy metric spaces*, Fuzzy Sets and Systems **90** (1997), 365-368.

[44] G. Gerla, *Fuzzy submonoids, fuzzy preorders and quasi-metrics*, Fuzzy Sets and Systems **157** (2006), 2356–2370.

[45] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, **27** (1988), 385–389.

[46] V. Gregori, J.A. Mascarell and A. Sapena, *On completion of fuzzy quasi-metric spaces*, Topology Appl. **153** (2005), 886–899.

[47] V. Gregori and S. Romaguera, *Fuzzy quasi-metric spaces*, Appl. Gen. Topol. **5** (2004), 129–136.

[48] C. Guerra and V. Pascucci, *Line-based object recognition using Hausdorff distance: from range images to molecular secondary structure*, Image and Vision Comput. J. **23** (2005), 405–415.

[49] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.

[50] J. J. Hull, *Document image similarity and equivalence detection*, International Journal on Document Analysis and Recognition **1** (1998), 37–42.

[51] D. P. Huttenlocher, G. A. Klanderman and W. J. Rucklidge, *Comparing images using the Hausdor distance*, IEEE Trans. Pattern Anal. Mach. Intelligence **15** (1993), 850–863.

[52] J. Isbell, *Uniform Spaces*, Providence, 1964.

[53] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets Syst. **12** (1984) 215–229.

[54] K. Keimel and W. Roth, *Ordered Cones and Approximation*, Springer-Verlag, Berlin (1992).

[55] J. L. Kelley, *Hyperspaces of continuum*, Trans. Amer. Math. Soc. **52** (1942), 23–26.

[56] J. C. Kelly, *Bitiopological spaces*, Proc. London Math. Soc. **13** (1963), 71–89.

[57] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika **11** (1975), 326–334.

[58] V. S. Krishnan, *A note on semiuniform spaces*, J. Madras Univ. Sect. B **25** (1955), 123–124

[59] R. Kruse, *Datastructures and program design*, Prentice-Hall, Inc.,(1984).

[60] H.P.A. Künzi, *An Introduction to Quasi-Uniform Spaces, in the book: Beyond Topology*, F. Mynard and E. Pearl editors; Contemporary Mathematics, American Mathematical Society, 2008, in press.

[61] H.P.A. Künzi, *Nonsymmetric topology*, Proc. Szekszárd Conference, Bolyai Soc. Math. Studies **4** 1993 Hungary (Budapest 1995), 303–338.

[62] H. P. A. Künzi, *Nonsymmetric distances and their associated topologies: about the origin of basic ideas in the area of asymmetric topology*, Handbook of the History of General Topology, ed. by C. E. Aull and R. lowen, Vol **3**, Hist. Topol. **3**, Kluwer Acad. Publ., Dordrecht, (2001), 853–968.

[63] H.P.A. Künzi and V. Kreinovich, *Static space-times naturally lead to quasi-pseudometrics*, Theoret. Comput. Sci. **405** (2008), 64–72.

[64] H.P.A. Künzi and S. Romaguera, *Left K-completeness of the Hausdorff quasi-uniformity*, Rostock. Math. Kolloq. **51** (1997), 287–298.

[65] H.P.A. Künzi and S. Romaguera, *Spaces of continuous functions and quasi-uniform convergence*, Acta Math. Hungar. **75** (1997), 287–298.

[66] H.P.A. Künzi and S. Romaguera, *Well-quasi-ordering and the Hausdorff quasi-uniformity*, Topology Appl. **85** (1998), 207–218.

[67] H.P.A. Künzi and C. Ryser, *The Bourbaki quasi-uniformity*, Topology Proc. **20** (1995), 161–183.

[68] C. Kuratowski, *Topology*, Vol **I** (transl.), Academi Press, New York, 1966.

[69] N. Levine and W.J. Stager Jr., *On the hyper-space of a quasi-uniform space*, Math. J. Okayama Univ. **15** (1971/72), 101–106.

[70] J.Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I and II*, Springer. Berlin. 1996.

[71] W.F. Lindgren and P. Fletcher, *A construction of the pair completion of a quasi-uniform space*, Canad. Math. Bull. **21** (1978), 53–59.

[72] Y. Liu, Z. Li, *Coincidence point theorems in probabilistic and fuzzy metric spaces*, Fuzzy Sets and Systems **158** (2007), 58–70.

[73] Y. Lu, C. L. Tan, W. Huang and L. Fan, *An Approach to Word Imagen Matching Based on Weighted Hausdorff Distance*, Sixth International Conference on Document Analysis and Recognition (ICDAR'01) (2001), 921–925.

[74] R. Lucchetti, *Convexity and Well-Posed Problems*, CMS Books in Mathematics, Springer, 2006.

[75] J. Marín, *An extension of Alaoglu's theorem for topological semicones*, Houston J. Math. **34** (2008), 795–806.

[76] S.G. Matthews, *Partial metric topology*, Ann. New York Acad. Sci. **728** (1994), 183–197.

[77] K. Menger, *Statistical metrics*, Proc. Nat. Acad. Sci. **28** (1942), 535–537.

[78] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–183.

[79] K. Morita, *Completions of hyperspaces of compact subsets and topological completions of open-closed maps*, General Topology and Appl. **4** (1974), 217–233.

[80] M.G. Murdeshwar and S.A. Naimpally, *Quasi-Uniform Topological Spaces*, Nordhoff, Groningen, 1966.

[81] C. Mustăta, *Extensions of semi-Lipschitz functions on quasi-metric spaces*, Ann. Numer. Theory Approx. **30** (2001), 61–67.

[82] C. Mustăta, *On the extremal semi-Lipschitz functions*, Ann. Numer. Theory Approx. **31** (2002), 103–108.

[83] C. Mustăta, *A Phelps type theorem for spaces with asymmetric norm*, Bul. Ştiinţ. Univ. Baia Mare, Ser. B, Matematică-Informatică, **18** (2002), 275–280.

[84] L. Nachbin, *Sur les espaces topologiques ordonnés*, C. R. Acad. Sci. **226** (1948), 774–775.

[85] V. Niemytzki, *Über die aciome des metrischen raumes*, Math. Ann. **104** (1931), 666–671.

[86] A. Peris, *Set-valued discrete chaos*, Chaos, Solitons and Fractals, **26** (2005), 19–23.

[87] W. J. Pervin, *Quasi-uniformization of topological spaces*, Math. Ann. **147** (1962), 316–317.

[88] D. Perrin; J.E. Pin. *Infinite Words. Automata, Semigroups, Logic and Games*. Pure and Appl. Math. Series, vol. **141**, Elsevier Acad. Press. 2004.

[89] I.L. Reilly, P.V. Subrahmanyam and M.K. Vamanamurthy, *Cauchy sequences in quasi-pseudo-metric spaces*, Monatsh. Math. **93** (1982), 127–140.

[90] R.T. Rockafellar, R.J.B. Wets, *Variational Analysis*, Comprehensive Studies in Mathematics, vol. **317**, Springer, 1998.

[91] J. Rodríguez-López, S. Romaguera, *Closedness of bounded convex sets of asymmetric normed linear spaces and the Hausdorff quasi-metric*, Bull. Belgian Math. Soc. - Simon Stevin **13** (2006), 551–562.

[92] J. Rodríguez-López and S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets Syst. **147** (2004), 273–283.

[93] J. Rodríguez-López, S. Romaguera, *The relationship between the Vietoris topology and the Hausdorff quasi-uniformity*, Topology Appl. **124** (2002), 451–464.

[94] J. Rodríguez-López, S. Romaguera, *Wijsman and hit-and-miss topologies of quasi-metric spaces*, Set-Valued Analysis **11** (2003), 323–344.

[95] J. Rodríguez-López, S. Romaguera and A. Sapena, *Casi-métricas difusas y dominios de computación*, Revista Iberomericana de Sistemas, Cibernética e Informática **2** (2005).

[96] J. Rodríguez-López, S. Romaguera, O. Valero, *Denotational semantics for programming languages, balanced quasi-metrics and fixed points*, Internat. J. Comput. Math. **85** (2008), 623–630.

[97] S. Romaguera, *Left K-completeness in quasi-metric spaces*, Math. Nachr. **157** (1992), 15–23.

[98] S. Romaguera, *On hereditary precompactness and completeness in quasi-uniform spaces*, Acta Math. Hungar. **73** (1996), 159–178.

[99] S. Romaguera, J. M. Sánchez-Álvarez and M. Sanchis, *On balancedness and* D*-completeness of the space of semi-Lipschitz functions*, Acta Math. Hugar. **120** (4) (2008), 383–390.

[100] S. Romaguera, E.A. Sánchez-Pérez, O. Valero, *Computing complexity distances between algorithms*, Kybernetika **36** (2003), 569–582.

[101] S. Romaguera and M. Sanchis, *Applications of utility functions defined on quasi-metric spaces*, J. Math. Anal. Appl. **283** (2003), 219–235.

[102] S. Romaguera and M. Sanchis, *On semi-Lipschitz functions and best approximation in quasi-metric spaces*, J. Approximation Theory **103** (2000), 292–301.

[103] S. Romaguera and M. Sanchis, *Properties of the normed cone of semi-Lipschitz functions*, Acta Math. Hungar. **108** (1-2) (2005), 55–70.

[104] S. Romaguera, A. Sapena, P. Tirado, *The Banach fixed point theorem in fuzzy quasi-metric spaces with application to the domain of words*, Topology Appl. **154** (2007), 2196–2203.

[105] S. Romaguera, A. Sapena, O. Valero, *Quasi-uniform isomorphisms in fuzzy quasi-metric spaces, bicompletion and D-completion*, Acta Math. Hungar. **114** (2007), 49–60

[106] S. Romaguera and M. Schellekens, *Duality and quasi-normability for complexity spaces*, Appl. Gen. Topol. **3** (1) (2002), 91–112.

[107] S. Romaguera and M. Schellekens, *Partial metric monoids and semivaluation spaces*, Topology Appl. **153** (2005), 948–962.

[108] S. Romaguera and M. Schellekens, *Quasi-metric properties of complexity spaces*, Topology Appl. **98** (1999), 311–322.

[109] S. Romaguera and M. Schellekens, *The quasi-metric of complexity convergence*, Quaestiones Math. **23** (2000), 359–374.

[110] S. Romaguera, O. Valero, *On the structure of the space of complexity partial functions*, Internat. J. Comput. Math. **85** (2008), 631–640.

[111] H. Román-Flores, *A note on transitivity in set-valued discrete systems*, Chaos, Solitons and Fractals, **17** (2003), 99–104.

[112] J.M. Sánchez-Álvarez, *On semi-Lipschitz functions with values in a quasi-normed linear space*, Appl. Gen. Topology Vol. **6**,(2) (2005), 217–228.

[113] M.A. Sánchez-Granero, *Weak completeness of the Bourbaki quasi-uniformity*, Appl. Gen. Topology **2** (2001), 101–112.

[114] M. Sanchis, M. Tkachenko, *Totally Lindelöf and totally ω-narrow paratopological vector spaces*, Topology Appl. **155** (2008), 322–334.

[115] M. Schellekens, *The correspondence between partial metrics and semivaluations*, Theoret. Comput. Sci. **315** (2004), 135–149.

[116] M. Schellekens, *The Smyth completion: A common foundation for denotational semantics and complexity analysis*, Proc. MFPS 11, Electron. Notes Comput. Sci. **1** (1995), 211–232.

[117] B. Schweizer, A. Sklar, *Probabilisitc Metric Spaces*, North-Holland, 1983.

[118] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 314–334.

[119] D. S. Scott, *Domains for Denotational Semantics*, Languages and Programming: Proceedings 1982. Ed. M. Nielson and E.M. Schmidt. Springer-Verlag, Berlin. Lecture Notes in Computer Science 140.

[120] A.K. Seda, P. Hitzler, *Generalized distance functions in the theory of computation*, The Computer Journal, in press.

[121] C. Sempi, *Hausdorff distance and the completion of probabilistic metric spaces*, Boll. U.M.I. **6-B** (1992), 317–327.

[122] B. Sendov, *Hausdorff distance and image processing*, Russian Math. Surveys **59** (2004), 319–328.

[123] J. Serra, *Image Analysis and Mathematical Morphology*, Academic Press, 1982.

[124] J.L. Sieber and W.J. Pervin, *Completeness in quasi-uniform spaces*, Math. Ann. **158** (1965), 79–81.

[125] W. Sierpinski, *General Topology*, 2nd Ed. (transl), University of Toronto Press, Toronto, 1956.

[126] R. Sikora, S. Piramuthu, *Efficient genetic algorithm based data mining using feature selection with Hausdorff distance*, Inf. Tech. Management **6** (2005), 315–331.

[127] M. B. Smyth, *Completeness of quasi-uniform and syntopological spaces*, J. London Math. Soc. **49** (1994), 385–400.

[128] M.B. Smyth, *Quasi-uniformities: Reconciling domains with metric spaces*, In: M. Main *et al.* (Eds), Mathematical Fondations of Programming Language Semantics, Third Workshop, Tulanem, 1987, Lecture Notes in Computer Science, vol. **298** (Berlin: Springer), 236-253.

[129] M. B. Smyth, *Totally bounded spaces and compact ordered spaces as domains of computation*, in Topology and Category Theory in Computer Science, G. M. Reed, A. W. Rosco, and R. F. Wachter, Eds., Clarendon, Oxford (1991), pp. 207–229.

[130] A.Stojmirovič, *Quasi-metric space with measure*, Topology Proc. **28** (2004), 655–671.

[131] R. A. Stoltenberg, *Some Properties of Quasi-Uniform Space*, Proc. London Math. Soc., **17** (1967), 226–240.

[132] Ph. Sünderhauf, *Quasi-uniform completeness in terms of Cauchy nets*, Acta Math. Hungar. **69** (1995), 47–54.

[133] R.M. Tardiff, *Topologies for probabilistic metric spaces*, Pacific J. Math. **65** (1976), 233–251.

[134] P. Veeramani, *Best approximation in fuzzy metric spaces*, J. Fuzzy Math. **9** (2001), 75–80.

[135] L. Vietoris, *Bereiche zweiter ordnung*, Monatshefte für Mathematik und Physik **33** (1923), 49–62.

[136] Y. Wang, G. Wei, *Characterizing mixing, weak mixing and transitivity of induced hyperspace dynamical systems*, Topology Appl. **155** (2007), 56–68.

[137] P. Waszkiewicz, *Partial metrisability of continuous posets*, Math. Struct. Comput. Sci. **16** (2006), 359–.

[138] W. A. Wilson, *On puasi-metric spaces*, American J. Math. **53** (1931), 675–684.

[139] L. A. Zadeh, *Fuzzy sets*, Inform and Control, **8** (1965), 338–353.

[140] C. Zhao, W. Shi and Y. Deng, *A new Hausdorff distance for image matching*, Pattern Recognition Letters **26** (2005), 581–586.