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Duality in spaces of $p-$integrable functions with respect to a vector measure

PhD Dissertation

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CERTIFICA que la presente memoria **Duality in spaces of $p-$integrable functions with respect to a vector measure** ha sido realizada bajo mi dirección en el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, por IRENE FERRANDO PALOMARES y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas.

Y para que así conste, en cumplimiento con la legislación vigente, presentamos ante el Departamento de Matemáticas de la Universidad Politécnica de Valencia, la referida Tesis Doctoral, firmando el presente certificado.

En Valencia, Septiembre de 2009

Fdo. Enrique A. Sánchez Pérez.

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Resumen

La tesis tiene como objetivo principal el estudio de la dualidad vectorial entre los espacios $L^p(m)$ y $L^q(m)$ de funciones integrables con respecto a una medida vectorial con valores en un espacio de Banach X, con $p, q > 1$ exponentes reales conjugados. La clave de la dualidad es la definición de una forma bilineal $\Phi : L^p(m) \times L^q(m) \to X$ dada por el operador integración, que a cada par (f, g) en $L^p(m) \times L^q(m)$ le asocia $\int_\Omega fgdm$. Mediante esta forma bilineal se definen dos topologías intermedias para el espacio $L^p(m)$. La más débil es la topología $m-$débil, que corresponde a la topología de la convergencia débil de la integrales. Además de estudiar sus propiedades, se prueba que para $p > 1$ esta topología coincide con la débil del espacio $L^p(m)$. La importancia de este resultado radica en que, al no conocerse una representación concreta del dual del espacio $L^p(m)$, es muy interesante describir la convergencia débil en términos de la convergencia débil de las integrales en el espacio de Banach X. La $m-$topología corresponde a la convergencia fuerte de las integrales en X, y puede coincidir en casos extremos con la débil y con la fuerte de $L^p(m)$. Se estudian sus propiedades, en particular se dan condiciones para asegurar que un subconjunto de $L^p(m)$ sea $m-$compacto.

Estas topologías, en particular la $m-$débil, son útiles para la descripción del predual del espacio $L^p(m)$ en términos de productos tensoriales. Esta construcción se describe de forma detalla en el tercer capítulo de la memoria de la tesis. Cabe destacar de éste un resultado que caracteriza aquellos operadores definidos en $L^p(m)$ con rango en X que se pueden escribir como una integral. Aunque sin duda el resultado más relevante es el que, bajo cierta hipótesis de compacidad de la bola unidad (equivalente a la reflexividad del espacio $L^p(m)$) ofrece una representación de $L^p(m)$ como el dual del producto tensorial $L^q(m) \otimes X^*$, dotado de una norma. Este resultado es clave para obtener una generalización de los resultados de dualidad para los espacios clásicos de funciones $p-$integrables.

La $m-$topología permite definir un concepto de sumabilidad en $L^p(m)$ basada en la dualidad vectorial, los llamados operadores $m-r-$sumantes

definidos en espacios de funciones integrables con respecto a una medida vectorial, que se estudian en el cuarto capítulo. Esta definición generaliza la sumabilidad clásica. Se estudian las propiedades de estos operadores, y se presentan ejemplos que ponen de manifiesto su interés. En la misma línea que en la teoría clásica, obtenemos teoremas de dominación y de factorización. La última sección de este capítulo está dedicada a la descripción de estos espacios de operadores como el dual de un espacio vectorial, extendiendo así la teoría clásica de Groethendieck, para el caso de operadores definidos en espacios $L^p(m)$.

En el último capítulo de la memoria, las técnicas de la dualidad vectorial se aplican a los espacios de Orlicz respecto a una medida vectorial, $L^\Phi(m)$, que generalizan a los $L^p(m)$. Se estudian propiedades de los espacios de Orlicz vectoriales y bajo la condición Δ_2 para la función de Young, se caracterizan el espacio de multiplicadores entre $L^\Phi(m)$ y $L^1(m)$. Como una aplicación de estos resultados, se caracterizan aquellos operadores que factorizan a través de un espacio de Orlicz vectorial.

Resum

Aquesta tesi té com a objetiu principal l'estudi de la dualitat vectorial entre els espais $L^p(m)$ i $L^q(m)$ de funcions integrables respecte a una mesura vectorial amb valors en un espai de Banach X, amb $p, q > 1$ exponents reals conjugats. La clau de la dualitat és la definició d'una forma bilineal $\Phi : L^p(m) \times L^q(m) \to X$ donada per l'operador integració, que a cada parell (f, g) en $L^p(m) \times L^q(m)$ li associa $\int_\Omega fgdm$. Mitjançant aquesta forma bilineal es defineixen dues topologies intermitges per a l'espai $L^p(m)$. La més dèbil és la topología $m-$dèbil, que correspon a la topologia de la convergència dèbil de les integrals. A banda d'estudiar les seues propietats, provem que per a $p > 1$ aquesta topologia coincideix amb la dèbil de l'espai $L^p(m)$. La importància d'aquest resultat es basa en el fet que, com que no es coneix una representació concreta del dual de l'espai $L^p(m)$, és molt interessant descriure la convergència dèbil en termes de la convergència dèbil de les integrals en l'espai de Banach X. L' $m-$topologia correspon a la convergència forta de las integrals en X, i pot coincidir en casos extrems amb la dèbil y amb la forta de $L^p(m)$. Estudiem les seues propietats i en particular, donem condicions suficients perquè un subconjunt de $L^p(m)$ siga $m-$compacte.

Aquestes topologies, en particular l' $m-$débil, són útils per a la descripció de l'espai predual de l'espai $L^p(m)$ en termes de productes tensorials. Aquesta construcció es descriu de forma detallada en el tercer capítol d' aquesta memòria. Hem de destacar d'aquest capítol un resultat que caracteritza aquells operadors definits a $L^p(m)$ amb rang en X que es poden escriure com una integral. Tanmateix, el resultat més rellevant és el que, sota certa hipòtesi de compacitat de la bola unitat (equivalent a la reflexivitat de l'espai $L^p(m)$), ens dóna una representació de $L^p(m)$ com al dual del producte tensorial $L^q(m) \otimes X^*$ dotat de certa norma. Aquest resultat és clau per a obtenir una generalització dels resultats de dualitat per als espais clàssics de funcions $p-$integrables.

L' $m-$topologia permet definir un concepte de sumabilitat a $L^p(m)$ basada en la dualitat vectorial. Als operadors corresponents els anome-

nem operadors $m - r-$sumants definits en espais de funcions integrables respecte a una mesura vectorial i els estudiem al quart capítol. Aquesta definició generalitza la sumabilitat clàssica. Estudiem les propietats d'aquests operadors i presentem exemples que posen de manifest el seu interés. En la mateixa línia que en la teoria clàssica, obtenem teoremes de dominació i de factorizació. L'última secció d'aquest capítol està dedicada a la descripció d'aquests espais d'operadors com al dual d'un espai vectorial, extenent així la teoria clàssica de Groethendieck per al cas d'operadors definits en espais $L^p(m)$.

En l'últim capítol de la memòria, les tècniques de la dualitat vectorial s'apliquen als espais d'Orlicz respecte a una mesura vectorial, $L^\Phi(m)$, que generalitzaran als espais $L^p(m)$. Estudiarem propietats dels espais d'Orlicz vectorials i sota la condició Δ_2 per a la funció de Young, caracteritzarem l'espai de multiplicadors entre $L^\Phi(m)$ i $L^1(m)$. Com a aplicació d'aquests resultats, caracteritzem aquells operadors que factorizen a través d'un espai d'Orlicz vectorial.

Summary

The main objective of this memoir is the study of the vector valued duality between the spaces $L^p(m)$ and $L^q(m)$ of integrable functions with respect to a vector measure with values in a Banach space X, with $p, q > 1$ conjugated real numbers. The key of this duality relationship is the definition of a bilinear map. Let $\Phi : L^p(m) \times L^q(m) \to X$ defined as follows, for $(f, g) \in L^p(m) \times L^q(m)$, $\Phi(f, g) := \int_\Omega f g dm$. Through this bilinear form we define two intermediates topologies for the space $L^p(m)$. The weakest one, the $m-$weak topology, corresponds to the topology of weak convergence of the integrals. We study the main properties and we show that, for $p > 1$, it coincides with the weak topology of $L^p(m)$. Since there is not a a concrete representation of the dual of $L^p(m)$, it is very interesting to describe the weak convergence in terms of the weak convergence of the integrals in the Banach space X. The $m-$topology corresponds to the topology of strong convergence of the integrals in X. It can coincide in extreme cases with the weak topology and with the norm topology of $L^p(m)$. We study some properties, particularly we give sufficient conditions to ensure the $m-$compactness of a subset of $L^p(m)$.

These topologies, in particular the $m-$weak topology, are extremely useful to describe the predual of $L^p(m)$ in terms of a tensor product. This construction is carefully described in Chapter 3 of this memoir. It is necessary to stand out in this chapter a result that characterizes those operators from $L^p(m)$ into X that can be represented as an integral. In fact, it is the key to prove the most relevant result in this chapter, where we represent the space $L^p(m)$ as the dual the tensor product $L^q(m) \otimes X^*$ endowed with a particular norm. In order to prove this result we assume an hypothesis of compactness of the unit ball of $L^p(m)$; this will be equivalent to the reflexivity of this space. This is the clue to obtain a generalization of some duality result for classical L^p-spaces.

The $m-$topology allows us to define a notion of summability for spaces of $p-$integrable functions. It s based in the vector valued duality. The so called $m - r-$summing operators, defined on spaces of $p-$integrable

functions with respect to a vector measure are studied in Chapter 4. This definition generalize the classical summability. We also study the spaces of sequences that are $m-r-$summable. We investigate the properties of these operator spaces and we present some revealing examples. Following the ideas of the classical theory of summing operators, we prove some domination and factorization theorems. The last section of this chapter is devoted to the description of this operator spaces as the dual of a tensor product. In this way we extend the classical Grothendieck's theory for operators defined on $L^p(m)$.

In the last chapter of the memoir, the vector duality techniques are applied to the study of Orlicz spaces with respect to a vector measure, that are the natural generalization of L^p spaces. We study fundamental properties of vector Orlicz spaces. Assuming the Δ_2-property for the Young function Φ, we characterize the space of multiplication operators between $L^\Phi(m)$ and $L^1(m)$. As an application of this result, we characterize those operators that factorize through a vector Orlicz space.

Contents

Introduction VIII

1. Preliminaries **1**

2. Weak topologies in $L^p(m)$ **13**
- 2.1. $m-$weak topology . . . 14
  - 2.1.1. Metrizability of the unit ball . . . 14
  - 2.1.2. Weak convergence in bounded sets of $L^p(m)$ . . . 17
- 2.2. $m-$topology . . . 22
  - 2.2.1. Metrizability of the unit ball . . . 22
  - 2.2.2. m-compactness in $L^p(m)$ . . . 24
- 2.3. Appendix . . . 30

3. Tensor product representation of the (pre)dual of $L^p(m)$ **35**
- 3.1. Tensor product representation . . . 40
- 3.2. Examples . . . 49

4. Summability in $L^p(m)$: $m-r-$summing operators **53**
- 4.1. Preliminaries . . . 53
- 4.2. Definitions and properties . . . 57
  - 4.2.1. Summability of sequences in $L^p(m)$ . . . 58
  - 4.2.2. Operators with range in $L^p(m)$ . . . 62
  - 4.2.3. Examples . . . 67
- 4.3. Comparing spaces of summing operators . . . 69
  - 4.3.1. Pietsch type theorems . . . 69
  - 4.3.2. Consequences . . . 78
- 4.4. Applications . . . 84
- 4.5. Tensor product representation . . . 88

5. Vector measure duality for Orlicz spaces **99**
- 5.1. Definition and properties . . . 100

5.2. Multiplication Operators 107
5.3. Applications 112

Index **114**

References **119**

Introduction

The main objective of this work is to develop a "vector duality" theory for the space $L^p(m)$ of integrable functions with respect to a Banach space valued measure m. When $p, q > 1$ are conjugated real numbers, the key for constructing such a vector duality is a bilinear form $\Phi : L^p(m) \times L^q(m) \to X$, with X the Banach space where the measure m takes its values. For a pair $(f, g) \in L^p(m) \times L^q(m)$ it is defined by $\Phi(f, g) := \int_\Omega fg dm$.

In the last fifty years a lot of mathematicians have worked in the development of a general theory of vector measures and particularly in the study of the spaces of integrable functions with respect to a vector measure. Remarkable references in this topic are the book of J. Diestel and J. J. Uhl, "Vector measures" and the seminal articles of R. G. Bartle, N. Dunford and J. Schwartz ([2]) and D. R. Lewis ([46]).

In the early nineteens, G.P. Curbera analyzed the spaces of integrable functions with respect to a vector measure from the point of view of the Banach lattices and the Banach function spaces. The natural duality in these spaces is given by the elements of the Köthe dual and is represented by an integral. The spaces of $p-$integrable functions were introduced by E. A. Sánchez Pérez in [70], and the natural duality defined on them is given by the bilinear form Φ defined above, that extends in a sense the integral representation of the duality given by the Köthe dual and the integral.

These spaces turn out be a useful tool to represent a large class of Banach lattices. For $p = 1$, G. P. Curbera proved in [12, Theorem 8] that every order continuous Banach lattice with weak order unit is lattice isometric to a space of integrable functions with respect to a vector measure. The corresponding result for $p > 1$ occurs in [31, Proposition 2.4]. A. Fernández et al. showed that every $p-$convex order continuous Banach lattice with weak unit is lattice isomorphic to an $L^p(m)$ of a vector measure.

Our work started with the objective of characterize the dual of the space $L^p(m)$ when $p > 1$. For $p = 1$ this study was first done by L. Egghe in [30]. In [53], S. Okada adapts the ideas of [30] in order to obtain a concrete representation of the elements of the dual of $L^1(m)$. In Theorem 3.1.6

we obtain a representation of the (pre)dual of the space $L^p(m)$ in terms of the dual of a normed tensor product.

G. Curbera in [13] and S. Okada in [53] studied independently a characterization of the weak convergence of nets on bounded subsets of $L^p(m)$ by the weak convergence of the integrals of the net elements in X. They concluded that this characterization was not possible in general. The natural question for $p > 1$ is answered in Theorem 2.1.7.

Our work is presented in five chapters. In the Preliminaries we establish the notation of the memoir and we recall the main definitions and properties of the theory of vector measure and integration that we will use later. We also introduce in this chapter the notion of weak $p-$integrability.

The second chapter is devoted to the study of two intermediate topologies defined for $L^p(m)$ when $p > 1$. These topologies are naturally defined when we consider the vector duality relationship between $L^p(m)$ and $L^q(m)$ given by the bilinear map Φ.

The $m-$weak topology corresponds to the topology of pointwise convergence on the norming subset of the unit ball of the dual of $L^p(m)$ defined by

$$\Gamma := \{|\gamma_{g,x^*}| : g \in L^q(m), x^* \in X^*\},$$

where $\gamma_{g,x^*}(f) := \int_\Omega fgd\langle m, x^*\rangle, f \in L^p(m)$. We study the metrizability of the closed unit ball of $L^p(m)$ when endowed with the $m-$weak topology and we give an explicit formula of the metric in Theorem 2.1.2. The interest of this result matters in the following fact: for $p > 1$, the $m-$weak topology coincides on bounded subsets of $L^p(m)$ with the weak topology, as shown in theorem 2.1.7. We also give in this section some conditions to ensure that the set Γ is a James boundary for the unit ball of the dual of $L^p(m)$.

The $m-$topology, coarser than the weak topology and weaker than the norm one, is generated by the family of seminorms

$$\Lambda := \{\zeta_g : g \in L^q(m)\},$$

where $\zeta_g(f) := \left\|\int_\Omega fgdm\right\|_X, f \in L^p(m)$. It corresponds to the topology of strong convergence of the integral on X. We study the main properties and we give conditions to ensure the compactness of a subset of $L^p(m)$ with respect to this topology in Proposition 2.2.5.

The $m-$weak topology is useful to describe the (pre)dual of $L^p(m)$ in terms of the dual of a tensor product. This construction is described in the third chapter of this memoir. In [72] a representation of the (pre)dual of the space of vector measure $p-$integrable functions was already obtained.

However this result gives only a partial answer to the general representation theorem, since it is only valid under certain restriction for the measure m –positivity– and the space $L^p(m)$ that sometimes are not easy to check, see for instance [70, Section 3].

So Chapter 3 of this memoir is presented as follows. In the first section we prove a characterization theorem for those operators $T : L^p(m) \to X$ that can be represented by an integral. This result is a consequence of a Radon-Nikodým theorem for scalarly dominated vector measures that was proved by Musial in [52]. In the second, and the main part of this third chapter we give three approaches in order to obtain the tensor product representation theorem of the (pre)dual of the space $L^q(m)$. For this aim we introduce some topologies for the tensor product of the space $L^p(m)$ and the dual of the Banach space X where the measure takes values. The main result of this part, Theorem 3.1.6, ensures that under a certain compactness condition for the unit ball of $L^q(m)$, the space $L^p(m)$ is isometrically isomorphic to the dual of a particular normed tensor product. We finish this part with a corollary of this theorem that gives us the natural "vector measure" version of the classical result that ensures that the dual of a Banach space with the norm topology coincides with the dual of the space with the weak topology.

In the last section of this chapter we give two examples. The first one deals with a measure with values in an Orlicz space; we provide an alternative formula to define the norm in the dual of the space of $q-$integrable functions with respect such a measure. In the second one we provide a characterization of the space of $q-$integrable functions with respect to a vector measure that is induced by a kernel operator.

The fourth chapter is is devoted to the study of summability in the spaces $L^p(m)$. In order to apply the vector duality we give a definition of summability related with the $m-$topology, that generalize the classical summability for the class of spaces of $p-$integrable functions with respect to a vector measure. In the first section we define the $m-r-$summing operators, that are those (linear and continuous) operators $T : L^p(m) \to Y$, such that for every finite choice of functions $f_1, \ldots, f_n$ in $L^p(m)$ there is a constant $C > 0$ so that

$$\left(\sum_{i=1}^{n} \|T(f_i)\|_Y^r\right)^{\frac{1}{r}} \leq C \cdot \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r\right)^{\frac{1}{r}}.$$

This definition is clearly related with the $m-$topology for the space $L^p(m)$, the topology of convergence of the integrals, and coincides with the classical definition whenever m is a scalar measure. As for the classical $r-$

summability, our notion is very close to the summability of $L^p(m)-$valued sequences; that is why we introduce the definition and the properties of the spaces of $m-r-$summable $L^p(m)-$valued sequences. The $m-r-$summing operators are exactly those that transform $m-r-$summable sequences into $L^p(m)$ into strongly summable $Y-$valued sequences, as proved in Theorem 4.2.2.

It is natural to consider those operators $S : Z \to L^p(m)$ that transform weakly $r-$summable sequences into $m-r-$summable ones. We define in section 4.2.2 the weak $m-r-$summing operators. In some sense, these operator spaces complete the set of technical tools that are necessary for the study of the summability associated to the $m-$topology of $L^p(m)$. In fact the composition of a weak $m-r-$summing operator $S : Z \to L^p(m)$ with an $m-r-$summing operator $T : L^p(m) \to Y$ is $r-$summing in the classical sense. For $r = 1$ we show that, under adequate assumptions, the converse is also true. Theorem 4.2.8 ensures that $1-$summing operators can be factorized through a weak $m-r-$summing operator and an $m-r-$summing one.

We finish this section with some examples of particular $m-r-$summing operators. The last one corresponds to a Hille-Tamarkin type operator which takes values in a space of Bochner integrable functions.

The third section of this chapter deals with the relationship between the classes of summing operators that have been introduced in the previous one. We begin by proving two Pietsch type theorems for $m-r-$summing operators. In the first one we assume a property on the space $L^p(m)$. The condition assumed is not very restrictive as shown in Examples 4.3.1 and 4.3.2. They represent in some sense two opposite extreme situations regarding the measure m; one of them has a "small" range and the other one "big" range. However, in both cases the associated space has the property. We also prove a generalized version of a Pietsch type domination theorem (see Theorem 4.3.6). Our domination and factorization theorems for $m-r-$summing operators are the key to find some relationships between different spaces of summing operators.

In the classical theory of operator ideal there are several results concerning the coincidence of the spaces of $r-$summing operators for different values of r. In order to generalize these results into our framework of operators defined on $L^p(m)$ spaces we prove in Proposition 4.3.7 a generalization of the classical inclusion theorem for $m-r-$summing operators. Proposition 4.3.9 provides a new version of a classical result due to Maurey (see [50]) that ensure the coincidence of the spaces of $2-$summing and $r-$summing operators between Banach spaces when $2 < r < \infty$ and the range of the operators has cotype 2.

We finish this section with some applications. The first one is given by the Maurey-Rosenthal factorization theory applied in this setting. We prove in Theorem 4.4.1 that, under the assumption of $r-$concavity of the Banach space where the measure m takes its values, if the identity is $m-r-$summing therefore $L^p(m)$ is isomorphic to an space of integrable functions with respect to a scalar measure. The second application corresponds to a generalization of the ideal of mixing operators. Our objective is to give a definition for operators defined in $L^p(m)$ in order to adapt the classical operator ideal scheme. For this aim we define the $(s, m-r)-$mixing operators that are those (linear and continuous) operators $T : L^p(m) \to Y$, such that for each $s-$summing operator S from Y into another Banach space Z, the composition $S \circ T$ is $m-r-$summing. In a similar way we define the $(m-s, r)-$mixing operators, in this case for $T \in \mathcal{L}(Y, L^p(m))$. We also obtain a domination theorem for this kind of operators.

The last section of this chapter is devoted to the study of the space of $m-r-$summing operators by means of a tensor product representation. The objective is to obtain a Grothendieck type representation of an operator ideal as the dual of a normed tensor product. Although our operator spaces $\Pi_r^m(L^p(m), Y)$ are not components of an operator ideal, we intend, through the definition of a particular norm for the tensor product $L^p(m) \otimes Y$ inspired in the Chevet-Saphard norms, to obtain a representation of the space of $m-r-$summing operators as the dual of this tensor product when we consider the trace duality.

The last chapter corresponds to an application of the vector duality theory developed above in a more generalized context. As happens in the classical case, Orlicz spaces provide an adequate setting for an extension of the ideas that hold for the spaces of $p-$integrable functions. In [21], O. Delgado define the Orlicz spaces with respect to a vector measure m, $L^\Phi(m)$, where ϕ is a Young's function; she studied some properties of these spaces in order to prove some results about the inclusion of them into the space $L^1(m)$. Our aim here is to continue the work done by A. Fernández et al. in [7] that deals with the study of multiplication operators between spaces $L^p(m)$.

We begin by recalling the basic notions about the construction of Orlicz spaces, and the natural extension to Orlicz spaces with respect to a vector measure. Lemma 5.2.2 provides a Hölder inequality for Orlicz spaces with respect to a vector measure, and is the key to apply similar arguments as those of the vector measure duality in a more general context. Theorems 5.2.7 and 5.2.9 correspond to the main results in this chapter, they characterize the (weak) Orlicz spaces with respect to a vector measure as an space of multiplication operators between (weak) Orlicz spaces.

As an application of this theory we characterize those operators between Banach function spaces that factorizes through Orlicz spaces with respect to a vector measure. In fact these spaces turns out to be optimal domains for such operators.

As a final comment for this introductory section, we must say that we have tried to present our analysis of the vector duality on spaces of integrable functions with respect to a vector measure following the "topological guide". We consider that this abstract presentation is useful also for explaining the consequences on the theory of Banach function spaces and operators on them as applications of the main results obtained in earlier chapters; a direct exposition of these results, from a different point of view, is also possible. Some of the papers on the results of this thesis have been written using this second scheme.

Chapter 1

Preliminaries

In this first chapter our aim is to introduce the framework of the integration with respect to a vector measure by presenting the main properties of the spaces of vector measure integrable functions, and also to recall some notions about operator theory.

The notation is standard. We deal with real Banach spaces. If X is a Banach space, we denote by X^* its dual space and by $B(X)$ its unit ball. Throughout this chapter we fix a *positive finite measure space* (Ω, Σ, μ), where Ω is a set, Σ is a $\sigma-$algebra of subsets of Ω and μ is a finite positive measure. We denote by χ_A the characteristic function of a set $A \in \Sigma$, and by $\mathcal{S}(\Sigma)$ the set of real simple functions. We say that a property $\mathcal{P}$ happens in Ω $\mu-$almost everywhere (briefly, $\mu-$a.e.) if $\mathcal{P}$ holds in $\Omega \setminus A$ where $A \in \Sigma$ with $\mu(A) = 0$.

By $L^0(\mu)$ we denote the space of equivalence classes of $\mu-$a.e. equal real $\Sigma-$measurable functions defined on Ω. $L^0(\mu)$ is a real vector lattice when it is endowed with the natural $\mu-$a.e. order.

A linear subspace $X(\mu)$ of $L^0(\mu)$ is an *order ideal* if $f \in X(\mu)$ whenever f is a function in $L^0(\mu)$ such that there is some $g \in X(\mu)$ satisfying $|f| \leq |g|$ $\mu-$a.e. A positive function $e \in X(\mu)$ is a *weak order unit* of $X(\mu)$ whenever $f \wedge (ne)_n \uparrow f$ for every $f \in X(\mu)$. A norm $\|\cdot\|_{X(\mu)}$ in $X(\mu)$ is a *lattice norm* if $\|f\|_{X(\mu)} \leq \|g\|_{X(\mu)}$ for $f, g \in X(\mu)$ such that $|f| \leq |g|$. A normed space that is complete when endowed with a lattice norm, is a *Banach lattice*.

An order ideal $X(\mu)$ of $L^0(\mu)$ is a *Banach function space* (briefly B.f.s.) based on the measure space (Ω, Σ, μ) if it contains the set of simple functions $\mathcal{S}(\Sigma)$ and if it is a complete space continuously included in the space of $\mu-$integrable functions, $L^1(\mu)$, when endowed with a lattice norm $\|\cdot\|_{X(\mu)}$. A B.f.s. $X(\mu)$ is said to be *order continuous* whenever every downward direct net $(f_\alpha)_\alpha$ in the positive cone $X(\mu)^+$ of $X(\mu)$ such that $f_\alpha \downarrow 0$, satisfies $\lim_\alpha \|f_\alpha\|_{X(\mu)} = 0$. For the particular case of Banach function spaces, order continuity is equivalent to $\sigma-$order continuity, for extended details see Remark 2.5 in [58].

A Banach lattice $(X, \|\cdot\|_X)$ has the *Fatou property* if every norm bounded increasing sequence $(x_n)_n$ in X^+ attempt his supremum $x \in X$. Moreover, if $\|x_n\|_X \uparrow \|x\|_X$, we say that X has the *$\sigma-$Fatou property*.

The definition above of Banach function space is the one in [48, Definition 1.b.17]. In [3, Definition I.1.3] a different one is given; although coming by different approaches, the definitions only differ in the Fatou property (assumed in [3, Definition I.1.3]), as shown in [3, Theorem I.1.7] and in [48, p.30]. In this memoir we will use notion of B.f.s. without this assumption. Nevertheless some properties of B.f.s. that appear in [3] will be useful in our framework, and it can be prove that they hold even for B.f.s. without the Fatou property. From [3] we have that a B.f.s. $X(\mu)$ is order continuous if and only if all the functions $f \in X(\mu)$ have *absolutely continuous norm*, that is $\|f\chi_A\|_{X(\mu)} \to 0$ when $\mu(A) \to 0$. When the density of simple functions is assumed in $X(\mu)$, we have that the B.f.s. $X(\mu)$ is order continuous if and only if the charateristic function χ_Ω has order continuous norm. These assertions correspond to [3, Proposition I.3.5, Theorem I.3.13] and their proofs hold for B.f.s without the Fatou property. As a consequence we have the following lemma, its prove can be found in [21, p.490], we give it for the aim of completeness.

Lemma 1.0.1. *When $X(\mu)$ is an order continuous B.f.s. we have that*

$$X(\mu) = \{f \in L^0(\mu) : \|f\chi_A\|_{X(\mu)} \to 0 \text{ when } \mu(A) \to 0\}. \tag{1.1}$$

Proof. The direct inclusion is clear. Suppose that $f \in L^0(\mu)$ satisfies that $\|f\chi_A\|_{X(\mu)} \to 0$ when $\mu(A) \to 0$ and define pointwise the sequence $f_n = |f|\chi_{A_n}$, where $A_n := \{w \in \Omega : |f(w)| \leq n\}$. Clearly $|f_n| \uparrow f$ and for $m > n$

$$\|f_m - f_n\|_{X(\mu)} = \|f\chi_{A_m \setminus A_n}\|_{X(\mu)}.$$

Since $A_m \setminus A_n = \{w \in \Omega : n < |f(w)| \leq m\}$, when $m > n \to \infty$ we have that $\mu(A_m \setminus A_n) \to 0$. So $\|f_n - f_m\|_{X(\mu)} \to 0$, and order continuity of $X(\mu)$ yields $f \in X(\mu)$. □

The *Köthe dual* of a B.f.s. $X(\mu)$ is defined as

$$X(\mu)' := \{h \in L^0(\mu) : fh \in L^1(\mu) \ \forall f \in X(\mu)\}.$$

Notice that when the B.f.s. $X(\mu)$ is $\sigma-$order continuous, its Köthe dual coincides with $X(\mu)^*$.

For $0 < p < \infty$ we define the *$p-$th power* of a B.f.s. $(X(\mu), \|\cdot\|_{X(\mu)})$ as the space

$$X(\mu)_{[p]} := \{f \in L^0(\mu) : |f|^{\frac{1}{p}} \in X(\mu)\} \tag{1.2}$$

when we endow it with the quasi norm defined by

$$\|f\|_{X(\mu)_{[p]}} := \left\| |f|^{\frac{1}{p}} \right\|_{X(\mu)}^p, \ \ f \in X(\mu)_{[p]}. \tag{1.3}$$

$X(\mu)_{[p]}$ becomes a quasi Banach function space, see for instance [58, Section 2.2]. Remark that the name "$p-$th power" is derived by the following fact

$$|f| \in X(\mu) \text{ if and only if } |f|^p \in X(\mu)_{[p]} \text{ whenever } f \in L^0(\mu).$$

The $\sigma-$order continuity is inherited by the $p-$th power, in fact if $\|\cdot\|_{X(\mu)}$ is an order continuous norm, the quasi norm $\|\cdot\|_{X(\mu)_{[p]}}$ is $\sigma-$order continuous. We cannot ensure that $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm; take for instance $1 < p < \infty$, the space $L^1(\mu)_{[p]} = L^{1/p}(\mu)$ is not normable. Nevertheless under some requirements the normability of $X(\mu)$ is assured.

Let $0 < p < \infty$, a Banach function space $X(\mu)$ is *$p-$convex* if there is a constant $c > 0$ such that for every finite family $f_1, \dots, f_n \in X(\mu)$, $n \in \mathbb{N}$,

$$\left\| \left(\sum_{i=1}^n |f_i|^p \right)^{\frac{1}{p}} \right\|_{X(\mu)} \leq c \left(\sum_{i=1}^n \|f_i\|_{X(\mu)}^p \right)^{\frac{1}{p}}. \tag{1.4}$$

The smallest constant c satisfying (1.4) for every $n \in \mathbb{N}$ and every choice of functions is the $p-$convexity constant of $X(\mu)$ and is denoted by $\mathbf{M}^{(p)}[X(\mu)]$. The following proposition appears in [58, page 43] .

Proposition 1.0.2. *Let $X(\mu)$ a B.f.s. with norm $\|\cdot\|_{X(\mu)}$.*

1. *If $0 < p \leq 1$ then, $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm, hence $(X(\mu)_{[p]}, \|\cdot\|_{X(\mu)_{[p]}})$ is a B.f.s.*
2. *Assume that $X(\mu)$ is $p-$convex for some $0 < p < \infty$. Then $\|\cdot\|_{X(\mu)_{[p]}}$ is a norm if and only if $\mathbf{M}^{(p)}[X(\mu)] = 1$.*

Vector measures. Let X be a real Banach space, Ω a set and Σ a $\sigma-$algebra of subsets of Ω. A set function $m : \Sigma \to X$ is called a *vector measure* whenever it is *$\sigma-$additive*, that is, for every countable collection of disjoint sets $(A_i)_i$ in Σ we have $m(\bigcup_i A_i) = \sum_i m(A_i)$. The *variation* of the measure m

is denoted by $|m|$ and corresponds to the set function $|m| : \Sigma \to [0, +\infty]$ given by $|m|(A) := \sup_\pi \sum_{E\in\pi} \|m(E)\|_X$ for $A \in \Sigma$, where the supremum is taken over all the finite (disjoint) partitions π of A. The variation corresponds to the smallest $[0, +\infty]-$valued measure dominating m, that is for each $A \in \Sigma$ one has $\|m(A)\|_X \leq |m|(A)$. For x^* in the topological dual of X, X^*, we denote by $\langle m, x^*\rangle : \Sigma \to \mathbb{R}$ the scalar measure given by

$$\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle, \ A \in \Sigma.$$

The *semivariation* of m is the set function $\|m\| : \Sigma \to [0, +\infty[$ defined by $\|m\|(A) := \sup_{x^*\in B(X^*)} |\langle m, x^*\rangle|(A)$.

A finite measure $\mu : \Sigma \to [0, \infty[$ is a *control measure* for the vector measure m when they are mutually absolutely continuous, that is $\mu(A) \to 0$ if and only if $m(A) \to 0$ in X. There is a special class of control measures for a vector measure m. Rybakov's Theorem (see [26, Chapter IX, Theorem 2.2]) ensures that there exists an element $x_0^* \in X^*$ so that $|\langle m, x_0^*\rangle|$ is a control measure for m. This particular class of scalar measures are called *Rybakov measures* for m. Throughout this work λ will stand as a Rybakov control measure for m.

A vector measure m is *scalarly dominated* by a measure $\widetilde{m} : \Sigma \to X$ if there exists a positive constant K such that $|\langle m, x^*\rangle|(A) \leq K |\langle \widetilde{m}, x^*\rangle|(A)$, for each $A \in \Sigma$ and each $x^* \in X^*$.

Integrability. Following the definition of D. R. Lewis in [46], a function $f \in L^0(\lambda)$ is integrable with respect to a vector measure m, briefly $m-$*integrable*, if

$(I1)$ it is $\langle m, x^*\rangle-$integrable for every $x^* \in X^*$ and,

$(I2)$ for every set $A \in \Sigma$ there is an element $m_f(A) \in X$ so that

$$\langle m_f(A), x^*\rangle = \int_A f d\langle m, x^*\rangle, \ x^* \in X^*.$$

In fact the vector $m_f(A)$ corresponds to the integral of f with respect to m over a set A, $\int_A f dm$. We denote by $\mathcal{L}^1(m)$ the space of integrable functions with respect to m.

This definition of integrability is not the first that appeared in this setting. In 1955, R. G. Bartle, N. Dunford and J. Schwartz introduced the notion of integrability with respect to a vector measure in [2]. Their aim was to extend the Riez Representation Theorem for weakly compact operators $T : \mathcal{C}(K) \to X$, defined on the space of real continuous functions on a compact set K, with range in a Banach space X. For this kind of operators the authors proved that there is a measure m defined on the $\sigma-$algebra of the Borel sets of K, $\mathcal{B}(K)$, with range in X so that $T(f) = \int_K f dm$ for every $f \in \mathcal{C}(K)$. To get this result, R. G. Bartle, N. Dunford and J. Schwartz had to build an integration theory. They began by introducing the integral of a simple function. For $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$, with $a_i \in \mathbb{R}$ and $A_i \in \Sigma$, $i = 1, \ldots, n$, the integral of Φ with respect to m over a set $A \in \Sigma$ is given by

$$\int_A \phi dm = \sum_{i=1}^{n} a_i m(A_i \cap A).$$

In general, a real function f is integrable with respect to the vector measure m (in the sense of R. G. Bartle, N. Dunford and J. Schwartz) whenever there is a sequence $(\phi_n)_n \subset \mathcal{S}(\Sigma)$ of simple functions so that $\phi_n \to f$ pointwise as $n \to +\infty$ and the sequence $\left(\int_A \phi_n dm\right)_n$ converges in X for every set $A \in \Sigma$. Moreover, if this holds, the integral of f with respect to m over a set $A \in \Sigma$ is given by

$$\int_A f dm = \lim_{n\to\infty} \int_A \phi_n dm.$$

D.R. Lewis proved in Theorem 2.4 in [46], that the definition of integrability given by conditions $(I1)$ and $(I2)$ is equivalent to the one of R. G. Bartle, N. Dunford and J. Schwartz. In [36] the authors provide another approach to integrability of real valued functions with respect to a vector measure. Their aim was to relate the $m-$integrability with the Birkhoff integrability and $\mathcal{S}^*-$integral. The notion of Birkhoff integrability for vector valued functions with respect to a nonnegative finite measure was introduced in [4]. As pointed by Lewis in [47, p.307], the adaptation of Birkhoff's definition for scalar function and vector measures is not trivial. The notion of the $\mathcal{S}^*$ integral was intensively studied by Dobrakov in [28] for vector valued functions and operator-valued vector measures. This is known as the Dobrakov integral, further researches on this subject can be found in [65] and [66].

Let $\mathcal{P}(\Omega)$ be the set of countable partitions of Ω. A partition $\mathcal{A} \in \mathcal{P}(\Omega)$ is finer that $\mathcal{B} \in \mathcal{P}(\Omega)$ (denoted $\mathcal{A} \succeq \mathcal{B}$) whenever for every $A \in \mathcal{A}$ there is a set $B \in \mathcal{B}$ so that $A \subset B$. For a function $f : \Omega \to \mathbb{R}$, let $\mathcal{P}(\Omega, f)$ denote the set of partitions $\mathcal{A} := (A_n)_n \in \mathcal{P}(\Omega)$ so that the $X-$valued sequence $(f(w_n)m(A_n))_n$ is unconditionally summable for every choice $(w_n)_n \in \Pi_{n\in\mathbb{N}} A_n$. A scalar function f (not necessarily measurable) is $\mathcal{S}^*-$*integrable* with integral $\mathcal{S}^* \int_\Omega f dm \in X$ if for every $\varepsilon > 0$ there is a partition $\mathcal{A}_0 \in \mathcal{P}(\Omega)$ such that $\mathcal{A} = (A_n)_n \in \mathcal{P}(\Omega, f)$ for every $\mathcal{A} \succeq \mathcal{A}_0$, and

$$\left\| \sum_{n=1}^{\infty} f(w_n) m(A_n) - \mathcal{S}^* \int_\Omega f dm \right\|_X \leq \varepsilon$$

for every choice of points $(w_n)_n \in \Pi_{n\in\mathbb{N}} A_n$. In order to compare the $m-$integrability with the S^*-integrability the authors introduced in [36, Definition 3] the concept of $B-$integrability which is an adaptation of the notion of Birkhoff integrability in the context of scalar functions and vector measures. A function $f \in L^0(\lambda)$ is $B-$*integrable* with respect to m if there is a partition $\mathcal{A}_0 \in \mathcal{P}(\Omega)$ so that $\mathcal{A} \in \mathcal{P}(\Omega, f)$ for each $\mathcal{A} \succeq \mathcal{A}_0$. As shown in [36, Theorem 6] the notion of $B-$integrability is stronger than the S^*-integrability. For a $\Sigma-$measurable function, $f : \Omega \to \mathbb{R}$ is $B-$integrable if and only if f is $m-$integrable, and in this case $\mathcal{S}^* \int_A f dm = \int f dm$ for every set $A \in \Sigma$ (see [36, Theorem 9]). Recall that a measure space (Ω, Σ, μ) is complete if it contains all the subsets of measure zero. Theorem 13 in [36] ensures that under assumption of completeness of the measure space $(\Omega, \Sigma, \lambda)$ (λ any control measure for m), the S^*-integrability coincides with the $m-$integrability.

In the space of integrable functions with respect to m we can identify the functions that are $\lambda-$a.e. equal, where λ is a Rybakov control measure for m, these are the functions that differ in a $\lambda-$null set. Notice that they coincide with functions that differ in a set of null semivariation of m; this is the reason we use both the expressions $\lambda-$a.e. and $m-$a.e. for the same notion. We denote by $L^1(m)$ the space of (classes of $\lambda-$a.e. equal) functions that are integrable with respect to m, and we endow this space with a norm given by

$$\|f\|_{L^1(m)} := \sup_{x^* \in B(X^*)} \int_\Omega |f| d|\langle m, x^* \rangle|, \ f \in L^1(m). \tag{1.5}$$

We have that $L^1(m)$ is a vector lattice with respect to the natural $m-$a.e. pointwise order and that $\|\cdot\|_{L^1(m)}$ is a lattice norm on $L^1(m)$. The following result, that can be found in [58, p.101], gives a version of the classical Dominated Convergence Theorem for $L^1(m)$ and states the main properties of this space. Recall that a Banach space X is *weakly compactly generated* (WCG for short) if X is the closed linear span of one of its weakly compact subsets (see for instance [51, page 366]).

Theorem 1.0.3. *For a vector measure $m : \Sigma \to X$,*

(i) Let g in the positive cone of $L^1(m)$. If $(f_n)_n$ is a sequence in $L^0(\lambda)$ so that

1. *$(f_n)_n$ converges $m-$a.e. to a function $f \in L^0(\lambda)$ and*
2. *$|f_n| \leq g$, pointwise $m-$a.e. for every $n \in \mathbb{N}$,*

then, $f \in L^1(m)$ and it is the limit of $(f_n)_n$ in the norm $\|\cdot\|_{L^1(m)}$.

(ii) The normed space $L^1(m)$ is complete, moreover it is a WCG space in which $\mathcal{S}(\Sigma)$ is dense.

(iii) For every Rybakov control measure λ for m, the space $L^1(m) \subset L^0(\lambda)$ is a Banach function space over the measure space $(\Omega, \Sigma, \lambda)$ and the norm is order continuous. The constant function χ_Ω is a weak order unit.

The spaces of integrable functions with respect to a vector measure are particularly interesting because they represent a large class of Banach lattices. The following result is due to G. Curbera (see [12, Theorem 8]).

Theorem 1.0.4. *Let X be an order continuous Banach lattice with weak order unit. There exists an $X-$valued positive measure m such that X and $L^1(m)$ are lattice isomorphic and isometric.*

As for the classical spaces of integrable functions, there is a natural extension of the definition to the $1/p-$th power integrable functions with respect to a vector measure. This definition was given by E. A. Sánchez Pérez in [70]. Let $1 < p < \infty$. A real function f is $p-$*integrable* with respect to m whenever $|f|^p$ is $m-$integrable. The space of (equivalence classes of $\lambda-$a.e.) $p-$integrable functions with respect to m is denoted by $L^p(m)$ and a natural norm for this space is given by

$$\|f\|_{L^p(m)} := \sup_{x^* \in B(X^*)} \left(\int_\Omega |f|^p d|\langle m, x^* \rangle| \right)^{\frac{1}{p}}, \; f \in L^p(m). \tag{1.6}$$

Notice that $L^p(m)$ is the $(1/p)-$th power of $L^1(m)$, with $0 < 1/p < 1$; it follows from Proposition 1.0.2 that $L^p(m)$ is a Banach function space with $\sigma-$order continuous norm included in $L^1(m)$. The definition of the norm $\|\cdot\|_{L^p(m)}$ implies directly that $L^p(m)$ is a $p-$convex B.f.s. over the measure space $(\Omega, \Sigma, \lambda)$ with $p-$convexity constant one. As for the case $p = 1$, there is a Representation Theorem for a large class of Banach lattices as an $L^p(m)$. The following theorem occurs in [31, Proposition 2.4].

Theorem 1.0.5. *Let $1 < p < \infty$. If X is a $p-$convex Banach lattice with weak order unit and order continuous norm, then there is an X valued positive vector measure m such that $L^p(m)$ and X are lattice and topologically isomorphic.*

For $p = \infty$, the space $L^\infty(m)$ consists of the real valued functions that are $\Sigma-$measurable and $m-$essentially bounded. When equipped with the essential supremum norm $\|\cdot\|_{L^\infty(m)}$, $L^\infty(m)$ is a Banach function space over $(\Omega, \Sigma, \lambda)$. Bounded $\Sigma-$measurable functions are integrable with respect to m, in fact we have the following chain of inclusions for $1 \leq p \leq p* \leq \infty$:

$$L^\infty(m) \subset L^{p^*}(m) \subset L^p(m) \subset L^1(m).$$

For $1 \leq p \leq \infty$, in the sequel q will denote its conjugated index, that is

$$q := \begin{cases} \infty & \text{if } p = 1, \\ \frac{p}{p-1} & \text{if } 1 < p < \infty, \\ 1 & \text{if } p = \infty. \end{cases} \tag{1.7}$$

In what follows we give the corresponding Hölder's inequalities for the spaces $L^p(m)$. First when $p = 1$, clearly by the definition of the norm (1.5) we have for $f \in L^1(m)$

$$\|fg\|_{L^1(m)} \leq \|f\|_{L^1(m)} \|g\|_{L^\infty(m)}, \ \ g \in L^\infty(m). \tag{1.8}$$

For $1 < p < \infty$, $fg \in L^1(m)$ for every $f \in L^p(m)$ and $g \in L^q(m)$ and $\|fg\|_{L^1(m)} \leq \|f\|_{L^p(m)} \|g\|_{L^q(m)}$. Moreover we have, as proved in [7, Lemma 2],

$$L^p(m) \cdot L^q(m) := \{fg : f \in L^p(m), g \in L^q(m)\} = L^1(m). \tag{1.9}$$

The space $L^p(m)$ is not reflexive in general. Recall first that $L^p(m)$ is reflexive if (and only if) it does not contain subspaces isomorphic to c_0 (combine [31, Corollary 3.10] and [51, Theorem 2.4.12]). For further characterizations of the reflexivity of $L^p(m)$, see [16] and [31]. We next present a simple example of a non reflexive $L^p(m)$ space.

Example 1.0.6. Construction of a non-reflexive $L^p(m)$. Take $\Omega := \mathbb{N}$, let Σ be the set of all subsets of $\mathbb{N}$ and consider the countably additive vector measure $m : \Sigma \to c_0$ given by $m(A) = \sum_{n \in A} (1/n) e_n$, where (e_n) is the canonical basis of c_0. It is not difficult to check that

$$L^p(m) = \left\{ f \in \mathbb{R}^{\mathbb{N}} : \ (n^{-1/p} f(n))_{n \in \mathbb{N}} \in c_0 \right\}$$

with $\|f\|_{L^p(m)} = \sup\{n^{-1/p}|f(n)| : n \in \mathbb{N}\}$ for all $f \in L^p(m)$. Clearly, c_0 is isomorphic to $L^p(m)$ and this space is not reflexive.

The next proposition is the key of what we call the *vector duality* between the spaces $L^p(m)$ and $L^q(m)$ for p and q conjugated real numbers. It proves that the unit ball of $L^q(m)$ is in some sense norming for $L^p(m)$, when the vector duality is used instead of the usual duality. A proof of this result can be found in [58, page 123].

Proposition 1.0.7. *Let $m : \Sigma \to X$ be a Banach space-valued vector measure and let $1 \leq p \leq \infty$. Then we have the identities*

$$\sup_{g \in B(L^q(m))} \left\| \int_\Omega fg dm \right\|_X = \|f\|_{L^p(m)} = \sup_{g \in B(L^q(m))} \|fg\|_{L^1(m)}, \ f \in L^p(m).$$

Thus, the duality relationship between the spaces $L^p(m)$ and $L^q(m)$ appears in a natural way through the $X-$valued bilinear map $\Phi : L^p(m) \times L^q(m) \to X$ defined by $\Phi(f,g) = \int_\Omega fg dm$, for $f \in L^p(m)$ and $g \in L^q(m)$.

Since $L^p(m)$ is order continuous, its dual $L^p(m)^*$ coincides with the Köthe dual of $L^p(m)$ (cf. [51, Corollary 2.6.5]), that is, $L^p(m)^* = \{\varphi_h : h \in \mathcal{H}\}$ where

$$\mathcal{H} := \{h : \Omega \to \mathbb{R}, \Sigma - \text{measurable} : \ fh \in L^1(\lambda) \text{ for all } f \in L^p(m)\}$$

and the continuous functionals φ_h are defined by $\langle f, \varphi_h \rangle := \int_\Omega fh \, d\lambda$.

Let us continue by showing some results regarding the spaces $L^p(m)$ as WCG spaces.

Theorem 1.0.8. *For $1 < p < \infty$, $L^p(m)^*$ is order continuous and has weak unit. In particular, $L^p(m)^*$ is WCG.*

Proof. Since $L^p(m)$ is $p-$convex and ℓ^1 is not $p-$convex, we can apply [48, Proposition 1.d.9] to conclude that no sublattice of $L^p(m)$ is order isomorphic to ℓ^1. Equivalently, $L^p(m)^*$ is order continuous, see [51, Theorem 2.4.14]. On the other hand, since $L^p(m)$ is an order continuous Banach function space over λ, the space $L^p(m)^*$ has weak unit (namely, the functional φ_{χ_Ω}). Therefore, $L^p(m)^*$ is order isomorphic to the L^1 space of some vector measure and so it is WCG. □

Subspaces of WCG Banach spaces are not WCG in general. The first example showing this phenomenon was built by H.P. Rosenthal [68] ([25, Chapter 5, §10]) over the L^1 space of certain probability measure. However, the property of being WCG is always inherited by subspaces having WCG dual, according to a result of W.B. Johnson and J. Lindenstrauss [41] ([25, Chapter 5, §8]). Since $L^p(m)$ is WCG and the dual of any subspace of $L^p(m)$ is WCG (because it is a quotient of the WCG space $L^p(m)^*$), we have the following corollary.

Corollary 1.0.9. *For $1 < p < \infty$, every subspace of $L^p(m)$ is WCG.*

A result of T. Kuo ([26, Corollary 7, p. 83]) states that every dual WCG Banach space has the Radon-Nikodým property. On the other hand, it is well known that a dual Banach space Y^* has the Radon-Nikodým property if and only if every separable subspace of Y has separable dual, see [26, Corollary 8, p. 198]. Bearing in mind these facts and Theorem 1.0.8, we get the following corollary. For further characterizations of the separability of $L^p(m)$, see [31].

Corollary 1.0.10. *Let $1 < p < \infty$. Every separable subspace of $L^p(m)$ has separable dual. In particular, $L^p(m)$ is separable if and only if $L^p(m)^*$ is separable.*

We continue with the definition and properties related to the space of scalarly integrable functions with respect to a Banach space valued measure. We say that a measurable real function f is *scalarly* $m-$ *integrable* whenever condition (I1) is satisfied; that is f is integrable with respect each scalar measure $\langle m, x^*\rangle$, with $x^* \in X^*$. We denote by $L^1_w(m)$ the space of equivalence classes of $\lambda-$a.e. equal scalarly $m-$integrable functions. Obviously $L^1(m) \subseteq L^1_w(m)$, and equality holds when X does not contain a copy of c_0 (see for instance in [42, Ch. II Thm. 5.1]). The first systematic study of $L^1_w(m)$ was done by G. Stefansson in [75], he showed that $L^1_w(m)$ endowed with the norm (1.5) is a B.f.s. containing $L^1(m)$ as a closed subspace. The space $L^1_w(m)$ has the $\sigma-$Fatou property. Indeed, take an increasing sequence in the positive cone $L^1_w(m)^+$ so that $\sup_n \|f_n\|_{L^1_w(m)} < \infty$, and define f as the $\lambda-$a.e. pointwise supremum of $(f_n)_n$, $f := \sup_n f_n$, thus

$$\begin{aligned}\|f\|_{L^1_w(m)} &= \sup_{x^*\in B(X^*)} \int_\Omega |f| d|\langle m, x^*\rangle| = \sup_{x^*\in B(X^*)} \sup_{n\in\mathbb{N}} \int_\Omega |f_n| d|\langle m, x^*\rangle| \\ &= \sup_{n\in\mathbb{N}} \sup_{x^*\in B(X^*)} \int_\Omega |f_n| d|\langle m, x^*\rangle| = \sup_n \|f_n\|_{L^1_w(m)} < \infty.\end{aligned}$$

For $1 \leq p < \infty$, let $L^p_w(m)$ denote the $(1/p)-$power of $L^1_w(m)$, then $L^p_w(m) := L^1_w(m)_{[1/p]} \subseteq L^1_w(m)$. By Proposition 1.0.2, $L^p_w(m)$ is a $\lambda-$B.f.s. when endowed with the norm $\|\cdot\|_{L^p_w(m)} := \|\cdot\|_{L^1_w(m)_{[1/p]}}$. The space $L^p_w(m)$ was firstly defined and studied by A. Fernández et al. in [31]. We clearly have again that $L^p(m)$ is a closed sublattice of $L^p_w(m)$. An appel to Proposition 1.0.2 yields the $p-$convexity of $L^p_w(m)$ with $p-$convexity constant equal to 1. For a general Banach lattice $(X, \|\cdot\|_X)$, its order continuous part is defined in [79] as

$$X_a := \{x \in X : |x| \geq |x_n| \downarrow 0 \text{ for } x_n \in X \text{ then } \|x_n\|_X \downarrow 0\}.$$

X_a is a closed ideal of X, in fact it is the largest order ideal of X so that its restriction has order continuous norm. In [16] the authors prove, for $p = 1$, that the order continuous part of $L^p_w(m)$ is exactly $L^p(m)$. The same arguments can be used to prove this for $p > 1$. This fact, and the $p-$convexity of $L^p_w(m)$, whose order unit χ_Ω belongs to its order continuous part, are the key to prove the following representation theorem, see for instance Theorem 4 in [17].

Theorem 1.0.11. *Let $1 \leq p < \infty$ and Z be a $p-$convex Banach lattice which has the $\sigma-$Fatou property and admits a weak order unit which belongs to its order continuous part Z_a. Then there is a Z_a-valued measure m such that $L^p(m)$ is lattice isomorphic to Z_a and $L^p_w(m)$ is lattice isomorphic to Z.*

Operator Theory. Let us finish this section by defining the notation and some basic concepts regarding the operator theory. For X and Y Banach spaces we will denote by $\mathcal{L}(X,Y)$ the collection of linear and continuous maps between X and Y, the elements in $\mathcal{L}(X,Y)$ are bounded operators . A linear map $T : X \to Y$ between Banach spaces is continuous if and only if

$$\|T\| := \sup_{x\in B(X)} \|T(x)\|_Y < \infty. \tag{1.10}$$

Notice that $\mathcal{L}(X,Y)$ is a Banach space when endowed with the norm given by (1.10), for $T \in \mathcal{L}(X,Y)$. The topology induced by this norm is called the *uniform operator topology*. The *strong operator topology* corresponds to the topology of pointwise convergence; a net $(T_\alpha) \subset \mathcal{L}(X,Y)$ converges to T in the strong operator topology whenever $\|(T_\alpha - T)(x)\|_Y \to 0$ for every $x \in X$.

A collection $\mathbf{I}$ of operators between Banach spaces is an *operator ideal* whenever for every $T \in \mathbf{I}$, the composition of T with S and U bounded operators, $S \circ T \circ U$ belongs to $\mathbf{I}$.

Let us recall some definitions about geometrical properties of operators between Banach lattices. Let X and Y be Banach lattices, and $0 < q < \infty$. A linear operator $T : X \to Y$ is said to be $q-$*convex* if there exists a constant $c > 0$ such that

$$\left\| \left(\sum_{i=1}^{n} |T(x_i)|^q \right)^{\frac{1}{q}} \right\|_Y \leq c \left(\sum_{i=1}^{n} \|x_i\|_X^q \right)^{\frac{1}{q}} \tag{1.11}$$

holds for every $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. The smallest constant c satisfying (1.11) is called the $q-$convexity constant of T, and denoted by $\mathbf{M}^{(q)}[T]$. A linear operator $S : X \to Y$ is said to be $q-$*concave* if there exists a constant $c > 0$ such that

$$\left(\sum_{i=1}^{n} \|S(x_i)\|_Y^q \right)^{\frac{1}{q}} \leq c \left\| \left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{1}{q}} \right\|_X \tag{1.12}$$

for every $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. The smallest constant c satisfying (1.12) is called the $q-$concavity constant of S and denoted by $\mathbf{M}_{(q)}[S]$. Recall that a Banach space X is $q-$convex ($q-$concave) whenever the identity map Id_X is a $q-$convex ($q-$concave) operator.

Chapter 2

Weak topologies in $L^p(m)$

This chapter is devoted to the study of two intermediates topologies for the space $L^p(m)$ when $1 < p < \infty$. These topologies were introduced by E. A. Sánchez Pérez in [70]. In the following $1 < p, q < \infty$ are conjugated real numbers, as in (1.7).

The *$m-$weak topology* corresponds to the topology of the weak convergence of integrals. It is defined by the family of seminorms

$$\Gamma := \{|\gamma_{g,x^*}| : g \in B(L^q(m)), x^* \in B(X^*)\}, \tag{2.1}$$

where $\gamma_{g,x^*}(f) := \int_\Omega fgd\langle m, x^*\rangle, f \in L^p(m)$. The $m-$weak topology will be denoted by $\sigma(L^p(m), \Gamma)$.

Since each pair of elements $x^* \in X^*$ and $g \in L^q(m)$ defines a functional $\gamma_{g,x^*} \in L^p(m)^*$, in general the $m-$weak topology is coarser than the weak topology of $L^p(m)$. The main result in the first section deals with the coincidence of the $m-$weak topology with the weak topology in bounded subsets of $L^p(m)$, for $1 < p < \infty$. We will also study some condition to ensure that Γ is a James boundary for the unit ball $B(L^p(m))$. Notice that for $f_0 \in L^p(m)$ and $\varepsilon > 0$, the neighborhoods of radius ε of f_0 for the $m-$weak topology are finite intersection of sets $V_{g,x^*,\varepsilon}(f_0)$, with $g \in B(L^q(m))$ and $x^* \in B(X^*)$ where

$$V_{g,x^*,\varepsilon}(f_0) := \{f \in L^p(m) : |\gamma_{g,x^*}(f - f_0)| \le \varepsilon\}. \tag{2.2}$$

The *$m-$topology* for the space $L^p(m)$ corresponds to the topology of norm convergence of the integrals. It is generated by the family of seminorms

$$\Lambda := \{\zeta_g : g \in B(L^q(m))\},$$

where $\varphi_g(f) := \left\| \int_\Omega fgdm \right\|_X$, for $f \in L^p(m)$. It is not difficult to see that the $m-$topology is coarser than the norm topology and finer than the weak topology on bounded sets. Let us show two simple examples that represent the extreme cases. When we consider a positive scalar measure, the $m-$topology coincides with the weak topology of the space $L^p(m)$ on bounded sets. If we consider a vector measure m with range in $L^r(\mu)$ defined by $m(A) := \chi_A$ for $A \in \Sigma$, we get that the $m-$topology coincides with the norm topology of the space $L^p(m)$. For more details about the $L^p(m)$ space of this particular vector measure we refer the reader to Example 3.1.4.

As for the $m-$weak topology, the neighborhoods of $f_0 \in L^p(m)$ of radius $\varepsilon > 0$ are finite intersections of sets $V_{g,\varepsilon}(f_0)$ with $g \in B(L^q(m))$ and

$$V_{g,\varepsilon}(f_0) := \{f \in L^p(m) : \zeta_g(f - f_0) \leq \varepsilon\}. \tag{2.3}$$

In the following $\sigma(L^p(m), \Lambda)$ denotes the $m-$topology.

2.1. $m-$weak topology

In the first part of this section we will give sufficient conditions to ensure that the unit ball of $L^p(m)$ endowed with the $m-$weak topology is metrizable. These conditions are related with the separability of $L^p(m)$. In the second part we show that, for $p > 1$ the weak convergence of bounded nets in $L^p(m)$ is characterized by the weak convergence of the integrals in the Banach space X. That means that the topology $\sigma(L^p(m), \Gamma)$ coincide with the weak topology on the bounded subsets of $L^p(m)$. For $f \in L^p(m)$, we can define the *integration operator*

$$I_f : L^q(m) \to X, \quad I_f(g) := \int_\Omega fg \, dm,$$

with $\|I_f\| = \|f\|_{L^p(m)}$, see Proposition 1.0.7. Notice that, as a consequence of the previous equality, the set $\Gamma \subset B(L^p(m)^*)$ defined in the introduction is norming.

2.1.1. Metrizability of the unit ball

Metrizability of $B(L^p(m))$ can be directly deduced from the following arguments assuming some conditions of separability. First notice that the map

$$\Phi : B(X^*) \times B(L^q(m)) \longrightarrow L^p(m)^*$$

defined by $\Phi(x^*, g)(f) := \int_\Omega fgd\langle m, x^*\rangle = \gamma_{g,x^*}(f)$ for $f \in L^p(m)$ is bilinear. Moreover we have $\|\Phi(x^*, g)\|_{L^p(m)^*} \leq \|x^*\|_{X^*}\|g\|_{L^q(m)}$ for every

$x^* \in B(X^*)$ and $g \in B(L^q(m))$, since

$$\begin{aligned} \|\Phi(x^*, g)\|_{L^p(m)^*} &= \sup_{f \in B(L^p(m))} |\Phi(x^*, g)(f)| \\ &\leq \|x^*\|_{X^*} \sup_{f \in B(L^p(m))} \left\| \int_\Omega fg dm \right\|_X \\ &\leq 1. \end{aligned}$$

Let $E := span\{\Phi(B(X^*) \times B(L^q(m)))\}$. Notice that E, when endowed with the topology induced by $L^p(m)^*$, can be identified with a subspace of $L^p(m)^*$. The (non injective) mapping given by Φ, yields that E is a normed separable space assuming that $L^q(m)$ and X^* are separable. Moreover $B(L^p(m))$ can be considered as a subset of $B(E^*)$, therefore $B(L^p(m))$ endowed with the weak* topology induced by $(L^p(m), E)$ is metrizable. The following proposition is a consequence of these arguments and of [43, (1) page 163].

Proposition 2.1.1. *Assuming that X^* and $L^q(m)$ are separable, $B(L^p(m))$ is metrizable when endowed with the topology $\sigma(L^p(m), \Gamma)$.*

In the appendix at the end of this chapter we give a proof of the following result in which an explicit definition for the metric of the space is given. Nevertheless the proof can be directly obtained as a consequence of the following fact. Let $S = \{x_i^* : i \in \mathbb{N}\}$ a sequence in $B(X^*)$ that separates points of X, then the topology in X of pointwise convergence in $\bar{S}$ is metrizable with the following distance

$$d(x, y) := \sum_{i \in \mathbb{N}} \frac{1}{2^i} |\langle x, x_i^* \rangle - \langle y, x_i^* \rangle|,$$

for x and y in X. When this argument is applied to the set Γ assuming the separability of $B(L^q(m))$ and $B(X^*)$, the following proposition is directly proved. Since in the following section we will prove the coincidence on bounded sets of the weak topology with the topology $\sigma(L^p(m), \Gamma)$, this proposition gives us a metric for the weak topology in the unit ball $B(L^p(m))$.

Proposition 2.1.2. *If X^* and $L^q(m)$ are separable, then the unit ball $B(L^p(m))$ endowed with the topology $\sigma(L^p(m), \Gamma)$ is metrizable, and the metric is given by the following formula:*

$$\rho(f_1, f_2) := \sum_{n=1}^{\infty} 2^{-n} \left(\sum_{k=1}^{\infty} 2^{-k} \left| \int_\Omega (f_1 - f_2) g_k d\langle m, x_n^* \rangle \right| \right),$$

for $f_1, f_2 \in B(L^p(m))$, where $S_1 = (g_k)_{k=1}^\infty$ and $S_2 = (x_n^*)_{n=1}^\infty$ are dense subsets of $B(L^q(m))$ and $B(X^*)$, respectively.

The main property assumed in the previous results is the separability of the space $L^p(m)$. For $p = 1$ separability of $L^p(m)$ is studied in [63], for $p > 1$ the corresponding results can be found in [60]. We say that the σ-algebra Σ is *$m-$essentially countably generated* (see definition in [42, Section II.6]) if there exists a countably generated sub$-\sigma-$algebra Σ_0 so that for each $A \in \Sigma$ there is $B \in \Sigma_0$ with $A - B$ and $B - A$ $m-$null sets. The following characterization of separability occurs in [31].

Proposition 2.1.3. *For $1 \leq p < \infty$, the following assertions are equivalent*

(i) $L^p(m)$ *is separable,*

(ii) $L^1(m)$ *is separable,*

(iii) Σ *is $m-$essentially countably generated,*

(iv) Σ *is $|\langle m, x^* \rangle|-$essentially countably generated for every Rybakov's measure for m.*

(iv) Σ *is $|\langle m, x^* \rangle|-$essentially countably generated for some Rybakov's measure for m.*

As a consequence we get the following corollary.

Corollary 2.1.4. *Let $1 < p < \infty$. If the $\sigma-$algebra Σ is $|\langle m, x^* \rangle|-$essentially countably generated for some Rybakov's measure $|\langle m, x^* \rangle|$ and X^* is separable, then $(B(L^p(m)), \sigma(L^p(m), \Gamma))$ is metrizable.*

We say that a subset A of a metric space X is *totally bounded* if for each $\varepsilon > 0$ there exists a finite subset $\{x_1, \ldots, x_n\} \subset X$ that is *$\varepsilon-$dense* in A. That is, the collection of balls centered in x_i with radius ε, for $i = 1, \ldots, n$, covers A. Obviously, every compact subset is totally bounded. The following result that holds for metric spaces can be found in [1].

Theorem 2.1.5. *For a metric space the following assertions are equivalent:*

(i) the space is compact,

(ii) the space is complete and totally bounded,

(iii) the space is sequentially compact, i.e. every sequence has a convergent subsequence.

Therefore, under the hypothesis of separability ensuring that the unit ball $B(L^p(m))$ is metrizable when endowed with topology $\sigma(L^p(m), \Gamma)$, we can establish the following equivalences.

(i) $(B(L^p(m)), \sigma(L^p(m), \Gamma))$ is compact.

(ii) $(B(L^p(m)), \sigma(L^p(m), \Gamma))$ is complete and totally bounded.

(iii) $(B(L^p(m)), \sigma(L^p(m), \Gamma))$ is sequentially compact.

Notice that, since we will prove that the $m-$ weak topology coincides with the weak topology on bounded sets, the equivalences above hold without the assumption of metrizability as a consequence of Eberlein-Smulyan Theorem.

2.1.2. Weak convergence in bounded sets of $L^p(m)$

Let μ be a probability measure and $1 < p < \infty$. In the classical space of $\mu-$integrable functions $L^p(\mu)$, the duality $L^p(\mu)^* \cong L^q(\mu)$ and the density of simple functions in $L^q(\mu)$ implies that a bounded net (f_α) is weakly convergent to $f \in L^p(\mu)$ if and only if $\int_A f_\alpha \, d\mu \to \int_A f \, d\mu$ for every measurable set A. For $p = 1$ and m a vector measure, G. P. Curbera in [13] and independently S. Okada in [53] showed that, assuming that $L^1(m)$ contains no complemented subspace isomorphic to ℓ^1, the weak convergence of bounded nets in $L^1(m)$ is characterized by the weak convergence in X of the integrals over arbitrary measurable sets. For bounded sequences in $L^1(m)$ such characterization of weak convergence holds whenever the range of m is norm relatively compact (see for instance [53]). Nevertheless this is not true in general, as showed in [14].

Our aim in this section is to obtain a positive result about the coincidence of the weak topology of $L^p(m)$ and the topology $\sigma(L^p(m), \Gamma)$ on bounded sets. In order to prove it we need the following lemma which might be well known, the proof is included for the aim of completeness.

Lemma 2.1.6. *Let Y be a Banach lattice such that both Y and Y^* are order continuous. Let $C \subset Y^*$ be a set which separates the points of Y. Then the ideal $\mathcal{I} \subset Y^*$ generated by C is norm dense in Y^*.*

Proof. The norm closure $\mathcal{I}'$ of $\mathcal{I}$ in Y^* is an ideal, cf. [51, Proposition 1.2.3]. Since Y^* is order continuous, every closed ideal of Y^* is a band, [51, Corollary 2.4.4]. On the other hand, the order continuity of Y ensures that any band of Y^* is w^*-closed, cf. [51, Corollary 2.4.7]. It follows that $\mathcal{I}'$ is w^*-closed. Finally, since $\mathcal{I}'$ is a linear subspace of Y^* which separates the points of Y, Hahn-Banach theorem yields that $\mathcal{I}' = Y^*$. □

The proof of the next result is inspired by some of the ideas in [13, Theorem 4].

Theorem 2.1.7. *Let $1 < p < \infty$. The weak topology and $\sigma(L^p(m), \Gamma)$ coincide on any bounded subset of $L^p(m)$. Consequently, a bounded net (f_α) in $L^p(m)$ converges weakly to $f \in L^p(m)$ if and only if $\int_A f_\alpha \, dm \to \int_A f \, dm$ weakly in X for every $A \in \Sigma$.*

Proof. Fix a bounded net (f_α) in $L^p(m)$ converging to $f \in L^p(m)$ in the topology $\sigma(L^p(m), \Gamma)$. We will show that $f_\alpha \to f$ weakly. Let $\mathcal{I} \subset L^p(m)^*$ be the ideal generated by Γ. Since $L^p(m)$ and $L^p(m)^*$ are order continuous (the latter by Theorem 1.0.8), we can apply Lemma 2.1.6 to conclude that $\mathcal{I}$ is norm dense in $L^p(m)^*$. Bearing in mind that (f_α) is bounded, it is clear that in order to prove that $f_\alpha \to f$ weakly it suffices to check that $\langle f_\alpha, \varphi \rangle \to \langle f, \varphi \rangle$ for every $\varphi \in \mathcal{I}$.

To this end, fix $\varphi \in \mathcal{I}$. There exist $g_1, \dots, g_n \in L^q(m)$ and $x_1^*, \dots, x_n^* \in X^*$ such that $|\varphi| \leq \sum_{i=1}^n |\gamma_{g_i, x_i^*}|$. An easy computation shows that $\gamma_{g_i, x_i^*} = \varphi_{h_i}$, where

$$h_i := g_i \frac{d\langle m, x_i^* \rangle}{d\lambda} \in \mathcal{H} \quad \text{for all } 1 \leq i \leq n.$$

As usual, $d\langle m, x_i^* \rangle / d\lambda$ denotes the Radon-Nikodým derivative of $\langle m, x_i^* \rangle$ with respect to λ. Take $g \in \mathcal{H}$ satisfying $\varphi = \varphi_g$. Then $\varphi_{|g|} = |\varphi| \leq \sum_{i=1}^n \varphi_{|h_i|} = \varphi_{\sum_{i=1}^n |h_i|}$ and therefore

$$|g| \leq \sum_{i=1}^n |h_i| \quad \lambda\text{-a.e.} \tag{2.4}$$

Let us consider the non-negative finite measures defined on Σ by $\mu(A) := \int_A |g| \, d\lambda$ and $\mu_i(A) := \int_A |h_i| \, d\lambda$ for all $1 \leq i \leq n$. Taking $\tilde{\mu} := \sum_{i=1}^n \mu_i$, inequality (2.4) ensures that $\mu \leq \tilde{\mu}$ and so we can define an operator $T : L^1(\tilde{\mu}) \to L^1(\mu)$ by $T(h) = h$. Notice that $f_\alpha, f \in L^1(\tilde{\mu})$ because $f_\alpha, f \in L^1(\mu_i)$ for all $1 \leq i \leq n$.

Claim.- $f_\alpha \to f$ weakly in $L^1(\mu_i)$ for every $1 \leq i \leq n$. Indeed, since (f_α) is bounded in $L^1(\mu_i)$ (because it is bounded in $L^p(m)$), we only have to check that $\int_A f_\alpha \, d\mu_i \to \int_A f \, d\mu_i$ for every $A \in \Sigma$. To this end, let us consider a Hahn decomposition $\{G, \Omega \setminus G\}$ of $\langle m, x_i^* \rangle$, that is, $G \in \Sigma$ and

$$|\langle m, x_i^* \rangle|(E) = \langle m, x_i^* \rangle(E \cap G) - \langle m, x_i^* \rangle(E \setminus G) \quad \text{for all } E \in \Sigma.$$

We have

$$\begin{aligned}
&\int_A f_\alpha d\mu_i = \int_A f_\alpha |g_i| \, d|\langle m, x_i^* \rangle| \\
&= \int_\Omega f_\alpha (|g_i| \chi_{A \cap G} - |g_i| \chi_{A \setminus G}) \, d\langle m, x_i^* \rangle \to \int_\Omega f (|g_i| \chi_{A \cap G} - |g_i| \chi_{A \setminus G}) \, d\langle m, x_i^* \rangle \\
&\qquad = \int_A f |g_i| \, d|\langle m, x_i^* \rangle| = \int_A f d\mu_i,
\end{aligned}$$

because $|g_i|\chi_{A\cap G} - |g_i|\chi_{A\setminus G} \in L^q(m)$ and $f_\alpha \to f$ in the topology $\sigma(L^p(m), \Gamma)$. This proves the *Claim*.

From the previous *Claim* it follows that $f_\alpha \to f$ weakly in $L^1(\tilde{\mu})$. Since T is weak-weak continuous, we infer that $f_\alpha \to f$ weakly in $L^1(\mu)$.

Set $A := \{\omega \in \Omega : g(\omega) \geq 0\} \in \Sigma$. Then

$$\begin{aligned}\langle f_\alpha, \varphi\rangle &= \int_\Omega f_\alpha g \, d\lambda = \int_A f_\alpha |g| \, d\lambda - \int_{\Omega\setminus A} f_\alpha |g| \, d\lambda \\ &= \int_A f_\alpha \, d\mu - \int_{\Omega\setminus A} f_\alpha \, d\mu \to \int_A f \, d\mu - \int_{\Omega\setminus A} f \, d\mu = \int_\Omega fg \, d\lambda = \langle f, \varphi\rangle.\end{aligned}$$

This finishes the proof of the first assertion of the theorem. The last part follows immediately bearing in mind that simple functions are dense in $L^q(m)$. □

Let $\mathcal{F} \subset L^p(m)$, the following lemma is a direct consequence of the Uniform Boundedness Principle applied to the family $\{I_f : f \in \mathcal{F}\}$ of operators from $L^q(m)$ to X.

Lemma 2.1.8. *A set $\mathcal{F} \subset L^p(m)$ is bounded if and only if the set of integrals $\{\int_\Omega fg \, dm : f \in \mathcal{F}\} \subset X$ is bounded for every $g \in L^q(m)$.*

Corollary 2.1.9. *A sequence $(f_n)_n$ in $L^p(m)$ converges weakly to $f \in L^p(m)$ if and only if $f_n \to f$ in the topology $\sigma(L^p(m), \Gamma)$.*

James boundaries of $B(L^p(m))$. Let K a set in a linear space E, we say that a subset $S \subset K$ is an *extremal set* of K, denoted $S = \mathrm{Ext}(K)$, when the following condition is satisfied

$$\text{if } x, y \in K, 0 < t < 1, \text{ and } tx + (1-t)y \in S, \text{ then } x, y \in S.$$

Let X be a Banach space, a subset $B \subset B(X^*)$ is a *James boundary* for $B(X^*)$ if for $x \in X$ we have $\|x\|_X = \max\{|\langle x, x^*\rangle| : x^* \in B\}$, i.e. for each $x \in X$ there is some $b^* \in B$ so that $\|x\|_X = |\langle x, b^*\rangle|$.

G. Manjabacas [49, Section 4.7] studied weak compactness in $L^1(m)$ with the help of the weaker topology $\sigma(L^1(m), B)$ of pointwise convergence on the norming set $B \subset B(L^1(m)^*)$ made up of all functionals of the form $f \mapsto \int_\Omega fh \, d\langle m, x^*\rangle$, where $h \in B(L^\infty(m))$ and $x^* \in B(X^*)$. The key point is that bounded $\sigma(L^1(m), B)-$compact sets are weakly compact whenever B is a James boundary for $B(L^1(m)^*)$, and this is the case, for instance, provided that m has norm relatively compact range.

A result of G. Godefroy (see [39, Theorem III.3]) ensures that if a dual Banach space Y^* is WCG, then

$$B(Y^*) = \overline{\mathrm{co}(C)}^{\mathrm{norm}} \tag{2.5}$$

for every James boundary $C \subset B(Y^*)$. Other cases where the previous equality holds can be found in [11]. Note that (2.5) implies that $\sigma(Y, C)$ coincides with the weak topology on any bounded subset of Y.

Bearing in mind that $L^p(m)^*$ is WCG when $1 < p < \infty$ (Theorem 1.0.8), we get the following corollary which, in particular, provides a different proof of Theorem 2.1.7 when Γ is a James boundary for $B(L^p(m)^*)$.

Corollary 2.1.10. *Let C be a James boundary for $B(L^p(m)^*)$. Then $\sigma(L^p(m), C)$ and the weak topology coincide on any bounded subset of $L^p(m)$.*

The rest of the section is essentially devoted to presenting a couple of sufficient conditions ensuring that Γ is a James boundary for $B(L^p(m)^*)$. We do not know whether this is always the case. In order to prove the first theorem regarding a condition to ensure that Γ is a James boundary of $B(L^p(m))$, we need the following technical lemma.

Lemma 2.1.11. *Suppose m has norm relatively compact range and let $1 < p < \infty$. Let $f \in L^p(m)$. Then the operator $I_f : L^q(m) \to X$ defined by $I_f(g) := \int_\Omega f g dm$, is compact.*

Proof. The norm relative compactness of $m(\Sigma)$ ensures that I_{χ_Ω} is compact, see [31, Theorem 3.6]. Clearly, this implies that I_{χ_A} is compact for every $A \in \Sigma$ and, consequently, I_f is compact whenever f is a simple function. For the general case, let $(f_n)_n$ be a sequence of simple functions converging to f in the norm topology of $L^p(m)$. Then $(I_{f_n})_n$ is a sequence of compact operators converging to I_f in the operator norm and, therefore, I_f is compact too. $\square$

Theorem 2.1.12. *Suppose m has norm relatively compact range and $L^p(m)$ is reflexive. Then:*

(i) Γ *is w^*−closed in $L^p(m)^*$,*

(ii) $\mathrm{Ext}(B(L^p(m)^*)) \subset \Gamma$. *In particular, Γ is a James boundary for $B(L^p(m)^*)$.*

Proof. Since Γ is norming and symmetric, the Hahn-Banach theorem ensures that $B(L^p(m)^*) = \overline{\mathrm{co}(\Gamma)}^{w^*}$. This equality and the so-called "converse" of the Krein-Milman theorem (cf. [29, Lemma 5, p. 440]) yield $\mathrm{Ext}(B(L^p(m)^*)) \subset \overline{\Gamma}^{w^*}$.

Since $\mathrm{Ext}(B(L^p(m)^*))$ is a James boundary for $B(L^p(m)^*)$, it only remains to prove that Γ is w^*−closed. To this end, let $(\gamma_{g_\alpha, x^*_\alpha})$ be a net in Γ which converges to some $\varphi \in B(L^p(m)^*)$ in the w^*−topology. We will check that $\varphi \in \Gamma$. By the reflexivity of $L^p(m)$, the space $L^q(m)$ is reflexive as well, see [31, Corollary 3.10]. Since $B(L^q(m))$ is weakly compact and $B(X^*)$ is w^*-compact, we can assume without loss of generality that $g_\alpha \to g \in B(L^q(m))$ weakly and $x_\alpha^* \to x^* \in B(X^*)$ in the w^*-topology. We claim that $\varphi = \gamma_{g,x^*}$.

To this end, fix $f \in L^p(m)$ and set $x_\alpha := \int_\Omega g_\alpha f \, dm \in X$ for every α. Since $g_\alpha \to g$ weakly in $L^q(m)$, we have

$$\langle x_\alpha, x^* \rangle = \int_\Omega g_\alpha f \, d\langle m, x^* \rangle \to \int_\Omega gf \, d\langle m, x^* \rangle = \gamma_{g,x^*}(f).$$

On the other hand, the set $\{x_\alpha\}$ is norm relatively compact (by Lemma 2.1.11), $x_\alpha^* \to x^*$ in the w^*-topology and (x_α^*) is bounded, so we have

$$|\langle x_\alpha, x_\alpha^* \rangle - \langle x_\alpha, x^* \rangle| \to 0.$$

Since $|\langle x_\alpha, x_\alpha^* \rangle - \gamma_{g,x^*}(f)| \leq |\langle x_\alpha, x_\alpha^* \rangle - \langle x_\alpha, x^* \rangle| + |\langle x_\alpha, x^* \rangle - \gamma_{g,x^*}(f)|$ for every α, we conclude that

$$\varphi(f) = \lim_\alpha \gamma_{g_\alpha, x_\alpha^*}(f) = \lim_\alpha \langle x_\alpha, x_\alpha^* \rangle = \gamma_{g,x^*}(f).$$

As $f \in L^p(m)$ is arbitrary, $\varphi = \gamma_{g,x^*}$ and the proof is over. □

Remark 2.1.13. Under the assumptions of the previous theorem, the fact that Γ is a James boundary for $B(L^p(m)^*)$ can be deduced in a more direct way. Let $f \in L^p(m)$, the operator $I_f : L^q(m) \to X$ is weak–weak continuous, hence the convex set $I_f(B(L^q(m)))$ is weakly compact and, in particular, norm closed. The compactness of I_f now ensures that $I_f(B(L^q(m)))$ is norm relatively compact, thus there is $g \in B(L^q(m))$ such that $\|I_f(g)\|_X = \|I_f\| = \|f\|_{L^p(m)}$. Clearly, we have $\|I_f(g)\| = \gamma_{g,x^*}(f)$ for some $x^* \in B(X^*)$, and the conclusion follows.

Recall that a vector measure ϑ taking values in a Banach lattice Y is said to be *positive* if $\vartheta(\cdot) \geq 0$. In this case, we have $|\langle \vartheta, y^* \rangle| \leq \langle \vartheta, |y^*| \rangle$ for every $y^* \in Y^*$ and the semivariation of ϑ can be computed in a simple way. In fact, for $A \in \Sigma$, $\|\vartheta\|(A) = \|\vartheta(A)\|_X$. This observation will be needed in the proof of the following theorem.

Theorem 2.1.14. *Suppose X is a Banach lattice and m is positive. Then Γ is a James boundary for $B(L^p(m)^*)$.*

Proof. Fix $f \in L^p(m) \setminus \{0\}$. Since m is positive, the vector measure $\vartheta : \Sigma \to X$ given by $\vartheta(A) := \int_A |f|^p \, dm$ is positive as well. The comments preceding the theorem can be applied to ϑ ensuring that

$$\|f\|_{L^p(m)}^p = \|\vartheta\|(\Omega) = \|\vartheta(\Omega)\| = \left\| \int_\Omega |f|^p \, dm \right\|_X.$$

Take $x^* \in B(X^*)$ such that $\|f\|_{L^p(m)}^p = \langle \int_\Omega |f|^p \, dm, x^* \rangle = \int_\Omega |f|^p \, d\langle m, x^* \rangle$. Set $h := \mathrm{sign}(f)|f|^{p-1}$ and note that $h \in L^q(m)$ and $\|h\|_{L^q(m)}^q = \|f\|_{L^p(m)}^p$. Define $g := (1/\|h\|_{L^q(m)})h \in B(L^q(m))$. We claim that $\gamma_{g,x^*}(f) = \|f\|_{L^p(m)}$. Indeed:

$$\int_\Omega fg \, d\langle m, x^* \rangle = \Big(\int_\Omega fh \, d\langle m, x^* \rangle \Big) \cdot \|h\|_{L^q(m)}^{-1}$$
$$= \Big(\int_\Omega |f|^p \, d\langle m, x^* \rangle \Big) \cdot \|f\|_{L^p(m)}^{-(p/q)} = \|f\|_{L^p(m)}^p \cdot \|f\|_{L^p(m)}^{-(p/q)} = \|f\|_{L^p(m)}.$$

This finishes the proof. □

2.2. $m-$topology

In this section we will introduce the $m-$topology of the space $L^p(m)$, also denoted by $\sigma(L^p(m), \Lambda)$. First, we will give the conditions to ensure that the unit ball $B(L^p(m))$ is metrizable when endowed with this topology. This topology will be extremely useful in Chapter 4, devoted to the study of a new class of $r-$summing operators. In order to prove some theorems regarding these new classes of operators, we will work with subsets of $L^p(m)$ that are compact with respect to the $m-$topology (in the following, $m-$compact). We will finish the section with a characterization of $m-$compact sets.

2.2.1. Metrizability of the unit ball

Notice that we can identify isometrically a subset K of $L^p(m)$ with a subset of $\mathcal{L}(L^q(m), X)$. Each function f in K is associated to the linear and continuous operator $I_f \in \mathcal{L}(L^q(m), X)$, this identification is an isometry as a consequence of Proposition 1.0.7. Under this identification the $m-$topology can be considered exactly as the strong operator topology for the space $\mathcal{L}(L^q(m), X)$.

The metrizability of the unit ball of $L^p(m)$ endowed with $m-$topology is a direct consequence of the previous comments and following classical argument. If X and Y are Banach spaces, with Y separable, let $\{y_i : i \in \mathbb{N}\}$ be a dense subset in $B(Y)$; the strong operator topology of $\mathcal{L}(Y, X)$ can be metrized with the distance

$$d(S, T) := \sum_{i \in \mathbb{N}} \frac{1}{2^i} \|T(y_i) - S(y_i)\|_X,$$

where S and T belong to $\mathcal{L}(L^q(m), X)$.

Proposition 2.2.1. *If $L^q(m)$ is separable, then $(B(L^p(m)), \sigma(L^p(m), \Lambda))$ is metrizable.*

Let ρ a metric for the space $L^p(m)$, $\varepsilon > 0$ and $f_0 \in L^p(m)$, we denote by $B_{\rho,g}(f_0)$ the ball (with respect to the metric ρ) centered in f_0 with radius ε, that is

$$B_{\rho,\varepsilon}(f_0) := \{f \in L^p(m) : \rho(f_0, f) < \varepsilon\}.$$

The following theorem can be directly proved with the arguments above, nevertheless in the appendix of this chapter we give an alternative prove, with the explicit construction of the metric.

Theorem 2.2.2. *Assuming the separability of $L^q(m)$, we have that $B(L^p(m))$ is metrizable when endowed with the $m-$topology, and the metric is given by*

$$\rho(f_1, f_2) := \sum_{n=1}^{\infty} 2^{-n} \left\| \int_\Omega (f_1 - f_2) g_n dm \right\|, \qquad f_1, f_2 \in B(L^q(m)).$$

where $S = (g_n)_{n=1}^{\infty}$ is a dense subset of $B(L^q(m))$.

As for the $m-$weak topology, we can apply the results regarding the separability of the space $L^p(m)$. The following corollary is consequence of Proposition 2.1.3.

Corollary 2.2.3. *Let $\langle m, x^* \rangle$ be a Rybakov's measure for m. If the $\sigma-$algebra Σ is $|\langle m, x^* \rangle|-$essentially countably generated and $p > 1$ then unit ball $(B(L^p(m))$ is metrizable when endowed with the topology $\sigma(L^p(m), \Lambda))$.*

We finish the study of metrizability of the unit ball endowed with the $m-$topology with a sort of converse of Theorem 2.2.2. In order to prove it, we must assume that the space $L^q(m)$ has a separation condition.We say that the space $L^q(m)$ has the *$m-$separation property* whenever for every proper closed subspace S of $L^q(m)$ there is some $f \in L^p(m)$, non null function $\lambda-$a.e. such that

$$\left\| \int_\Omega hg dm \right\|_X = 0$$

for every $h \in S$.

Proposition 2.2.4. *If $B(L^p(m))$ is metrizable when endowed with the $m-$topology and $L^q(m)$ has the separation property, then $L^q(m)$ is separable.*

Proof. Metrizability of the unit ball $B(L^p(m))$ guaranties the existence of a sequence of neighborhoods of 0 (with respect to the metric ρ), $(B_{\frac{1}{n}}(0))_n$. Clearly $\bigcap_{n \in \mathbb{N}} B_{\frac{1}{n}}(0) = \{0\}$. The coincidence of the $m-$topology with the topology induced by the metric ρ implies the following. For each $n \in \mathbb{N}$ there is some neighborhood of 0 for the $m-$topology $V_{g_1^n,\ldots,g_{m_n}^n,\varepsilon_n}(0)$, so that

$$B(L^p(m)) \cap V_{g_1^n,\ldots,g_{m_n}^n,\varepsilon_n}(0) = \cap_{i=1}^{m_n}(B(L^p(m)) \cap V_{g_i{}^n,\varepsilon_n}(0)) \subset B_{\frac{1}{n}}(0).$$

Notice that, for every $g \in L^q(m)$ and each $\varepsilon > 0$

$$\begin{aligned} B(L^p(m)) \cap V_{g,\varepsilon}(0) &= \left\{ f \in B(L^p(m)) : \left\| \int_\Omega fgdm \right\| < \varepsilon \right\} \\ &= \left\{ f \in B(L^p(m)) : \left\| \int_\Omega \frac{g}{\varepsilon} fdm \right\| < 1 \right\} \\ &= B(L^p(m)) \cap V_{\frac{g}{\varepsilon},1}(0). \end{aligned}$$

Let $\overline{g_i^n} = \frac{g_i^n}{\varepsilon_n}$, for every $i = 1, \ldots, n_m$ and take $\{g_1, \ldots\} = \left\{ \overline{g_1^1}, \ldots \right\}$. We have

$$\{0\} = \bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}(0) \supset \bigcap_{n\in\mathbb{N}} B(L^p(m)) \cap V_{g_n,1}(0) = \bigcap_{n\in\mathbb{N}} V_n. \tag{2.6}$$

Denote by Y the closure of the linear subspace generated by $\{g_1, g_2, \ldots\}$. We will show that $Y = L^q(m)$ by contradiction. Suppose that $Y \neq L^q(m)$. Separation property of $L^q(m)$ yields the existence of a non null function $f \in L^p(m)$ so that $\left\| \int_\Omega fgdm \right\|_X = 0$ for every $g \in Y$. We can assume without loss of generality that $\|f\|_{L^p(m)} = 1$. Then we get that $f \in V_n$ for every $n \in \mathbb{N}$. By equality (2.6) we conclude $f = 0$, a contradiction. Therefore $Y = L^q(m)$. To obtain the separability of $L^q(m)$ it suffices to notice that the set of finite linear combinations of $\{g_1, g_2, \ldots\}$ with rational coefficients is countable and dense in $L^q(m)$.

□

2.2.2. m-compactness in $L^p(m)$

Our aim in this section is to characterize those subsets of $L^p(m)$ that are compact with respect to the $m-$topology ($m-$compact sets). The point is that each subset K of $L^p(m)$ can be identified isometrically with a subset of $\mathcal{L}(L^q(m), X)$ as follows. Each $f \in L^p(m)$ is associated to the integration operator $I_f : L^q(m) \to X$, isometry is a consequence of Proposition 1.0.7. In fact the $m-$topology coincides with the strong operator topology of $\mathcal{L}(L^q(m), X)$ when restricted to those operators that are defined by an integral with respect to m, I_f with $f \in L^q(m)$.

We say that a subset $K \subset L^p(m)$ is $m-$*complete* if every Cauchy net with respect to the $m-$topology ($m-$Cauchy) contained in K is $\sigma(L^p(m), \Lambda)-$convergent ($m-$convergent) in K.

Proposition 2.2.5. *Let $K \subseteq L^p(m)$. The following statements are equivalent.*

(i) K is $m-$compact.

(ii) The set K is closed in $\mathcal{L}(L^q(m), X)$ for the strong operator topology and for every $g \in L^q(m)$, the set

$$K_g := \left\{ \int_\Omega fgdm \in X : f \in K \right\}$$

is (norm) compact.

(iii) K is $m-$complete and for every $g \in L^q(m)$, the set

$$K_g := \left\{ \int_\Omega fgdm \in X : f \in K \right\}$$

is (norm) compact.

Proof. The proof of $(i) \Rightarrow (ii)$ is obvious, having in mind that $L^p(m)$ endowed with the $m-$topology is a Hausdorff space. For the converse, consider the Cartesian Product $\Pi_{g \in L^q(m)} X$. Notice that $\mathcal{L}(L^q(m), X)$ with the strong operator topology is isomorphic to the set

$$\left\{ (T(g))_{g \in L^q(m)} : T \in \mathcal{L}(L^q(m), X) \right\} \subseteq \Pi_{g \in L^q(m)} X$$

with the product topology. Let $i : K \to \Pi_{g \in L^q(m)} X$ be the map defined by $i(f) := (\int_\Omega fgdm)_{g \in L^q(m)}$, $f \in K$. Clearly, the $m-$topology in K coincides with the restriction of the product topology to the set

$$i(K) = \left\{ \left(\int_\Omega fgdm \right)_{g \in L^q(m)} : f \in K \right\} \subseteq \Pi_{g \in L^q(m)} X.$$

Thus, i is an isomorphism. By the compactness of the sets K_g and Tychonov theorem, the set

$$\left\{ \left(\int_\Omega f_g gdm \right)_{g \in L^q(m)} : f_g \in K \right\} = \Pi_{g \in L^q(m)} K_g =: W$$

is a compact subset of $\Pi_{g \in L^q(m)} X$ for the product topology. Consequently, $i(K)$ is relatively compact, and its closure is compact in the product. Since $i(K)$ endowed with the product topology restricted to W is isomorphic to K endowed with the $m-$topology, to show the $m-$compactness of K it is enough to prove that $i(K)$ is closed in W.

For this aim take a convergent net $((\int_\Omega f_\tau g)_g)_{\tau \in \mathcal{T}}$ in $i(K)$. Consider two elements $g_1, g_2 \in L^q(m)$. Then the linearity of the integral and the fact that the topology in the product is given by the pointwise convergence give

$$\lim_\tau \int_\Omega f_\tau(g_1+g_2)dm = \lim_\tau \int_\Omega f_\tau g_1 dm + \lim_\tau \int_\Omega f_\tau g_2 dm.$$

Therefore, $\lim_\tau(\int_\Omega f_\tau g dm)_g$ can be identified with the range of a linear map $R : L^q(m) \to X$. Since the sets K_g are compact and the operators $I_{f_\tau} : L^q(m) \to X$ are continuous, the set $\{f_\tau | \tau \in \mathcal{T}\}$ is pointwise bounded and the pointwise limit of the net is also a continuous map (see [29, Theorem II, 1.18]). Moreover, since the net converges in $\mathcal{L}(L^q(m), X)$ for the strong operator topology and K is closed for this topology, we obtain that there is a function $f \in K$ such that $R(g) = \int_\Omega fgdm$ for every $g \in L^q(m)$. Therefore, the set $i(K)$ is closed in W, and thus it is $m-$compact. This gives (i).

The proof of the fact that (*iii*) and (*i*) are equivalent follows the same lines. To prove $(iii) \Rightarrow (i)$, take an $m-$Cauchy net $(f_\tau)_{\tau\in\mathcal{T}}$ in K. Obviously, for every $g \in L^q(m)$, the net $\left(\int_\Omega f_\tau g dm\right)_{\tau\in\mathcal{T}}$ converges in X. So the limit of the net is pointwise defined by the compactness of the sets K_g. The $m-$completeness of the set K gives a function f such that $m - \lim_\tau f_\tau = f$. Since this element can be identified with the element $(\int_\Omega fgdm)_{g\in L^q(m)}$ of the product, we obtain that $i(K)$ is a closed subset of the product, and thus it is compact –see the argument given in the proof of (*ii*) $\Longrightarrow$ (*i*)–. Therefore, as a consequence of the fact that i is an isomorphism, we obtain that K is an $m-$compact set.

□

Let m be a positive vector measure with range in a Banach lattice and E a finite dimensional subspace of $L^p(m)$. In the following example we show the construction of an $m-$compact $m-$*norming* set $K_E \subset B(L^q(m))$ for E. That is, an $m-$compact set K_E so that for every $f \in E$, $\|f\|_{L^p(m)} = \sup_{g\in K_E} \left\|\int_\Omega fgdm\right\|$. Recall that when the vector measure m is positive, that is $m(\Sigma) \subset X^+$, the norm of $f \in L^p(m)$ is given by

$$\|f\|_{L^p(m)} = \left\|\int_\Omega |f|^p dm\right\|_X^{\frac{1}{p}}.$$

Example 2.2.6. Define $\Phi : \delta B(L^p(m)) \to B(L^q(m))$ by $\Phi(f) = |f|^{\frac{p}{q}} sig\{f\}$, where $\delta B(L^p(m)) = \{f \in L^p(m) : \|f\|_{L^p(m)} = 1\}$. The map Φ is well defined since for $f \in \delta B(L^p(m))$ we have by the positivity of m,

$$\|\Phi(f)\|_{L^q(m)} = \left\|\int_\Omega |f|^p dm\right\|_X^{\frac{1}{q}} = 1.$$

Moreover, for each $f \in \delta B(L^p(m))$ we have

$$\left\| \int_\Omega f\Phi(f)dm \right\|_X = \left\| \int_\Omega |f|^p dm \right\|_X = \|f\|^p_{L^p(m)} = 1.$$

Let E be a finite dimensional subspace of $L^p(m)$, by the equality above we have that $\Phi(\delta B(E))$ is $m-$norming for E. We will prove that Φ is continuous in order to show that the $m-$norming set for E, $\Phi(\delta(B(E)))$ is norm compact, then $m-$compact. For this aim, let $(f_n)_n \in \delta B(L^p(m))$ such that $\lim_{n\to\infty} f_n = f \in \delta B(L^p(m))$, that is $\lim_{n\to\infty} \left\| \int_\Omega |f_n - f|^p dm \right\|_X = 0$. We must show that $\lim_{n\to\infty} \left\| \int |\Phi(f_n) - \Phi(f)|^q dm \right\|_X = 0$. For each $n \in \mathbb{N}$, we define

$$\begin{aligned} A_n &:= \{w \in \Omega : sig\{f_n(w)\} = sig\{f(w)\}\}, \\ B_n &:= \{w \in \Omega : sig\{f_n(w)\} \neq sig\{f(w)\}\}, \end{aligned}$$

then

$$\begin{aligned} \left\| \int_\Omega \left| |f_n|^{\frac{p}{q}} sig\{f_n\} - |f|^{\frac{p}{q}} sig\{f\} \right|^q dm \right\|_X &\leq \underbrace{\left\| \int_{A_n} \left| |f_n|^{\frac{p}{q}} - |f|^{\frac{p}{q}} \right|^q dm \right\|_X}_{(I)} \\ &+ \underbrace{\left\| \int_{B_n} \left| |f_n|^{\frac{p}{q}} + |f|^{\frac{p}{q}} \right|^q dm \right\|_X}_{(II)}. \end{aligned}$$

In order to work with (I) and (II), the following inequalities for positive real numbers a and b will be used:

$$(a+b)^r \leq a^r + b^r, \ \ r \leq 1, \tag{2.7}$$

and

$$(a+b)^r \leq 2^{r-1}(a^r + b^r), \ \ r \geq 1. \tag{2.8}$$

We will distinguish two cases.

Case 1. Suppose that $p \leq q$. We can assume without lost of generality that $|f_n|(w) > |f|(w)$ for $w \in A_n$. Having in mind inequality (2.7) above, we get

$$\begin{aligned} |f_n|(w) &= (|f_n|(w) - |f|(w)) + |f|(w) \\ &= \left(((|f_n|(w) - |f|(w)) + |f|(w))^{\frac{p}{q}} \right)^{\frac{q}{p}} \\ &\leq \left((|f_n|(w) - |f|(w))^{\frac{p}{q}} + |f|(w)^{\frac{p}{q}} \right)^{\frac{q}{p}}, \end{aligned}$$

thus $\left||f_n|(w)^{\frac{p}{q}}-|f|(w)^{\frac{p}{q}}\right| \leq ||f_n|(w)-|f|(w)|^{\frac{p}{q}}$, therefore, when $n\to\infty$

$$\left\|\int_{A_n}\left||f_n|^{\frac{p}{q}}-|f|^{\frac{p}{q}}\right|^q dm\right\|_X \leq \left\|\int_\Omega ||f_n|-|f||^p\, dm\right\|_X \to 0.$$

In order to study part (II), an application of (2.8) yields

$$\left(|f_n|(w)^{\frac{p}{q}}+|f|(w)^{\frac{p}{q}}\right)^{\frac{q}{p}} \leq 2^{\frac{q}{p}-1}(|f_n|(w)+|f|(w)).$$

Since for $w\in B_n$ we have that $sig\{f_n(w)\}\neq sig\{f(w)\}$, the norms of the integrals in (II) are bounded by $\left\|\int_\Omega |f_n-f|^p dm\right\|_X=\|f_n-f\|_{L^p(m)}$, that tends to 0, and the conclusion follows.

Case 2. Suppose now that $p>q$ we have, for $w\in A_n$,

$$\left||f_n|(w)^{\frac{p}{q}}-|f|(w)^{\frac{p}{q}}\right|^q \leq \left(\frac{p}{q}\right)^q \left||f_n|(w)^{\frac{p}{q}-1}+|f|(w)^{\frac{p}{q}-1}\right|^q ||f_n|(w)-|f|(w)|^q,$$

thus, Hölder's inequality with k the conjugated exponent of $\frac{p}{q}$ (we obtain it by solving $\frac{q}{p}+\frac{1}{k}=1$, we get $k=\frac{p}{p-q}$, and $\frac{p}{q}-1=\frac{p/q}{k}$), yields

$$\left\|\int_{A_n}\left(|f_n|(w)^{\frac{p}{q}}-|f|(w)^{\frac{p}{q}}\right)^q dm\right\|_X \leq$$

$$\left(\frac{p}{q}\right)^q \underbrace{\left\|\int_{A_n}\left(|f_n|(w)^{\frac{\frac{p}{q}}{k}}+|f|(w)^{\frac{\frac{p}{q}}{k}}\right)^{qk} dm\right\|_X^{\frac{1}{k}}}_{(III)} \underbrace{\left\|\int_{A_n}(|f_n|(w)-|f|(w))^p\, dm\right\|_X^{\frac{q}{p}}}_{\downarrow 0}$$

In order to find a bound for (III) we distinguish two cases. If $\frac{1}{kq}\geq 1$ by (2.7) we get $\left(|f_n|(w)^{p\frac{1}{kq}}+|f|(w)^{p\frac{1}{kq}}\right)^{qk} \leq |f_n|(w)^p+|f|(w)^p$. For $\frac{1}{kq}\leq 1$, then $qk\geq 1$ and we can apply inequality (2.8) and we get

$$\left(|f_n|(w)^{\frac{p}{kq}}+|f|(w)^{\frac{p}{kq}}\right)^{qk} \leq 2^{qk-1}(|f_n|(w)^p+|f|(w)^p).$$

Thus we have that (III) is bounded therefore, the limit of (I) when n tends to ∞ is 0.

Let us study now the part (II). An application of (2.7), yields

$$\left(|f_n|(w)^{\frac{p}{q}}+|f|(w)^{\frac{p}{q}}\right)^q \leq \left((|f_n|(w)+|f|(w))^{\frac{p}{q}}\right)^q.$$

So, again, the norms of the integrals in (II) are bounded by

$$\left\|\int_\Omega |f_n-f|^p dm\right\|_X=\|f_n-f\|_{L^p(m)},$$

that tends to 0, and the conclusion follows also in this case.

We have obtained that Φ is continuous with respect to the norm topology. Then for any finite dimensional subspace $E \subset L^p(m)$, $\Phi(\delta B(E))$ is an $m-$norming norm compact (then $m-$compact) subset of $B(L^q(m))$.

For a general vector measure it is not possible to construct a norming $m-$compact set. Nevertheless, for every vector measure m (not necessarily with range in a Banach lattice), for E a finite dimensional subspace of $L^p(m)$ and for every $0 < \varepsilon < \frac{1}{2}$ we can obtain an $m-$compact $m-\varepsilon-$*norming* set $K_{E,\varepsilon} \subset L^q(m)$. That is an $m-$compact set $K_{E,\varepsilon}$ so that

$$(1-\varepsilon)\|f\|_{L^p(m)} \leq \sup_{g \in K_{E,\varepsilon}} \left\| \int_\Omega fgdm \right\|_X \leq \|f\|_{L^p(m)},$$

for each $f \in E$. ,In the following example, we show the details of the construction of a finite (then compact) subset in $B(L^q(m))$ that is $m-\varepsilon-$norming for E.

Example 2.2.7. Denote by $\delta B(E) = \{f \in E : \|f\|_{L^p(m)} = 1\}$ the boundary of the unit ball of E. For each natural number n we can cover $\delta B(E)$ with a finite number of balls of radius $\frac{1}{n}$, with centers $\{f_1^n, \ldots, f_{m_n}^n\}$. Moreover, for each center f_i^n, $i = 1, \ldots, m_n$ and each $\varepsilon_0 > 0$, there is some $g_{i,\varepsilon_0}^n \in B(L^q(m))$ so that $\|f_i^n\|_{L^p(m)} \leq \varepsilon_0 + \left\| \int_\Omega f_i^n g_{i,\varepsilon_0}^n dm \right\|_X$. Thus for $f \in \delta B(E)$, we have

$$\begin{aligned}
\|f\|_{L^p(m)} = 1 &\leq \frac{1}{n} + \varepsilon_0 + \left\| \int_\Omega f_i^n g_{i,\varepsilon_0}^n dm \right\|_X \\
&\leq \frac{1}{n} + \varepsilon_0 + \left\| \int_\Omega (f_i^n - f) g_{i,\varepsilon_0}^n dm \right\|_X + \left\| \int_\Omega f g_{i,\varepsilon_0}^n dm \right\|_X \\
&\leq \frac{2}{n} + \varepsilon_0 + \sup_{g \in K_{E,\varepsilon}} \left\| \int_\Omega fgdm \right\|_X
\end{aligned}$$

where $K_{E,\varepsilon}$ is the set of functions g_{i,ε_0}^n, $i = 1, \ldots, m_n$ for $2/n \leq \varepsilon/4$ and $\varepsilon_0 = \frac{\varepsilon}{2}$. Then we get for $f \in \delta B(E)$,

$$1 - \varepsilon \leq \sup_{g \in K_{E,\varepsilon}} \left\| \int fgdm \right\|_X,$$

so for any $f \in E$ we have that

$$(1-\varepsilon)\|f\|_{L^p(m)} \leq \sup_{g \in K_{E,\varepsilon}} \left\| \int fgdm \right\|_X \leq \|f\|_{L^p(m)}.$$

Obviously the set $K_{E,\varepsilon}$ is norm compact (because it is finite), therefore it is m-compact.

2.3. Appendix

The following corresponds to the proof of Proposition 2.1.1.

Proof. Notice that ρ defines a metric, indeed for $f_1, f_2 \in L^p(m)$, we have

$$\begin{aligned} \rho(f_1, f_2) &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\sum_{k=1}^{\infty} 2^{-k} \left\| \int_\Omega (f_1 - f_2) g_k dm \right\|_X \|x_n^*\| \right) \\ &\leq \sum_{n=1}^{\infty} 2^{-n} \left(\sum_{k=1}^{\infty} 2^{-k} \|f_1 - f_2\|_{L^p(m)} \|g_k\|_{L^q(m)} \right) \\ &\leq \|f_1 - f_2\|_{L^p(m)} \leq \|f_1\|_{L^p(m)} + \|f_2\|_{L^p(m)}. \end{aligned}$$

In order to prove the continuity of ρ with respect to the $m-$topology, let $(f_\alpha)_\alpha \subset B(L^p(m))$ be a net converging to f with respect to the $m-$weak topology. We claim that this net also converges to f in the topology induced by the metric ρ. Let $\varepsilon > 0$. By the convergence of the serie $\sum_{n=1}^{\infty} 2^{-n}$, there is $N \in \mathbb{N}$ so that $\sum_{n=N}^{\infty} 2^{-n} < \frac{\varepsilon}{4}$. Therefore

$$\sum_{n=N}^{\infty} 2^{-n} \left(\sum_{k=1}^{\infty} 2^{-k} \left| \left\langle \int_\Omega (f_\alpha - f) g_k dm, x_n^* \right\rangle \right| \right) \leq \frac{\varepsilon}{2}.$$

Let M in $\mathbb{N}$ such that $2 \sum_{k=M}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{4}$, we have

$$\sum_{n=1}^{N-1} \frac{1}{2^n} \left(\sum_{k=1}^{\infty} \left| \left\langle \int_\Omega (f_\alpha - f) g_k dm, x_n^* \right\rangle \right| \right) \leq$$

$$\leq \sum_{n=1}^{N-1} \frac{1}{2^n} \left(\sum_{k=1}^{M-1} \frac{1}{2^k} \left| \left\langle \int_\Omega (f_\alpha - f) g_k dm, x_n^* \right\rangle \right| \right) + \frac{\varepsilon}{4}.$$

Therefore,

$$\rho(f_\alpha, f) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \sum_{n=1}^{N-1} \frac{1}{2^n} \left(\sum_{k=1}^{M-1} \frac{1}{2^k} \left| \left\langle \int_\Omega (f_\alpha - f) g_k dm, x_n^* \right\rangle \right| \right).$$

The convergence of the net implies that for $n \leq N-1$ and $k \leq M-1$ there is an index $\alpha_{n,k}$ such that $|\langle \int_\Omega (f_{\alpha_{n,k}} - f) g_k dm, x_n^* \rangle| < \frac{\varepsilon}{4}$. Let α_0 be an index so that $\alpha_0 \geq \alpha_{n,k}$ for every $n \leq N-1$ and $k \leq M-1$; we have

$$\sum_{n=1}^{N-1} \frac{1}{2^n} \left(\sum_{k=1}^{M-1} \frac{1}{2^k} \left| \left\langle \int_\Omega (f_\alpha - f) g_k dm, x_n^* \right\rangle \right| \right) < \frac{\varepsilon}{4}.$$

Therefore, for $\alpha \geq \alpha_0$, we get $\rho(f, f_\alpha) < \varepsilon$. The metric ρ is continuous with respect to the $m-$weak topology. In order to show that they coincide, fix

$f_0 \in B(L^p(m))$, $g_0 \in B(L^q(m))$, $x_0^* \in B(X^*)$ and $\varepsilon > 0$ and consider the neighborhood of f_0

$$V_{g_0,x_0^*,\varepsilon}(f_0) = \left\{ f \in L^p(m) : \left| \left\langle \int_\Omega (f_0 - f) g_0 dm, x_0^* \right\rangle \right| < \varepsilon \right\}.$$

We claim that this set contains a ball (with respect to ρ) centered in f_0. We prove it in two steps.

First, we search $\overline{g_0} \in S_1$, $\overline{x_0^*} \in S_2$, so that inclusion

$$U := V_{\overline{g_0},\overline{x_0^*},\frac{\varepsilon}{3}}(f_0) \cap B(L^p(m)) \subset V_{g_0,x_0^*,\varepsilon}(f_0) \cap B(L^p(m)) =: W. \tag{2.9}$$

holds, where

$$U = \left\{ f \in B(L^p(m)) : \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, \overline{x_0^*} \right\rangle \right| < \frac{\varepsilon}{3} \right\}.$$

and

$$W = \left\{ f \in B(L^p(m)) : \left| \left\langle \int_\Omega (f_0 - f) g_0 dm, x_0^* \right\rangle \right| < \varepsilon \right\},$$

Since S_1 and S_2 are dense subsets in $B(L^q(m))$ and $B(X^*)$ respectively, there are $\overline{g_0} := g_{k_0} \in S_1$ and $\overline{x_0^*} := x_{n_0}^* \in S_2$ so that

$$\|g_0 - \overline{g_0}\|_{L^q(m)} < \frac{\varepsilon}{6\|x_0^*\|}, \qquad \text{and} \qquad \|x_0^* - \overline{x_0^*}\|_{X^*} < \frac{\varepsilon}{6\|\overline{g_0}\|_{L^q}(m)}.$$

Let $f \in U$, we have

$$\left| \left\langle \int_\Omega (f_0 - f) g_0 dm, x_0^* \right\rangle \right| \leq$$

$$\begin{aligned}
&\leq \left| \left\langle \int_\Omega (f_0 - f)(g_0 - \overline{g_0}) dm, x_0^* \right\rangle \right| + \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, x_0^* \right\rangle \right| \\
&\leq \left| \left\langle \int_\Omega (f_0 - f)(g_0 - \overline{g_0}) dm, \frac{x_0^*}{\|x_0^*\|} \right\rangle \right| \|x_0^*\| + \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, x_0^* \right\rangle \right| \\
&\leq \|f_0 - f\|_{L^p(m)} \|g_0 - \overline{g_0}\|_{L^q(m)} \|x_0^*\| \\
&+ \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, x_0^* - \overline{x_0^*} \right\rangle \right| + \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, \overline{x_0^*} \right\rangle \right| \\
&< 2 \frac{\varepsilon}{6\|x_0^*\|} \|x_0^*\| + \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, x_0^* - \overline{x_0^*} \right\rangle \right| + \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \left| \left\langle \int_\Omega (f_0 - f) \overline{g_0} dm, \frac{x_0^* - \overline{x_0^*}}{\|x_0^* - \overline{x_0^*}\|} \right\rangle \right| \|x_0^* - \overline{x_0^*}\| + \frac{\varepsilon}{3} \\
&\leq \frac{\varepsilon}{3} + \|f_0 - f\|_{L^p(m)} \|\overline{g_0}\|_{L^q(m)} \|x_0^* - \overline{x_0^*}\| + \frac{\varepsilon}{3} \\
&< \frac{\varepsilon}{3} + 2\|\overline{g_0}\|_{L^q(m)} . \frac{\varepsilon}{6.\|\overline{g_0}\|_{L^q(m)}} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

then $f \in W$. We finish by proving $B_{\rho,\frac{\varepsilon}{3\cdot 2^{n_0+k_0}}}(f_0) \cap B(L^p(m)) \subset U$. Let $f \in B_{\rho,\frac{\varepsilon}{3\cdot 2^{n_0+k_0}}}(f_0) \cap B(L^p(m))$, therefore

$$\rho(f_0, f) = \sum_{n=1}^{\infty} 2^{-n} \sum_{k=1}^{\infty} 2^{-k} \left| \int (f_0 - f) h_k d\langle m, x_n^* \rangle \right| < \frac{\varepsilon}{3 \cdot 2^{n_0+k_0}}.$$

That means, for every pair of indexes k, n, in particular for k_0 and n_0 we have

$$2^{-(k+n)} \left| \left\langle \int (f_0 - f) h_k dm, x_n^* \right\rangle \right| < \frac{\varepsilon}{3 \cdot 2^{n_0+k_0}},$$

thus, $f \in U$, and therefore $f \in V_{g_0, x_0^*, \varepsilon}(f_0) \cap B(L^p(m))$ by inclusion (2.9). $\square$

We finish this chapter with the proof of Theorem 2.2.2.

Proof. Since each $g_n \in B(L^q(m))$ for $n \in \mathbb{N}$, we have that $\rho(f_1, f_2) \leq \|f_1\| + \|f_2\|$. The metric is consistent.

In order to show that ρ is continuous with respect to the $m-$topology let $(f_\alpha)_\alpha \subset B(L^p(m))$ so that $f_\alpha \longrightarrow f$ in $(B(L^p(m)), \sigma(L^p(m), \Lambda))$ and take $\varepsilon > 0$.

Let $k \in \mathbb{N}$ such that $\sum_{n=k+1}^{\infty} 2^{-n} < \varepsilon/4$. There is an index α_0 so that for every $\alpha \geq \alpha_0$ we get

$$\left\| \int_\Omega (f_\alpha - f) g_n dm \right\|_X < \frac{\varepsilon}{2} \text{ , for each } 1 \leq n \leq k$$

Such an index α_0 can be found following the same arguments as those in the proof of Proposition 2.1.1 (see appendix at the end of this chapter). Therefore,

$$\rho(f_\alpha, f) < \frac{\varepsilon}{2} + \sum_{n=k+1}^{\infty} 2^{-n} \left\| \int_\Omega (f_\alpha - f) g_n dm \right\|_X < \varepsilon.$$

In order to prove the coincidence of the topologies, let $f_0 \in B(L^p(m))$, $\varepsilon > 0$ and $g \in B(L^q(m))$ and consider the neighborhood

$$V_{g,\varepsilon}(f_0) := \left\{ f \in L^p(m) : \left\| \int_\Omega (f - f_0) g dm \right\|_X \leq \varepsilon \right\}.$$

We must find some $g_0 \in S$ so that

$$U := V_{g_0, \frac{\varepsilon}{2}}(f_0) \cap B(L^p(m)) \subset V_{g,\varepsilon}(f_0) \cap B(L^p(m)) =: W.$$

Since S is a dense subset in $B(L^q(m))$, there is an element $g_0 := g_{n_0} \in S$ so that $\|g - g_0\|_{L^q(m)} < \varepsilon/4$. Let $f \in U$, in fact we have $f \in W$, indeed

$$\begin{aligned} \left\| \int_\Omega (f_0 - f) g \, dm \right\|_X &< \left\| \int_\Omega (f_0 - f)(g - g_0) \right\|_X + \left\| \int_\Omega (f_0 - f) g_0 \, dm \right\|_X \\ &< \|f_0 - f\|_{L^p(m)} \|g - g_0\|_{L^q(m)} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

We claim that $B_{\rho, \frac{\varepsilon}{2^{n_0+1}}}(f_0) \subset V_{g_0, \frac{\varepsilon}{2}}(f_0) \cap B(L^p(m))$; in fact we have

$$\rho(f_0, f) = \sum_{n=1}^{\infty} 2^{-n} \left\| \int_\Omega (f_0 - f) g_n dm \right\|_X < \frac{\varepsilon}{2^{n_0+1}},$$

therefore $2^{-n} \left\| \int_\Omega (f_0 - f) g_n dm \right\|_X < \varepsilon/2^{n_0+1}$ for every $n \in \mathbb{N}$; particularly for $n = n_0$ we get $\left\| \int_\Omega (f_0 - f) g_n dm \right\|_X < \frac{\varepsilon}{2}$ and $f \in V_{g_0, \frac{\varepsilon}{2}}(f_0) \cap B(L^p(m))$ as claimed. This directly yields the conclusion, since each basic neighborhood for the $m-$topology corresponds to the finite intersection of those appearing in the definition of W.

$\square$

Chapter 3

Tensor product representation of the (pre)dual of $L^p(m)$

It seems natural to represent the dual space of $L^p(m)$ in terms of the space $L^q(m)$, as in the case of classical L^p-spaces. However, two facts suggest that this representation cannot be direct. The first one is that it is well-known that the dual of $L^p(m)$ coincides with $L^q(m)$ only in the trivial cases (i.e. when $L^p(m)$ is isomorphic to $L^p(\mu)$ of a scalar measure μ). The second one is that $L^p(m)$ can be a weighted c_0-space as in Example 1.0.6, and then reflexivity cannot be expected in general for these spaces. From the technical point of view, the natural weak topology associated with the integration map –the so called $m-$weak topology– is the keystone of our arguments. We have proved in Chapter 2 that it coincides with the weak topology of $L^p(m)$ on bounded sets. However, for the aim of this chapter –and also for a lot of applications– it is better to use this description. For the case $p = 1$, a representation of the elements of the dual space of $L^1(m)$ has been given in [53].

Integration operators. Our aim in this section is to characterize those operators $G : L^p(m) \to X$ that can be written as an integral. The following Radon-Nikodým theorem for scalarly dominated measures is proved in [52, Theorem 1] and provides an important tool for our work. We write an adapted version in the following lemma.

Lemma 3.0.1. *Let m and $\tilde{m}$ be vector measures defined in the same measurable space and with range in a Banach space X. The following assertions are equivalent:*

(i) *There exists a bounded measurable function θ such that*

$$m(E) = \int_E \theta d\tilde{m}, \quad E \in \Sigma.$$

(ii) *m is scalarly dominated by $\tilde{m}$, that is, there exists a positive constant K such that $|\langle m, x^*\rangle|(A) \leq K |\langle \tilde{m}, x^*\rangle|(A)$, for each $A \in \Sigma$ and each $x^* \in X^*$.*

Theorem 3.0.2. *The following assertions are equivalent for an operator $G : L^p(m) \to X$.*

(i) *There is a function $g \in L^q(m)$ such that $G(f) = \int_\Omega fg dm$ for every $f \in L^p(m)$.*

(ii) *There are $g_1, \ldots, g_n$ in $L^q(m)$ such that for all $x^* \in X^*$:*

$$|\langle G(f), x^*\rangle| \leq \sum_{i=1}^{n} \left| \left\langle \int_\Omega f g_i dm, x^* \right\rangle \right|, \quad f \in L^p(m).$$

(iii) *There is a function g_0 in $L^q(m)$ such that for all $x^* \in X^*$:*

$$|\langle G(f), x^*\rangle| \leq \int_\Omega |f g_0| d|\langle m, x^*\rangle|, \quad f \in L^p(m). \tag{3.1}$$

Moreover, the subspace of all the operators G of $\mathcal{L}(L^p(m), X)$ that satisfy (i), (ii) or (iii) is isometrically isomorphic to $L^q(m)$.

Proof. By the representation of the operator G of $\mathcal{L}(L^p(m), X)$ as an integral, it is obvious that (i) implies (ii).

The proof of (ii) $\Rightarrow$ (iii) is a direct consequence of the following inequalities. Let $G : L^p(m) \to X$ be an operator satisfying (ii). For all x^* in X^* and f in $L^p(m)$,

$$|\langle G(f), x^*\rangle| \leq \sum_{i=1}^{n} \left| \left\langle \int_\Omega f g_i dm, x^* \right\rangle \right|$$

$$\leq \sum_{i=1}^{n} \int_\Omega |f g_i| d|\langle m, x^*\rangle| = \int_\Omega \left(\sum_{i=1}^{n} |g_i| \right) |f| d|\langle m, x^*\rangle|.$$

Since $\sum_{i=1}^{n} |g_i| \in L^q(m)$, we obtain ($iii$).

For the proof of (iii) $\Rightarrow$ (i), suppose that there is a function $g_0 \in L^q(m)$ so that (3.1) holds and define the set function $m_G : \Sigma \to X$ by

$$m_G(A) := G(\chi_A), \quad A \in \Sigma. \tag{3.2}$$

It is easy to see that m_G is a countably additive vector measure, since $L^p(m)$ is order continuous. Let us define the measure $m_1 : \Sigma \to X$ by $m_1(A) := \int_A g_0 \, dm$, $A \in \Sigma$. For each $f \in L^p(m)$ the following inequality holds

$$|\langle G(f), x^* \rangle| \leq \int_\Omega |f| d|\langle m_1, x^* \rangle|$$

for all x^* in X^*. Therefore, for each set $A \in \Sigma$ and every $x^* \in X^*$ we have

$$|\langle m_G(A), x^* \rangle| = |\langle G(\chi_A), x^* \rangle| \leq \int_\Omega \chi_A d|\langle m_1, x^* \rangle| = |\langle m_1, x^* \rangle|(A).$$

Hence, m_G is scalarly dominated by m_1. By Lemma 3.0.1, there is a bounded measurable function θ such that

$$m_G(A) = G(\chi_A) = \int_A \theta dm_1 = \int_A \theta g_0 dm$$

for each $A \in \Sigma$. Note that the product θg_0 is also in $L^q(m)$. If $I_{\theta g_0}$ is the integration operator from $L^p(m)$ into X defined by $I_{\theta g_0}(f) = \int_\Omega f \theta g_0 dm$, we have that $I_{\theta g_0}$ and G coincides in the set of simple functions. Since this set is dense in $L^p(m)$ we obtain $G(f) = I_{\theta g_0}(f)$ for all f in $L^p(m)$ which gives (i) for $g = \theta g_0$. Finally, the isometry is a consequence of Proposition 1.0.7.

□

In order to prove a characterization theorem for those operators defined in $L^p_w(m)$ that can be represented as an integral, recall that, for p, q conjugated real numbers (as in 1.7) we have

$$L^p_w(m) \cdot L^q(m) = L^p(m) \cdot L^q(m) = L^1(m),$$

as proved in Lemma 1 in [7].

Lemma 3.0.3. *Let $G \in \mathcal{L}(L^p_w(m), X)$ so that G is null at $L^p(m)$, and there is some $g_0 \in L^q(m)$ such that*

$$|\langle G(h), x^* \rangle| \leq \int_\Omega |hg_0| d|\langle m, x^* \rangle|, \tag{3.3}$$

for every $h \in L^p(m)$ and $x^* \in X^*$. Then, the operator $G = 0$.

Proof. Let $f \in L^p_w(m)$ and $x^* \in X^*$. We define a functional T of $L^p_w(m)^*$ as $T(\cdot) := \langle G(\cdot), x^* \rangle$. Let $(f_n)_n$ be a sequence of simple functions that converges to f in $L^p(|\langle m, x^* \rangle|)$. Therefore

$$\int_\Omega |(f_n - f) g_0| d|\langle m, x^* \rangle| \to 0,$$

and condition (3.3) ensures that $|T(f_n - f)| \to 0$. Since each simple function f_n belongs to $L^p(m)$, we have that $T(f_n) = 0$ for every $n \in \mathbb{N}$, then $T(f) = 0$. Since the element $x^* \in X^*$ is arbitrary, we have $G(f) = 0$. □

The following theorem gives a similar result for operators on $L^p_w(m)$. We write it separately because for its proof it is necessary to use a slightly different argument, due to the fact that the set of simple functions is not in general dense in $L^p_w(m)$.

Theorem 3.0.4. *The following assertions are equivalent for an operator $G : L^p_w(m) \to X$.*

(*i*) *There is a function $g \in L^q(m)$ such that $G(f) = \int_\Omega fg dm$ for every $f \in L^p_w(m)$.*

(*ii*) *There are $g_1, \ldots, g_n$ in $L^q(m)$ such that for all x^* in X^*:*

$$|\langle G(f), x^* \rangle| \leq \sum_{i=1}^{n} \left| \left\langle \int_\Omega f g_i dm, x^* \right\rangle \right|, \quad f \in L^p_w(m).$$

(*iii*) *There is a function g_0 in $L^q(m)$ such that for all x^*:*

$$|\langle G(f), x^* \rangle| \leq \int_\Omega |f g_0| d|\langle m, x^* \rangle|, \quad f \in L^p_w(m).$$

Moreover, the subspace of $\mathcal{L}(L^p_w(m), X)$ of all the operators G satisfying (i), (ii) or (iii) is isometrically isomorphic to $L^q(m)$.

Proof. We only show the proof of $(iii) \Rightarrow (i)$, which is different that the one in the previous theorem. First note that the restriction of G to $L^p(m)$ is well defined and continuous.

By the previous theorem, we have that $G|_{L^p(m)}$ can be represented as an integration operator with some $g \in L^q(m)$. We consider the operator I_g associated to these function g, $I_g \in \mathcal{L}(L^p_w(m), X)$ defined by $I_g(f) := \int_\Omega fg dm$. Both G and I_g are 'scalarly dominated by a function in $L^q(m)$' in the sense of inequality (3.3), therefore the operator defined by the difference $G - I_g$, also follows this boundedness condition. A direct application of the previous lemma yields the conclusion.

The isometry is a consequence of Proposition 1.0.7. □

Even for the finite dimensional case, the domination requirement given in (iii) Theorem 3.0.4 cannot be replaced by a domination in norm, i.e. the condition given there for G is not equivalent to the existence of a function $g \in L^q(m)$ such that for every $f \in L^p_w(m)$, $\|G(g)\|_X \leq \left\| \int_\Omega fg \, dm \right\|_X$. Let us show this with a simple example of a vector measure with values in $\mathbb{R}^2$.

Example 3.0.5. Define the vector measure m on the σ-algebra of Borel subset of $[0,1]$, $\mathcal{B}([0,1])$ over $\mathbb{R}^2$ endowed with the euclidean norm

$$m(A) := \left(\mu\left(A \cap \left[0, \frac{1}{2}\right]\right), \mu\left(A \cap \left[\frac{1}{2}, 1\right]\right)\right)$$

for $A \in \mathcal{B}([0,1])$ with μ the Lebesgue measure on $[0,1]$. Since the dual of $\mathbb{R}^2$ endowed with the euclidean norm is itself, we have that the spaces $L^p(m)$ and $L^p_w(m)$ will necessarily coincide for $1 < p < \infty$. Notice that a function f is integrable with respect to m whenever the restrictions of f to $[0,1/2]$ and to $[1/2,1]$ are Lebesgue integrable. That means that, for $1 < p < \infty$, $L^p(m)$ is the direct sum of the spaces $L^p(\mu|_{[0,\frac{1}{2}]})$ and $L^p(\mu|_{[\frac{1}{2},1]})$. Clearly, the norm of a function $f \in L^p(m)$ is given by

$$\|f\|_{L^p(m)} = \left(\left(\int_0^{\frac{1}{2}} |f|^p d\mu\right)^2 + \left(\int_{\frac{1}{2}}^1 |f|^p d\mu\right)^2\right)^{1/2p}.$$

Let $\Phi : L^p(m) \to \mathbb{R}^2$ be the operator defined by $\Phi(f) = \left(\int_{1/2}^1 f d\mu, \int_0^{1/2} f d\mu\right)$. Note that, for $g_0 = \chi_\Omega \in L^q(m)$ we have, for all $f \in L^p(m)$

$$\|\Phi(f)\| = \left\|\int f g_0 dm\right\| = \left\|\int f dm\right\|.$$

But clearly Φ is not an integral operator: there is no function g such that $\Phi(f) = \int fg dm$ for all $f \in L^p(m)$. Take for example $f_0 = \chi_{[0,1/2]}$, that gives $\Phi(f_0) = (0, 1/2)$; but for every $g \in L^q(m)$, $\int f_0 g dm = (k, 0)$ for some $k \in \mathbb{R}$ depending on g.

Remark 3.0.6. Notice that an operator $G : L^p_w(m) \to X$ satisfying the requirements of Theorem 3.0.4 factorizes through the space $L^1(m)$. Indeed, in this case there is a function $g \in L^q(m)$ such that

$$\begin{tikzcd} L^p_w(m) \arrow[r, "G"] \arrow[d, "M_g"'] & X \\ L^1(m) \arrow[ur, "I"'] & \end{tikzcd}$$

where $M_g(f) = fg$ for all $f \in L^p_w(m)$ and $I(h) = \int h dm$ for all $h \in L^1(m)$.

This leads us to apply the results of [7] concerning some properties of the multiplication operator M_g from $L^p_w(m)$ into $L^1(m)$. An operator T between a Banach lattice E and a Banach space F is said to be *M–weakly* *compact* whenever $\|T(f_n)\|_F \to 0$ for all disjoint sequences $(f_n)_n$ in $B(E)$. This space of operators is denoted by $\mathcal{M}(E,F)$. Notice that the composition $S \circ T$ of an M–weakly compact operator $T : E \to G$ with a bounded operator $S : G \to F$ belongs to $\mathcal{M}(E,F)$. We denote by $\mathcal{W}(E,F)$ the ideal of weakly compact operators. It is known (see [51, Proposition 3.6.12]) that $\mathcal{M}(E,F) \subseteq \mathcal{W}(E,F)$.

A. Fernández et al. proved in [7, Theorem 7] that for $g \in L^q(m)$, the multiplication operator $M_g : L^p_w(m) \to L^1(m)$ is M–weakly compact (and then weakly compact). The following corollary is a direct consequence of this result and of the factorization given in Remark 3.0.6.

Corollary 3.0.7. *Let $T : L^p_w(m) \to X$ satisfy the requirements of Theorem 3.0.4. Then $T \in \mathcal{M}\left(L^p_w(m), X\right)$. In particular T is weakly compact and its norm coincides with the norm of the function g that is given by Theorem 3.0.4.*

3.1. Tensor product representation

In this section we develop a representation technique for spaces $L^q(m)$ based on topological tensor products. A first approximation to this matter has been done in [72]. In this paper this kind of identification is done, although under strong restrictions on the spaces $L^q(m)$; the topologies introduced there are different that the ones that we consider here, which lead to a general representation for *any* $L^q(m)$. We will prove that in fact $L^q(m)$ can be always written as a dual space of a particular topological tensor product. The main tool that we use for this representation is given by Theorem 3.0.2. From the technical point of view, it is necessary to define several tensor product topologies. We will introduce these topologies in three approaches in order to get, in the last one, the representation of $L^p(m)$ as the dual of a normed space.

First approach. We establish the topological framework regarding the tensor product $L^p(m) \otimes X^*$. If $g \in L^q(m)$, we define the seminorm p_g by

$$p_g(z) := \left| \sum_{i=1}^n \left\langle \int_\Omega f_i g \, dm, x_i^* \right\rangle \right|, \quad z = \sum_{i=1}^n f_i \otimes x_i^* \in L^p(m) \otimes X^*.$$

The definition does not depend on the particular representation of z. Using this family of seminorms we can provide a topology (in general not Hausdorff) on the tensor product $L^p(m) \otimes X^*$. We will denote it by τ and it corresponds to the one generated by the family of seminorms $\{p_g : g \in L^q(m)\}$.

Let $g \in L^q(m)$ and consider its associated integration map $I_g : L^p(m) \to X$ given by $I_g(f) := \int_\Omega fg\, dm$ for all f in $L^p(m)$. Define the functional

$$\varphi_g : L^p(m) \otimes X^* \longrightarrow \mathbb{R}$$

by $\varphi_g(z) := \sum_{i=1}^n \langle I_g(f_i), x_i^* \rangle$, where $z = \sum_{i=1}^n f_i \otimes x_i^*$ is any representation of the tensor z in $L^p(m) \otimes X^*$; again note that the definition does not depend on the particular representation of z. The following result shows that this relation provides a procedure to identify the set of $q-$integrable functions with respect to m with the dual space $(L^p(m) \otimes_\tau X^*)^*$. As usual, we denote by τ_{weak^*} the weak topology generated on a dual space by the elements of the original space. Recall that the $m-$weak topology, $\sigma(L^q(m), \Gamma)$ is generated by the family of seminorms Γ, defined by

$$\Gamma := \{|\gamma_{f,x^*}| : f \in B(L^p(m)), x^* \in B(X^*)\}, \tag{3.4}$$

with $\gamma_{f,x^*}(g) := \int_\Omega fg d\langle m, x^* \rangle, g \in L^p(m)$.

Proposition 3.1.1. *The map*

$$\Upsilon : (L^q(m), \sigma(L^q(m), \Gamma)) \to ((L^p(m) \otimes_\tau X^*)^*, \tau_{weak^*})$$

given by $\Upsilon(g) := \varphi_g$*, is a linear isomorphism.*

Proof. We start by proving that Υ is well defined and injective. Clearly, if $g \in L^q(m)$,

$$|\varphi_g(z)| = \left| \sum_{i=1}^n \left\langle I_g(f_i), x_i^* \right\rangle \right| = p_g(z)$$

for any tensor $z = \sum_{i=1}^n f_i \otimes x_i^*$, and then φ_g belongs to $(L^p(m) \otimes_\tau X^*)^*$. Since the set Γ is norming in $L^q(m)$, for $h \in L^q(m)$ and $h \neq g$, there are $f \in L^p(m)$ and $x^* \in X^*$ such that $\langle \int_\Omega fh\, dm, x^* \rangle \neq \langle \int_\Omega fg\, dm, x^* \rangle$, and then the identification $g \mapsto \varphi_g$ given by Υ is injective. Note that Υ is also linear.

To prove that the map is also surjective, consider a functional ϕ in $(L^p(m) \otimes_\tau X^*)^*$. Since it is continuous with respect to τ, there are functions $g_1, \dots g_n \in L^q(m)$ such that $|\phi(z)| \leq \sum_{i=1}^n p_{g_i}(z)$ for any tensor $z \in L^p(m) \otimes_\tau X^*$. In particular, for a simple tensor $z = f \otimes x^*$,

$$|\phi(z)| \leq \sum_{i=1}^n p_{g_i}(f \otimes x^*) = \sum_{i=1}^n \left| \left\langle \int_\Omega fg_i\, dm, x^* \right\rangle \right|. \tag{3.5}$$

Now fix a $p-$integrable function f and define the map $F_f : X^* \longrightarrow \mathbb{R}$ by $F_f(x^*) := \phi(f \otimes x^*)$. Note that F_f is well defined and linear; by (3.5), we also have

$$|F_f(x^*)| = |\phi(f \otimes x^*)| \leq \sum_{i=1}^n \left| \left\langle \int_\Omega fg_i\, dm, x^* \right\rangle \right|$$

for every $x^* \in X^*$; since for all $i = 1, ..., n$, $\int_\Omega f g_i dm \in X$, it follows that F_f is continuous with respect to the weak* topology of X^*. Therefore F_f is an element of the dual space $(X^*, \tau_{weak^*})^*$ that coincides with X.

Thus, we can define the operator $T_\phi : L^p(m) \to X$ by $T_\phi(f) := F_f$. Note that T_ϕ is linear and $\langle T_\phi(f), x^* \rangle = \phi(f \otimes x^*)$ for all f in $L^p(m)$, and then

$$|\langle T_\phi(f), x^* \rangle| = |\phi(f \otimes x^*)| \leq \sum_{i=1}^{n} \left| \left\langle \int_\Omega f g_i \, dm, x^* \right\rangle \right|.$$

Therefore, the operator T_ϕ satisfies the inequalities in (ii) of Theorem 3.0.2. Thus, there is a function g_0 in $L^q(m)$ such that $T_\phi(f) = \int_\Omega g_0 f dm$ for all f in $L^p(m)$. Hence, $\varphi_{g_0}(f \otimes x^*) = \langle \int_\Omega g_0 f dm, x^* \rangle = \phi(f \otimes x^*)$ for every simple tensor in $L^p(m) \otimes X^*$, which implies $\varphi_{g_0} = \phi$, and then Υ is surjective. The topological isomorphism is obvious because of the definitions of the topologies $\sigma(L^q(m), \Gamma)$ and τ_{weak^*}; the action of the tensors of $L^p(m) \otimes_\tau X^*$ on the functionals of its dual space is given by evaluations of a finite set of the functionals that define the topology $\sigma(L^q(m), \Gamma)$.

□

Although Proposition 3.1.1 provides a representation of the space $L^p(m)$ as the dual of a certain topological linear space, this space is not in general Hausdorff. The following trivial example shows this. Consider the Lebesgue measure space (Ω, Σ, μ) and the vector measure $m_0 : \Sigma \to \ell^2$ given by $m_0(A) := \mu(A) e_1$, $A \in \Sigma$, where e_1 is the first element of $\{e_i : i \in \mathbb{N}\}$, the canonical basis of ℓ^2. Clearly, if we consider a simple tensor $f \otimes e_i$, $f \in L^p(m)$, $i > 1$, we obtain $p_g(f \otimes e_i) = 0$ for every $g \in L^q(m)$. The same argument can be used for any vector measure m to show that for every $x^* \in X^*$ that satisfies $\langle \int_\Omega h \, dm, x^* \rangle = 0$ for each $h \in L^1(m)$ and every $f \in L^p(m)$, the equality $p_g(f \otimes x^*) = 0$ is obtained, and then the induced topology cannot be Hausdorff.

Second approach. The following approaches are devoted to improve the representation of $L^q(m)$ as a dual of a Hausdorff topological vector space. The first step is to construct a (Hausdorff) quotient space preserving the duality properties with respect to $L^q(m)$. As usual, if $g \in L^q(m)$, we define the kernel of p_g as

$$\ker p_g = \{z \in L^p(m) \otimes X^* : p_g(z) = 0\}.$$

The set $\cap \ker p_g$, where the intersection is defined by the set of functions g in $L^q(m)$, is a linear subspace of the tensor product. Consider the quotient space (defined algebraically) $(L^p(m) \otimes X^*)/(\cap \ker p_g)$, for $g \in L^q(m)$. We define in this space the topology $\tilde{\tau}$ generated by the family of quotient seminorms $\{\widetilde{p_g} : g \in L^q(m)\}$, that are given by

$$\widetilde{p_g}([z]) = \left| \sum_{i=1}^{n} \left\langle \int_\Omega f_i g \, dm, x_i^* \right\rangle \right|$$

for any $\sum_{i=1}^n f_i \otimes x_i^* \in [z]$, since the quotient is defined using the family of seminorms $\{p_g : g \in L^q(m)\}$. The next result, together with Proposition 3.1.1, provides a representation of the space $L^q(m)$ as the dual space of a *Hausdorff* topological vector space.

Proposition 3.1.2. *The map*

$$Q : ((L^p(m) \otimes_\tau X^*)^*, \tau_{weak^*}) \to \left(\left(\frac{L^p(m) \otimes X^*}{\cap \ker p_g}, \tilde{\tau} \right)^*, \tau_{weak^*} \right)$$

given by $Q(\phi) = \tilde{\phi}$*, where* $\tilde{\phi}([z]) = \phi(z)$ *for each tensor* z *in* $L^p(m) \otimes X^*$*, is a linear isomorphism.*

Proof. Let ϕ be a functional in the dual space $(L^p(m) \otimes_\tau X^*)^*$. By the continuity of ϕ with respect to τ there are n $q-$integrable functions $g_1, \dots, g_n$ such that

$$|\phi(z)| \leq \sum_{i=1}^{n} p_{g_i}(z) \text{ for every tensor } z \text{ in } L^p(m) \otimes X^*. \tag{3.6}$$

Let $\tilde{\phi}$ be the linear map from $(L^p(m) \otimes X^*)/(\cap \ker p_g)$ into $\mathbb{R}$ given by $\tilde{\phi}([z]) := \phi(z)$, for $z \in [z] \in (L^p(m) \otimes X^*)/(\cap \ker p_g)$. Since $[z_1] = [z_2]$ implies $p_g(z_1 - z_2) = 0$ for each $g \in L^q(m)$, then $\phi(z_1 - z_2) = 0$ by (3.6). Thus by the linearity of ϕ, $\tilde{\phi}([z_1]) = \tilde{\phi}([z_2])$. Obviously $\tilde{\phi}$ is continuous with respect to the topology $\tilde{\tau}$, since we obtain $|\tilde{\phi}([z])| \leq \sum_{i=1}^n \tilde{p}_{g_i}([z])$ for every $[z]$ by (3.6). Therefore $Q(\phi) := \tilde{\phi}$ is well-defined, linear and injective. To see that it is also a surjection, consider a ($\tilde{\tau}$-continuous) functional $\tilde{\phi} : \frac{L^p(m) \otimes X^*}{\cap \ker p_g} \to \mathbb{R}$ and define $\phi : L^p(m) \otimes X^* \to \mathbb{R}$ by $\phi(z) := \tilde{\phi}([z])$. Direct computations as those in the previous part of the proof show that ϕ belongs to the space $(L^p(m) \otimes X^*, \tau)^*$; clearly $Q(\phi) = \tilde{\phi}$. The equivalence between the weak* topologies of both spaces is also clear.

□

Notice that the previous proposition repeats a general argument about locally convex spaces, in the particular case of the tensor product $(L^p(m) \otimes X^*, \tau)$.

Third approach. In what follows we introduce the uniform topology associated to τ in the tensor product $L^p(m) \otimes X^*$ in order to find a representation of $L^q(m)$ as the dual space of a *normed* space. We denote this topology by τ_u; it is the one generated by the seminorm

$$u(z) = \sup_{\|g\|_{L^q(m)} \leq 1} \left| \sum_{i=1}^{n} \left\langle \int_\Omega f_i g dm, x_i^* \right\rangle \right|,$$

being $z = \sum_{i=1}^n f_i \otimes x_i^*$ an element of $L^p(m) \otimes X^*$. If ϕ is a functional in $(L^p(m) \otimes X^*, \tau_u)^*$, we define

$$\|\phi\|_u := \sup_{u(z) \leq 1} |\phi(z)|,$$

where the supremum is computed over all tensors $z \in L^p(m) \otimes X^*$ satisfying $u(z) \leq 1$.

Clearly $\ker u = \cap \ker p_g$, where the intersection is defined for the whole set of integrable functions in $L^q(m)$; as in the previous case we will work with the quotient space $L^p(m) \otimes X^* / \ker u$. We define also in this case the quotient topology $\tau_{\tilde{u}}$ generated by the seminorm $\tilde{u}([z]) := u(z)$, for $z \in L^p(m) \otimes X^*$, respectively. The corresponding norm on the dual of the quotient space is given by

$$\|\tilde{\phi}\|_{\tilde{u}} := \sup_{\tilde{u}([z]) \leq 1} |\tilde{\phi}([z])|, \;\; \tilde{\phi} \in \left(\frac{L^p(m) \otimes X^*}{\ker u}, \tau_{\tilde{u}} \right)^*,$$

where the elements $[z]$ belong to $\frac{L^p(m) \otimes X^*}{\ker u}$.

We omit the proof of the next proposition, that follows the lines of the one of Proposition 3.1.2.

Proposition 3.1.3. *The function*

$$Q_u : \left((L^p(m) \otimes_{\tau_u} X^*)^*, \|\cdot\|_u \right) \longrightarrow \left(\left(\frac{L^p(m) \otimes X^*}{\ker u}, \tau_{\tilde{u}} \right)^*, \|\cdot\|_{\tilde{u}} \right)$$

defined by $Q_u(\phi) = \tilde{\phi}$*, where* $\tilde{\phi}([z]) = \phi(z)$ *for each tensor* z *in* $L^p(m) \otimes X^*$*, is an isometric isomorphism.*

Let us now show with an easy example the representation procedure developed in this section

Example 3.1.4. Let $1 < r < \infty$, $1 < p < \infty$ and s, q their corresponding conjugated exponents, and let $([0,1], \Sigma, \mu)$ be Lebesgue measure space. We define the vector measure $m : \Sigma \to L^r(\mu)$ as $m(A) := \chi_A$, $A \in \Sigma$. It is easy to see that $L^p(m) = L^{pr}(\mu)$, $L^q(m) = L^{qr}(\mu)$, and $(L^r(\mu))^* = L^s(\mu)$. Take $z \in L^p(m) \otimes (L^r(\mu))^* = L^{pr}(\mu) \otimes L^s(\mu)$, $z = \sum_{i=1}^n f_i \otimes h_i$. Then

$$
\begin{aligned}
u(z) &= \sup_{g \in B(L^q(m))} \left| \sum_{i=1}^n \left\langle \int_0^1 f_i g dm, h_i \right\rangle \right| \\
&= \sup_{g \in B(L^{qr}(\mu))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^1 f_i g dm \right) h_i d\mu \right| \\
&= \sup_{g \in B(L^{qr}(\mu))} \left| \int_0^1 g \sum_{i=1}^n f_i h_i d\mu \right| .
\end{aligned}
$$

Since $\frac{1}{pr} + \frac{1}{s} = (1 - \frac{1}{q})\frac{1}{r} + \frac{1}{s} = 1 - \frac{1}{qr}$, then $\sum_{i=1}^n f_i h_i \in (L^{qr}(\mu))^*$ and $u(z) = \|\sum_{i=1}^n f_i h_i\|_{(L^{qr}(\mu))^*}$. Remark that

$$
\ker u = \{z = \sum_{i=1}^n f_i \otimes h_i \in L^p(m) \otimes L^s(\mu) : \sum_{i=1}^n f_i h_i = 0 \ \mu - \text{a.e.}\}.
$$

Therefore, the space $\left(\frac{L^p(m)\otimes L^s(\mu)}{\ker u}, \tau_u\right)$ can be identified isometrically with $(L^q(m))^* = L^t(\mu)$, where $\frac{1}{qr} + \frac{1}{t} = 1$, and the formulae for u provides an equivalent representation of the norm of $L^t(\mu)$.

The following theorem constitutes the main result of this section. It shows that a certain compactness assumption for the unit ball of $L^q(m)$ gives the key for obtaining a satisfactory generalization of the duality results that hold for the case of L^p-spaces (scalar measure). Actually, it provides a description of a suitable normed predual of the space $L^q(m)$, and consequently of the dual space $(L^q(m))^*$. Notice that the compactness condition is equivalent to the assumption of reflexivity of the space $L^q(m)$, by the coincidence of the $m-$ weak topology with the weak topology on bounded sets of $L^q(m)$.

The proof of the main result uses Ky Fan's Lemma, see for instance [61, E.4.].

Lemma 3.1.5. *(Ky Fan) Let W a compact convex subset of a Hausdorff topological vector space and let Ψ be a concave family of lower semicontinuous, convex real functions on W. Let $C \in \mathbb{R}$. Suppose that, for every $\psi \in \Psi$ there exists $x_\psi \in W$ such that $\psi(x_\psi) \le C$. Then there exists $x \in W$ such that $\psi(x) \le C$ for every $\psi \in \Psi$.*

Theorem 3.1.6. *The space $\left(\left(\frac{L^p(m)\otimes X^*}{\ker u}, \tau_{\tilde{u}}\right)^*, \|\cdot\|_{\tilde{u}}\right)$ and $(L^q(m), \|\cdot\|_{L^q(m)})$ are isometrically isomorphic if and only if the unit ball of $L^q(m)$ is $m-$weakly compact.*

Proof. We start by showing the direct implication. If $L^q(m)$ is the topological dual of $\left(\frac{L^p(m)\otimes X^*}{\ker u}, \tau_{\tilde{u}}\right)$ then this space defines the weak* topology on bounded sets of $L^q(m)$. By Alaoglu's Theorem the unit ball of $L^q(m)$ is weakly* compact; but in fact the weak* topology coincides with the $m-$weak topology of $L^q(m)$ on its unit ball. Therefore, $B(L^q(m))$ is $m-$weakly compact.

To prove the converse, first note that the function space $L^q(m)$ can be identified with a subspace of $\left(\frac{L^p(m)\otimes X^*}{\ker u}, \tau_{\tilde{u}}\right)^*$, that by Proposition 3.1.3 coincides with $(L^p(m)\otimes X^*, \tau_u)^*$. The inclusion is given by the identification explained before Proposition 3.1.1, i.e. by the map

$$i : L^q(m) \to (L^p(m)\otimes X^*, \tau_{\tilde{u}})^*,$$

where $i(g) := \varphi_g$ for g in $L^q(m)$ with

$$\varphi_g\left(\sum_{i=1}^n f_i \otimes x_i^*\right) := \sum_{i=1}^n \left\langle \int_\Omega f_i g dm, x_i^* \right\rangle.$$

Clearly, i is well defined, and direct computations show that it is continuous. To prove that it is an isomorphism, we choose an element $\tilde{\phi} \in \left(\frac{L^p(m)\otimes X^*}{\ker u}, \tau_{\tilde{u}}\right)^*$ and we must prove that in fact $\tilde{\phi}$ belongs to $\left(\frac{L^p(m)\otimes X^*}{\ker u}, \tilde{\tau}\right)^*$, which by Proposition 3.1.1 and proposition 3.1.2 can be identified with $L^q(m)$; our aim is to show that every functional $\tilde{\phi} : \frac{L^p(m)\otimes X^*}{\ker u} \to \mathbb{R}$ that is continuous with respect to $\tau_{\tilde{u}}$ is also continuous with respect to the topology $\tilde{\tau}$. Thus, we search for a function g_0 in $L^q(m)$ such that

$$|\tilde{\phi}([z])| \le \|\tilde{\phi}\|_{\tilde{u}} \cdot \tilde{p}_{g_0}([z])$$

for every $[z] \in \frac{L^p(m)\otimes X^*}{\ker u}$. In order to find this element it is necessary to use a separation argument; we choose one based on Ky Fan's Lemma (see Lemma 3.1.5). For a fixed $z = \sum_{i=1}^n f_i \otimes x_i^*$ in $L^p(m)\otimes X^*$ we define the function Φ_z over the unit ball of $L^q(m)$ with range in $\mathbb{R}$ as follows:

$$\Phi_z(g) := \tilde{\phi}([z]) - \|\tilde{\phi}\|_{\tilde{u}}\varphi_g(z) = \sum_{i=1}^n \phi(f_i\otimes x_i^*) - \|\tilde{\phi}\|_{\tilde{u}}\left(\sum_{i=1}^n \left\langle \int_\Omega f_i g dm, x_i^* \right\rangle\right),$$

where ϕ is a functional satisfying $Q(\phi) = \tilde{\phi}$ given by proposition 3.1.2, and $\sum_{i=1}^n f_i \otimes x_i^*$ is any representation of z (note that the definition of the function Φ_z do not depend neither on the particular representation of $[z]$ nor on the one of $\tilde{\phi}$). Thus let $\mathcal{F}$ be the family of functions Φ_z for z in $L^p(m)\otimes X^*$. We need to prove that $\mathcal{F}$ satisfies all the hypothesis of Ky Fan's Lemma.

All the functions Φ_z are defined on the unit ball of $L^q(m)$ that is by assumption a compact set with respect to the m-weak topology. Remark that the space $L^q(m)$ with the m-weak topology is a Hausdorff space.

The family of functions $\mathcal{F}$ is concave; if z_1 and z_2 are in $L^p(m) \otimes X^*$ then for a real number α, $0 \leq \alpha \leq 1$, there is an element z_0 in $L^p(m) \otimes X^*$ such that

$$\alpha \Phi_{z_1} + (1-\alpha)\Phi_{z_2} = \Phi_{z_0};$$

take $z_0 = \alpha z_1 + (1-\alpha) z_2$.

Let us show now that for every tensor z in $L^p(m) \otimes X^*$, the function Φ_z is convex. By the linearity of Φ it is enough to prove that this is true for a simple tensor $z = f \otimes x^*$. Let g_1 and g_2 be in $B(L^q(m))$ and α a positive real number, $0 \leq \alpha \leq 1$. Then

$$\begin{aligned}\Phi_z(\alpha g_1 + (1-\alpha) g_2) &= \phi(f \otimes x^*) - \|\tilde{\phi}\|_{\tilde{u}} \left\langle \int_\Omega f(\alpha g_1 + (1-\alpha) g_2) dm, x^* \right\rangle \\ &= \alpha \Phi_z(g_1) + (1-\alpha)\Phi_z(g_2).\end{aligned}$$

Moreover, by the construction, for all z in $L^p(m) \otimes X^*$, Φ_z is continuous with respect to the $m-$weak topology of $L^p(m)$.

Finally, we must prove that for all z in the tensor product $L^p(m) \otimes X^*$, there is a function g_z in the unit ball of $L^q(m)$ such that $\Phi_z(g_z) \leq 0$; this is a consequence of the fact that Φ_z is a continuous function defined on a compact set. In fact,

$$\phi(z) := \tilde{\phi}([z]) \leq |\tilde{\phi}([z])| \leq \|\tilde{\phi}\|_{\tilde{u}} \cdot \tilde{u}([z]) = \|\tilde{\phi}\|_{\tilde{u}} \cdot \sup_{\|g\|_{L^q(m)} \leq 1} \left(\left\langle \sum_{i=1}^n f_i g \, dm, x_i^* \right\rangle \right)$$

and this supremum is attained for some g_z of the unit ball of $L^q(m)$. Then for all z in $L^p(m) \otimes X^*$ there is some g_z in $B(L^q(m))$ such that

$$\phi\left(\sum_{i=1}^n f_i \otimes x_i^*\right) \leq \|\tilde{\phi}\|_{\tilde{u}} \left(\sum_{i=1}^n \left\langle \int_\Omega f_i g_z \, dm, x_i^* \right\rangle \right).$$

We can conclude by Ky Fan's Lemma that there is some g_0 in the unit ball of $L^q(m)$ such that for all $z = \sum_{i=1}^n f_i \otimes x_i^* \in L^p(m) \otimes X^*$

$$\phi\left(\sum_{i=1}^n f_i \otimes x_i^*\right) \leq \|\tilde{\phi}\|_{\tilde{u}} \left(\sum_{i=1}^n \left\langle \int_\Omega f_i g_0 \, dm, x_i^* \right\rangle \right). \tag{3.7}$$

Thus $\tilde{\phi}$ is continuous with respect to $\tilde{\tau}$ and we have that $\tilde{\phi}$ is in $\left(\frac{L^p(m) \otimes X^*}{\ker u}, \tilde{\tau}\right)^*$; the identification is clearly bijective, since this space is isomorphic to $L^q(m)$

by proposition 3.1.1 and proposition 3.1.2. Moreover, a direct computation using inequality (3.7) shows that actually the function $g_0^* := g_0 \|\tilde{\phi}\|_{\tilde{u}} \in L^q(m)$ can be identified with $\tilde{\phi}$; clearly $\|g_0^*\|_{L^q(m)} \le \|\tilde{\phi}\|_{\tilde{u}}$. The converse inequality follows by a simple calculation: if $\tilde{\phi}_{g_0^*}$ and $\phi_{g_0^*}$ are the functionals defined by g_0^* and $z = \sum_{i=1}^n f_i \otimes x_i^* \in L^p(m) \otimes X^*$,

$$|\tilde{\phi}([z])| = |\tilde{\phi}_{g_0^*}([z])| = |\phi_{g_0^*}(z)| = \left| \sum_{i=1}^n \langle \int_\Omega f_i \|g_0^*\|_{L^q(m)} \frac{g_0^*}{\|g_0^*\|_{L^q(m)}} \, dm, x_i^* \rangle \right|$$

$$\le \|g_0^*\|_{L^q(m)} \cdot \sup_{\|h\|_{L^q(m)} \le 1} \left| \sum_{i=1}^n \langle \int_\Omega f_i h \, dm, x_i^* \rangle \right|$$

$$= \|g_0^*\|_{L^q(m)} \cdot u(z) = \|g_0^*\|_{L^q(m)} \cdot \tilde{u}([z]).$$

This proves the isometry and finishes the proof.

□

After the results of Chapter 2, it is known that the $m-$weak topology coincides with the weak topology of the space on bounded subsets of $L^p(m)$. Thus the compactness property required in Theorem 3.1.6 is satisfied if and only if the space $L^q(m)$ is *reflexive*; the reader can find some results regarding reflexivity of this space in [31].

We isolate in the following corollary a relevant result concerning duality of the space $L^q(m)$ that has been implicitly shown in the proof of theorem 3.1.6; in particular, this theorem gives a sufficient and necessary condition to assure that the topological dual of the space $\frac{L^p(m)\otimes X^*}{\ker u}$ with the topologies $\tilde{\tau}$ and $\tau_{\tilde{u}}$ coincide. This assertion is the natural "vector measures" version of one of the main results of the duality theory of Banach spaces: the dual of a Banach space with the norm topology coincides with the dual of the space with the weak topology.

Corollary 3.1.7. *The following assertions are equivalent.*

(i) *$L^q(m)$ is reflexive.*

(ii) *The unit ball of $L^q(m)$ is compact with respect to the m-weak topology.*

(iii) $\left(\frac{L^p(m)\otimes X^*}{\ker u}, \tilde{\tau} \right)^* = \left(\frac{L^p(m)\otimes X^*}{\ker u}, \tau_{\tilde{u}} \right)^*.$

(iv) $(L^p(m) \otimes X^*, \tau)^* = (L^p(m) \otimes X^*, \tau_u)^*.$

Let us finish this chapter by illustrating our procedure with two examples. In the first one we obtain an alternative formula to define the norm in the dual of $L^q(m)$ of a vector measure over an Orlicz space. In the second one we provide a characterization of the dual of $L^q(m)$ of the measure induced by a kernel operator.

3.2. Examples

Orlicz spaces. Let (Ω, Σ, μ) be a measure space. Take a Young function Φ with the Δ_2-property (see Section 5.1. for basic definitions about Orlicz spaces). We define the vector measure $m : \Sigma \to L^{\Phi}(\mu)$ by $m(A) = \chi_A$. Since $L^{\Phi}(\mu)$ is order continuous, equality $L^1(m) = L^{\Phi}(\mu)$ holds, and then

$$\begin{aligned} L^p(m) &= \{f \in L^0(\mu) : |f|^p \in L^1(m)\} \\ &= \{f \in L^0(\mu) : |f|^p \in L^{\Phi}(\mu)\} \\ &= \{f \in L^0(\mu) : \Phi(|f|^p) \in L^1(\mu)\}. \end{aligned}$$

Notice that the function $\Phi \circ p : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\Phi \circ p(t) = \Phi(t^p)$ is a Young function, and that Δ_2-property for Φ implies Δ_2-property for $\Phi \circ p$ since for all $t > 0$ there is some b such that

$$\Phi \circ p(2t) = \Phi(2^p t^p) \leq b^{[p]+1}\Phi(2^{p-[p]-1}t^p) \leq b^{[p]+1}\Phi \circ p(t),$$

where $[p] = \max\{n \in \mathbb{Z} : n \leq p\}$. Therefore we have that $L^p(m) = L^{\Phi \circ p}(\mu)$. Let Ψ be the conjugated Young function of Φ. Since $L^{\Phi}(\mu)$ is order continuous we have $(L^{\Phi}(\mu))' = L^{\Psi}(\mu)$. Take $z = \sum_{i=1}^n f_i \otimes h_i \in L^p(m) \otimes L^{\Psi}(\mu)$. Then

$$\begin{aligned} u(z) &= \sup_{g \in B(L^q(m))} \left| \sum_{i=1}^n \left\langle \int f_i g \, dm, h_i \right\rangle \right| \\ &= \sup_{g \in B(L^q(m))} \left| \int g(\sum_{i=1}^n \int f_i h_i) d\mu \right| \\ &= \left\| \sum_{i=1}^n f_i h_i \right\|^o_{(L^q(m))^*}, \end{aligned}$$

where $(L^q(m))^*$ is again an Orlicz space and $\| \cdot \|^o_{(L^q(m))^*}$ is the corresponding Orlicz norm. Assume now that $L^q(m)$ is reflexive. Since a Banach space Z is reflexive if and only if Z^* is reflexive (see [77, II.A.14]), we have, as a consequence of theorem 3.1.6, that there is an isometric isomorphism between the spaces $(L^q(m), \| \cdot \|^o)^*$ and $\left(\frac{L^p(m) \otimes L^{\Psi}(\mu)}{\ker u}, \tau_{\tilde{u}}\right)$. Thus we can represent a dense subset of elements of the dual space of $L^q(m)$ as equivalence classes of elements $\sum_{i=1}^n f_i \otimes h_i \in L^p(m) \otimes L^{\Psi}(m)$.

Kernel operators. Fix $1 < p < \infty$ and $1 < r < \infty$ and let q and v be their respective conjugated exponents. Let $([0, 1], \Sigma, \mu)$ be the Lebesgue measure space and $V : L^r(\mu) \to L^r(\mu)$ the kernel operator defined by

$$V(f)(t) := \int_0^t f(s) K(s, t) ds,$$

where $K : [0,1] \times [0,1] \to \mathbb{R}^+$ is a bounded integrable function. We define the vector measure $m_V : \Sigma \to L^r(\mu)$ by $m_V(A) := V(\chi_A)$. Notice that for $\phi = \sum_{i=1}^n a_i \chi_{A_i}$,

$$\int \phi \, dm_V = \sum_{i=1}^n a_i m_V(A_i) = V(\phi).$$

For $0 \leq f \in L^1(m)$, there is a sequence $(\phi_n) \in \mathcal{S}(\Sigma)$ such that $\phi_n \uparrow f$. By order continuity of $L^1(m_V)$, $\phi_n \to f$ in $L^1(m_V)$, and then $\int \phi_n \, dm_V \to \int f \, dm_V$ in $L^r(\mu)$. There is a subsequence $(\Phi_{n_k})_k$ such that $0 \leq \phi_{n_k} \uparrow f$ and

$$\begin{aligned} \int f \, dm_V &= \lim_k \int \phi_{n_k} \, dm_V = \lim_k V(\phi_{n_k}) \\ &= \lim_k \int_0^t \phi_{n_k}(s) K(s,t) ds. \end{aligned}$$

Fix $t \in [0,1]$. Since the kernel $K(s,t)$ is positive in its first variable, $0 \leq \phi_{n_k}(s)K(s,t) \uparrow f(s)K(s,t)$. A direct application of the Monotone Convergence Theorem yields

$$\int_0^t f(s)K(s,t)ds = \lim_k \int_0^t \phi_{n_k}(s)K(s,t)ds.$$

Since every function $f \in L^1(m_V)$ can be written as a difference of positive functions, we obtain $\int f \, dm_V = V(f)$ for every $f \in L^1(m_V)$. We get, for each representation of $z = \sum_{i=1}^n f_i \otimes h_i$ in $L^p(m) \otimes L^r(\mu)^* = L^p(m_V) \otimes$

$L^v(\mu)$, as a consequence of Fubini's theorem

$$\begin{aligned}
u(z) &= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \left\langle \int_0^1 f_i g\, dm_V, h_i \right\rangle \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^1 f_i g\, dm_V \right) h_i d\mu \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 V(f_i g) h_i d\mu \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^t f_i(s) g(s) K(s,t) d\mu(s) \right) h_i(t) d\mu(t) \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 \left(\int_0^1 f_i(s) g(s) K(s,t) \chi_{[0,t]}(s) d\mu(s) \right) h_i(t) d\mu(t) \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \sum_{i=1}^n \int_0^1 g(s) f_i(s) \left(\int_s^1 K(s,t) h_i(t) d\mu(t) \right) d\mu(s) \right| \\
&= \sup_{g\in B(L^q(m_V))} \left| \int_0^1 g(s) \sum_{i=1}^n \phi_i d\mu(s) \right| \\
&= \| \sum_{i=1}^n \phi_i \|_{L^q(m_V)^*}
\end{aligned}$$

where $\phi_i(s) = f_i(s) \int_s^1 K(s,t) h_i(t) d\mu(t)$. Then, we obtain a representation of a dense subset of the elements of the predual space $L^q(m_V)$ as equivalence classes of functions defined by means of elements of $L^p(m_V)$ and $L^v(\mu)$.

Chapter 4

Summability in $L^p(m)$: $m-r-$summing operators

Our aim in this chapter is to study the summability properties of series with respect to the $m-$topology, i.e. the convergence of series of functions in $L^p(m)$ associated to the norm convergence of the integrals $\int_\Omega (\cdot) g dm$, for $g \in L^q(m)$. In order to do this we use an operator ideal type approach. We define and study the operators that transform $m-r-$summable sequences into strongly $r-$summable ones. We also define those that send $r-$summable sequences into $m-r-$summable ones. We will extend in this way the class of classical $r-$summing operators for operators defined on $L^p(m)$ with the aim of developing the theory of summing operators in general $p-$convex Banach lattices. We will show that these extensions preserves some properties of the classical operator ideal of $r-$summing operators. We begin by introducing the basic definitions and results about classical $r-$summing operators. The references we have used are the books [27], [69] and [22]. The monograph by A. Pietsch about Operators Ideals [61] is also a complete guide for this subject.

4.1. Preliminaries

Let $1 \leq r < \infty$. An operator $T : X \to Y$ between Banach spaces is *$r-$summing* whenever there is a positive constant C such that for every finite choice of elements $x_1, \ldots, x_n \in X$, the following inequality holds,

$$\left(\sum_{i=1}^{n} \|T(x_i)\|_Y^r \right)^{\frac{1}{r}} \leq C \cdot \sup_{x^* \in B(X^*)} \left(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^r \right)^{\frac{1}{r}}. \tag{4.1}$$

The set of $r-$summing operators between the Banach spaces X and Y is denoted by $\Pi_r(X,Y)$, and by $\pi_r(T)$ the least of the positive constants C so that (4.1) holds. Clearly $\Pi_r(X,Y)$ is a linear subspace of $\mathcal{L}(X,Y)$, and π_r defines a norm in $\Pi_r(X,Y)$. Moreover the space $\Pi_r(X,Y)$ is an operator ideal and a Banach space when endowed with the π_r norm.

In order to study the behavior of $r-$summing operators, several spaces of vector valued sequences are introduced. An $X-$valued sequence $(x_n)_n$ is *strongly $r-$summable* whenever the scalar sequence $(\|x_n\|)_n$ is $r-$summable. Let $\ell_r(X)$ be the vector space of $X-$valued strongly $r-$summable sequences. When it is endowed with the natural norm

$$\|(x_n)_n\|_{\ell_r(X)} := \|(\|x_n\|_X)_n\|_{\ell_r} = \left(\sum_{n=1}^{\infty}\|x_n\|^r\right)^{\frac{1}{r}}, \tag{4.2}$$

$\ell_r(X)$ is a Banach space. Clearly, strong $r-$summability is related with the norm topology of the space X. The following definition corresponds to the analogue for the weak topology of X. A vector valued sequence $(x_n)_n$ is *weakly$-r-$summable* if the scalar sequence $(\langle x_n, x^*\rangle)_n$ is $r-$summable for every $x^*\in X^*$. Let $\ell_r^w(X)$ be to the vector space of all weakly$-r-$summable $X-$valued sequences. It is a Banach space when endowed with a norm defined by

$$\|(x_n)_n\|_{\ell_r^w(X)} := \sup\left\{\left(\sum_{n=1}^{\infty}|\langle x_n, x^*\rangle|^r\right)^{\frac{1}{r}} : x^*\in B(X^*)\right\}. \tag{4.3}$$

Notice that the case $p=\infty$ has been excluded. In fact for a bounded $X-$valued sequence $(x_n)_n$ we have

$$\sup_{n\in\mathbb{N}}\|x_n\|_X = \|(\|x_n\|_X)_n\|_{\ell_\infty} = \sup_{x^*\in B(X^*)}\sup_{n\in\mathbb{N}}|\langle x_n, x^*\rangle|,$$

then the spaces $\ell_\infty(X)$ and $\ell_\infty^w(X)$ coincide and $\|(x_n)_n\|_{\ell_\infty(X)} = \|(x_n)_n\|_{\ell_\infty^w(X)}$ for $(x_n)_n\in\ell_\infty(X)$.

These spaces of vector valued sequences are intimately related with summability of operators between Banach spaces. First notice that $\ell_r(X)$ is a linear subspace of $\ell_r^w(X)$. A linear and continuous operator $T: X\to Y$ between Banach spaces, induces a bounded linear operator $\hat{T}:\ell_r^w(X)\to \ell_r^w(Y)$ by the correspondence

$$\hat{T}((x_n)_n) := (T(x_n))_n$$

and also a bounded linear operator from $\ell_r(X)$ into $\ell_r(Y)$. The following proposition characterizes the summability of an operator T through the behavior of $\hat{T}$.

Proposition 4.1.1. *A bounded linear operator $T : X \to Y$ is $r-$summing if and only if $\hat{T}(\ell_r^w(X)) \subset \ell_r(Y)$. In this case $\pi_r(T) = \|\hat{T} : \ell_r^w(X) \to \ell_r(Y)\|$.*

There is an inclusion relationship between the ideals of $r-$summing operators. In fact once we know that a map is $1-$summing we can conclude that it is $r-$summing for every $1 < r < \infty$. This occurs in the following inclusion theorem when different values of r are considered.

Theorem 4.1.2. *If $1 \leq r < s < \infty$, then $\Pi_r(X,Y) \subset \Pi_s(X,Y)$. Moreover, $\pi_s(T) \leq \pi_r(T)$ for every $T \in \Pi_r(X,Y)$.*

The following basic result about $r-$summing operators is due to A. Pietsch, and it characterizes the $r-$summability by means of a domination property.

Theorem 4.1.3. *Let $1 \leq r < \infty$, $T : X \to Y$ a bounded operator between Banach spaces and K a weak$^*-$compact norming subset of $B(X^*)$. Then T is $r-$summing if and only if there is a positive constant C and a probability Borel measure μ in K so that*

$$\|T(x)\|_Y \leq C \left(\int_K |\langle x, x^* \rangle|^r d\mu(x^*) \right)^{\frac{1}{r}}, \quad x \in X. \tag{4.4}$$

In this case, $\pi_r(T)$ is the least of all the constants C such that (4.4) *holds.*

In order to adapt the previous result into a factorization theorem, we present a basic example of an $r-$summing operator. Let K be a compact set, and μ a probability measure defined on the Borel subsets of K. Then the canonical map $I_r : \mathcal{C}(K) \to L^r(\mu)$ is $r-$summing. Indeed, for every finite choice $f_1, \dots, f_n \in \mathcal{C}(K)$ we have

$$\begin{aligned} \sum_{i=1}^n \|I_r(f_i)\|^r_{L^r(\mu)} &\leq \sum_{i=1}^n \int_K |f_i(x)|^r d\mu(x) \\ &= \int_K \sum_{i=1}^n |f_i(x)|^r d\mu(x) \\ &= \int_K \sum_{i=1}^n |\langle f_i, \delta_x \rangle|^r d\mu(x) \\ &\leq \mu(K) \|(f_i)_{i=1}^n\|^r_{\ell_r^w(\mathcal{C}(K))}. \end{aligned}$$

Theorem 4.1.3 can be rewritten as a factorization theorem, it is sometimes called Grothendieck-Pietsch Factorization Theorem. If K is a weak* compact norming subset of X, let $i_X : X \to \mathcal{C}(K)$, defined by $i_X(x)(\varphi) := \langle x, \varphi \rangle$ for $x \in X$ and $\varphi \in K$.

Corollary 4.1.4. *Let $1 \leq r < \infty$ and $T \in \mathcal{L}(X,Y)$, the following assertions are equivalent for K weak$^*-$compact norming subset $K \subset B(X^*)$*

(i) T is $r-$summing.

(ii) There are a probability measure μ defined on the Borel subsets of K, a (closed) subspace $X_r \subset L^r(\mu)$, and an operator $\widetilde{T} : X_r \to Y$

(a) $I_r i_X(X) \subset X_r$ and

(b) $\widetilde{T} I_r i_X(x) = T(x)$ for every $x \in X$.

That is, the following diagram conmutes:

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "i_X"'] & Y \\
i_X(X) \arrow[r, "I_r^X"] \arrow[d, phantom, "\bigcap"] & X_r \arrow[u, "\widetilde{T}"'] \arrow[d, phantom, "\bigcap"] \\
\mathcal{C}(K) \arrow[r, "I_r"] & L^p(\mu)
\end{tikzcd}$$

We may choose μ and $\widetilde{T}$ so that $\|\widetilde{T}\| = \pi_r(T)$.

A Banach space Z is *injective* if whenever W_0 is a subspace of some Banach space W, any operator $T \in \mathcal{L}(W_0, Z)$ has an extension $\widetilde{T} \in \mathcal{L}(W,Z)$ preserving its norm, $\|T\| = \|\widetilde{T}\|$. The canonical example of injective space is ℓ_∞^K (bounded sequences indexed in K) where K is a weak*-dense norming subset of $B(X^*)$. Notice that the Banach space Y can be embedded in $\ell_\infty^{B(Y^*)}$ as follows, $y \in Y \mapsto i_Y(y) := (\varphi(y))_{\varphi \in B(Y^*)} \in \ell_\infty^{B(Y^*)}$. This directly proves the corollary below.

Corollary 4.1.5. *Let $1 \leq r < \infty$, $T \in \mathcal{L}(X,Y)$, the following assertions are equivalent for K weak$^*-$compact norming subset $K \subset B(X^*)$*

(i) T is $r-$summing.

(ii) There exist a probability measure μ defined on K and an operator $\widetilde{T}$: $L^r(\mu) \to \ell_\infty^{B(Y^*)}$ such that the following diagram conmutes:

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[dd, "i_X"'] & Y \arrow[dr, "i_Y"] & \\
& & \ell_\infty^{B(Y^*)} \\
\mathcal{C}(K) \arrow[r, "I_r"] & L^r(\mu) \arrow[ur, "\widetilde{T}"'] &
\end{tikzcd}$$

(iii) Assuming that the Banach space Y is injective, there is a regular probability measure μ on K and $\widetilde{T} \in \mathcal{L}(L^r(\mu), Y)$ so that the following diagram conmutes

$$
\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "i_X"'] & Y \\
\mathcal{C}(K) \arrow[r, "I_r"] & L^r(\mu) \arrow[u, "\widetilde{T}"']
\end{tikzcd}
.
$$

We may choose μ and $\widetilde{T}$ so that $\|\widetilde{T}\| = \pi_r(T)$.

4.2. Main definitions and properties of $m-r-$summing operators

Let $m : \Sigma \to X$ be a vector measure and $p, q > 1$ so that $1/p + 1/q = 1$. Suppose $1 \le r < \infty$, and $T : L^p(m) \to Y$ is a bounded operator. We will adapt the definition of summability for operators defined on the space $L^p(m)$. This definition is related with the $m-$topology of the space $L^p(m)$. In [74] the author provides a similar definition in order to study the summability of operators defined on the space of bounded operators between Banach spaces X and Y, $\mathcal{L}(X, Y)$, when it is endowed with the strong operator topology.

We say that T is *$m-r-$summing* if there is some constant $C \ge 0$ such that for every natural number n and regardless the choice of functions $f_1, \dots, f_n$ in $L^p(m)$ we have

$$\left(\sum_{i=1}^{n} \|T(f_i)\|_Y^r \right)^{\frac{1}{r}} \le C \cdot \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r \right)^{\frac{1}{r}}. \qquad (4.5)$$

The least C for which the inequality (4.5) always holds is denoted by $\pi_r^m(T)$. We shall write $\Pi_r^m(L^p(m), Y)$ for the set of $m-r-$summing operators in $\mathcal{L}(L^p(m), Y)$. We clearly have that $\Pi_r^m(L^p(m), Y)$ is a linear subspace of $\mathcal{L}(L^p(m), Y)$ and that π_r^m defines a norm in $\Pi_r^m(L^p(m), Y)$ with

$$\|T\| \le \pi_r^m(T), \ T \in \Pi_r^m(L^p(m)).$$

Notice that if m is a scalar measure, the notion of $m-r-$summability coincides with classical $r-$summability; for a general vector measure m the inclusion $\Pi_r(L^p(m), Y) \subset \Pi_r^m(L^p(m), Y)$ always holds.

As in the classical study of summing operators, it is necessary to develop, for $L^p(m)-$valued sequences, a summability theory with respect to the $m-$topology.

4.2.1. Summability of sequences in $L^p(m)$

In order to adapt the classical results that relate summability of sequences with summing operators, we introduce a space of summable $L^p(m)-$valued sequences.

A sequence $(f_n)_n \subset L^p(m)$ is $m-r-$*summable* whenever for each $g \in L^q(m)$, the $X-$valued sequence $\left(\int_\Omega f_n g dm\right)_n$ is strongly $r-$summable in X. That is, the sequence $\left(\|\int_\Omega f_n g dm\|_X\right)_n$ is $r-$summable for each $g \in L^q(m)$. In what follows we denote by $\ell_r^m(L^p(m))$ the space of $m-r-$summable sequences in $L^p(m)$. As we will prove a suitable norm is given by

$$\|(f_n)\|_{\ell_r^m(L^p(m))} := \sup\left\{\left(\sum_n \left\|\int_\Omega f_n g dm\right\|_X^r\right)^{\frac{1}{r}} : g \in B(L^q(m))\right\}. \tag{4.6}$$

The first step is to show that this quantity is finite. For this aim we use the Closed Graph Theorem. Let $(f_n)_n$ be a sequence in $\ell_r^m(L^p(m))$ and define the associated map $u : L^q(m) \to \ell_r(X)$ given by $u(g) := \left(\int_\Omega f_n g dm\right)_n$. Obviously, u is well defined and linear. Take a sequence $(g_k)_k$ convergent to g_0 in $L^q(m)$. For a fixed $n \in \mathbb{N}$ we have

$$\begin{aligned}\left\|\int_\Omega f_n g_k dm - \int_\Omega f_n g_0 dm\right\|_X &= \left\|\int_\Omega f_n(g_k - g_0)dm\right\|_X \\ &\leq \|f_n\|_{L^p(m)}\|g_k - g_0\|_{L^q(m)} \to 0\end{aligned}$$

then u has a closed graph, that means

$$\|u\| = \sup\left\{\left(\sum_n \left\|\int_\Omega f_n g dm\right\|_X^r\right)^{\frac{1}{r}} : g \in B(L^q(m))\right\} < \infty,$$

as wanted.

Proposition 4.2.1. *The space $(\ell_r^m(L^p(m)), \|\cdot\|_{\ell_r^m(L^p(m))})$ is a Banach space.*

Proof. To prove the completeness of the norm take a Cauchy sequence $\left(f^{(k)}\right)_k = \left(\left(f_n^{(k)}\right)_n\right)_k$ of elements in $\ell_r^m(L^p(m))$, we search for a candidate for the limit of $(f^{(k)})_k$. For a fixed $\varepsilon > 0$, there is some positive index k_0 such that if $k, k^* \geq k_0$

$$\left\|f^{(k)} - f^{(k^*)}\right\|_{\ell_r^m(L^p(m))} = \sup_{g\in B(L^q(m))}\left(\sum_n \left\|\int_\Omega g\left(f_n^{(k)} - f_n^{(k^*)}\right)dm\right\|_X^r\right)^{\frac{1}{r}} \leq \varepsilon.$$

That means, for $g \in B(L^q(m))$ and $k, k^* \geq k_0$

$$\sum_n \left\| \int_\Omega g \Big(f_n^{(k)} - f_n^{(k^*)} \Big) dm \|_X^r \leq \varepsilon^r. \tag{4.7}$$

Each term in this series is dominated by ε^r, so for every $n \in \mathbb{N}$ and $g \in B(L^q(m))$

$$\left\| \int_\Omega g \left(f_n^{(k)} - f_n^{(k^*)} \right) dm \right\|_X \leq \varepsilon,$$

for all $k, k^* \geq k_0$. Thus $\left\| f_n^{(k)} - f_n^{(k^*)} \right\|_{L^p(m)} \leq \varepsilon$ for $k, k^* \geq k_0$, and every $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$ the sequences $\left(f_n^{(k)} \right)_k$ are Cauchy sequences in $L^p(m)$ and so convergent to a function f_n in $L^p(m)$. We have a candidate $(f_n)_n$ for the limit of $\left(f^{(k)} \right)_k$, we must show that $(f_n)_n$ is really a limit in $\ell_r^m(L^p(m))$.

Since all the terms in the sum (4.7), we have, for every $N \in \mathbb{N}$, $g \in B(L^q(m))$ and $k, k^* \geq k_0$

$$\sum_{n=1}^{N} \left\| \int_\Omega g \left(f_n^{(k)} - f_n^{(k^*)} \right) dm \right\|_X^r \leq \varepsilon^r. \tag{4.8}$$

Now, let k^* tend to infinity in (4.8). We get, for all $g \in B(L^p(m))$, $N \in \mathbb{N}$ and $k \geq k_0$

$$\left(\sum_{n=1}^{N} \left\| \int_\Omega g \left(f_n^{(k)} - f_n \right) dm \right\|_X^r \right)^{\frac{1}{r}} \leq \varepsilon.$$

Clearly, this implies, that the infinite sum is also bounded as follows, for every $g \in B(L^q(m))$ and $k \geq k_0$,

$$\left(\sum_n \left\| \int_\Omega g \left(f_n^{(k)} - f_n \right) dm \right\|_X^r \right)^{\frac{1}{r}} \leq \varepsilon.$$

That means that $\left(f_n^{(k)} \right)_n - (f_n)_n \in \ell_r^m(L^p(m))$ for each $k \geq k_0$, and hence $f = (f_n)_n$ belongs to $\ell_r^m(L^p(m))$, with $\left\| \left(f_n^{(k)} \right)_n - (f_n)_n \right\|_{\ell_r^m(L^p(m))} \leq \varepsilon$ for all $k \geq k_0$. Since this happens for an arbitrary $\varepsilon > 0$ we conclude that $\left(f_n^{(k)} \right)_n \to (f_n)_n$ in $\ell_r^m(L^p(m))$. $\square$

Let $T : L^p(m) \to Y$ be a bounded linear operator into a Banach space Y. It induces a bounded linear map $\hat{T} : (f_n)_n \mapsto (T(f_n))_n$ between the spaces $\ell_r^m(L^p(m))$ and $\ell_r^w(Y)$. Indeed, for a sequence $(f_n)_n \subset \ell_r^m(L^p(m))$, we have that $\left(\|\int_\Omega f_n g dm\|_X\right)_n \in \ell_r$ for all $g \in L^q(m)$. Fix $y^* \in Y^*$. Recall that $\Gamma \subset L^p(m)^*$ is norming (by Proposition 1.0.7) and that the weak norm of a sequence can be computed by norming subsets (see for instance [27] page 36). Since $\frac{y^*\circ T}{\|y^*\circ T\|} \in B((L^p(m))^*)$ we get the following

$$\begin{aligned}
\left(\sum_n |\langle T(f_n), y^*\rangle|^r\right)^{\frac{1}{r}} &= \|y^*\circ T\| \left(\sum_n \left|\left(\frac{y^*\circ T}{\|y^*\circ T\|}\right)(f_n)\right|^r\right)^{\frac{1}{r}} \\
&\leq \|y^*\circ T\| \sup_{\varphi\in B(L^p(m)^*)} \left(\sum_n |\langle f_n, \varphi\rangle|^r\right)^{\frac{1}{r}} \\
&\leq \|y^*\circ T\| \sup_{g\in B(L^q(m)), x^*\in B(X^*)} \left(\sum_n \left|\left\langle \int_\Omega f_n g dm, x^*\right\rangle\right|^r\right)^{\frac{1}{r}} \\
&\leq \|y^*\circ T\| \sup_{g\in B(L^q(m))} \left(\sum_n \left\|\int_\Omega f_n g dm\right\|_X^r\right)^{\frac{1}{r}},
\end{aligned}$$

that is finite because $\left(\|\int f_n g dm\|_X\right)_n$ is $r-$summable.

Basic arguments and inequalities above (taking T the identity map in $L^p(m)$) yield the following chain of containments:

$$\ell_r(L^p(m)) \subset \ell_r^m(L^p(m)) \subset \ell_r^w(L^p(m)).$$

As happens in Proposition 4.1.1 for $r-$summing operators, $m-r-$summing operators are characterized by their behavior over summable sequences. The following result shows that $m-r-$summing operators are exactly those that transform $m-r-$summable sequences in $L^p(m)$ into strongly $r-$summable ones in the range Y.

Theorem 4.2.2. *An operator $T \in \mathcal{L}(L^p(m), Y)$ is $m-r-$summing if and only if $\hat{T}(\ell_r^m(L^p(m))) \subset \ell_r(Y)$. Moreover $\|\hat{T}\| = \pi_r^m(T)$.*

Proof. Suppose first that T is $m-r-$summing, then for each finite collection $f_1, \ldots, f_k \in L^p(m)$ we have

$$\left(\sum_{i=1}^k \|T(f_i)\|_Y^r\right)^{\frac{1}{r}} \leq \pi_r^m(T) \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^k \left\|\int_\Omega f_i g dm\right\|_X^r\right)^{\frac{1}{r}}.$$

Take a sequence $(f_n)_n \in \ell_r^m(L^p(m))$. We claim that $\hat{T}((f_n)_n) \in \ell_r(Y)$, hence

$$\begin{aligned}
\|\hat{T}((f_n)_n)\|_{\ell_r(Y)} &= \|(T(f_n))_n\|_{\ell_r(Y)} = \sup_{k\in\mathbb{N}}\left(\sum_{n\leq k}\|T(f_n)\|_Y^r\right)^{\frac{1}{r}}\\
&\leq \pi_r^m(T)\sup_{k\in\mathbb{N}}\sup_{g\in B(L^q(m))}\left(\sum_{n\leq k}\left\|\int_\Omega f_n g dm\right\|_X^r\right)^{\frac{1}{r}}\\
&= \pi_r^m(T)\sup_{g\in B(L^q(m))}\left(\sum_{n=1}^{\infty}\left\|\int_\Omega f_n g dm\right\|_X^r\right)^{\frac{1}{r}}\\
&= \pi_r^m(T)\|(f_n)_n\|_{\ell_r^m(L^p(m))},
\end{aligned}$$

therefore $\|\hat{T}\| \leq \pi_r^m(T)$.

We prove the converse implication by a closed graph argument. Suppose $\hat{T}(\ell_r^m(L^p(m))) \subset \ell_r(Y)$. Since $\hat{T} : \ell_r^w(L^p(m)) \to \ell_r(Y)$ is continuos and the $\ell_r(Y)$ norm dominates the $\ell_r^w(Y)$ norm we have that the corresponding operator $\hat{T} : \ell_r^m(L^p(m)) \to \ell_r(Y)$ has closed graph and is bounded. Thus for a finite sequence $(f_i)_{i=1}^k \subset L^p(m)$ we get

$$\|(T(f_i))_{i=1}^k\|_{\ell_r(Y)} \leq \|\hat{T}\|\|(f_i)_{i=1}^n\|_{\ell_r^m(L^p(m))}.$$

Therefore T is $m-r-$summing and $\pi_r^m(T) \leq \|\hat{T}\|$. $\square$

As a consequence of the previous characterization we prove that the space of $m-r-$summing operators endowed with their respective norms are Banach spaces.

Theorem 4.2.3. *Let Y be a Banach space, and $1 \leq r < \infty$. The space of $m-r-$summing operators, $\Pi_r^m(L^p(m), Y)$, endowed with the norm π_r^m is a Banach space,*

Proof. Let $(T_n)_n$ be a π_r^m-Cauchy sequence in $\Pi_r^m(L^p(m), Y)$. Since the operator norm is dominated by π_r^m, we have that $(T_n)_n$ is a Cauchy sequence in $\mathcal{L}(L^p(m), Y)$, thus convergent to an operator $T \in \mathcal{L}(L^p(m), Y)$.

We claim that T is in fact $m-r-$summing and $\pi_r^m(T_n - T) \to 0$ when $n \to \infty$, these facts are consequences of Theorem 4.2.2. Since each T_n is $m-r$–summing we have that $(\hat{T}_n)_n$ is a Cauchy sequence in $\mathcal{L}(\ell_r^m(L^p(m)), \ell_r(Y))$ and therefore, convergent to $\hat{T} \in \mathcal{L}(\ell_r^m(L^p(m)), \ell_r(Y))$. We directly get that T is $m-r-$summing and $\pi_r^m(T_n - T) \to 0$. $\square$

4.2.2. Operators with range in $L^p(m)$

In order to continue the analysis of the summability properties associated to the $m-$topology on $L^p(m)$, it is natural to investigate the operators satisfying that transform weakly $r-$summable sequences in $m-r-$summable ones. As in the previous section, we define and characterize the corresponding class of operators, that we call weak $m-r-$summing operators. In this section we prove that these operators, together with the $m-r-$summing ones, complete in a sense the tools for the study of the summability associated to the $m-$topology. Moreover this study provides information about those vector measures that satisfy that the identity map in $L^p(m)$ is $m-r-$summing. Finiteness of the dimension of subspaces where its restriction satisfy this property is proved. Also, for finishing this section, we analyze operators between Banach spaces that can be factorized through an $L^p(m)$ space in such a way that one of the factors is a weak $m-r-$summing operator and the other one is $m-p-$summing. In fact, we show that under the adequate assumptions $1-$summing operators can always be factorized through a weak $m-1-$summing operator and a $m-1-$summing one. As a consequence, we prove a Dvoretsky-Rogers type result regarding finiteness of the dimension of the Banach spaces in which the identity map factorizes in such a way.

We say that an operator $T: Y \to L^p(m)$ is *weak $m-r-$summing* if there is a constant $C>0$ such that for every finite set of elements $y_1, ..., y_n \in Y$,

$$\sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\| \int_\Omega T(y_i) g \, dm \right\|^r \right)^{1/r} \leq C \sup_{y^*\in B(Y^*)} \left(\sum_{i=1}^n |\langle y_i, y^* \rangle|^r \right)^{1/r}. \tag{4.9}$$

We write $\pi_r^{w-m}(T)$ for the least constant so that the inequality above holds.

Examples of such kind of operators are easy to find. More interesting for the aim of this section are such $L^p(m)$ spaces in which the identity map satisfies this property. The canonical one is given for the case where m is a scalar positive finite measure μ. Obviously, the identity $Id: L^p(\mu) \to L^p(\mu)$ satisfies this property since in this case the integrals in the left hand side term of inequality (4.9) give exactly the usual duality, the one that appears in the right hand side term.

For a linear and continuous operator T between spaces of integrable functions with respect to a vector measure, $T: L^{p_1}(m_1) \to L^{p_2}(m_2)$, with $p_1, p_2 > 1$, we have that T is weak $m-r-$summing whenever it is $m-r-$summing.

Cleary every $r-$summing operator $T: Y \to L^p(m)$ is weak $m-r-$summing, and $\pi_r^{w-m}(T) \leq \pi_r(T)$. The characterization of weak $m-$

$r-$summing operators in terms of a Pietsch type domination theorem is in this case also easy to prove. As the following result shows, it is required that the composition of T with the integration map for every $g \in L^q(m)$ is $r-$summing, with some sort of uniform behavior of the associated $r-$summing norms.

Proposition 4.2.4. *Let $T : Y \to L^p(m)$ with Y a Banach space. The following statements are equivalent.*

(i) T is weak $m - r-$summing.

(ii) There is a constant $C > 0$ such that for every $g \in B(L^q(m))$, the operator $I_g \circ T : Y \to X$ is $r-$summing, and

$$\pi_r(I_g \circ T) \leq C.$$

(iii) There is a constant $C > 0$ such that for every $g \in B(L^q(m))$, there is a probability measure η_g defined on the $\sigma-$algebra of Borel subsets of $B(Y^*)$ (endowed with the weak$^*-$topology) such that, for every $y \in Y$,

$$\left\| \int_\Omega T(y) g \, dm \right\|_X \leq C \left(\int_{B(Y^*)} |\langle y, y^* \rangle|^r \, d\eta_g(y^*) \right)^{1/r}. \tag{4.10}$$

Moreover, the least C appearing in (i), (ii) and (iii) coincides with

$$\sup_{g \in B(L^q(m))} \pi_r(I_g \circ T) = \pi_r^{w-m}(T).$$

Proof. For the implication $(i) \Rightarrow (ii)$ it is enough to use the definition of $p-$summing operator. The converse is also obvious. The equivalence between (iii) and (ii) is obtained just by applying Pietsch Domination Theorem to each one of the maps $I_g \circ T$. The formula for the norm is also a direct consequence of the definitions.

□

Remark 4.2.5. The lattice properties of the sets of Pietsch measures appearing in (iii) of Proposition 4.2.4 provide a criterion for an operator to be weak $m - r-$summing. Let $T \in \Pi_r^{w-m}(Y, L^p(m))$. Let $\mathcal{M}(B(Y^*))$ be the usual space of Radon measures over the $\sigma-$algebra of Borel subsets of $B(Y^*)$, where Y^* is endowed with the weak$^*-$topology. Then $\mathcal{M}(B(Y^*)) = \mathcal{C}(B(Y^*))^*$. As a consequence of Proposition 4.2.4 there is a set of Pietsch measures $\{\eta_g : g \in B(L^q(m))\}$ associated to the operator T so that for each $g \in L^q(m)$ inequality (4.10) holds. Assuming that

$\{\eta_g : g \in B(L^q(m))\}$ is order bounded in $\mathcal{M}(B(Y^*))$ by an element η, we obtain that for every $y \in Y$,

$$\|T(y)\| \leq K \left(\int_{B_{Y^*}} |\langle y, y^* \rangle|^r \, d\eta \right)^{1/r}.$$

Consequently, T is $r-$summing. The converse is also obvious, since every $r-$summing operator $T : Y \to L^p(m)$ is weak $m-r-$summing. When this argument is applied to the case of the identity map $I_d : L^p(m) \to L^p(m)$, we obtain that it is weak $m-r-$summing with a set of Pietsch measures that is uniformly order bounded if and only if $L^p(m)$ is finite dimensional, as a consequence of Dvoretsky-Rogers Theorem and the following calculations. If η is the required order bound, for every $f \in L^p(m)$,

$$\begin{aligned}
\|f\|_{L^p(m)} &= \sup_{g \in B(L^q(m))} \left\| \int f g \, dm \right\|_X \\
&\leq K \sup_{g \in B(L^q(m))} \left(\int_{B((L^p(m))^*)} |\langle f, h \rangle|^r d\eta_g \right)^{1/r} \\
&\leq K \left(\int_{B((L^p(m))^*)} |\langle f, h \rangle|^r d\eta \right)^{1/r}.
\end{aligned}$$

The previous remark shows that uniform boundedness of the integrals $\left\| \int_\Omega (\cdot) g \, dm \right\|_X$ by an integral $\left(\int_{B((L^p(m))^*)} |\langle f, h \rangle|^r d\eta \right)^{1/r}$ only holds for finite dimensional $L^p(m)$ spaces. In the same direction, the following result shows that $L^p(m)$ spaces where $m-r-$summable sequences and weak $r-$summable sequences coincide (i.e. the identity map is weak $m-r-$summing) for some $1 \leq r < \infty$, have strong restrictions on the properties of the integration maps $\int_\Omega (\cdot) g \, dm$, $g \in L^q(m)$.

Proposition 4.2.6. *If $Id : L^p(m) \to L^p(m)$ is weak $m-r-$summing for some $1 \leq r < \infty$, then there is no function $g \in L^q(m)$ such that there is an infinite dimensional subspace $S \subseteq L^p(m)$ satisfying that the restriction $I_g|_S : S \to X$ is an isomorphism onto the range.*

Proof. Suppose that there is a subspace S such that the restriction $I_g|_S : S \to X$ is an isomorphism onto the range. Let us write i for the inclusion map $i : S \to L^p(m)$ and $R : I_g|_S(S) \to S$ for the inverse map $(I_g|_S)^{-1} : I_g|_S(S) \to S$. Since Id is weak $m-r-$summing, each I_g is $r-$summing as a consequence of (ii) in Proposition 4.2.4. Therefore, $I_g|_S = I_g \circ i : S \to L^p(m) \to X$ is a r-summing isomorphism onto the range, and since the identity in S can be factorized as

$$R \circ I_g|_S : S \to I_g(i(S)) \to S,$$

therefore the ideal property of r-summing operators and Dvoretsky-Rogers Theorem yields that S is finite dimensional.

$\square$

In the following result we show that the compositions of weak $m-r-$summing maps and $m-r-$summing maps give $r-$summing operators; this motivates the definitions of this section. Notice that the definition of $m-r-$summing operator can be extended to those operators defined on closed subspaces of $L^p(m)$ in a natural way.

Proposition 4.2.7. *The composition $T = R \circ U$ of a weak $m-r-$summing operator $U : Y \to L^p(m)$ and an $m-r-$summing one $R : S \to Z$, where S is a subspace of $L^p(m)$ such that $U(Y) \subseteq S$, is r-summing.*

Proof. Let $y_1, ..., y_n \in Y$, then

$$\begin{aligned}\left(\sum_{i=1}^{n} \|T(y_i)\|_Z^r\right)^{1/r} &= \left(\sum_{i=1}^{n} \|R(U(y_i))\|_Z^r\right)^{1/r} \\ &\leq K \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^{n} \left\|\int_\Omega U(y_i) g\, dm\right\|_X^r\right)^{1/r} \\ &\leq KQ \sup_{y^*\in B(Y^*)} \left(\sum_{i=1}^{n} |\langle U(y_i), y^*\rangle|^r\right)^{1/r}.\end{aligned}$$

and the conclusion follows. $\square$

The following result gives a sort of converse of the previous one and provides a new factorization theorem for summing operators. It shows that in a sense, regarding the structure properties of $L^1(m)$ spaces and factorizations through them, $1-$summability can be decomposed in $m-1-$summability and weak $m-1-$summability. Notice that the definitions of (weak) $m-r-$summability for operators defined (or with range) in $L^1(m)$ have sense when we take $q=\infty$, the conjugated index. Recall that $L^\infty(m) = L^\infty(\lambda)$.

Theorem 4.2.8. *Let $T : Y \to Z$ be an operator between Banach spaces. The following statements are equivalent.*

(i) T is $1-$summing.

(ii) There is a vector measure m such that T factorizes through a subspace of $L^1(m)$ as $T = R \circ U$, where U is weak $m-1-$summing and R is $m-1-$summing.

Proof. For the proof of (*i*) $\Rightarrow$ (*ii*), consider the factorization of T as $1-$ summing operator through the map $i:\mathcal{C}(B(Y^*))\to L^1(B(Y^*),\eta)$ given by the classical Piesth domination theorem. Recall that we consider $B(Y^*)$ endowed with the weak$^*-$topology. Here η is a Radon probability measure and $i(f)=f$ is the identification map of continuous functions as integrable functions. Take the vector measure defined on $\mathcal{B}$, the $\sigma-$algebra of the Borel subsets of $B(Y^*)$, with range in $L^1(B(Y^*),\eta)$ given by $m(A)=\chi_A$, $A\in\mathcal{B}$. Then $L^1(m)=L^1(B(Y^*),\eta)$ isometrically (see Example 3.1.4). Consider the map $U:Y\to F\subset L^1(m)$ given by $U(y)=\langle y,\cdot\rangle$, where F is the closure of the functions $\langle y,\cdot\rangle$ in $L^1(\eta)$. Recall that $L^\infty(m)=L^\infty(\eta)$. The following calculations show that U is weak $m-1-$summing. For a finite set $y_1,...,y_n\in Y$,

$$\sup_{g\in B(L^\infty(m))}\left(\sum_{i=1}^n\left\|\int U(y_i)g\,dm\right\|_{L^1(\eta)}\right)$$

$$\begin{aligned}
&= \sup_{g\in B(L^\infty(m))}\left(\sum_{i=1}^n\left\|\int \langle y_i,\cdot\rangle g\,dm\right\|_{L^1(\eta)}\right)\\
&= \sup_{g\in B(L^\infty(\eta))}\left(\sum_{i=1}^n\left(\int_{B(Y^*)}|\langle y_i,\cdot\rangle g|\,d\eta\right)\right)\\
&= \left(\sum_{i=1}^n\int_{B(Y^*)}|\langle y_i,\cdot\rangle|\,d\eta\right)\\
&\le \sup_{y^*\in B(Y^*)}\left(\sum_{i=1}^n|\langle y_i,y^*\rangle|\right).
\end{aligned}$$

Now take the map $R:F\to Z$ given by $R(\langle x,\cdot\rangle)=T(x)$ and extended by density to the elements of the closure of the range of U. Let us show that it is $m-1-$summing. It is enough to prove it for elements of the range of U. Take $\langle y_1,\cdot\rangle,...,\langle y_n,\cdot\rangle$. Then, having in mind that there is a constant K such that for every $y\in Y$, $\|T(y)\|_Z\le K\|\langle y,\cdot\rangle\|_{L^1(\eta)}$, we obtain

$$\begin{aligned}
\left(\sum_{i=1}^n\|R(\langle y_i,\cdot\rangle)\|_Z\right) &= \left(\sum_{i=1}^n\|T(y_i)\|\right)\le K\left(\sum_{i=1}^n\|\langle y_i,\cdot\rangle\|_{L^1(\eta)}\right)\\
&\le K\sup_{g\in B(L^\infty(m))}\left(\sum_{i=1}^n\|\langle y_i,\cdot\rangle g\|_{L^1(\eta)}\right).
\end{aligned}$$

Consequently, the map is $m-1-$summing.

Implication (*ii*) $\Rightarrow$ (*i*) follows directly by Proposition 4.2.7. □

4.2.3. Examples

We finish this section with some examples of $m-r-$summing operators.

Example 4.2.9. The canonical $m-r-$summing operator is the integration map $I_g : L^p(m) \to X$, defined by $I_g(f) := \int_\Omega fgdm$ where g is a fixed $q-$integrable function with respect to m. Indeed we have, for $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in L^p(m)$,

$$\begin{aligned}
\sum_{i=1}^{n} \|I_g(f_i)\|^r &= \sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r \\
&= \|g\|_{L^q(m)}^r \sum_{i=1}^{n} \left\| \int_\Omega f_i \frac{g}{\|g\|_{L^q(m)}} dm \right\|_X^r \\
&\le \|g\|_{L^q(m)}^r \sup_{h \in B(L^q(m))} \sum_{i=1}^{n} \left\| \int_\Omega f_i h dm \right\|_X^r,
\end{aligned}$$

therefore I_g is $m-r-$summing and $\pi_r^m(I_g) \le \|g\|_{L^q(m)}$. Note that for $g \in L^q(m)$ and $\varepsilon > 0$, there is always a function $f_\varepsilon \in B(L^p(m))$ such that $\|g\|_{L^q(m)} \le (1+\varepsilon)\|I_g(f_\varepsilon)\|_X$. We obtain that, taking into account the inequality for $m-r-$summing operators for the single function f_ε, we get

$$\begin{aligned}
\|g\|_{L^q(m)} &\le (1+\varepsilon)\|I_g(f_\varepsilon)\| \\
&\le (1+\varepsilon)\pi_r^m(I_g) \sup_{h \in B(L^q(m))} \left\| \int_\Omega f_\varepsilon h dm \right\|_X \\
&\le (1+\varepsilon)\pi_r^m(I_g).
\end{aligned}$$

Thus $\|g\|_{L^q(m)} \le \pi_r^m(I_g)$.

The following example follows the lines of [27, Example 2.11]. Recall that a Banach space valued strongly measurable function $f : \Omega \to X$ is *Bochner $r-$integrable* whenever $\int_\Omega \|f(w)\|_X^r d\mu(w) < \infty$. We denote by $L^r(\mu, X)$ the space of equivalence classes of $\mu-$a.e. $X-$valued functions that are Bochner $r-$integrable. Clearly, for each $x^* \in X^*$, the function $w \mapsto \langle f(w), x^* \rangle$ is in $L^r(\mu)$.

Example 4.2.10. Let $m : \Sigma \to X$ be a vector measure and λ a Rybakov's control measure for m. Fix $(\bar{\Omega}, \bar{\Sigma}, \bar{\mu})$ a finite positive measure space and $f_0 : \bar{\Omega} \to L^p(m)$ a Bochner r-integrable function with respect to the measure $\bar{\mu}$.

Let us show the operator $u_{f_0} : L^q(m) \to L^r(\bar{\mu}, X)$ defined by $u_{f_0}(g) : \bar{\Omega} \to X$ with $u_{f_0}(g)(\bar{w}) := \int_\Omega g f_0(\bar{w}) dm$ $\bar{\mu}-$a.e., is $m-r-$summing.

In order to show that the definition of u_{f_0} is consistent, we prove the following claim.

*Claim.*Let $f:\Omega\to X$ be a Bochner $\mu-$integrable function, and $T: X\to Y$ a linear and continuous operator. The composition Tf is Bochner integrable. Since f is strongly measurable, there is a sequence of simple functions $(s_n)_n\mathcal{S}(\Sigma)$ converging to f $\mu-$a.e. The sequence $(Ts_n)_n\subset\mathcal{S}(\Sigma)$ and converges pointwise to Tf $\mu-$a.e. Since $\|Tf\|^r\le\|T\|^r\|f\|^r$, and f is Bochner integrable, we have $\int\|Tf\|^r d\mu\le+\infty$. This proves the claim.

Since $u_{f_0}(g)=I_g(f_0)$, and $f_0\in L^r(\bar{\mu},X)$, a direct application of the previous claim yields $u_{f_0}(g)\in L^r(\bar{\mu},X)$.

It remains to prove the $m-r-$summability of the operator u_{f_0}. Take a finite collection of functions $g_1,\dots,g_n\in L^q(m)$. Then we have

$$\left(\sum_{i=1}^n\|u_{f_0}(g_i)\|^r_{L^r(\bar{\mu},X)}\right)^{\frac{1}{r}}=\left(\sum_{i=1}^n\int_{\bar{\Omega}}\|u_{f_0}(g_i)(\bar{w})\|^r_X d\bar{\mu}(\bar{w})\right)^{\frac{1}{r}}$$

$$\begin{aligned}
&= \left(\sum_{i=1}^n\int_{\bar{\Omega}}\left\|\int_\Omega g_i f_0(\bar{w})dm\right\|^r_X d\bar{\mu}(\bar{w})\right)^{\frac{1}{r}}\\
&= \left(\int_{\bar{\Omega}}\sum_{i=1}^n\left\|\int_\Omega g_i\frac{f_0(\bar{w})}{\|f_0(\bar{w})\|_{L^p(m)}}\|f_0(\bar{w})\|_{L^p(m)}dm\right\|^r_X d\bar{\mu}(\bar{w})\right)^{\frac{1}{r}}\\
&= \left(\int_{\bar{\Omega}}\|f_0(\bar{w})\|^r_{L^p(m)}\sum_{i=1}^n\left\|\int_\Omega g_i\frac{f_0(\bar{w})}{\|f_0(\bar{w})\|_{L^p(m)}}dm\right\|^r_X d\bar{\mu}(\bar{w})\right)^{\frac{1}{r}}\\
&\le \left(\sup_{\bar{w}\in\bar{\Omega}}\sum_{i=1}^n\left\|\int_\Omega g_i\frac{f_0(\bar{w})}{\|f_0(\bar{w})\|_{L^p(m)}}dm\right\|^r_X\int_{\bar{\Omega}}\|f_0(\bar{w})\|^r_{L^p(m)}d\bar{\mu}(\bar{w})\right)^{\frac{1}{r}}\\
&= \|(g_i)_{i=1}^n\|_{\ell_r^m(L^q(m))}\|f_0\|_{L^r(\bar{\mu},L^p(m))}.
\end{aligned}$$

We conclude that u_{f_0} is $m-r-$summing with $\pi_r^m(u_{f_0})\le\|f_0\|_{L^r(\mu,L^p(m))}$.

A particular case of this example is given when the operator u_{f_0} is defined by a kernel. Kernel operators have been largely studied in this setting, see for instance the articles by G. Curbera, O. Delgado and W. Ricker ([16, 20]). Take a function $k:\bar{\Omega}\times\Omega\to\mathbb{R}$, $\bar{\mu}\times\lambda-$ measurable, so that

$$c_{p,r}=\left(\int_{\bar{\Omega}}\left(\int_\Omega|k(\bar{w},w)|^p dm(w)\right)^{\frac{r}{p}}d\bar{\mu}(\bar{w})\right)^{\frac{1}{r}} \tag{4.11}$$

is finite. Then we have that the kernel operator defined by

$$K:L^q(m)\to L^r(\bar{\mu},X)\ \text{ with }\ g\mapsto K(g):=\int_\Omega k(.,w)g(w)dm(w)$$

for $g \in L^q(m)$ is $m-r-$summing. Indeed if we consider a strongly measurable function $f_0 : \bar{\Omega} \to L^p(m)$ given by $f_0(\bar{w}) := k(\bar{w}, \cdot)$, we have by condition (4.11) that $f_0 \in L^r(\bar{\mu}, L^p(m))$, and $K(g) = u_{f_0}(g)$ $\bar{\mu}-$a.e. in $\bar{\Omega}$.

4.3. Comparing spaces of summing operators

4.3.1. Pietsch type theorems

In order to prove a a Pietsch type theorem for $m-r-$summing operator we need to assume a condition on the space $L^p(m)$ that will be called property $(r-\star)$. In the following we give two examples that are in some sense extreme cases in which the spaces $L^p(m)$ have property $(r-\star)$. But firstly let us introduce this property.

A family $\mathcal{S} := \{S_i : i \in I\}$ of finite dimensional subspaces of $L^p(m)$ is a *dense family of subspaces* if it satisfies that for every $f_1, ..., f_n \in L^p(m)$ and every $\varepsilon > 0$ there is an $i_0 \in I$ such that there are functions $f_1^0, ..., f_n^0 \in S_{i_0}$ satisfying that

$$\|f_i - f_i^0\|_{L^p(m)} < \varepsilon,\ i = 1, \dots, n.$$

Let $1 \leq r < \infty$. We say the space $L^p(m)$ has the *property* $(r-\star)$ over $\mathcal{S}$ if there is a dense family of subspaces $\mathcal{S}$ so that for each subspace $S \in \mathcal{S}$ and for each $0 < \varepsilon < \frac{1}{2}$ there exists an $m-$compact set $K \subset B(L^q(m))$ so that for every finite choice of functions $f_1, \dots, f_n \in S$ the following inequality holds:

$$(1-\varepsilon) \sup_{g \in B(L^q(m))} \sum_{i=1}^n \left\| \int_\Omega f_i g dm \right\|_X^r \leq \sup_{g \in K} \sum_{i=1}^n \left\| \int_\Omega f_i g dm \right\|_X^r. \tag{4.12}$$

In this case we say that the set K is $r-\varepsilon-$*norming* for the subspace S.

In the following we give two examples of vector measures so that the associated $L^p(m)$ space satisfies the property $(r-\star)$. In the first one, the span of the range of the vector measure is one dimensional. In the second example, we find what is in a sense the canonical case of the opposite situation, which also satisfies the property $(r-\star)$.

Example 4.3.1. When the vector measure is a positive finite scalar measure, the $m-$topology coincides with the weak topology. So for $1 < q < \infty$ the unit ball of $L^q(m)$ is compact with respect to this topology, that gives an easy example of $m-$compact norming subset. Moreover, the equality

$$\sup_{h \in B(L^q(m))} \left(\sum_{i=1}^n |\langle f_i, h \rangle|^r \right)^{1/r} = \sup_{(\lambda_i) \in B(\ell^{r'})} \left\| \sum_{i=1}^n \lambda_i f_i \right\|_{L^p(m)}$$

holds for every finite sequence $f_1,...,f_n \in L^p(m)$. Consequently, the weak $r-$norm expression is evaluated by computing norms of elements that belong to the finite dimensional subspace generated by $f_1,...,f_n$. Therefore, a *finite* (and then $m-$compact) set of elements of the unit ball of $B(L^q(m))$ is enough to approximate the weak $r-$norm "up to an ε".

Example 4.3.2. Let us give now an example of a vector measure for which this property is also satisfied but has a large range. Let $1 \leq r < \infty$ and consider a finite measure space (Ω, Σ, μ) and the vector measure $m : \Sigma \to L^r(\mu)$ given by $m(A) := \chi_A$, that has been used in several examples before. Notice that the linear span of the range of this vector measure is dense in $L^r(\mu)$ and $\int_\Omega h dm = h$ for every $h \in L^1(m) = L^r(\mu)$. In this case, for every finite set $f_1,...,f_n \in L^p(m)$,

$$\begin{aligned}
\sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\| \int_\Omega f_i g dm \right\|_{L^r(\mu)}^r \right)^{\frac{1}{r}} &= \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \|f_i g\|_{L^r(\mu)}^r \right)^{\frac{1}{r}} \\
&= \sup_{g\in B(L^q(m))} \left\| \left(\sum_{i=1}^n |f_i|^r \right)^{\frac{1}{r}} |g| \right\|_{L^r(\mu)} \\
&= \left\| \left(\sum_{i=1}^n |f_i|^r \right)^{\frac{1}{r}} \right\|_{L^p(m)}
\end{aligned}$$

Now assume that the elements $f_1,...,f_n$ are simple functions. Then a simple calculation shows that there is a finite partition $\{A_k : k = 1,...,t\}$ of Ω such that the expression $(\sum_{i=1}^n |f_i|^r)^{1/r}$ can be written as $\sum_{k=1}^t \tau_k \chi_{A_k}$ for some non-negative τ_k. Take $S = \{\chi_{A_k} : k = 1,...,t\}$. Since the measure m is positive, we can obtain a compact (and then $m-$compact) subset of $B(L^q(m))$ that is $r-\varepsilon-$norming for S, as showed in example 2.2.6. Since the subspaces generated by finite sets of characteristic functions of disjoint sets define a dense family for $L^p(m)$, we obtain the result. Notice that for this particular measure m and every Banach space Y, the operator $T : L^p(m) \to Y$ is $m-r-$summing if and only if it is $r-$concave. Indeed, for $f_1, \ldots, f_n \in L^p(m)$ we have, by the computations above

$$\left(\sum_{i=1}^n \|T(f_i)\|_Y^r \right)^{\frac{1}{r}} \leq K \left\| \left(\sum_{i=1}^n |f_i|^r \right)^{\frac{1}{r}} \right\|_{L^p(m)}.$$

A classical result (see [48, Theorem 1.d.10]) ensures that every $r-$summing operator is $r-$concave. In this example we give a sort of converse; we reproduce this geometrical property of the operator T ($r-$concavity) in terms of a summability property.

Notice that disjoint sums of scalar measures and measures as in example 4.3.2 provides more non trivial examples of spaces having property $(r-\star)$.

The following result is obtained by means of straightforward calculations, we give the proof for the aim of completeness.

Lemma 4.3.3. *Let $\mathcal{S} := \{S_i : i \in I\}$ be a dense family of subspaces of $L^p(m)$, and $T : L^p(m) \to Y$ a linear and continuous operator. T is $m-r-$summing if and only if all the restrictions $T|_{S_i}$ are uniformly $m-r-$summing.*

Proof. The direct implication is trivial, in order to prove the converse, let $f_1, \ldots, f_n \in L^p(m)$ and $\varepsilon > 0$. There is an index $i_0 \in I$ such that there are $f_1^0, \ldots, f_n^0$ in S_{i_0} such that $\|f_i^0 - f_i\|_{L^p(m)} \leq \varepsilon/(n^{\frac{1}{r}} \max\{C, \|T\|\})$, where $C > 0$ is the uniform bound that appears by the the summability of $T|_{S_i}$, for $i = 1, \ldots, n$. We get

$$\left(\sum_{i=1}^n \|T(f_i)\|_Y^r\right)^{\frac{1}{r}} \leq \left(\sum_{i=1}^n \|T(f_i - f_i^0)\|_Y^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n \|T(f_i^0)\|_Y^r\right)^{\frac{1}{r}}$$

$$\begin{aligned}
&\leq \|T\| \left(\sum_{i=1}^n \|f_i - f_i^0\|_{L^p(m)}^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n \|T(f_i^0)\|_Y^r\right)^{\frac{1}{r}} \\
&\leq \|T\| \left(\sum_{i=1}^n \|f_i - f_i^0\|_{L^p(m)}^r\right)^{\frac{1}{r}} + C \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\|\int_\Omega f_i^0 g dm\right\|_X^r\right)^{\frac{1}{r}} \\
&\leq \varepsilon + C \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\|\int_\Omega (f_i^0 - f_i) g dm\right\|_X^r\right)^{\frac{1}{r}} \\
&+ C \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\|\int_\Omega f_i g dm\right\|_X^r\right)^{\frac{1}{r}} \\
&\leq \varepsilon + C \left(\sum_{i=1}^n \|f_i - f_i^0\|_{L^p(m)}^r\right)^{\frac{1}{r}} + C \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\|\int_\Omega f_i g dm\right\|_X^r\right)^{\frac{1}{r}} \\
&\leq 2\varepsilon + C \sup_{g\in B(L^q(m))} \left(\sum_{i=1}^n \left\|\int_\Omega f_i g dm\right\|_X^r\right)^{\frac{1}{r}}
\end{aligned}$$

for every $\varepsilon > 0$. Therefore T is $m-r-$summing. □

Theorem 4.3.4. *Fix $1 \leq r < \infty$ and $1 < p < \infty$. Let $T : L^p(m) \to Y$ be a linear and continuous operator taking values in a Banach space Y, and suppose that $L^p(m)$ has property $(r-\star)$ over a dense family of subspaces $\mathcal{S}$. The following assertions are equivalent,* *(i) T is $m-r-$summing, that is, there is $Q>0$ such that for every finite choice of functions $f_1,\dots,f_n$*

$$\left(\sum_{i=1}^{n}\|T(f_i)\|_Y^r\right)^{\frac{1}{r}} \leq Q \sup_{g\in B(L^q(m))}\left(\sum_{i=1}^{n}\left\|\int_\Omega f_i g dm\right\|_X^r\right)^{\frac{1}{r}}. \qquad (4.13)$$

(ii) There is a positive constant Q so that for every $m-$compact set $K\subset B(L^q(m))$ and for each subspace $S\in\mathcal{S}$ of $L^p(m)$, that is $r-\varepsilon-$normed by K for some $0<\varepsilon<\frac{1}{2}$, there is a probability measure $\delta_{K,\varepsilon}$ defined on the Borel subsets of K such that the following holds for every $f\in S$

$$\|T(f)\|_Y^r \leq \frac{Q^r}{1-\varepsilon}\int_K\left\|\int_\Omega f g dm\right\|_X^r d\delta_{K,\varepsilon}(g). \qquad (4.14)$$

Proof. We begin by proving $(i)\Rightarrow(ii)$. We fix an $m-$compact set K in $B(L^q(m))$ and $0<\varepsilon<\frac{1}{2}$, now let S be a subspace belonging to a dense family $\mathcal{S}$ of subspaces of $L^p(m)$ and $r-\varepsilon-$normed by K. We will apply Ky Fan's Lemma 3.1.5 to obtain a probability measure so that (4.14) holds in S, we will prove that the constant Q is given by the $m-r-$summability of the operator T and that it is independent of K.

We define a family of functions Ψ defined on the space of probability measures over the Borel subsets of K, $\mathcal{P}(K)$. Notice that $\mathcal{P}(K)$ is a subset of the unit ball of the Radon measures over K, denoted by $B(\mathcal{M}(K))$. Since $B(\mathcal{M}(K))=B(\mathcal{C}(K)^*)$, $\mathcal{P}(K)$ is w^*-compact and convex. Each finite family $\{f_1,\dots,f_n\}$ in S defines a function of Ψ in the following way: $\phi_{f_1,\dots,f_n}:\mathcal{P}(K)\to\mathbb{R}$ is given by

$$\phi_{f_1,\dots,f_n}(\eta):=\sum_{i=1}^{n}\|T(f_i)\|_Y^r-\frac{Q^r}{1-\varepsilon}\int_K\sum_{i=1}^{n}\left\|\int_\Omega f_i g dm\right\|_X^r d\eta(g).$$

We will show that the family $\Psi=\{\phi_{f_1,\dots,f_n}: f_1,\dots,f_n\in S\}$ satisfies the requirements of Ky Fan's Lemma. For every $f\in S$ the function $\psi_f: g\mapsto \|\int fgdm\|$ is continuous, therefore $\phi_{f_1,\dots,f_n}$ is lower semi continuous in the weak*-topology of $\mathcal{P}(K)$. Each $\phi_{f_1,\dots,f_n}$ is clearly convex, and the convex combination of two functions of Ψ stays in Ψ.

Since K is compact and the function $g\mapsto\sum_{i=1}^n\|\int_\Omega f_i g dm\|_X^r$ is continuous, there is some $g_0\in K$ so that

$$\sup_{g\in K}\sum_{i=1}^{n}\left\|\int_\Omega f_i g dm\right\|_X^r=\sum_{i=1}^{n}\left\|\int_\Omega f_i g_0 dm\right\|_X^r,$$

therefore, for $\delta_{g_0} \in \mathcal{P}(K)$ the Dirac measure associated to g_0 we have, as a consequence of (4.12) and (4.13)

$$\begin{aligned}
\phi_{f_1,\ldots,f_n}(\delta_{g_0}) &= \sum_{i=1}^{n} \|T(f_i)\|_Y^r - \frac{Q^r}{1-\varepsilon}\int_K \sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r d\delta_{g_0}(g) \\
&= \sum_{i=1}^{n} \|T(f_i)\|_Y^r - \frac{Q^r}{1-\varepsilon} \sup_{g\in K} \sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r \\
&\leq \sum_{i=1}^{n} \|T(f_i)\|_Y^r - Q^r \sup_{g\in B(L^q(m))} \sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r \leq 0.
\end{aligned}$$

An application of Ky Fan's lemma gives a probability measure $\eta_{K,\varepsilon}$ so that $\phi_{f_1,\ldots,f_n}(\eta_{K,\varepsilon}) \leq 0$ for every finite choice $f_1,\ldots,f_n$ in S, therefore (4.14) holds.

In order to prove $(ii) \Rightarrow (i)$, let $\mathcal{S}$ be a dense family of subspaces of $L^p(m)$. We will show that all the restrictions of T to the subspaces of the family $\mathcal{S}$ are uniformly $m-r-$summing. Take a subspace S in $\mathcal{S}$ and $0 < \varepsilon < \frac{1}{2}$. Property $(r-\star)$ of $L^p(m)$ ensures the existence of an $m-$compact set $K_{S,\varepsilon}$ that is $r-\varepsilon-$norming for S. Take now a finite sequence of functions $f_1,\ldots,f_n$ in S, by (ii) there is a probability measure $\delta_{K_{S,\varepsilon}}$

$$\begin{aligned}
\left(\sum_{i=1}^{n} \|T(f_i)\|_Y^r\right)^{\frac{1}{r}} &\leq \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}} \left(\sum_{i=1}^{n} \int_{K_{S,\varepsilon}} \left\| \int_\Omega f_i g dm \right\|^r d\delta_{K_{S,\varepsilon}}(g)\right)^{\frac{1}{r}} \\
&= \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}} \left(\int_{K_{S,\varepsilon}} \sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|^r d\delta_{K_{S,\varepsilon}}(g)\right)^{\frac{1}{r}} \\
&\leq \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}} \left(\sup_{g\in B(L^q(m))} \sum_{i=1}^{n} \left\| \int f_i g dm \right\|_X^r\right)^{\frac{1}{r}},
\end{aligned}$$

where the constant Q is clearly uniform for every subspace S. Since this holds for every $\varepsilon > 0$ we obtain that T is uniformly $m-r-$summing. An appel to Lemma 4.3.3 gives us the conclusion.

□

The following result corresponds to the Factorization Theorem for $m-r-$summing operators when property $(r-\star)$ is assumed for the space $L^p(m)$.

Theorem 4.3.5. *Let $T \in \mathcal{L}(L^p(m), Y)$ with Y Banach space and $L^p(m)$ with property $(r-\star)$ over a dense family of subspaces $\mathcal{S}$. For $1 \leq r < \infty$ the following assertions are equivalent,* *(i)* $T \in \Pi_r^m(L^p(m), Y)$,

(ii) *There is a constant* $Q > 0$ *so that for every* $K \subset B(L^q(m))$ *that is compact with respect to the* $m-$*topology and for every subspace of* $L^p(m)$, $S \in \mathcal{S}$, *that is* $r-\varepsilon-$*normed by* K *for some* $0 < \varepsilon < 1$, *there exists a probability measure* $\delta_{K,\varepsilon}$ *defined on the Borel subsets of* K *so that*

$$\begin{array}{ccccc} & S & \xrightarrow{T|_S} & Y & \\ & {\scriptstyle i_{K,\varepsilon}}\downarrow & & \uparrow{\scriptstyle R_{K,\varepsilon}} & \\ \mathcal{C}(K,X) \supset & E & \xrightarrow{j_{K,\varepsilon}} & \overline{j_{K,\varepsilon}(E)} & \subset L^r(K, \delta_{K,\varepsilon}, X) \end{array}$$

where $(1-\varepsilon)\|f\|_{L^p(m)} \leq \|i_{K,\varepsilon}(f)\|_{\mathcal{C}(K,X)} \leq \|f\|_{L^p(m)}$ *for every* $f \in S$ *and* $\|R_{K,\varepsilon}\| \leq Q$ *uniformly in* K, S *and* ε.

Proof. The domination property given in (4.14) directly implies the factorization. In fact the map $i_{K,\varepsilon} : S \to \mathcal{C}(K,X)$ given by $i_{K,\varepsilon}(f) = \int_\Omega f(\cdot)dm$ is continuous and satisfies the inequalities

$$(1-\varepsilon)\|f\|_{L^p(m)} \leq \|i_{K,\varepsilon}(f)\|_{\mathcal{C}(K,X)} \leq \|f\|_{L^p(m)}.$$

The map $j_{K,\varepsilon}$ corresponds to the natural inclusion of the space $\mathcal{C}(K,X)$ into the space of Bochner $\delta_{K,\varepsilon}$-integrable functions with values in X, $L^r(K, \delta_{K,\varepsilon}, X)$. Take $R_{K,\varepsilon}(f) = T(f)$ for every $f \in j_{K,\varepsilon}(E)$. $R_{K,\varepsilon}$ is well defined and continuous by the definition of $i_{K,\varepsilon}$ and $j_{K,\varepsilon}$, and by the boundedness property for $T|_S$ we get

$$\|T|_S(f)\|_Y^r \leq \frac{Q^r}{1-\varepsilon} \int_K \left\| \int_\Omega fg dm \right\|_X^r d\delta_{K,\varepsilon}(g).$$

□

We finish this section with a generalized version of Theorem 4.3.4. Here no conditions on the space $L^p(m)$ are needed, the theorem holds even for finite sequences or for finite dimensional subspaces of $L^p(m)$. We begin with the description of the construction.

1. Take a family $\{A_i\}_{i=1}^n$ of subsets of $L^p(m)$, so that each set A_i is *positively balanced*, that is, for every $\theta \in [0,1]$, we have $\theta A_i \subset A_i$, for each $i = 1, \ldots, n$.

2. We define $\mathcal{A}$ to be the family of $n-$tuples

$$\mathcal{A} := \left\{ \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} : f_i \in A_i, i = 1, \ldots, n \right\} \tag{4.15}$$

3. Associated to $\mathcal{A}$, let $\Lambda_{\mathcal{A}}$ be the set of formal matrices α defined in the following way,

$$\alpha := \begin{pmatrix} f_1^1 \lambda_1^{\frac{1}{r}} & f_1^2 \lambda_2^{\frac{1}{r}} & \ldots & f_1^m \lambda_m^{\frac{1}{r}} \\ \vdots & \vdots & & \vdots \\ f_n^1 \lambda_1^{\frac{1}{r}} & f_n^2 \lambda_2^{\frac{1}{r}} & \ldots & f_n^m \lambda_m^{\frac{1}{r}} \end{pmatrix} \tag{4.16}$$

for $m \in \mathbb{N}$, where $f_i^j \in A_i$, $i = 1, \ldots, n$, $\lambda_j \in [0, 1]$ and $\sum_{j=1}^m \lambda_j = 1$.

For $0 \leq \varepsilon \leq 1$, we say that $\mathcal{A}$ is $m - r - \varepsilon-$*normed* by an $m-$compact set $K \subset B(L^q(m))$ if

$$\sup_{g \in B(L^q(m))} \sum_{i=1}^n \sum_{j=1}^m \left\| \int_\Omega \lambda_j^{\frac{1}{r}} f_i^j g dm \right\|_X^r \leq \varepsilon^r + (1+\varepsilon)^r \sup_{g \in K} \sum_{i=1}^n \sum_{j=1}^m \left\| \int_\Omega \lambda_j^{\frac{1}{r}} f_i^j g dm \right\|_X^r$$

for every $\alpha \in \Lambda_{\mathcal{A}}$ defined as in (4.16).

Theorem 4.3.6. *Let $T : L^p(m) \to Y$ be a linear and continuous operator, the following assertions are equivalent.*

(i) T is $m - r-$summing,

(ii) there is $Q > 0$ so that for every $0 < \varepsilon < 1$ and each family $\mathcal{A}$ as in (4.15), there is an $m-$compact subset $K \subset B(L^q(m))$ that is $m - r - \varepsilon-$norming for $\mathcal{A}$, and there exists a probability measure η_K defined on the Borel subsets of K such that

$$\|T(f)\|_Y^r \leq Q^r \varepsilon^r + (1+\varepsilon)^r Q^r \int_K \left\| \int_\Omega f g dm \right\|_X^r d\eta_K(g). \tag{4.17}$$

for every $f \in \bigcup_{i=1}^n A_i$.

Proof. We begin by proving $(i) \Rightarrow (ii)$. We fix $0 < \varepsilon < 1$, and a family $\mathcal{A}$ of $n-$tuples of positively balanced subsets of $\{A_1, \ldots, A_n\}$ of $L^p(m)$. First we must find an $m-$compact $K \subset B(L^q(m))$ that is an $m - r - \varepsilon-$norming subset for $\mathcal{A}$. We define in the $n-$cube $[0, 1]^n \subset \mathbb{R}^n$ a pseudo-distance (a distance without the separation condition, that is, the pseudo-distance of two different points can be zero) as follows

$$\bar{d}((\lambda_1,\ldots,\lambda_n),(\lambda'_1,\ldots,\lambda'_n)) := \sup_{g\in B(L^q(m))}\sum_{i=1}^n |\lambda_i-\lambda'_i| \left\|\int_\Omega f_i g dm\right\|_X^r .$$

The topology induced by the usual distance d in $[0,1]^n$ is finer than the one induced by the pseudo-distance $\bar{d}$, $\tau_{\bar{d}}$. Then $[0,1]^n$ is compact when endowed with the topology $\tau_{\bar{d}}$. Let $(\lambda_1,\ldots,\lambda_n)\in[0,1]^n$ and $0<\varepsilon<1$, there is $(\tau_1,\ldots,\tau_n)\in[0,1]^n$ so that $d((\lambda_i)_{i=1}^n-(\tau_i)_{i=1}^n)<\varepsilon^r$.We can choose $\tau_i\le\lambda_i$ for every $i=1,\ldots,n$. We get

$$\sup_{g\in B(L^q(m))}\sum_{i=1}^n \left\|\int_\Omega \lambda_i^{\frac{1}{r}} f_i g dm\right\|_X^r = \sup_{g\in B(L^q(m))}\sum_{i=1}^n \lambda_i \left\|\int_\Omega f_i g dm\right\|_X^r$$

$$\begin{aligned}
&\le \sup_{g\in B(L^q(m))}\sum_{i=1}^n(\lambda_i-\tau_i) \left\|\int_\Omega f_i g dm\right\|_X^r + \sup_{g\in B(L^q(m))}\sum_{i=1}^n \tau_i \left\|\int_\Omega f_i g dm\right\|_X^r \\
&\le \varepsilon^r + \sup_{g\in B(L^q(m))}\sum_{i=1}^n \lambda_i \left\|\int_\Omega f_i g dm\right\|_X^r \\
&\le \varepsilon^r + (1+\varepsilon)^r\left(\sum_{i=1}^n \lambda_i \left\|\int_\Omega f_i g_0 dm\right\|_X^r\right)
\end{aligned}$$

where $g_0\in B(L^q(m))$. Since $[0,1]^n$ is compact, a finite number of functions $g_0\in L^q(m)$ is enough to obtain inequalities above for every $(\lambda_i)_{i=1}^n\in[0,1]^n$. This finite set of $q-$integrable functions is the compact set we search for.

We apply Ky Fan's Lemma in order to obtain a probability measure so that (4.17) holds in $\bigcup_{i=1}^n A_i$. We define a family $\Psi=\{\Phi_\alpha:\alpha\in\Lambda_{\mathcal{A}}\}$, where each Φ_α is defined as follows over a probability measure $\eta\in\mathcal{P}(K)$:

$$\begin{aligned}
\Phi_\alpha(\eta) &:= \sum_{i=1}^n\sum_{j=1}^m \|T(\lambda_j^{\frac{1}{r}} f_i^j)\|_Y^r - Q^r\varepsilon^r \\
&- Q^r(1+\varepsilon)^r\sum_{i=1}^n\sum_{j=1}^m\int_K \left\|\int_\Omega \lambda_j^{\frac{1}{r}} f_i^j g dm\right\|_X^r d\eta(g).
\end{aligned}$$

We will show that the family Ψ satisfies the requirements of Ky Fan's Lemma. For every $f\in\bigcup_{i=1}^n A_i$, the function $\psi_f: g\mapsto \|\int_\Omega fgdm\|$ is continuous, therefore Φ_α is lower semi continuous in the w^*-topology of $\mathcal{P}(K)$, each Φ_α is clearly convex. In what follows we show that the convex combination of two functions in Ψ stays in Ψ, for this aim, take $0 \leq \theta \leq 1$, α and β in $\Lambda_{\mathcal{A}}$ as follows

$$\alpha := \begin{pmatrix} f_1^1 \lambda_1^{\frac{1}{r}} & \dots & f_1^m \lambda_m^{\frac{1}{r}} \\ \vdots & & \vdots \\ f_n^1 \lambda_1^{\frac{1}{r}} & \dots & f_n^m \lambda_m^{\frac{1}{r}} \end{pmatrix}, \quad \beta := \begin{pmatrix} \widetilde{f_1^1} \widetilde{\lambda_1}^{\frac{1}{r}} & \dots & \widetilde{f_1^{\bar{m}}} \widetilde{\lambda_{\bar{m}}}^{\frac{1}{r}} \\ \vdots & & \vdots \\ \widetilde{f_n^1} \widetilde{\lambda_1}^{\frac{1}{r}} & \dots & \widetilde{f_n^{\bar{m}}} \widetilde{\lambda_{\bar{m}}}^{\frac{1}{r}} \end{pmatrix}.$$

Take $\eta \in \mathcal{P}(K)$, we have that

$$\theta \Phi_\alpha(\eta) + (1-\theta)\Phi_\beta(\eta) =$$

$$\sum_{i=1}^n \left(\sum_{j=1}^m \left\| T((\theta\lambda_j)^{\frac{1}{r}} f_i^j) \right\|^r + \sum_{j=1}^{\bar{m}} \left\| T((\theta\widetilde{\lambda_j})^{\frac{1}{r}} \widetilde{f_i^j}) \right\|^r \right) - Q^r \varepsilon^r$$

$$-Q^r(1+\varepsilon)^r \left[\int_K \sum_{i=1}^n \left(\left(\sum_{j=1}^m \left\| \int_\Omega f_i^j (\theta\lambda_j)^{\frac{1}{r}} g dm \right\| \right) + \left(\sum_{j=1}^{\bar{m}} \left\| \int_\Omega \widetilde{f_i^j} (\theta\widetilde{\lambda_j})^{\frac{1}{r}} g dm \right\| \right) \right) d\eta(g) \right].$$

That is $\theta\Phi_\alpha(\eta) + (1-\theta)\Phi_\beta(\eta) = \Phi_\lambda(\eta)$ for λ given by:

$$\begin{pmatrix} f_1^1(\theta\lambda_1)^{\frac{1}{r}} & \dots & f_1^m(\theta\lambda_m)^{\frac{1}{r}} & \widetilde{f_1^1}((1-\theta)\widetilde{\lambda_1})^{\frac{1}{r}} & \dots & \widetilde{f_1^{\bar{m}}}((1-\theta)\widetilde{\lambda_{\bar{m}}})^{\frac{1}{r}} \\ \vdots & & \vdots & \vdots & & \vdots \\ f_n^1(\theta\lambda_1)^{\frac{1}{r}} & \dots & f_n^m(\theta\lambda_m)^{\frac{1}{r}} & \widetilde{f_n^1}((1-\theta)\widetilde{\lambda_1})^{\frac{1}{r}} & \dots & \widetilde{f_n^{\bar{m}}}((1-\theta)\widetilde{\lambda_{\bar{m}}})^{\frac{1}{r}} \end{pmatrix}$$

clearly, $\lambda \in \Lambda_{\mathcal{A}}$, therefore Ψ is a concave family of functions. We must find, for each $\alpha \in \Lambda_{\mathcal{A}}$, a probability measure η_α so that $\Phi_\alpha(\eta_\alpha) \leq 0$.

Since K is compact, for $\alpha \in \Lambda_{\mathcal{A}}$ there is some $g_\alpha \in K$ so that

$$\sup_{g\in K} \sum_{i=1}^n \sum_{j=1}^m \left\| \int_\Omega \lambda_j^{\frac{1}{r}} f_i^j g dm \right\|_X^r = \sum_{i=1}^n \sum_{j=1}^m \left\| \int_\Omega \lambda_j^{\frac{1}{r}} f_i^j g_\alpha dm \right\|_X^r \tag{4.18}$$

Let δ_{g_α} the Dirac measure associated to g_α, inequality (4.18) and $m-r-$summability of T implies that $\Phi_\alpha(\delta_{g_\alpha}) \leq 0$. Direct application of Ky Fan's Lemma ensures that there is a probability measure defined on K so that (4.17) holds.

For the proof of $(ii) \Rightarrow (i)$ take a finite quantity of functions $f_1, \dots, f_n$ in $L^p(m)$. Fix $\varepsilon > 0$ and take $A_i = \{\lambda f_i : 0 \leq \lambda \leq 1\}$. A direct application of (ii) to the family $\mathcal{A} = [A_1, \dots, A_n]$ implies the existence of an $m-$compact set and a probability measure η_K so that

$$\|T(f_i)\|_Y^r \leq Q^r \varepsilon^r + (1+\varepsilon)^r Q^r \int_K \left\| \int f_i g dm \right\|_X^r d\eta_K(g),$$

therefore

$$\sum_{i=1}^{n} \|T(f_i)\|_Y^r \leq Q^r \varepsilon^r n + (1+\varepsilon)^r Q^r \sup_{g \in B(L^q(m))} \sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r,$$

since this happens for every $\varepsilon > 0$, we get that T is $m-r-$summing. □

4.3.2. Consequences

The last part of this section is devoted to the study of the relationship between different spaces of summing operators defined on spaces of $p-$integrable functions with respect to a vector measure. The first result is the adaptation of Theorem 4.1.2, we prove that once we know that an operator $T \in \mathcal{L}(L^p(m), Y)$ is $m-1-$summing we can conclude that it is $m-r-$summing for every $1 < r < \infty$.

Proposition 4.3.7. *If $1 \leq r < s < \infty$, we have for every Banach space Y and each vector measure $m : \Sigma \to X$, $\Pi_r^m(L^p(m), Y) \subset \Pi_s^m(L^p(m), Y)$. Moreover, for each operator $T \in \Pi_r^m(L^p(m), Y)$ we have $\pi_s^m(T) \leq \pi_r^m(T)$.*

Proof. Let $T : L^p(m) \to Y$ be an $m-r-$summing operator. Take $f_1, \ldots, f_n \in L^p(m)$. Define, for $i = 1, \ldots, n$, $\lambda_i = \|T(f_i)\|_Y^{\frac{s}{r}-1}$, clearly, for each i we have $\|T(f_i)\|^s = \|T(\lambda_i f_i)\|^r$. Since T is an $m-r-$summing operator, we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \|T(f_i)\|_Y^s \right)^{\frac{1}{r}} &= \left(\sum_{i=1}^{n} \|T(\lambda_i f_i)\|_Y^r \right)^{\frac{1}{r}} \\ &\leq \pi_r^m(T) \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega \lambda_i f_i g \right\|_X^r \right)^{\frac{1}{r}}. \end{aligned}$$

Since $s > r$, then $\frac{s}{r}$ and $\frac{s}{s-r}$ are conjugated indexes, therefore applying Hölder's inequality we get

$$\left(\sum_{i=1}^{n} \|T(f_i)\|_Y^s \right)^{\frac{1}{r}} \leq \pi_r^m(T) \left(\sum_{i=1}^{n} \lambda_i^{\frac{sr}{s-r}} \right)^{\frac{s-r}{sr}} \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^s \right)^{\frac{1}{s}}$$

$$= \pi_r^m(T) \left(\sum_{i=1}^{n} \|T(f_i)\|_Y^s \right)^{\frac{1}{r}-\frac{1}{s}} \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^s \right)^{\frac{1}{s}},$$

thus rearranging we get

$$\left(\sum_{i=1}^{n} \|T(f_i)\|_Y^s \right)^{\frac{1}{s}} \leq \pi_r^m(T) \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^s \right)^{\frac{1}{s}}.$$

Then T is $m-s-$summing with norm $\pi_s^m(T) \leq \pi_r^m(T)$. □

In the classical theory of summing operators there are several results concerning the coincidence of the spaces Π_r for different values of r. For example a result due to Maurey (see [50]) ensures that when the Banach space Y has cotype 2, then for each Banach space X, $\Pi_r(X,Y) = \Pi_2(X,Y)$, for every $2 < r < \infty$. In the following we recall some definitions about geometrical properties of general Banach spaces.

Definition 4.3.8. A Banach space X has *type* p, for $1 \le p \le 2$ whenever there is a constant $C \ge 0$ so that for every finite choice of elements $x_1, \dots, x_n \in X$

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_X dt \le C \left(\sum_{i=1}^n \|x_i\|_X^p \right)^{\frac{1}{p}}, \tag{4.19}$$

holds, where r_i are the *Rademacher functions*, defined by

$$r_i(t) = sign(\sin(2^i \pi t)),$$

for $i = 1, \dots, n$ and $0 \le t \le 1$. The infimum of the constant C so that (4.19) holds for every finite choice of elements in X is denoted by $T_p(X)$.

A Banach space X has *cotype* q, with $2 \le q < \infty$ whenever there is $C \ge 0$ so that

$$\left(\sum_{i=1}^n \|x_i\|_X^q \right)^{\frac{1}{q}} \le C \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_X dt, \tag{4.20}$$

for every $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$. The infimum of C so that 4.20 holds is denoted by $C_q(X)$.

Some types and cotypes only have sense for trivial spaces, as a consequence od Kinchin's inequality (see [27, 1.10]) only $X = \{0\}$ can have type > 2 and cotype < 2. For X a Banach space, and $x_1, \dots, x_n \in X$ and $0 < p \le 1$ we have

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|^2 dt \right)^{\frac{1}{2}} \le \sum_{i=1}^n \|x_i\| \le \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}.$$

Thus, every Banach space has type p for every $0 < p \le 1$. Also, every Banach space has cotype ∞, see for instance [27, Remark 11.5, (d)].

The following proposition extends the result of Maurey for $m - r-$ summing operators. The proof uses *Kahane's Inequality* (see [27], 11.1). If $0 < r < s < \infty$, then there is a constant $K_{r,s} > 0$ so that regardless the choice of a Banach space X and of finitely many vectors $x_1, \dots, x_n \in X$ the following inequality holds

$$\left(\int_0^1 \left\|\sum_{k\leq n} r_k(t)x_k\right\|_X^s dt\right)^{\frac{1}{s}} \leq K_{r,s}\cdot\left(\int_0^1 \left\|\sum_{k\leq n} r_k(t)x_k\right\|_X^r dt\right)^{\frac{1}{r}}. \tag{4.21}$$

Proposition 4.3.9. *Let $m : \Sigma \to X$ a vector measure with X a Banach space with type 2 and $L^p(m)$ with property $(r-\star)$. For each Banach space Y with cotype 2 and each $2 < r < \infty$, we have $\Pi_r^m(L^p(m), Y) = \Pi_2^m(L^p(m), Y)$.*

Proof. Since $2 < r$, Theorem 4.3.7 yields $\Pi_2^m(L^p(m), Y) \subset \Pi_r^m(L^p(m), Y)$. Let $T \in \Pi_r^m(L^p(m), Y)$. In order to prove that $T \in \Pi_2^m(L^p(m), Y)$, take a finite number of functions $f_1, \dots, f_n \in L^p(m)$. Let $\mathcal{S} := \{S_i : i \in I\}$ a dense family of subspaces of $L^p(m)$, therefore for $\bar{\varepsilon} > 0$, there is $i_0 \in I$ and $f_1^0, \dots, f_n^0 \in S_{i_0}$ such that $\|f_i - f_i^0\|_{L^p(m)} \leq \bar{\varepsilon}/(n\|T\|C_2(Y))$ for $i = 1, \dots, n$. Since $L^p(m)$ has property $(r-\star)$, for $0 < \varepsilon < 1/2$, there is a $m-$compact subset K in $B(L^q(m))$ that is $r-\varepsilon-$norming for S_{i_0}.

Since T is $m-r-$summing, by (ii) in Theorem 4.3.4, for each $0 < \varepsilon < 1/2$, there is $Q > 0$ and a probability measure $\delta_{K,\varepsilon}$ defined on the Borel subsets of K so that

$$\|T(f)\|_Y^r \leq \frac{Q^r}{1-\varepsilon}\int_K \left\|\int_\Omega fg\,dm\right\|_X^r d\delta_{K,\varepsilon}(g), \quad f \in S_{i_0}. \tag{4.22}$$

Therefore

$$\left(\sum_{i=1}^n \|T(f_i)\|_Y^2\right)^{\frac{1}{2}} \leq C_2(Y)\int_0^1 \left\|\sum_{i=1}^n r_i(t)T(f_i)\right\|_Y dt \tag{4.23}$$

$$
\begin{aligned}
&\leq C_2(Y)\left(\int_0^1\left\|\sum_{i=1}^n r_i(t)T(f_i-f_i^0)+\sum_{i=1}^n r_i(t)T(f_i^0)\right\|_Y^r\right)^{\frac{1}{r}}\\
&\leq C_2(Y)\left(\int_0^1\left\|\sum_{i=1}^n r_i(t)T(f_i-f_i^0)\right\|_Y^r dt\right)^{\frac{1}{r}} \qquad (4.24)\\
&+ C_2(Y)\left(\int_0^1\left\|\sum_{i=1}^n r_i(t)T(f_i^0)\right\|_Y^r dt\right)^{\frac{1}{r}}\\
&\leq C_2(Y)\|T\|\sum_{i=1}^n\|f_i-f_i^0\|_{L^p(m)}+C_2(Y)\left(\int_0^1\left\|\sum_{i=1}^n r_i(t)T(f_i^0)\right\|_Y^r dt\right)^{\frac{1}{r}}\\
&\leq \bar{\varepsilon}+C_2(Y)\left(\int_0^1\frac{Q^r}{1-\varepsilon}\int_K\left\|\int_\Omega\left(\sum_{i=1}^n r_i(t)f_i^0\right)gdm\right\|_X^r d\delta_{K,\varepsilon}(g)dt\right)^{\frac{1}{r}} \qquad (4.25)\\
&\leq \bar{\varepsilon}+C_2(Y)\frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}\sup_{g\in K}\left(\int_0^1\left\|\sum_{i=1}^n r_i(t)\int f_i^0 gdm\right\|_X^r dt\right)^{\frac{1}{r}}\\
&\leq \bar{\varepsilon}+C_2(Y)\frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}K_{1,r}\sup_{g\in K}\left(\int_0^1\left\|\sum_{i=1}^n r_i(t)\int_\Omega f_i^0 gdm\right\|_X dt\right) \qquad (4.26)\\
&\leq \bar{\varepsilon}+C_2(Y)\frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}K_{1,r}T_2(X)\sup_{g\in K}\left(\sum_{i=1}^n\left\|\int_\Omega f_i^0 gdm\right\|_X^2\right)^{\frac{1}{2}} \qquad (4.27)\\
&\leq \bar{\varepsilon}+C_2(Y)\frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}K_{1,r}T_2(X)\sup_{g\in B(L^q(m))}\left(\sum_{i=1}^n\left\|\int_\Omega f_i^0 gdm\right\|_X^2\right)^{\frac{1}{2}},
\end{aligned}
$$

where in (4.23) and in (4.27) we use the cotype 2 of Y and the type 2 of X respectively. Since $r\geq 1$ inequality (4.24) yields from inequality (2.7). The inequality (4.25) is a consequence of (4.22) and (4.26) yields from of Kahane's inequality. Since we have

$$
\begin{aligned}
&\sup_{g\in B(L^q(m))}\left(\sum_{i=1}^n\left\|\int_\Omega f_i^0 gdm\right\|_X^2\right)^{\frac{1}{2}}\leq\\
&\leq \sup_{g\in B(L^q(m))}\left\{\left(\sum_{i=1}^n\left\|\int_\Omega (f_i^0-f_i)gdm\right\|_X^2\right)^{\frac{1}{2}}+\left(\sum_{i=1}^n\left\|\int_\Omega f_i gdm\right\|_X^2\right)^{\frac{1}{2}}\right\}\\
&\leq \frac{\bar{\varepsilon}}{n^{\frac{1}{2}}\|T\|C_2(Y)}+\sup_{g\in B(L^q(m))}\left(\sum_{i=1}^n\left\|\int_\Omega f_i gdm\right\|_X^2\right)^{\frac{1}{2}}
\end{aligned}
$$

the conclusion follows. □

In the following we extend the Extrapolation Theorem (see [27], 3.17) for the particular case of $m-r-$summing operators. It is a consequence of the Theorem 4.3.4.

Theorem 4.3.10. *Let $1<r<s<\infty$, $m:\Sigma\to X$ a vector measure and $1<p<\infty$ so that $L^p(m)$ has condition $(r-\star)$. If for every probability measure μ there is a uniform constant C so that for every operator $T:L^p(m)\to L^s(\mu,X)$,*

$$\pi_r^m(T)\leq C\pi_s^m(T), \tag{4.28}$$

then for every Banach space Y,

$$\Pi_s^m(L^p(m),Y)=\Pi_1^m(L^p(m),Y). \tag{4.29}$$

Proof. Let Y be a Banach space and $T:L^p(m)\to Y$ an $m-s-$summing operator, we have to prove that T is $m-1-$summing. For this aim we will apply Theorem 4.3.4. Since $L^p(m)$ has condition $(r-\star)$, there is a dense family of subspaces $\mathcal{S}:=\{S_i:i\in I\}$ such that for each $S\in\mathcal{S}$ there is an $m-$compact set $K\subset B(L^q(m))$ that is $r-\varepsilon-$norming for S for some $0<\varepsilon<1/2$. Theorem 4.3.4 ensures the existence of a probability measure δ_1 (depending on K and ε) defined on the Borel subsets of K so that

$$\|T(f)\|_Y\leq\frac{\pi_s^m}{(1-\varepsilon)^{\frac{1}{s}}}(T)\left(\int_K\left\|\int_\Omega fgdm\right\|_X^s d\delta_1(g)\right)^{\frac{1}{s}} \tag{4.30}$$

for every $f\in S$.

We consider the operator $T_{\delta_1}:L^p(m)\to L^s(\delta_1,X)$ given by $T_{\delta_1}(f)(g)=\int_\Omega fgdm$ for each $f\in L^p(m)$ and each $g\in K$. T_{δ_1} is well defined, indeed, for $f\in L^p(m)$

$$\begin{aligned}\|T_{\delta_1}(f)\|_{L^s(\delta_1,X)} &= \left(\int_K\|T_{\delta_1}(f)(g)\|_X^s d\delta_1(g)\right)^{\frac{1}{s}}\\ &= \left(\int_K\left\|\int_\Omega fgdm\right\|_X^s d\delta_1(g)\right)^{\frac{1}{s}}\\ &\leq \|f\|_{L^p(m)}.\end{aligned}$$

Moreover since for every finite choice $f_1,\dots,f_n\in L^p(m)$ we have

$$\begin{aligned}\sum_{i=1}^n\|T_{\delta_1}(f_i)\|_{L^s(\delta_1,X)}^s &= \sum_{i=1}^n\int_K\left\|\int_\Omega f_igdm\right\|_X^s d\delta_1(g)\\ &\leq \sup_{g\in B(L^q(m))}\sum_{i=1}^n\left\|\int_\Omega f_igdm\right\|_X^s,\end{aligned}$$

then $T_{\delta_1} \in \Pi_s^m(L^p(m), Y)$ and by (4.28), T_{δ_1} is also $m-r-$summing with $\pi_r^m(T_{\delta_1}) \leq C$.

We apply again Theorem 4.3.4 to the $m-r-$summing operator T_{δ_1}. For the $m-$compact subset K of $B(L^q(m))$ and the subspace S of $L^p(m)$ that is $r-\varepsilon-$normed by K, we obtain a probability measure δ_2 so that

$$\|T_{\delta_1}(f)\|_{L^s(\delta_1,X)} \leq \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \left(\int_K \left\| \int fgdm \right\|_X^r d\delta_2(g) \right)^{\frac{1}{r}}$$

for every $f \in S$. As previously we can consider the operator associated to the measure δ_2 as follows. $T_{\delta_2} : L^p(m) \to L^r(\delta_2, X)$, with $T_{\delta_2}(f)(g) := \int_\Omega fgdm$ for $f \in L^p(m)$ and $g \in K$. Clearly, for each $f \in S$, $\|T_{\delta_1}(f)\|_{L^s(\delta_1,X)} \leq \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \|T_{\delta_2}(f)\|_{L^r(\delta_2,X)}$. Proceeding in this way, we obtain a sequence of probability measures (δ_n) defined on the $\sigma-$algebra of subsets of K and so that

$$\|T_{\delta_n}(f)\|_{L^s(\delta_n,X)} \leq \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \|T_{\delta_{n+1}}(f)\|_{L^r(\delta_{n+1},X)}, \quad f \in S. \tag{4.31}$$

Since $1 < r < s < \infty$, there is some $\alpha \in]0,1[$ such that $\frac{1}{r} = \alpha + \frac{1-\alpha}{s}$. A direct application of Hölder's inequality yields for $n \in \mathbb{N}$ and $f \in S$,

$$\|T_{\delta_n}(f)\|_{L^r(\delta_n,X)} \leq \|T_{\delta_n}(f)\|_{L^1(\delta_n,X)}^\alpha \cdot \|T_{\delta_n}(f)\|_{L^s(\delta_n,X)}^{1-\alpha}. \tag{4.32}$$

An application of (4.31), (4.32) and Hölder's inequality for series gives, for each $f \in S$,

$$\begin{aligned}
\sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_n}(f)\|_{L^s(\delta_n,X)} &\leq \sum_{n=1}^\infty \frac{1}{2^n} \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \|T_{\delta_{n+1}}(f)\|_{L^r(\delta_{n+1},X)} \\
&\leq \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_{n+1}}(f)\|_{L^1(\delta_{n+1},X)}^\alpha \|T_{\delta_{n+1}}(f)\|_{L^s(\delta_{n+1},X)}^{1-\alpha} \\
&\leq \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \left(\sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_{n+1}}(f)\|_{L^1(\delta_{n+1},X)} \right)^\alpha \left(\sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_{n+1}}(f)\|_{L^s(\delta_{n+1},X)} \right)^{1-\alpha} \\
&\leq \frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \left(\sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_{n+1}}(f)\|_{L^1(\delta_{n+1},X)} \right)^\alpha \left(2 \sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_n}(f)\|_{L^s(\delta_n,X)} \right)^{1-\alpha}.
\end{aligned}$$

A direct computation yields for $f \in S$ and $n \in \mathbb{N}$

$$\sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_n}(f)\|_{L^s(\delta_n,X)} \leq \left(\frac{C}{(1-\varepsilon)^{\frac{1}{r}}} \right)^{\frac{1}{\alpha}} 2^{\frac{1-\alpha}{\alpha}} \sum_{n=1}^\infty \frac{1}{2^n} \|T_{\delta_{n+1}}(f)\|_{L^1(\delta_{n+1},X)}. \tag{4.33}$$

Then, we can define a probability measure λ over the Borel subsets of K as $\lambda := \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_n$. Therefore for every $f \in S$ we get

$$\begin{aligned} \frac{1}{2}\|T_{\delta_1}(f)\|_{L^s(\delta_1,X)} &\leq \left(2\frac{C}{(1-\varepsilon)^{\frac{1}{r}}}\right)^{\frac{1}{\alpha}} \sum_{n=1}^{\infty} \frac{1}{2^n}\|T_{\delta_{n+1}}(f)\|_{L^1(\delta_{n+1},X)} \\ &= \left(2\frac{C}{(1-\varepsilon)^{\frac{1}{r}}}\right)^{\frac{1}{\alpha}} \|T_\lambda(f)\|_{L^1(\lambda,X)} \end{aligned}$$

where $T_\lambda : L^p(m) \to L^1(\lambda, X)$ is defined in the natural way. Thus, by (4.30) we have for $f \in E$

$$\|T(f)\|_Y \leq \pi_s^m(T)\|T_{\delta_1}(f)\|_{L^s(\delta,X)} \leq \frac{\pi_s^m(T)}{(1-\varepsilon)^{\frac{1}{s}}}\left(2\frac{C}{(1-\varepsilon)^{\frac{1}{r}}}\right)^{\frac{1}{\alpha}} \int_K \left\|\int_\Omega fg dm\right\| d\lambda(g). \tag{4.34}$$

By Theorem 4.3.4, this directly implies that T is $m-1-$summing. □

4.4. Applications

Maurey-Rosenthal Theory relates the geometrical properties of Banach lattices, norms inequalities for operators and factorization through L^p-spaces. The original works were done by Rosenthal, Krivine and Maurey, see [45], [50], [67] and [48], their purposes were related to the study of the structure of Banach lattices and operators. Nowadays this theory is the keystone in several areas of functional analysis with applications in interpolation of Banach spaces, operator ideal theory (see [27] and [22]) and geometry of Banach lattices, [23] and [24].

In [58, Chapter 6] the authors relate factorization results for $p-$th factorable operators with Maurey-Rosenthal factorization Theory. Here we find thar Maurey-Rosenthal Theory provides, under certain assumptions, a factorization theorem for $q-$concave operators. For $1 \leq q < \infty$, let $X(\mu)$ be a $\sigma-$order continuous q-convex Banach function space. Consider a Banach space E and a q-concave linear operator $T : X(\mu) \to E$, then there exist $g_0 \in M(X(\mu), L^q(m)) := \{g \in L^0(\mu) : g \cdot f \in L^q(m)$ for every $f \in X(\mu)\}$ so that T factorizes as follows:

$$\begin{tikzcd} X(\mu) \arrow[rr, "T"] \arrow[dr, "M_g"'] & & E \\ & L^q(\mu) \arrow[ur, "S"'] & \end{tikzcd}$$

where S is a continuous linear operator, M_g is the continuous operator of multiplication by g and $\|M_g\|\|S\| \leq \mathbf{M}_{(q)}[T]\mathbf{M}^{(q)}[X(\mu)]$.

Proposition 4.4.1. *Let $p \geq r$ and $m : \Sigma \to X$ be a positive vector measure with values in an $r-$concave Banach lattice X. If the identity $I : L^p(m) \to L^p(m)$ is $m - r-$summing, then there is some x_0^* defining a Rybakov control measure for m so that $L^p(m) = L^p(\langle m, x_0^* \rangle)$.*

Proof. Since I is $m - r-$summing, for every finite choice $f_1, \ldots, f_n \in L^p(m)$ we have

$$\begin{aligned} \left(\sum_{i=1}^{n} \|f_i\|^r\right)^{\frac{1}{r}} &\leq K \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^{n} \left\| \int_\Omega f_i g dm \right\|_X^r\right)^{\frac{1}{r}} \\ &\leq K\mathbf{M}_r(X) \sup_{g \in B(L^q(m))} \left\| \left(\sum_{i=1}^{n} \left| \int_\Omega |g f_i| dm \right|^r\right)^{\frac{1}{r}} \right\|_X \\ &\leq K\mathbf{M}_r(X) \sup_{g \in B(L^q(m))} \left\| \int_\Omega \left(\sum_{i=1}^{n} |f_i|^r\right)^{\frac{1}{r}} |g| dm \right\|_X \\ &= K \left\| \left(\sum_{i=1}^{n} |f_i|^r\right)^{\frac{1}{r}} \right\|_{L^p(m)} \end{aligned}$$

where the last inequality is a consequence of [48, Proposition 1.d.9] ensuring that a positive operator with range in an $r-$concave Banach space is $r-$concave.

Then, I is $r-$concave, therefore $p-$concave (see [58] Prop. 2.54). A direct application of Maurey Rosenthal theory yields the following factorization scheme for a particular Rybakov measure $\langle m, x_0^* \rangle$,

$$\begin{tikzcd} L^p(m) \arrow[rr, "I"] \arrow[dr, "i"'] & & L^p(m) . \\ & L^p(\langle m, x_0^* \rangle) \arrow[ur, "S_{x_0^*}"'] & \end{tikzcd}$$

Therefore, a direct density argument yields that $S_{x_0^*}$ is also the identity and $L^p(m) = L^p(\langle m, x_0^* \rangle)$. □

Remark 4.4.2. It is easy to see that the same arguments can be adapted to prove factorization theorems for $m - r-$summing operators $T : L^p(m) \to Y$ through $L^p(\langle m, x_0^* \rangle)$ spaces. The key is that under the adequate restrictions these operators are $r-$concave and then the factorization theory of [58, Ch.6] applies.

In what follows we study of those operators T that have the following property: $S \circ T$ is $r-$summing ($m-r-$summing) whenever S is $s-$summing (respectively $m-s-$summing). In the classical Operator Ideal Theory, these operators are known as mixing operators. Our aim is to adapt the study made by Pietsch in [61]; a good reference in this topic is also [22]. For $1 \leq r < s < \infty$, an operator $T \in \mathcal{L}(X,Y)$ between Banach spaces is said to be $(s,r)-$*mixing* if for each Banach space Z and every $S \in \Pi_s(Y,Z)$ the composition $S \circ T$ is $r-$summing. The restriction $r < s$ is made to avoid the trivial case. The following characterization is a consequence of Pietsch domination theorem, the proof can be found in [22] page 419.

Proposition 4.4.3. *Let $1 \leq r < s \leq \infty$ and $T \in \mathcal{L}(X,Y)$, the following assertions are equivalent.*

(i) T is $(s,r)-$mixing.

(ii) There is a constant $c \geq 0$ so that for every probability measure μ defined on $B(Y^*)$ there is a probability measure ν defined on $B(X^*)$ such that

$$\left(\int_{B(Y^*)} |\langle T(x), y^*\rangle|^s d\mu(y^*)\right)^{\frac{1}{s}} \leq c\left(\int_{B(X^*)} |\langle x, x^*\rangle|^r d\nu(x^*)\right)^{\frac{1}{r}}$$

holds for every $x \in X$.

(iii) There is a constant $c \geq$ such that

$$\left(\sum_{j=1}^{m}\left(\sum_{k=1}^{n} |\langle T(x_j), y_k^*\rangle|^s\right)^{\frac{r}{s}}\right)^{\frac{1}{r}} \leq c\|(x_j)_j\|_{\ell_r^w(X)}\|(y_k^*)_k\|_{\ell_s(Y^*)},$$

for all finite sequence $x_1,\ldots,x_m \in X$ and $y_1^*,\ldots,y_n^* \in Y^*$.

As for the summability, in order to study operators defined (or with range) in spaces of $p-$integrable functions with respect to a vector measure, we will give a new definition in the framework of the duality relationship between $L^p(m)$ and $L^q(m)$.

Definition 4.4.4. Let $1 \leq r < s < \infty$, $m : \Sigma \to X$ and Y a Banach space. An operator $T \in \mathcal{L}(L^p(m),Y)$ is $(s,m-r)-$*mixing* whenever for every Banach space Z and every $S \in \Pi_s(Y,Z)$, the composition $S \circ T$ is $m-r-$summing.

The following result gives a characterization for the $(s,m-r)-$mixing operators. The proof is based in Theorem 4.3.4.

Proposition 4.4.5. *Let $1 \leq r < s < \infty$, $m : \Sigma \to X$ and $T \in \mathcal{L}(L^p(m),Y)$, the following statements are equivalent assuming that the space $L^p(m)$ has property $(r-\star)$,* *(i)* *T is* $(s, m-r)-$*summing,*

(ii) *There is a positive constant* Q *so that for each probability measure* μ *defined on* $B(Y^*)$*, each* $m-$*compact* $K \subset L^q(m)$ *and for every* $S \in \mathcal{S}$*, where* $\mathcal{S}$ *is a dense family of subspaces of* $L^p(m)$*, so that* S *is* $r-\varepsilon-$*normed by* K *for some* $0 < \varepsilon < \frac{1}{2}$*, there exists a probability measure* δ_K *defined on the Borel subsets of* K *so that*

$$\left(\int_{B(Y^*)} |\langle T(f), y^* \rangle|^s d\mu(y^*)\right)^{\frac{1}{s}} \leq \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}} \left(\int_K \left\| \int_\Omega fg dm \right\|_X^r d\delta_K(g)\right)^{\frac{1}{r}} \tag{4.35}$$

for every $f \in S$*.*

(iii) *There is* $Q \geq 0$ *so that for every* K *and* S *as is (ii) the following holds*

$$\left(\sum_{j=1}^m \left(\sum_{k=1}^n |\langle T(f_j), y_k^* \rangle|^s\right)^{\frac{r}{s}}\right)^{\frac{1}{r}} \leq \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}} \|(f_j)_j\|_{\ell_r^m(L^p(m))} \|(y_k^*)_k\|_{\ell_s(Y^*)} \tag{4.36}$$

for every finite choice $f_1, \ldots, f_m \in S$ *and* $y_1^*, \ldots, y_n^* \in Y^*$*.*

Proof. We begin by proving (i) implies (ii). Let μ be a probability measure defined on $B(Y^*)$. The operator $I_\mu : Y \to L^s(\mu)$ given by $I_\mu(y)(y^*) = \langle y, y^* \rangle$ for each $y \in Y$ and $y^* \in B(Y^*)$ is $s-$summing. Indeed, for $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in Y$

$$\begin{aligned}\left(\sum_{i=1}^n \|I_\mu(y_i)\|_{L^s(\mu)}^s\right)^{\frac{1}{s}} &= \left(\sum_{i=1}^n \int_{B(Y^*)} |\langle y_i, y^* \rangle|^s d\mu(y^*)\right)^{\frac{1}{s}} \\ &\leq \sup_{y^* \in B(Y^*)} \left(\sum_{i=1}^n |\langle y_i, y^* \rangle|^s\right)^{\frac{1}{s}}.\end{aligned}$$

Since T is $(s, m-r)-$mixing, the composition $I_\mu \circ T$ is $m-r-$summing. For an $r-\varepsilon-$norming set $K \subset B(L^q(m))$ for a subspace S of $L^p(m)$, a direct application of Domination Theorem 4.3.4 yields the existence of a probability measure δ_K defined on the Borel subsets of K and a constant $Q \geq 0$ such that

$$\|I_\mu \circ T(f)\|_{L^s(\mu)} \leq \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}} \left(\int_K \left\| \int_\Omega fg dm \right\|^r d\delta_K(g)\right)^{\frac{1}{r}}$$

for every $f \in S$. Since $\|I_\mu \circ T(f)\|_{L^s(\mu)} = \left(\int_{B(Y^*)} |\langle T(f), y^* \rangle|^s d\mu(y^*)\right)^{\frac{1}{s}}$ for every $f \in L^p(m)$, (4.35) holds in S.

In order to prove (ii) implies (iii), let K be an $m-$compact subset in $B(L^q(m))$, and S a subspace of $L^p(m)$ that is $r-\varepsilon-$normed by K for $0<\varepsilon<\frac{1}{2}$. Fix two families $f_1,\ldots,f_m \in S$ and $y_1^*,\ldots,y_n^* \in Y^*$. We can assume without loss of generality that $\|(y_k^*)\|_{\ell_s(Y^*)}=1$. We define a probability measure on $B(Y^*)$ $\mu := \sum_{k=1}^n \|y_k^*\|_{Y^*}\delta_{\|y_k^*\|_{Y^*}^{-1}y_k^*}$, then (ii) ensures the existence of a probability measure δ_K defined on K such that

$$\begin{aligned}
\sum_{j=1}^m\left(\sum_{k=1}^n |\langle T(f_j),y_k^*\rangle|^s\right)^{\frac{r}{s}} &\leq \sum_{j=1}^m\left(\frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}\left(\int_K\left\|\int_\Omega f_j g dm\right\|_X^r d\delta_K(g)\right)^{\frac{1}{r}}\right)^r\\
&= \frac{Q^r}{1-\varepsilon}\sum_{j=1}^m\int_K\left\|\int_\Omega f_j g dm\right\|_X^r d\delta_K(g)\\
&\leq \frac{Q^r}{1-\varepsilon}\|(f_j)_j\|^r_{\ell_r^m(L^p(m))}.
\end{aligned}$$

We finish with the proof of (iii) implies (i). Remark that (iii) ensures that every discrete probability measure μ defined on $B(Y^*)$ satisfies

$$\left(\sum_{j=1}^m\left(\int_{B(Y^*)}|\langle T(f_j),y^*\rangle|^s d\mu(y^*)\right)^{\frac{r}{s}}\right)^{\frac{1}{r}} \leq \frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}\|(f_j)_j\|_{\ell_r^m(L^p(m))} \tag{4.37}$$

for every $f_1,\ldots,f_n\in S$ and for some $Q\geq 0$. Since the set of discrete probability measures on $B(Y^*)$ is $\sigma(\mathcal{M}(B(Y^*)),\mathcal{C}(B(Y^*)))-$dense in the set of probability measures defined on $B(Y^*)$, (4.37) holds for every probability measure μ. Let $S: Y\to Z$ be an $s-$summing operator. An appel to the classical Domination Theorem for $s-$summing operators and inequality (4.37) directly implies the $m-r-$summability of $S\circ T$. Indeed for every $n\in\mathbb{N}$ and $f_1,\ldots,f_n$ in $L^p(m)$ we have

$$\begin{aligned}
\left(\sum_{j=1}^n\|S\circ T(f_j)\|^r\right)^{\frac{1}{r}} &\leq \left(\sum_{j=1}^n\left(C\left(\int_{B(Y^*)}|\langle T(f_j),y^*\rangle|^s d\mu(y^*)\right)^{\frac{1}{s}}\right)^r\right)^{\frac{1}{r}}\\
&\leq C\cdot\frac{Q}{(1-\varepsilon)^{\frac{1}{r}}}\|(f_j)_j\|_{\ell_r^m(L^p(m))}.
\end{aligned}$$

□

4.5. Tensor product representation

Some of the deep ideas of A. Grothendieck that appear in his "Résumé de la théorie métrique des produits tensoriels topologique" (see [40]) were used later in the study of operator ideals. G. Pisier's work, starting around 1975, gives a first approach to the idea of the relationship between tensor product and operator ideals. This idea corresponds to the Representation Theorem for maximal operator ideals. It ensures that there is a one–to–one correspondence between finitely generated tensor norms and maximal normed operator ideals. All the details of this representation technique for general operator ideals can be found in Chapter 17 of the book by A. Defant and K. Floret, see [22]. An easier version, for the particular representation of the ideal of $r-$summing operators, can be found in Chapter 6 of [69]. In the following we give the main definitions and properties of this representation.

A norm α defined in a tensor product $X \otimes Y$ of Banach spaces X and Y is a *reasonable crossnorm* if it has the following natural properties:

1. $\alpha(x \otimes y) \leq \|x\|_X \|y\|_Y$ for every $x \in X$ and $y \in Y$.
2. For each $x^* \in X^*$ and $y^* \in Y^*$ the linear functional $x^* \otimes y^* : X \otimes Y \to \mathbb{R}$ defined by
$$x^* \otimes y^*(u) := \sum_{i=1}^{n} \lambda_i \langle x_i, x^* \rangle \langle y_i, y^* \rangle$$
for $u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \in X \otimes Y$ is bounded and $\|x^* \otimes y^*\| \leq \|x^*\| \|y^*\|$.

The basic norms for tensor products of Banach spaces are the projective and the injective norms. The *projective norm* for the tensor product $X \otimes Y$ is defined as follows, for $u \in X \otimes Y$,

$$\pi(u) := \inf\{\sum_{i=1}^{n} \|x_i\|_X \|y_i\|_Y : u = \sum_{i=1}^{n} x_i \otimes y_i\}.$$

The *injective norm* is induced by the canonical algebraical embedding of $X \otimes Y$ into the space of bilinear forms defined on $X^* \times Y^*$, $\mathcal{B}(X^* \times Y^*)$. It is defined as follows for a tensor $u \in X \otimes Y$,

$$\varepsilon(u) := \sup\left\{ \left| \sum_{i=1}^{n} \langle x_i, x^* \rangle \langle y_i, y^* \rangle \right| : x^* \in B(X^*), y^* \in B(Y^*) \right\},$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is any representation of the tensor u.

Clearly the projective and injective norms satisfy the conditions to be reasonable crossnorms. In fact they lie at extremum of the spectrum of reasonable crossnorms, see for instance [69, Prop. 6.1].

Proposition 4.5.1. *Let X and Y be Banach spaces. A norm α on $X \otimes Y$ is a reasonable crossnorm if and only if $\varepsilon(u) \leq \alpha(u) \leq \pi(u)$ for every $u \in X \otimes Y$.*

We say that a reasonable crossnorm is *uniform* if it behaves well with respect to the formation of tensor products of operators; if $S: X \to W$ and $T: Y \to Z$ are bounded linear operators, then $S \otimes T: X \otimes_\alpha Y \to W \otimes_\alpha Z$ is bounded too and satisfies $\|S \otimes T\| \leq \|S\|\|T\|$.

A uniform crossnorm is *finitely generated* if the behavior of α is completely determined by its values on tensor products of finite dimensional spaces. That is, for every pair of Banach spaces X and Y and each $u \in X \otimes Y$ we have

$$\alpha(u; X \otimes Y) = \inf\{\alpha(u; M \otimes N) : u \in M \otimes N, \dim M, \dim N < \infty\},$$

where M and N are finite dimensional subspaces of X and Y. A reasonable crossnorm is a *tensor norm* whenever it is finitely generated and uniform. In the following we recall the definition of some tensor norms that are useful to represent the ideal of $r-$summing operators as the dual of a tensor product.

Let $1 \leq r \leq \infty$ and s its conjugated index (see (1.7)). The *Chevet-Saphard norm* is defined as follows for $u \in X \otimes Y$,

$$d_r(u) = \inf\left\{ \|(x_i)\|_{\ell_s^w(X)} \|(y_i)\|_{\ell_r(Y)} : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}, \tag{4.38}$$

where the infimum is taken over all the representations of $u \in X \otimes Y$.

In order to represent the ideal of $r-$summing operators as the dual of a tensor product the dual of $X\hat{\otimes}_{d_r}Y$ can be considered as a space of operators from X into Y^* using the *trace duality*. That is, if T is a linear and continuous operator between the spaces X and Y^*, $T \in \mathcal{L}(X, Y^*)$, T is a linear functional on $X\hat{\otimes}_{d_r}Y$ when the action over a tensor $u = \sum_{i=1}^n x_i \otimes y_i$ is given by

$$\langle u, T \rangle = \sum_{i=1}^{n} \langle y_i, T(x_i) \rangle,$$

and the boundedness of T means that there is some $C \geq 0$ such that

$$\left| \sum_{i=1}^{n} \langle y_i, T(x_i) \rangle \right| \leq C \|(x_i)\|_{\ell_s^w(X)} \|(y_i)\|_{\ell_r(Y)}.$$

The minimum value of such C is the norm of T as a functional in the dual space $(X\hat{\otimes}_{d_r}Y)^*$. In this context, the following representation theorem ensures that the space of $s-$summing operators between the Banach spaces X and Y^*, $\Pi_s(X, Y^*)$, is isometrically isomorphic to $(X \otimes_{d_r} Y)^*$.

Theorem 4.5.2. *Let $1 < r, s \leq \infty$, conjugated numbers and X, Y Banach spaces. An operator $T \in \mathcal{L}(X, Y^*)$ defines a functional of $(X\hat{\otimes}_{d_r}Y)^*$ if and only if T is $s-$summing. Moreover the norm of T in $(X\hat{\otimes}_{d_r}Y)^*$ coincides with $\pi_s(T)$.*

In our context of $m-r-$summing operators we are interested into obtaining representations of these operators spaces in terms of particular tensor products. For this aim we first have to adapt classical definitions to the tensor product of an $L^p(m)$ space of a vector measure $m:\Sigma\to X$ with a Banach space Y. In this framework we say that a norm α in $L^p(m)\otimes Y$ is an $m-$*reasonable crossnorm* whenever it is a reasonable crossnorm for the particular tensor products of $L^p(m)$ of a vector measure and a Banach space. We say that α is a *uniform* $m-$*crossnorm* whenever is behaves well with respect the formation of tensor products of operators, that is, for $T\in\mathcal{L}(L^{p_1}(m_1),L^{p_2}(m_2))$ and $S\in\mathcal{L}(Y_1,Y_2)$, the operator $T\otimes S: L^{p_1}(m_1)\otimes_\alpha Y_1\to L^{p_2}(m_2)\otimes_\alpha Y_2$ is bounded with respect to the norm α and $\|T\otimes S\|\leq\|T\|\|S\|$. An $m-$*tensor norm* is an m-reasonable crossnorm which is uniform and finitely generated. We define an m-tensor norm in the tensor product $L^p(m)\otimes Y$ inspired in the Chevet-Saphard tensor norm as follows. Let p,q,r,s be real numbers such that

$$\frac{1}{p}+\frac{1}{q}=1=\frac{1}{r}+\frac{1}{s},\ \text{ with } 1<p,q<\infty \text{ and } 1\leq r,s\leq\infty. \tag{4.39}$$

For $u=\sum_{i=1}^n f_i\otimes y_i\in L^p(m)\otimes Y$, we define

$$d_r^m(u):=\inf\{\|(f_i)_i\|_{\ell_s^m(L^p(m))}\|(y_i)\|_{\ell_r(Y)}:u=\sum_{i=1}^n f_i\otimes y_i\} \tag{4.40}$$

Notice that when m is a scalar measure, the norm d_r^m for $L^p(m)\otimes Y$ coincides with the Chevet-Saphard tensor norm. Obviously d_r^m is not a tensor norm, since it is only defined for tensor products where the first space is an L^p-space of a vector measure. Although we will show that d_r^m still preserves some properties of a tensor norm in the classical sense. We say that an $m-$tensor norm defined for the tensor product $L^p(m)\otimes Y$ is a *generalized tensor norm* if it stays between the π and the ε norms.

Remark 4.5.3. In order to prove some properties of d_r^m we will provide an alternative formula to compute (4.40). For $u\in L^p(m)\otimes Y$, define

$$\delta_r^m(u)=\inf\{\|(\lambda_i)_i\|_{\ell_r}\|(f_i)_i\|_{\ell_s^m(L^p(m))}\|(y_i)_i\|_{\ell_\infty(Y)}:u=\sum_{i=1}^n\lambda_i f_i\otimes y_i\},$$

then $\delta_r^m(u)=d_r^m(u)$ for $u\in L^p(m)\otimes Y$. Indeed, since each representation

$u=\sum_{i=1}^n \lambda_i f_i\otimes y_i$ can be written as $u=\sum_{i=1}^n f_i\otimes \lambda_i y_i$, then

$$\begin{aligned} d_r^m(u) &\le \|(f_i)_i\|_{\ell_s^m(L^p(m))}\|(\lambda_i y_i)_i\|_{\ell_r(Y)} = \|(f_i)_i\|_{\ell_s^m(L^p(m))}\left(\sum_{i=1}^n \|\lambda_i y_i\|_Y^r\right)^{\frac{1}{r}} \\ &\le \|(f_i)_i\|_{\ell_s^m(L^p(m))}\left(\sum_{i=1}^n |\lambda_i|^r \sup_i \|y_i\|_Y^r\right)^{\frac{1}{r}} \\ &= \|(\lambda_i)_i\|_r \|(f_i)_i\|_{\ell_s^m(L^p(m))}\|(y_i)_i\|_\infty . \end{aligned}$$

And this is true for all representations of u, thus $d_r^m(u)\le \delta_{r,m}(u)$. Conversely, let $u=\sum_{i=1}^n f_i\otimes y_i$, then $u=\sum_{i=1}^n \lambda_i f_i\otimes z_i$, where $\lambda_i=\|y_i\|_Y$ and $z_i=y_i/\|y_i\|_Y$ for all $i=1,\dots,n$. Then the infimum $\delta_{r,m}(u)$ is less or equal than the infimum in $d_r^m(u)$ for each $u\in L^p(m)\otimes Y$.

Proposition 4.5.4. *For p and r as in (4.39) and Y a Banach space, we have*

- (i) *d_r^m is a generalized tensor norm over $L^p(m)\otimes Y$,*
- (ii) *if $r_1\le r_2$ then $d_{r_1,m}\ge d_{r_2,m}$*

Proof. To prove (i), let $u_1,u_2\in L^p(m)\otimes Y$ and $\varepsilon>0$. We can choose particular representations $u_1=\sum_{i=1}^n f_j^1\otimes y_j^1$ and $u_2=\sum_{i=1}^n f_j^2\otimes y_j^2$ such that for $i=1,2$ we have

$$\|(f_j^i)_j\|_{\ell_s^m(L^p(m))}\le (d_r^m(u_i)+\varepsilon)^{\frac{1}{r}} \text{ and } \|(y_j^i)_j\|_{\ell_r(Y)}\le (d_r^m(u_i)+\varepsilon)^{\frac{1}{s}}.$$

Therefore $\sum_{i=1}^2\sum_{i=1}^n f_j^i\otimes y_j^i$ is a representation of u_1+u_2 such that

$$\begin{aligned} &\|(f_j^i)_j\|_{\ell_s^m(L^p(m))}\|(y_j^i)_j\|_{\ell_r(Y)} \le \\ &\le (d_r^m(u_1)+d_r^m(u_2)+2\varepsilon)^{\frac{1}{r}}(d_r^m(u_1)+d_r^m(u_2)+2\varepsilon)^{\frac{1}{s}} \\ &= d_r^m(u_1)+d_r^m(u_2)+2\varepsilon, \end{aligned}$$

therefore, when $\varepsilon\to 0$ we get $d_r^m(u_1+u_2)\le d_r^m(u_1)+d_r^m(u_2)$.

Let $\lambda\in\mathbb{K}$, since $\|\cdot\|_{\ell_s^m(L^p(m))}$ and $\|\cdot\|_{\ell_r(Y)}$ are homogeneous we have that $d_r^m(\lambda u)=|\lambda| d_r^m(u)$ for every tensor u in $L^p(m)\otimes Y$.

It least to prove that d_r^m is an $m-$reasonable crossnorm. For this aim we will prove that $\varepsilon(u)\le d_r^m(u)\le \pi(u)$ for all $u\in L^p(m)\otimes Y$. The first inequality is consequence of Hölder's inequality and the fact that the set $\Gamma:=\{\gamma_{g,x^*}: g\in B(L^q(m)), x^*\in B(X^*)\}\subset (L^p(m))^*$ is norming for $\gamma_{g,x^*}(f):=\int_\Omega fg d\langle m,x^*\rangle$. We get, for all the representations for $u=$

$\sum_{i=1}^n f_i \otimes y_i$,

$$
\begin{aligned}
\varepsilon(u) &= \sup \left\{ \left\| \sum_{i=1}^n \varphi(f_i) y_i \right\|_Y : \varphi \in B((L^p(m))^*) \right\} \\
&\leq \sup \left\{ \left(\sum_{i=1}^n |\varphi(f_i)|^s \right)^{\frac{1}{s}} : \varphi \in B((L^p(m))^*) \right\} \left(\sum_{i=1}^n \|y_i\|_Y^r \right)^{\frac{1}{r}} \\
&\leq \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^n \left(\sup_{x^* \in B(X^*)} \left| \left\langle \int_\Omega f_i g dm, x^* \right\rangle \right| \right)^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^n \|y_i\|_Y^r \right)^{\frac{1}{r}} \\
&= \|(f_i)_i\|_{\ell_r^m(L^p(m))} \|(y_i)_i\|_{\ell_r(Y)},
\end{aligned}
$$

therefore $\varepsilon(u) \leq d_r^m(u)$ for every $u \in L^p(m) \otimes Y$. For the second inequality, note that for all $u \in L^p(m) \otimes Y$ and $\varepsilon > 0$ there is a representation $u = \sum_{i=1}^n f_i \otimes y_i$ such that $\sum_{i=1}^n \|f_i\|_{L^p(m)} \|y_i\|_Y \leq \pi(u) + \varepsilon$. We define, for each $i = 1, \ldots, n$,

$$
\tilde{f}_i = \frac{\|y_i\|_Y^{\frac{1}{s}}}{\|f_i\|_{L^p(m)}^{\frac{1}{r}}} f_i \quad \text{and} \quad \tilde{y}_i = \frac{\|f_i\|_{L^p(m)}^{\frac{1}{r}}}{\|y_i\|_Y^{\frac{1}{s}}} y_i.
$$

Then $\sum_{i=1}^n \tilde{f}_i \otimes \tilde{y}_i$ is also a representation of the tensor u, thus

$$
\begin{aligned}
d_r^m(u) &\leq \|(\tilde{f}_i)_i\|_{\ell_s^m(L^p(m))} \|(\tilde{y}_i)_i\|_{\ell_r(Y)} \\
&= \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^n \left\| \int \tilde{f}_i g dm \right\|_X^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^n \|\tilde{y}_i\|_Y^r \right)^{\frac{1}{r}} \\
&\leq \left(\sum_{i=1}^n \sup_{g \in B(L^q(m))} \left\| \int \tilde{f}_i g dm \right\|_X^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^n \|\tilde{y}_i\|_Y^r \right)^{\frac{1}{r}} \\
&\leq \left(\sum_{i=1}^n \|\tilde{f}_i\|_{L^p(m)}^s \right)^{\frac{1}{s}} \left(\sum_{i=1}^n \|\tilde{y}_i\|_Y^r \right)^{\frac{1}{r}} \\
&= \left(\sum_{i=1}^n \|f_i\|_{L^p(m)} \|y_i\|_Y \right)^{\frac{1}{s}} \left(\sum_{i=1}^n \|f_i\|_{L^p(m)} \|y_i\|_Y \right)^{\frac{1}{r}} \\
&= \left(\sum_{i=1}^n \|f_i\|_{L^p(m)} \|y_i\|_Y \right) \\
&= \pi(u) + \varepsilon.
\end{aligned}
$$

This construction is possible for all $\varepsilon > 0$, therefore the conclusion follows.

Clearly, d_r^m is uniform, and since the $m - s-$weak and the strong$-\ell_r$ norms are unchanged if the range of the space is enlarged, we get that it is also finitely generated.

To prove (ii), let $1 < r_1 < r_2 < \infty$ and let s_1, s_2 be their corresponding conjugated exponents, $u \in L^p(m) \otimes Y$ and $\varepsilon > 0$. We want to prove that $d^m_{r_2}(u) \leq d^m_{r_1}(u) + \varepsilon$. Having in mind Remark 4.5.3, there is a representation of $u = \sum_{i=1}^n \lambda_i f_i \otimes y_i$ such that $\|(\lambda_i)_i\|_{\ell_{r_1}} \|(f_i)_i\|_{\ell^m_{s_1}} \|(y_i)_i\|_{\ell_\infty(Y)} \leq d^m_{r_1}(u) + \varepsilon$ and all the coefficients λ_i are non-negative. We can rewrite this representation as $\sum_{i=1}^n \lambda_i f_i \otimes y_i = \sum_{i=1}^n \lambda_i^{\frac{r_1}{r_2}} (\lambda_i^{1-\frac{r_1}{r_2}} f_i) \otimes y_i$, therefore

$$d^m_{r_2}(u) \leq \left\| \left(\lambda_i^{\frac{r_1}{r_2}} \right)_i \right\|_{\ell_{r_2}} \left\| \left(\lambda_i^{1-\frac{r_1}{r_2}} f_i \right)_i \right\|_{\ell^m_{s_2}(L^p(m))} \|(y_i)_i\|_{\ell_\infty(Y)}.$$

We clearly have, by definition

$$\begin{aligned} \left\| \left(\lambda_i^{1-\frac{r_1}{r_2}} f_i \right)_i \right\|_{\ell^m_{s_2}(L^p(m))} &= \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^n \left\| \int_\Omega |\lambda_i^{1-\frac{r_1}{r_2}} f_i g| dm \right\|_X^{s_2} \right)^{\frac{1}{s_2}} \\ &= \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^n |\lambda_i|^{s_2\left(1-\frac{r_1}{r_2}\right)} \left\| \int_\Omega f_j g dm \right\|_X^{s_2} \right)^{\frac{1}{s_2}}, \end{aligned}$$

where, by applying Hölder's inequality in the sum with the conjugated exponents $\frac{s_1}{s_1-s_2}$ and $\frac{s_1}{s_2}$, we get the following

$$\|(\lambda_i^{1-\frac{r_1}{r_2}} f_i)_i\|_{\ell^m_{s_2}(L^p(m))} \leq \left(\sum_{i=1}^n |\lambda_i|^{r_1} \right)^{\frac{1}{r_1}-\frac{1}{r_2}} \sup_{g \in B(L^q(m))} \left(\sum_{i=1}^n \left\| \int_\Omega f_i g dm \right\|_X^{s_1} \right)^{\frac{1}{s_1}}.$$

Thus we get

$$d^m_{r_2}(u) \leq \|(f_i)_i\|^m_{s_1} \|(y_i)_i\|_\infty \left\| \left(\lambda_i^{\frac{r_1}{r_2}} \right)_i \right\|_{\ell_{r_2}} \left(\sum_{i=1}^n |\lambda_i|^{r_1} \right)^{\frac{1}{r_1}-\frac{1}{r_2}} \leq d^m_{r_1}(u) + \varepsilon,$$

for all $\varepsilon > 0$, as wanted.

□

We say that an $m-$tensor norm α is *$m-$right projective* if for every Banach space Z and every quotient operator $Q : Z \to Y$, the tensor product operator $Id \otimes_\alpha Q : L^p(m) \otimes_\alpha Z \to L^p(m) \otimes_\alpha Y$ is a quotient operator. Note that $Q \in \mathcal{L}(X, Y)$ is a *quotient operator* whenever it is surjective and $Q(B(X)) \subset B(Y)$, in such a case Y will be isomorphic to $X/\ker Q$ (as a direct consequence of open mapping theorem).

Proposition 4.5.5. *The $m-$tensor norm d^m_r defined in $L^p(m) \otimes Y$ is $m-$right projective.*

Proof. Let Z be a Banach space and $Q : Z \to Y$ a quotient operator. Clearly $(Id \otimes Q)(B(L^p(m)) \otimes_{d_r^m} Z) \subset B(L^p(m) \otimes_{d_r^m} Y)$. Take $u \in L^p(m) \otimes Y$ such that $d_r^m(u) < 1$. Take $\varepsilon > 0$ so that $(1+\varepsilon)d_r^m(u) < 1$, and a representation of $u = \sum_{i=1}^n f_i \otimes y_i$ such that $\|(f_i)_i\|_{\ell_s^m}\|(y_i)_i\|_{\ell_r(Y)} < 1/(1+\varepsilon)$.

Since Q is a quotient operator for each y_i, $i = 1, \ldots, n$, there is some $z_i \in Z$ such that $Q(z_i) = y_i$ and $\|z_i\|_Z \leq (1+\varepsilon)\|y_i\|_Y$. Therefore, taking $v = \sum_{i=1}^n f_j \otimes z_j \in L^p(m) \otimes Z$ the conclusion follows since $Id \otimes Q(v) = u$ and $d_r^m(v) \leq (1+\varepsilon)\|(f_i)_i\|_{\ell_s^m(L^p(m))}\|(y_i)_i\|_{\ell_r(Y)} < 1$. □

In the sequel we give the description of the completion of the normed tensor product $L^p(m) \otimes_{d_r^m} Y$. We need a technical lemma.

Lemma 4.5.6. *Let $1 < r < \infty$, $(f_n)_n \in \ell_s^m(L^p(m))$ and $(y_n)_n \in \ell_r(Y)$. Then, $\sum_{n=1}^\infty f_n \otimes y_n$ exists in $L^p(m)\hat{\otimes}_{d_r^m} Y$, that is the sequence $\left(\sum_{i=1}^n f_i \otimes y_i\right)_n$ is d_r^m convergent.*

Proof. Since $L^p(m)\hat{\otimes}_{d_r^m} Y$ is complete, we just have to prove that $(s_n)_n = \left(\sum_{i=1}^n f_i \otimes y_i\right)_n$ is a d_r^m-Cauchy sequence. Take for $n \leq n_0$, $d_r^m(s_{n_0} - s_n) = d_r^m(\sum_{i=n}^{n_0} f_i \otimes Y_i)$ small. Since we have the following inequality

$$
\begin{aligned}
d_r^m\Big(\sum_{i=n}^{n_0} f_i \otimes y_i\Big) &\leq \|(f_i)_{i=n}^{n_0}\|_{\ell_s^m(L^p)}\|(y_i)_{i=n}^{n_0}\|_{\ell_r(Y)} \\
&= \sup_{g \in B(L^q(m))} \left(\sum_{i=n}^{n_0} \left\| \int_\Omega f_i g dm \right\|_X^s\right)^{\frac{1}{s}} \left(\sum_{i=n}^{n_0} \|y_i\|_Y^r\right)^{\frac{1}{r}},
\end{aligned}
$$

the completeness of the spaces $\ell_s^m(L^p(m))$ and $\ell_r(Y)$ ensure that the sequence $(s_n)_n$ satisfies the Cauchy condition as wanted. □

Proposition 4.5.7. *For each $u \in L^p(m)\hat{\otimes}_{d_r^m} Y$ there are sequences $(f_n)_n \in \ell_s^m(L^p(m))$ and $(y_n)_n \in \ell_r(Y)$ such that the series $\sum_{n=1}^\infty f_n \otimes y_n$ is d_r^m-convergent to u in $L^p(m)\hat{\otimes}_{d_r^m} Y$. Moreover, for every $\varepsilon > 0$ there are $(f_n)_n$ and $(y_n)_n$ such that*

$$
d_r^m(u) \leq \|(f_n)_n\|_{\ell_s^m(L^p(m))}\|(y_n)_n\|_{\ell_r(Y)} \leq d_r^m(u) + \varepsilon. \tag{4.41}
$$

Proof. By definition $L^p(m) \otimes Y$ is dense in the Banach space $L^p(m)\hat{\otimes}_{d_r^m} Y$, thus, for each $\eta > 0$ and $u \in L^p(m)\hat{\otimes}_{d_r^m} Y$ we can find a sequence $(u_m)_m \in L^p(m) \otimes Y$ so that

$$
u = \sum_{m=1}^\infty u_m, \; d_r^m(u_1) < d_r^m(u) + \eta \; \text{ and } \; d_r^m(u_m) < \frac{\eta^2}{4^m} \; \text{ for } \; m \geq 2.
$$

For each $u_m \in L^p(m) \otimes Y$ we can do the following construction. For $m = 1$, take a representation of $u_1 = \sum_{i^1}^{k_1} f_i^1 \otimes y_i^1$ so that

$$
\|(f_i^1)_{i=1}^{k_1}\|_{\ell_s^m(L^p(m))} < d_r^m(u) + \eta, \; \text{ and } \; \|(y_i^1)_{i=1}^{k_1}\|_{\ell_r(Y)} \leq 1.
$$

For $m \geq 2$, we can write $u_m = \sum_{i=1}^{k_m} f_i^m \otimes y_i^m$ so that

$$\|(f_i^m)_{i=1}^{k_m}\|_{\ell_s^m(L^p(m))} < \frac{\eta}{2^m} \quad \text{and} \quad \|(y_i^m)_{i=1}^{k_m}\|_{\ell_r(Y)} \leq \frac{\eta}{2^m}.$$

Joining the sequences we get

$$\begin{aligned} (f_n)_n &= (f_1^1, f_2^1, \ldots, f_{k_1}^1, f_1^2, \ldots, f_1^n, \ldots, f_{k_n}^n, \ldots) \\ (y_n)_n &= (y_1^1, y_2^1, \ldots, y_{k_1}^1, y_1^2, \ldots, y_1^n, \ldots, y_{k_n}^n, \ldots), \end{aligned}$$

that is, there is an increasing sequence of natural numbers $l_1 < l_2 < \ldots < l_n < \ldots$ with $l_1 = 1$, $l_2 = k_1 + 1, \ldots, l_{m+1} = k_m + l_m$ so that for $l_m \leq n < l_{m+1}$ then $f_n \in (f_i^m)_{i=1}^{k_m}$ and $y_n \in (y_i^m)_{i=1}^{k_m}$. We finish by proving that $\sum_{n=1}^{\infty} f_n \otimes y_n$ converges to u in $L^p(m) \otimes_{d_r^m} Y$. For this aim, we will use lemma 4.5.6. We must show that the norms of $(y_n)_n$ and $(f_n)_n$ are bounded in $\ell_r(Y)$ and $\ell_s^m(L^p(m))$ respectively.

$$\begin{aligned} \|(y_n)_n\|_{\ell_r(Y)} &= \left(\sum_{n=1}^{\infty} \|y_n\|_Y^r \right)^{\frac{1}{r}} = \left(\sum_{m=1}^{\infty} \sum_{i=1}^{k_m} \|y_i^m\|_Y^r \right)^{\frac{1}{r}} \\ &= \left(\sum_{i=1}^{k_1} \|y_i^1\|_Y^r + \sum_{i=1}^{k_2} \|y_i^2\|_Y^r + \ldots + \sum_{i=1}^{k_m} \|y_i^m\|_Y^r + \ldots \right) \\ &< \left(1 + \left(\frac{\eta}{2^2}\right)^r + \ldots + \left(\frac{\eta}{2^m}\right)^r + \ldots \right)^{\frac{1}{r}} \\ &= \left(1 + \sum_{m=2}^{\infty} \left(\frac{\eta}{2^m}\right)^r \right)^{\frac{1}{r}}, \end{aligned}$$

that is

$$\|(y_n)_n\|_{\ell_r(Y)} < \left(1 + \eta^r \sum_{m=2}^{\infty} \left(\frac{1}{2^m}\right)^r \right)^{\frac{1}{r}}. \tag{4.42}$$

To make easier the computation of the norm $\|(f_n)_n\|_{\ell_s^m(L^p(m))}$, let us fix a function $g \in B(L^q(m))$. We get

$$\begin{aligned} \left(\sum_{n=1}^{\infty} \left\| \int_\Omega g f_n dm \right\|_X^s \right)^{\frac{1}{s}} &= \left(\sum_{m=1}^{\infty} \sum_{i=1}^{k_m} \left\| \int_\Omega g f_i^m dm \right\|_X^s \right)^{\frac{1}{s}} \\ &= \left(\sum_{i=1}^{k_1} \left\| \int_\Omega g f_i^1 dm \right\|_X^s + \ldots + \sum_{i=1}^{k_m} \left\| \int_\Omega g f_i^m dm \right\|_X^s + \ldots \right)^{\frac{1}{s}} \\ &\leq \left(\left(\|(f_i^1)_{i=1}^{k_1}\|_{\ell_s^m(L^p(m))} \right)^s + \left(\|(f_i^2)_{i=1}^{k_2}\|_{\ell_s^m(L^p(m))} \right)^s + \ldots \right)^{\frac{1}{s}} \\ &< \left((d_r^m(u) + \eta)^s + \sum_{m=2}^{\infty} \left(\frac{\eta}{2^m}\right)^s \right)^{\frac{1}{s}}. \end{aligned}$$

Since this holds for every $g \in B(L^q(m))$, we get

$$\|(f_n)_n\|_{\ell_s^m(L^p(m))} < \left((d_r^m(u) + \eta)^s + \eta^s \sum_{m=2}^{\infty} \frac{1}{2^{ms}} \right)^{\frac{1}{s}}. \tag{4.43}$$

Therefore we have that $(y_n)_n \in \ell_r(Y)$ and $(f_n)_n \in \ell_s^m(L^p(m))$ thus the conclusion (4.41) holds directly from (4.42)and (4.43). Moreover, taking an adequate η for the product of the right hand side of (4.42) and (4.43) to be less or equal that $d_r^m(u) + \varepsilon$, we obtain the final bound given in the proposition. □

In what follows we will characterize the dual of $L^p(m) \otimes_{d_r^m} Y$ as the space of $m - r-$summing operators. For this aim we will use the trace duality. We say that $T \in \mathcal{L}(L^p(m), Y^*)$ is a bounded linear functional on $L^p(m) \otimes_{d_r^m} Y$ whenever it acts as follows over a tensor $u = \sum_{i=1}^n f_i \otimes y_i \in L^p(m) \otimes_{d_r^m} Y$:

$$\langle u, T \rangle := \sum_{i=1}^{n} \langle y_i, T(f_i) \rangle, \tag{4.44}$$

with the following boundedness condition

$$\left| \sum_{i=1}^{n} \langle y_i, T(f_i) \rangle \right| \leq C \|(f_i)_i\|_{\ell_s^m(L^p(m))} \|(y_i)_i\|_{\ell_r(Y)}, \tag{4.45}$$

where C is a positive constant, the minimum value of which is the norm of T in $(L^p(m) \otimes_{d_r^m} Y)^*$.

Proposition 4.5.8. *Let $T \in \mathcal{L}(L^p(m), Y^*)$, the following assertions are equivalent.*

(i) T is an $m - s-$summing operator, that is, for each $n \in \mathbb{N}$ and every finite choice of functions $f_1, \dots, f_n$ in $L^p(m)$, there is a positive constant Q such that the following inequality holds

$$\left(\sum_{i=1}^{n} \|T(f_i)\|_{Y^*}^s \right)^{\frac{1}{s}} \leq Q \left(\sup_{g \in B(L^q(m))} \sum_{i=1}^{n} \left\| \int f_i g dm \right\|_X^s \right)^{\frac{1}{s}}. \tag{4.46}$$

(ii) $T \in \left(L^p(m) \otimes_{d_r^m} Y \right)^*$.

Moreover the norm of T as an operator in $(L^p(m) \otimes_{d_r^m} Y)^*$ coincides with its norm in the space $\Pi_s^m(L^p(m), Y^*)$ of $m - s-$summing operators, that is the minimum of the constants such that (4.46) *holds for every finite choice of functions.*

Proof. We begin by proving the direct implication. Let $T \in \Pi_s^m(L^p(m), Y^*)$, we will prove that inequality (4.45) holds for every $u = \sum_{i=1}^n f_i \otimes y_i$ in $L^p(m) \otimes_{d_r^m} Y$. A direct application of trace duality and Hölder's inequality give us

$$\begin{aligned}
|\langle u, T\rangle| &= \left|\sum_{i=1}^n \langle y_i, T(f_i)\rangle\right| \leq \sum_{i=1}^n |\langle y_i, T(f_i)\rangle| \\
&\leq \sum_{i=1}^n \|y_i\|_Y \|T(f_i)\|_{Y^*} \leq \left(\sum_{i=1}^n \|y_i\|_Y^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^n \|T(f_i)\|_{Y^*}^s\right)^{\frac{1}{s}} \\
&\leq \left(\sup_{g\in B(L^q(m))} \sum_{i=1}^n \left\|\int f_i g dm\right\|_X^s\right)^{\frac{1}{s}} \left(\sum_{i=1}^n \|y_i\|_Y^r\right)^{\frac{1}{r}} \\
&= Q\|(f_i)_i\|_{\ell_s^m(L^p(m))} \|(y_i)_i\|_{\ell_r(Y)},
\end{aligned}$$

that is true for every representation of u, therefore $T \in (L^p(m) \otimes_{d_r^m} Y)^*$.

For the converse, take $\varphi \in (L^p(m) \otimes_{d_r^m} Y)^*$ and $u = \sum_{i=1}^n f_i \otimes y_i \in L^p(m) \otimes_{d_r^m} Y$, applying again trace duality, we can associate to φ an operator $T_\varphi : L^p(m) \to Y^*$ given by $\langle T_\varphi(f), y^*\rangle = \varphi(f \otimes y^*)$such that $\varphi(u) = \sum_{i=1}^n \langle y_i, T_\varphi(f_i)\rangle$ with condition (4.45). Let $\varepsilon > 0$, for each $i = 1, \ldots, n$, there is some $y_i \in Y$ so that

$$\langle y_i, T_\varphi(f_i)\rangle = \|T_\varphi(f_i)\|_{Y^*}^s \text{ and } \|y_i\|_Y \leq (1+\varepsilon)\|T_\varphi(f_i)\|_{Y^*}^{s-1}.$$

We will show that (4.46) holds,

$$\begin{aligned}
\sum_{i=1}^n \|T_\varphi(f_i)\|_{Y^*}^s &= \sum_{i=1}^n \langle y_i, T_\varphi(f_i)\rangle \leq |\varphi(u)| \\
&\leq Q d_r^m(u) \leq Q\|(f_i)_i\|_{\ell_s^m(L^p(m))} \|(y_i)_i\|_{\ell_r(Y^*)} \\
&= Q\|(f_i)_i\|_{\ell_s^m(L^p(m))} \left(\sum_{i=1}^n \|y_i\|_{Y^*}^r\right)^{\frac{1}{r}} \\
&\leq Q(1+\varepsilon)\|(f_i)_i\|_{\ell_s^m(L^p(m))} \left(\sum_{i=1}^n \|T_\varphi(f_i)\|_{Y^*}^{(s-1)r}\right)^{\frac{1}{r}},
\end{aligned}$$

since $(s-1)r = s$ we get $\left(\sum_{i=1}^n \|T_\varphi(f_i)\|_{Y^*}^s\right)^{\frac{1}{s}} \leq Q(1+\varepsilon)\|(f_i)_i\|_{\ell_s^m(L^p(m))}$, and the conclusion follows. $\square$

Chapter 5

Vector measure duality for Orlicz spaces with respect to a vector measure

Orlicz spaces with respect to a vector measure are the natural generalization of $L^p(m)$ spaces. This spaces were defined and studied firstly by O. Delgado in [21]. Another reference in this topic is the work of M. J. Rivera (see [64]). Or aim in this chapter is to study the multiplication operators between Orlicz spaces of integrable functions with respect to a vector measure. The study of multiplication operators $M_g : f \in \mathcal{F} \to M_g(f) := fg \in \mathcal{G}$ has been already done when the spaces $\mathcal{F}$ and $\mathcal{G}$ are spaces of continuous, holomorphic or analytic functions. But the study of multiplication operators between Banach spaces of measurable functions is relatively little. In [76], H. Takagi and K. Yokouchi studied multiplication operators between L^p spaces over a $\sigma-$finite measure space, they particularly studied the continuity and the closedness of range. For multiplication operators between spaces of $p-$integrable functions with respect to a vector measure, the corresponding study was done by R. del Campo et al. in [7, 8]. Notice that the Köthe dual of a $\mu-$Banach function space W, can be considered as a space of multiplication operator, $W' = M(W, L^1(\mu))$ under the identification $g \in W' \mapsto M_g \in M(W, L^1(\mu))$. Then the spaces of multiplication operators are the natural generalization of Köthe dual spaces.

Our aim in this section is to generalize this work, using the tools of vector measure duality, for operators defined on Orlicz spaces with respect to a vector measure. We begin the chapter with an introduction on classical Orlicz spaces, and some general properties of the multiplications operators between Banach functions spaces. Following the definition of O. Delgado, we study Orlicz spaces with respect to a vector measure defined by conjugated Young's functions, and the relation therein. The vector measure duality relationship is the key to study multiplication operators between those spaces. We finish the chapter with an application of this theory, a characterization of those operators factorizing trough vector measure Orlicz spaces.

5.1. Definition and properties

In this first section we introduce the notions on Orlicz spaces. Our basic references in this topic are the books [3, 44]. A function $\phi : [0,+\infty) \to [0,+\infty)$ is *admissible* whenever it is monotonically increasing, right– continuous, $\phi(u) = 0$ if and only if $u = 0$, and $\lim_{u\to\infty} \phi(u) = \infty$. In this case the function Φ defined by

$$\Phi(s) = \int_0^s \phi(u)du, \qquad s \geq 0,$$

and called a *Young's function* is strictly increasing, continuous and convex. Moreover, $\Phi(s) = 0$ if and only if $s = 0$ and $\lim_{s\to\infty} \Phi(s) = \infty$.

The conditions defining what we have called an admissible function are those used in [44, (1.12)] and they differ from the ones in [3, Def. 4.8.1], where a more general context is considered; nevertheless, the generated Young's function is the same.

Let Φ be a Young's function given by the admissible function ϕ. Next, we define

$$\psi(v) = \sup\{u : \phi(u) \leq v\}, \text{ for } 0 \leq v < \infty. \tag{5.1}$$

Then $\psi : [0,+\infty) \to [0,+\infty)$ is also an admissible function. Thus

$$\Psi(t) = \int_0^t \psi(v)dv \qquad 0 \leq t < \infty,$$

is a Young's function, which is called the *conjugated Young's function* of Φ. It turns out that Φ is the conjugated function of Ψ [44, p. 11]. In the following Φ will always be a Young's function and Ψ its conjugated Young's function.

Remark 5.1.1. Let Φ be defined by the admissible function ϕ. Instead of using the function ψ defined in (5.1), the construction in [3] of its conjugated Young´s function employs another "generating" function. Nevertheless the conjugated function so obtained coincides with Ψ, since both of them can be characterized directly in terms of Φ as shown in [44, (2.9)] and [3, Thm. 4.8.12].

The following inequality gives the fundamental relation between Φ and Ψ; it is called *Young's inequality*:

$$uv \leq \Phi(u) + \Psi(v), \ \forall u, v \geq 0. \tag{5.2}$$

Equality holds when $u = \psi(v)$ or $v = \phi(u)$, see for instance [44, (2.8)].

Since the function $\Phi : [0, +\infty) \to [0, +\infty)$ is continuous, strictly increasing and $\lim_{x\to\infty} \Phi(x) = \infty$, its inverse function $\Phi^{-1} : [0, +\infty) \to [0, +\infty)$ is also continuous and strictly increasing. This is also true for its conjugated function Ψ. The following inequality relates Φ^{-1} and Ψ^{-1}, the proof can be found in [3, Lemma 4.8.16].

Lemma 5.1.2. *If $0 \leq w < \infty$ then $w \leq \Phi^{-1}(w)\Psi^{-1}(w) \leq 2w$.*

The *Luxemburg norm* corresponding to the Young's function Φ is defined by

$$\|f\|_{L^\Phi(\mu)} = \inf \left\{ k > 0 : \int_\Omega \Phi\left(\frac{|f|}{k}\right) d\mu \leq 1 \right\}, \ f \in L^0(\mu). \tag{5.3}$$

The *Orlicz space* $L^\Phi(\mu)$ consists of those (classes of μ-a.e. equal) functions $f \in L^0(\mu)$ so that $\|f\|_{L^\Phi(\mu)} < \infty$. This space is a B.f.s. having the Fatou property when endowed with the Luxemburg norm, as proved in [3, Thm. 4.8.9].

There is a duality relationship between the spaces $L^\Phi(\mu)$ and $L^\Psi(\mu)$. In fact we have that the Köthe dual of $L^\Phi(\mu)$ is $L^\Psi(\mu)$, that is

$$\begin{aligned} L^\Phi(\mu)' = M(L^\Phi(\mu), L^1(\mu)) &= \{g \in L^0(\mu) : fg \in L^1(\mu), \ \forall f \in L^\Phi(\mu)\} \\ &= L^\Psi(\mu). \end{aligned}$$

This duality relationship provides another norm for the space $L^\Phi(\mu)$, the *Orlicz norm*:

$$\|f\|^o_{L^\Phi(\mu)} = \sup \left\{ \int_\Omega |fg| d\mu : \|g\|_{L^\Psi(\mu)} \leq 1 \right\}, \qquad f \in L^\Phi(\mu). \tag{5.4}$$

Notice that the Orlicz norm is equivalent to the Luxemburg norm, in fact we have:

$$\|f\|_{L^\Phi(\mu)} \leq \|f\|^o_{L^\Phi(\mu)} \leq 2\|f\|_{L^\Phi(\mu)}, \qquad f \in L^\Phi(\mu). \tag{5.5}$$

From (5.4) it follows the so called Hölder's inequality for Orlicz spaces (see for example [3, Sect. 4.8]). If $f \in L^\Phi(\mu)$ and $g \in L^\Psi(\mu)$, then fg is integrable and

$$\int_\Omega |fg| \, d\mu \leq \|f\|^o_{L^\Phi(\mu)} \|g\|_{L^\Psi(\mu)}. \tag{5.6}$$

The *Orlicz class* corresponding to the Young's function Φ is defined as

$$O^{\Phi}(\mu) := \{f \in L^0(\mu) : \rho_{\mu,\Phi}(f) < \infty\},$$

where $\rho_{\mu,\Phi}$ is the *Orlicz functional* defined as

$$\rho_{\mu,\Phi}(f) := \int_{\Omega} \Phi(|f|)\, d\mu = \|\Phi(|f|)\|_{L^1(\mu)}.$$

Notice that, for $1 < p < \infty$, $\Phi_p(s) = s^p,\ s \geq 0$ is a Young's function. Moreover $L^{\Phi_p}(\mu) = L^p(\mu) = O^{\Phi_p}(\mu)$ and its conjugated Young's function is $\Psi_q(s) = s^q,\ s \geq 0$, where the relation between p and q is given by equality (1.7).

The following lemma, that corresponds to [3, Lemma 4.8.8] relates the Orlicz functional and the Luxemburg norm.

Lemma 5.1.3. *For $f \in L^0(\mu)$ and Φ a Young's function.*

(i) If $\|f\|_{L^{\Phi}(\mu)} \leq 1$, then $\rho_{\mu,\Phi}(f) \leq \|f\|_{L^{\Phi}(\mu)}$.

(ii) If $\|f\|_{L^{\Phi}(\mu)} > 1$, then $\rho_{\mu,\Phi}(f) \geq \|f\|_{L^{\Phi}(\mu)}$.

(iii) $\|f\|_{L^{\Phi}(\mu)} \leq 1$ if and only if $\rho_{\mu,\Phi}(f) \leq 1$.

In general, the Orlicz class and the Orlicz space are not equal, but we always have the following inclusion

$$O^{\Phi}(\mu) \subset L^{\Phi}(\mu). \tag{5.7}$$

In order to assure the equality the following condition is introduced (see for instance [44, (9.1), p. 75]). A Young's function Φ has the Δ_2-*property* whenever there are real numbers $b > 0$ and $s_0 \geq 0$ such that

$$\Phi(2s) \leq b\Phi(s),\ \forall\, s \geq s_0. \tag{5.8}$$

Assuming that Φ has the Δ_2-property, the space $L^{\Phi}(\mu)$ can be represented as

$$L^{\Phi}(\mu) = O^{\Phi}(\mu). \tag{5.9}$$

Remark 5.1.4. Let Φ have the Δ_2-property and assume (5.8) holds with $s_0 > 0$. Notice that for any $0 < s_1 < s_0$ we can find $b_1 > 0$ such that $\Phi(2s) \leq b_1\Phi(s),\ \forall\, s \geq s_1$. Indeed, take

$$c = \max\left\{\frac{\Phi(2s)}{\Phi(s)} : s_1 \leq s \leq s_0\right\} < \infty,$$

it suffices to choose $b_1 := \max\{b, c\}$ and the conclusion follows.

Given a Young's function Φ with the Δ_2-property, one may wonder whether its conjugated function Ψ also has this property. As we show in the following example this is not always true.

Example 5.1.5. Take for instance $\phi(u) = \log(1+u)$ for $u \geq 0$, as in [44, p. 28]. Then ϕ generates the Young's function

$$\Phi(s) = (1+s)\log(1+s) - s, \; s \geq 0.$$

Since ϕ is strictly increasing and continuous, we have $\psi(v) = \phi^{-1}(v) = e^v - 1, \; v \geq 0$. Hence the conjugated Young's function of Φ is

$$\Psi(t) = e^t - t - 1, \; t \geq 0.$$

For $s \geq 0$, after changing variables we find

$$\begin{aligned} \Phi(2s) &= \int_0^{2s} \log(1+u)du = 2\int_0^s \log(1+2w)dw \\ &\leq 4\int_0^s \log(1+v)dv = 4\Phi(s). \end{aligned}$$

Therefore Φ has the Δ_2-property. Moreover we have

$$\frac{\Psi(2t)}{\Psi(t)} = \frac{e^t - e^{-t}(2t+1)}{1 - e^{-t}(t+1)} \to \infty, \text{ when } t \to \infty.$$

Hence Ψ does not have the Δ_2-property.

We introduce now the Orlicz spaces with respect to a vector measure $m : \Sigma \to X$. Recall that a map $\rho : X^* \times L^0(\mu) \to [0,\infty]$ is an $m-$*norm* (in the sense of [21, Def. 3.1]) if it has the following properties:

(a) For each $x^* \in X^*$, each map $\rho_{x^*} : L^0(\mu) \to [0,\infty]$ given by $\rho_{x^*}(f) := \rho(x^*, f)$, for $f \in L^0(\mu)$, satisfies

- (a1) $\rho_{x^*}(f) = 0$ if and only if $f = 0$ $|\langle m, x^*\rangle|-$a.e.,
- (a2) $\rho_{x^*}(f) = |a|\rho_{x^*}(f)$ for every $a \in \mathbb{R}$ and $f \in L^0(\mu)$,
- (a3) $\rho_{x^*}(f+g) \leq \rho_{x^*}(f) + \rho_{x^*}(g)$, for f, g in $L^0(\mu)$.
- (a4) if $f, g \in L^0(\mu)$ such that $|f| \leq |g|$, $|\langle m, x^*\rangle|-$a.e., then $\rho_{x^*}(f) \leq \rho_{x^*}(g)$,
- (a5) if f, f_n such that $0 \leq f_n \uparrow f$ $|\langle m, x^*\rangle|-$a.e., then $\rho_{x^*}(f_n) \uparrow \rho_{x^*}(f)$,
- (a6) $\rho_{x^*}(\chi_\Omega) < \infty$,

(a7) there is some $C = C(x^*)$ such that for every $f \in L^0(\mu)$,

$$\int_\Omega |f| d|\langle m, x^*\rangle| \leq C\rho_{x^*}(f).$$

(b) For each $f \in L^0(\mu)$, the map $\rho_f : X^* \to [0, \infty]$ satisfies:

(b1) $|a|\rho_f(x^*) \leq \rho_f(ax^*)$, for all $a \in \mathbb{R}$, $a \leq 1$ and $x^* \in X^*$,

(b2) for $f = \chi_\Omega$ we have $\sup_{x^* \in B(X^*)} \rho_f(x^*) < \infty$.

Following the work of O. Delgado in [21] we consider the $m-$norm $\rho : X^* \times L^0(m) \to [0, +\infty]$ defined by

$$\rho(x^*, f) := \|f\|_{L^\Phi(|\langle m, x^*\rangle|)} = \inf_{k>0} \left\{ \int_\Omega \Phi\left(\frac{|f|}{k}\right) d|\langle m, x^*\rangle| \leq 1 \right\}. \tag{5.10}$$

Note that this definition is motivated by the Luxemburg norm for classical Orlicz spaces $L^\Phi(\mu)$. In this setting, the *weak Orlicz space* with respect to the vector measure m is defined as

$$L^\Phi_w(m) = \left\{ f \in L^0(m) : \|f\|_{m,\Phi} < \infty \right\} \tag{5.11}$$

where the norm is given by

$$\|f\|_{m,\Phi} := \sup\{\rho(x^*, f) : x^* \in B(X^*)\}.$$

The closure $\overline{\mathcal{S}(\Sigma)}^{\|\cdot\|_{m,\Phi}}$ is the *Orlicz space with respect to the vector measure* m, and will be denoted by $L^\Phi(m)$.

Example 5.1.6. Notice that, for $\Phi_p(s) = s^p,\ s \geq 0$, the spaces $L^{\Phi_p}_w(m)$ and $L^{\Phi_p}(m)$ correspond, respectively, to the spaces of weakly $p-$integrable and $p-$integrable functions with respect to the vector measure m.

Recall λ is a Rybakov measure for the vector measure m. By Propositions 3.5, 3.8 and 4.1 in [21] we have:

(i) $L^\Phi_w(m)$ is a $\lambda -$ B.f.s. with the Fatou property and is continuously included in $L^1_w(m)$, (5.12)

(ii) $L^\Phi(m)$ is an order continuous $\lambda -$ B.f.s. and is continuously included in $L^1(m)$, (5.13)

(iii) $L^\Phi_w(m) = \{f \in L^0(m) : \rho(x^*, f) < \infty$ for all $x^* \in X^*\}$. (5.14)

As in the scalar situation, we now define the Orlicz classes

$$\begin{aligned} O^\Phi_w(m) &:= \{f \in L^0(m) : \Phi(|f|) \in L^1_w(m)\}, \\ O^\Phi(m) &:= \{f \in L^0(m) : \Phi(|f|) \in L^1(m)\}. \end{aligned}$$

and the associated Orlicz functional $\rho_{m,\Phi}(f) \equiv \|\Phi(|f|)\|_{L^1(m)}$. The following result is the corresponding extension of Lemma 5.1.3 to Orlicz spaces with respect to a vector measure.

Lemma 5.1.7. *Let $f \in L^0(m)$.*

(i) If $\|f\|_{m,\Phi} \leq 1$, then $\rho_{m,\Phi}(f) \leq \|f\|_{m,\Phi}$.

(ii) If $\|f\|_{m,\Phi} > 1$, then $\rho_{m,\Phi}(f) \geq \|f\|_{m,\Phi}$.

(iii) $\|f\|_{m,\Phi} \leq 1$ if and only if $\rho_{m,\Phi}(f) \leq 1$.

(iv) $O^\Phi_w(m) \subset L^\Phi_w(m)$.

Proof. In order to prove (*i*), let $f \in L^0(m)$ satisfy $\|f\|_{m,\Phi} \leq 1$ and take $x^* \in B(X^*)$. Then $\|f\|_{L^\Phi(|\langle m,x^*\rangle|)} \leq 1$. By (*i*) in lemma 5.1.3 this implies

$$\|\Phi(|f|)\|_{L^1(|\langle m,x^*\rangle|)} \leq \|f\|_{L^\Phi(|\langle m,x^*\rangle|)} \leq \|f\|_{m,\Phi},$$

and the conclusion follows immediately.

To show (*i*), assume that $\|f\|_{m,\Phi} > 1$ and take ε such that $0 < 2\varepsilon < \|f\|_{m,\Phi} - 1$. Next choose $x^* \in B(X^*)$ such that $\|f\|_{L^\Phi(|\langle m,x^*\rangle|)} \geq \|f\|_{m,\Phi} - \varepsilon \geq 1 + \epsilon$. Assertion (*ii*) of Lemma 5.1.3 yields $\rho_{|\langle m,x^*\rangle|,\Phi}(f) \geq \|f\|_{m,\Phi} - \varepsilon$. If we take $\varepsilon \to 0$ we obtain the conclusion.

Assertion (*iii*) is a direct consequence of (*i*) and (*ii*).

To prove (*iv*) let $f \in O^\Phi_w(m)$ and take $x^* \in B(X^*)$. Then we have $\Phi(|f|) \in L^1(|\langle m, x^*\rangle|)$. Using now (5.7), we have $f \in L^\Phi(|\langle m, x^*\rangle|)$. By (5.14) this implies $f \in L^\Phi_w(m)$. □

Assuming that the Young's function Φ has the Δ_2-property with $s_0 = 0$, O. Delgado established in [21, Props. 4.2 and 4.4] the following results that we extend to the case $s_0 > 0$.

Proposition 5.1.8. *Let Φ have the Δ_2-property. Then:*

(i) $L^\Phi_w(m) = O^\Phi_w(m)$.

(ii) A sequence $(f_n)_n \subset L^0(\mu)$ converges to 0 in $L^\Phi_w(m)$ if and only if $(\Phi(|f_n|))_n$ converges to 0 in $L^1_w(m)$.

(iii) $L^\Phi(m) = O^\Phi(m)$.

Proof. Assertion (*i*) follows immediately from the corresponding scalar case (5.9).

In order to prove (*ii*) assume first that $\|f_n\|_{m,\Phi} \to 0$. Then from (*i*) in Lemma (5.1.7) we conclude that $\|\Phi(|f_n|)\|_{L^1(m)} \to 0$. Now let us assume that $\|\Phi(|f_n|)\|_{L^1(m)} \to 0$. For $\varepsilon > 0$, there is some $j \in \mathbb{N}$ such that $2^{-j} < \varepsilon$. Moreover we can choose $s_1 > 0$ small enough so that

$$\Phi(2^j s_1)\|m\|(\Omega) < \frac{1}{2}. \tag{5.15}$$

By Remark 5.1.4 there is some $C > 0$ such that

$$\Phi(2s) \leq C\Phi(s), \ \forall\, s \geq s_1. \tag{5.16}$$

We can now find $N \in \mathbb{N}$ so that

$$C^j\|\Phi(|f_n|)\|_{L^1(m)} \leq \frac{1}{2}, \ \forall\, n \geq N. \tag{5.17}$$

Let $n \geq N$ and $x^* \in B(X^*)$. Take $\Omega_n := \{x \in \Omega : |f_n(x)| \geq s_1\}$, previous inequalities (5.15)-(5.17), yields

$$\begin{aligned}
\int_\Omega \Phi(2^j|f_n|)\, d|\langle m, x^*\rangle| &= \int_{\Omega\setminus\Omega_n} \Phi(2^j|f_n|)\, d|\langle m, x^*\rangle| \\
&\quad + \int_{\Omega_n} \Phi(2^j|f_n|)\, d|\langle m, x^*\rangle| \\
&\leq \Phi(2^j s_1)\|m\|(\Omega) + C^j \int_\Omega \Phi(|f_n|) d|\langle m, x^*\rangle| \\
&\leq \frac{1}{2} + C^j\|\Phi(|f_n|)\|_{L^1(m)} \leq 1.
\end{aligned}$$

It follows that $\|f_n\|_{L^\Phi(|\langle m,x^*\rangle|)} \leq 2^{-j}, \ \forall\, n \geq N, \ \forall\, x^* \in B(X^*)$. Hence, $\|f_n\|_{m,\Phi} \leq 2^{-j} < \varepsilon, \ \forall\, n \geq N$.

To show (iii) let $f \in L^\Phi(m)$. Since $L^\Phi(m)$ is order continuous, from Lemma 1.0.1 we have

$$L^\Phi(m) = \{f \in L^0(\mu) : \lim_{\mu(A)\to 0} \|f\chi_A\|_{m,\Phi} = 0\} \tag{5.18}$$

By (ii) this implies

$$\lim_{\mu(A)\to 0} \|\Phi(f)\chi_A\|_{L^1(m)} = \|\Phi(f)\chi_A\|_{m,\Phi} = 0.$$

Using (5.18) we conclude that $\Phi(f) \in L^1(m)$. The other containment can be proven in a similar way. $\square$

5.2. Multiplication Operators

As claimed in the introduction, we are interested in the study of multiplication operators between Orlicz spaces. We begin this section with some basic properties of $M(W,Y)$, the space of multiplication operators from W into Y, where W and Y are Banach function spaces with respect to μ,

$$M(W,Y) := \{g \in L^0(\mu) : fg \in Y \text{ for every } f \in W\}.$$

Notice that if $g \in M(W,Y)$ can be identified to the multiplication operator $M_g : W \to Y$ defined as $M_g(f) := fg$ for $f \in W$.

Proposition 5.2.1. *The following asssertions holds for $W \subset Y$ Banach functions spaces.*

(i) *If $g \in M(W,Y)$, then $M_g \in \mathcal{L}(W,Y)$.*

(ii) *The function $\|g\|_M := \|M_g\|$ defines a norm on $M(W,Y)$.*

(iii) *$M(W,Y)$ is a B.f.s. with respect to μ.*

(iv) *If Y has the Fatou property, then $M(W,Y)$ also has it.*

Proof. Since W and Y are Banach spaces, to prove (i) it is enough to show that M_g has closed graph. So, let us assume that $(f_n) \subset W,\ f \in W,\ h \in Y$ and $f_n \to f$ in W, $gf_n \to h$ in Y. Lattice property of the Banach function spaces yields the existence of subsequence of (f_n) such that $f_{n_k} \to f\ \mu-$a.e. and $gf_{n_k} \to h\ \mu$-a.e. It follows that $gf = h\ \mu-$a.e. Therefore the graph of M_g is closed.

In order to prove (ii) notice that the association $g \to M_g$ is linear. It follows that the function $\|\cdot\|_M$ is a seminorm. Assume that $g \in M(W,Y)$ and $gw = 0, \forall w \in W$. Since $\chi_\Omega \in W$, we have $g = g\chi_\Omega = 0$.

To show (iii) let $f \in L^0(\mu)$ and $g \in M(W,Y)$ be such that $0 \le |f| \le |g|$ and take $w \in W$. Define $h(x) = f(x)/g(x)$ when $g(x) \ne 0$ and $h(x) = 0$ when $g(x) = 0$. Then $h \in L^0(\mu)$ and $|h| \le 1$. Since W has the lattice property, we get $hw \in W$. Therefore $fw = ghw \in Y$. Thus $f \in M(W,Y)$. Moreover, for each $w \in W$ we get

$$\|fw\|_Y = \|ghw\|_Y \le \|g\|_M \|hw\|_W \le \|g\|_M . \|w\|_W,$$

This implies that $\|f\|_M \le \|g\|_M$. And the space of multiplication operators has the lattice property.

Since $W \subset Y$, we have that $\chi_\Omega w = w \in Y$ for all $w \in W$. This shows that $\chi_\Omega \in M(W,Y)$. Using now the lattice property we just established, we conclude that $\chi_A \in M(W,Y)$ for all $A \in \Sigma$.

Take $g \in M(W, Y)$. Since $\chi_\Omega \in W$, we have

$$\|g\|_Y = \|g\chi_\Omega\|_Y \leq \|g\|_M \|\chi_\Omega\|_W. \tag{5.19}$$

This shows that $M(W, Y)$ is continuously included in Y. Since Y is also continuously included in $L^1(\mu)$, it follows that $M(W, Y)$ is continuously included in $L^1(\mu)$.

It only rests to prove that $M := M(W, Y)$ is complete. For this aim take a sequence $(g_n)_n$ in M satisfying $\sum_{n=1}^\infty \|g_n\|_M < \infty$, and take $f \in W$. Therefore

$$\sum_{n=1}^\infty \|fg_n\|_Y \leq \|f\|_W \sum_{n=1}^\infty \|g_n\|_M < \infty$$

As a direct application of Riez–Fisher criterion (see [78, Section 64]), the inequality above implies $\sum_{n=1}^\infty |f||g_n| \in Y$. Thus $\sum_{n=1}^\infty |g_n| \in M$. Another application of Riez–Fisher criterion yields the completeness of M.

To prove assertion (*iv*) let $(g_n)_n \in M$ be an increasing sequence so that $\|g_n\|_M \leq C$ for some positive constant C. If $0 \leq f \in W$, then $0 \leq g_n f \uparrow$ is an increasing sequence in Y, with $\|g_n f\|_Y \leq C\|f\|_W$. Since Y has the Fatou property, this implies that the limit $gf \in Y$ and $\|gf\|_Y \leq C\|f\|_W$. Hence $g \in M(W, Y)$ and $\|g\|_M \leq \sup_n \|g_n\|_m$, and the proof is complete. □

Multiplication operators on vector measure Orlicz spaces. In this section we will study the spaces of multiplication operators $M(L^\Phi(m), L^1(m))$, $M(L^\Phi(m), L^1_w(m))$, $M(L^\Phi_w(m), L^1_w(m))$ and $M(L^\Phi_w(m), L^1(m))$. We begin by proving some results that show the vector measure duality between the spaces $L^\Phi(m)$ and $L^\Psi(m)$. We start with the following lemma, which provides a Hölder's inequality for vector measure Orlicz spaces.

Lemma 5.2.2. *Let $f \in L^\Phi_w(m)$ and $g \in L^\Psi_w(m)$, then $fg \in L^1_w(m)$ and*

$$\|fg\|_{L^1(m)} \leq 2\|f\|_{m,\Phi} \cdot \|g\|_{m,\Psi}.$$

Proof. Using inequality (5.6), for $f \in L^\Phi_w(m)$ and $g \in L^\Psi_w(m)$ we obtain:

$$\begin{aligned} \|fg\|_{L^1(m)} &= \sup_{x^* \in B(X^*)} \int_\Omega |fg| d|\langle m, x^* \rangle| \\ &\leq \sup_{x^* \in B(X^*)} \Big(2\|f\|_{L^\Phi(|\langle m, x^* \rangle|)} \cdot \|g\|_{L^\Psi(|\langle m, x^* \rangle|)} \Big) \\ &\leq 2\|f\|_{m,\Phi} \cdot \|g\|_{m,\Psi}. \end{aligned}$$

□

In (5.12) we stated that $L^\Phi_w(m)$ is continuously included en $L^1_w(m)$, the following proposition strengthens this result.

Proposition 5.2.3. *$L_w^\Phi(m)$ is continuously included in $L^1(m)$.*

Proof. Take $f \in L_w^\Phi(\mu)$ and $A \in \Sigma$. Direct application of inequality (1.9) and inequality (5.5) we get

$$\|f\chi_A\|_{L^1(m)} \leq 2\|f\|_{m,\Phi}\|\chi_A\|_{m,\Psi}.$$

Recall that $\chi_\Omega \in L^\Psi(\mu)$ and $L^\Psi(\mu)$ is an order continuous B.f.s. with respect to λ, a Rybakov's control measure for m. From (1.1) it follows that $\|\chi_A\|_{m,\Psi} \to 0$ when $\lambda(A) \to 0$. Using this in the above inequality, we conclude that $\|f\chi_A\|_{L^1(m)} \to 0$ when $\lambda(A) \to 0$. Since $L^1(m)$ is order continuous, using again (1.1) we conclude that $f \in L^1(m)$. The continuity of the inclusion is obtained having in mind that

$$\|f\|_{L^1(m)} = \|f\chi_\Omega\|_{L^1(m)} \leq 2\|\chi_\Omega\|_{m,\Psi}\|f\|_{m,\Phi}.$$

□

The following proposition generalize Lemma 2 in [7].

Proposition 5.2.4. *For Φ and Ψ conjugated Young's functions*

(i) $L^\Psi(m) \cdot L_w^\Phi(m) \subset L^1(m), \;\; L^\Phi(m) \cdot L_w^\Psi(m) \subset L^1(m)$.

(ii) $L_w^\Psi \cdot L_w^\Phi = L_w^1(m)$.

(iii) *If Ψ has the Δ_2-property, then $L^\Psi(m) \cdot L_w^\Phi(m) = L^1(m)$.*

(iv) *If Φ and Ψ have the Δ_2- property, then $L^\Psi(m) \cdot L^\Phi(m) = L^1(m)$.*

Proof. To prove (i) take $f \in L_w^\Phi(m)$ and $g \in L^\Psi(m)$. Since by construction simple functions are dense in $L^\Psi(m)$, there is a sequence $(g_n)_n \subset \mathcal{S}(\Sigma)$ such that $\|g_n - g\|_{m,\Psi} \to 0$. By Lemma 5.2.3 we have $fg_n \in L_w^\Phi(m) \subset L^1(m)$, for all $n \in \mathbb{N}$. Using Lemma 5.2.2 we obtain $fg \in L_w^1(m)$ and

$$\|fg - fg_n\|_{L^1(m)} \leq 2\|f\|_\Phi \cdot \|g - g_n\|_{m,\Psi}.$$

So we conclude that $fg_n \to fg$ in $L_w^1(m)$. Since $L^1(m)$ is closed in $L_w^1(m)$ and $fg_n \in L^1(m)$ for all $n \in \mathbb{N}$, then $fg \in L^1(m)$.

The other containment is obtained by interchanging Φ with Ψ in what we have just proved.

To show equality (iii), by (i) it remains to prove inclusion $L^1(m) \subset L_w^\Phi(m) \cdot L^\Psi(m)$. For this aim fix $f \in L^1(m)$, by Lemma 5.1.2 we have $|f| \leq \Phi^{-1}(|f|)\Psi^{-1}(|f|)$. From Lemma 5.1.7 it follows that $\Phi^{-1}(|f|) \in$

$L^{\Phi}_w(m)$. Since Ψ has the Δ_2-property, by (*iii*) in Proposition 5.1.8 we have $\Psi^{-1}(|f|) \in L^{\Psi}(m)$. Thus $|f| = f_1 f_2$ where

$$f_1 = \frac{|f|}{\Phi^{-1}(|f|)\Psi^{-1}(|f|)}\Phi^{-1}(|f|) \quad \text{and} \quad f_2 = \Psi^{-1}(|f|).$$

Since $\dfrac{|f|}{\Phi^{-1}(|f|)\Psi^{-1}(|f|)} \leq 1$, we have $f_1 \in L^{\Phi}_w(m)$. Hence we have the decomposition $f = \text{sign}(f) f_1 f_2$.

Using similar arguments we can prove (*ii*) and (*iv*). □

The next result is the vector measure case corresponding to the scalar situation [44, Lemma 9.1]. Recall that ψ is the admissible function defining Ψ.

Theorem 5.2.5. *Let $g \in L^0(m)$ so that $\|g\|_{M(L^{\Phi}_w(m), L^1_w(m))} < 1$, then:*

(i) $f := \psi(|g|) \in L^{\Phi}_w(m)$ *and* $\|f\|_{m,\Phi} \leq 1$.

(ii) $\|\Psi(|g|)\|_{L^1(m)} \leq \|g\|_{M(L^{\Phi}_w(m), L^1_w(m))}$.

Proof. To prove (*i*) let $a > 0$, the function ψ is monotonically increasing, it follows that $\{u \geq 0 : \psi(u) < a\} \subset \mathbb{R}$ is an interval. Since $|g| : \Omega \to \mathbb{R}^+$ is a measurable function, we have that $f = \psi(|g|)$ is also measurable.

Let us suppose that $\|f\|_{m,\Phi} > 1$. Then, by (*ii*) in Lemma 5.1.7 we have $\|\Phi(f)\|_{L^1(m)} > 1$. For $n \in \mathbb{N}$, take $A_n := \{w \in \Omega : |g(w)| \leq n\}$ and $g_n := |g|\chi_{A_n}$. Then for each $n \in \mathbb{N}$, $g_n \in L^{\infty}(m)$, $0 \leq g_n \leq g_{n+1}$ and $g_n \to |g|$ $\lambda-$a.e.; therefore, since ψ is monotonically increasing, we have $0 \leq \psi(g_n) \leq \psi(g_{n+1})$. Consider $w \in \Omega$. If $|g(w)| < \infty$ then, for large enough $n \in \mathbb{N}$ we have $\psi(g_n) = \psi(|g|)$. If $|g(w)| = \infty$, then $g_n(w) \to \infty$ and so $\psi(g_n) \to \infty = \psi(|g(x)|)$. Thus, $\psi(g_n) \to \psi(|g|) = f$ $\lambda-$a.e. in Ω.

Since $L^1_w(m)$ has the Fatou property and $\|\Phi(f)\|_{L^1(m)} > 1$, it follows from above that $\|\Phi(\psi(g_{n_0}))\|_{L^1(m)} > 1$ for some $n_0 \in \mathbb{N}$.

By Young's inequality (see (5.2)), we have

$$0 \leq \Phi(\psi(g_{n_0})) \leq \Phi(\psi(g_{n_0})) + \Psi(g_{n_0}) = g_{n_0}\psi(g_{n_0}). \tag{5.20}$$

Let $M := M(L^{\Phi}_w(m), L^1_w(m))$. From (5.20) and (*ii*) in Lemma 5.1.7 follows

$$\begin{aligned} \|\Phi(\psi(g_{n_0})\|_L^1(m) &\leq \|g_{n_0}\psi(g_{n_0})\|_L^1(m) \qquad (5.21)\\ &\leq \|g_{n_0}\|_M \|\psi(g_{n_0})\|_{m,\Phi} \leq \|g_{n_0}\|_M \|\Phi(\psi(g_{n_0}))\|_{L^1(m)}. \end{aligned}$$

Since g_{n_0} is bounded, it follows that $\Phi(\psi(g_{n_0}))$ is also bounded and so $\|\Phi(\psi(g_n))\|_{L^1(m)} < \infty$. By (5.21), this implies $\|g_{n_0}\|_M \geq 1$. On the hand side, since M is a Banach lattice, from $|g_{n_0}| \leq |g|$ we conclude that $\|g_{n_o}\|_M < 1$. Thus, we have a contradiction.

(ii) From Young's inequality it follows that $0 \leq \Psi(|g|) \leq |g|\psi(|g|)$. Using ($i$), this implies $\|\Psi(g)\|_L^1(m) \leq \||g|\psi(|g|)\|_L^1(m) \leq \|g\|_M$. □

Corollary 5.2.6. *For Φ and Ψ conjugated Young's functions:*

(i) $\|g\|_{m,\Psi} \leq \|g\|_{M(L^\Phi_w(m),L^1_w(m))} \leq 2\|g\|_{m,\Psi}$, *for* $g \in L^\Psi_w(m)$,

(ii) $\|g\|_{M(L^\Phi_w(m),L^1_w(m))} = \|g\|_{M(L^\Phi(m),L^1(m))}$, *for* $g \in L^\Psi_w(m)$.

Proof. We will take $M := M(L^\Phi_w(m), L^1_w(m)),\ M_0 := M(L^\Phi(m), L^1(m))$.

In order to prove (i) let $g \in L^\Psi_w(m)$. From Lemma 5.2.2 we get $\|g\|_M \leq 2\|g\|_{m,\Psi}$. It only rests to establish the first inequality. Let $0 < r < 1$. By (ii) in Theorem 5.2.5, we have $\left\|\Psi(r\frac{|g|}{\|g\|_M})\right\|_{L^1(m)} \leq 1$. Take $x^* \in B(X^*)$, then $\int_\Omega \Psi\left(\frac{r|g|}{\|g\|_M}\right) d(|\langle m, x^*\rangle|) \leq 1$. By the definition of the Luxemburg norm this implies $\|rg\|_{L^\Psi(|\langle m,x^*\rangle|)} \leq \|g\|_M$. Letting $r \to 1$, we conclude that $\|g\|_{m,\Psi} \leq \|g\|_M$.

(ii) Let $g \in L^\Psi(m), g \neq 0$ and take $0 < r < 1$. Using what we have just established in (i) we obtain $f \in B(L^\Phi_w(m))$ such that $\|gf\|_{L^1(m)} > r\|g\|_M$. Given $n \in \mathbb{N}$, take $A_n := \{w \in \Omega : |g(w)| \leq n\}$ and $f_n := |f|\chi_{A_n}$. Then $f_n \in L^\infty(m),\ f_n \in B(L^\Phi(m)),\ 0 \leq f_n \leq f_{n+1}$ and $f_n \to |f|$. Since $L^1_w(m)$ has the Fatou property, this implies $\|gf_n\|_{L^1(m)} \to \|gf\|_{L^1(m)}$. Hence $\|gf_n\|_{L^1(m)} > r\|gf\|_{L^1(m)} > r^2\|g\|_M$, for some $n \in \mathbb{N}$. Therefore $\|g\|_{M_0} \geq r^2\|g\|_M$. Letting $r \to 1$, we conclude that $\|g\|_{M_0} \geq \|g\|_M$. The conclusion follows having in mind that $\|g\|_M \geq \|g\|_{M_0}$. □

Theorem 5.2.7. *The following equalities hold for Ψ and Φ conjugated Young's functions and $m : \Sigma \to X$ a vector measure:*

$$\begin{aligned} L^\Psi_w(m) &= M(L^\Phi(m), L^1(m)) \\ &= M(L^\Phi(m), L^1_w(m)) \\ &= M(L^\Phi_w(m), L^1_w(m)). \end{aligned}$$

Proof. In Proposition 5.2.4 we established $L^\Psi_w(m) \subset M(L^\Phi(m), L^1(m))$. Clearly $M(L^\Phi(m), L^1(m)) \subset M(L^\Phi(m), L^1_w(m))$.

From Lemma 5.2.2 we have $L^\Psi_w(m) \subset M(L^\Phi_w(m), L^1_w(m))$ and clearly

$$M(L^\Phi_w(m), L^1_w(m)) \subset M(L^\Phi(m), L^1_w(m)).$$

It only rests to prove $M := M(L^\Phi(m), L^1_w(m)) \subset L^\Psi_w(m)$. Take $g \in L^0(m)$ and, for $n \in \mathbb{N}$, consider $A_n := \{w \in \Omega : |g(w)| \leq n\}$ and $g_n \equiv$

$|g|\chi_{A_n}$. Then $g_n \in L^\infty(m) \subset L^\Psi_w(\mu)$, $0 \le g_n \le g_{n+1}$ and $g_n \to |g|$. Since $0 \le g_n \le |g|$, by Proposition 5.2.1 we have $\|g_n\|_M \le \|g\|_M$, for all $n \in \mathbb{N}$. Applying (*i*) of Theorem 5.2.6, this implies (g_n) is also bounded in $L^\Psi_w(\mu)$. Since $L^\Psi_w(\mu)$ has the Fatou property, the above conditions imply that $g \in L^\Psi_w(\mu)$. □

The following result corresponds to two of the equalities we have just established.

Corollary 5.2.8. *Let $g \in L^0(m)$. Then:*

i) $gf \in L^1_w(m)$, $\forall f \in L^\Phi_w(\mu)$ if, and only if, $gf \in L^1(m)$, $\forall f \in L^\Phi(\mu)$.

ii) $gf \in L^1_w(m), \forall f \in L^\Phi(\mu)$ if, and only if, $gf \in L^1(m), \forall f \in L^\Phi(\mu)$.

Theorem 5.2.9. *If Ψ has the Δ_2-property, then*

$$L^\Psi(m) = M(L^\Phi_w(m), L^1(m)).$$

Proof. From Proposition 5.2.4 we have $L^\Psi(m) \subset M(L^\Phi_w(m), L^1(m))$.

Take now $g \in M(L^\Phi_w(m), L^1(m))$. Then $g \in M(L^\Phi_w(m), L^1_w(m))$. So we can apply Theorem 5.2.7 to conclude that $g \in L^\Psi_w(m)$. Since Ψ has the Δ_2-property $\Psi(|g|) \in L^1_w(m)$ and then by Lemma 5.1.7 $\Phi^{-1}(\Psi(|g|)) \in L^\Phi_w(m)$. Thus $g\Phi^{-1}(\Psi(|g|)) \in L^1(m)$.

Applying Lemma 5.1.2, we get

$$\Psi(|g|) \le \Psi^{-1}(\Psi(|g|))\Phi^{-1}(\Psi(|g|)) = |g|\Phi^{-1}(\Psi(|g|)).$$

Since $L^1(m)$ is a Banach lattice, it follows that $\Psi(|g|) \in L^1(m)$. Using now (*iii*) in Proposition 5.1.8, we get $g \in L^\Psi(m)$. □

5.3. Applications: operators factorizing through vector measure Orlicz spaces

In this last section our aim is to characterize the class of operators defined in a B.f.s. with range in a Banach space that factorize through a vector measure Orlicz space, indeed these spaces turn out to be the optimal domains for such operators. The theory of optimal domains for continuous operators defined on B.f.s. has been developed recently: see for instance the book by S. Okada, W. Ricker and E. A. Sánchez-Pérez, [58]. We begin with a technical lemma that might be known, but whose proof we include for the sake of completeness.

Lemma 5.3.1. *Let Z be a B.f.s. continuously included in $L^1(m)$. If $f \in M = M(Z, L^1(m))$ then $\|f\|_M = \sup\left\{\left\|\int_\Omega fgdm\right\|_X : g \in B(Z)\right\}$.*

Proof. We will apply Lemma 3.11 in [58] which indicates that the norm of $h \in L^1(m)$ can be computed as

$$\|h\|_{L^1(m)} = \sup\left\{\left\|\int_\Omega shdm\right\| : s \in \mathcal{S}(\Sigma) \cap B(L^\infty(m))\right\}.$$

Hence

$$\|f\|_M = \sup_{r \in B(Z)} \sup\left\{\left\|\int_\Omega sfrdm\right\| : s \in \mathcal{S}(\Sigma) \cap B(L^\infty(m))\right\},\ f \in M.$$

Recall that Z has the lattice property. Hence, if $r \in B(Z)$ and $s \in \mathcal{S}(\Sigma) \cap B(L^\infty(m))$, then we have $sr = g \in B(Z)$. From the above equality this implies the conclusion. $\square$

Throughout what follows we will assume that W is an order continuous Banach function space with respect to μ and $T \in \mathcal{L}(W, X)$. It follows that the set function $m_T : \Sigma \to X$ defined by $m_T(A) = T(\chi_A)$ for $A \in \Sigma$ is a countably additive vector measure. We will suppose the operator T is *$\mu-$determined*, that is, $\mu(A) \to 0$ whenever $\|m_T\|(A) \to 0$. Note that $\int_\Omega fdm_T = T(f)$, $\forall f \in W$, holds. The following proposition provides a characterization of bounded operators that factorize through a vector measure Orlicz space.

Proposition 5.3.2. *Let $T : W \to X$ be a $\mu-$determined bounded operator. Then, the following assertions are equivalent.*

(i) There is a constant $K > 0$ such that

$$\|T(fg)\|_X \leq K\|f\|_W\|g\|_{m_T,\Psi},\ \forall f \in W, g \in \mathcal{S}(\Sigma). \tag{5.22}$$

(ii) T satisfies the following factorization diagram

$$\begin{tikzcd}
W \arrow[r, "T"] \arrow[dr, "i"'] & X \\
& L^\Phi(m_T) \arrow[u, "I"']
\end{tikzcd}$$

h where i and I are the respective inclusion and integration maps.

Moreover $L^\Phi(m_T)$ is the optimal domain, in the sense that if Z is a μ-B.f.s. such that $W \hookrightarrow Z$ and (5.22) *holds with Z instead of W, then Z will be continuously included in $L^\Phi(m_T)$.*

Proof. Let $M_\Phi := M(L^\Phi(m_T), L^1(m_T))$ and $M_\Psi := M(L^\Psi(m_T), L^1(m_T))$.

We will first prove that (*ii*) implies (*i*). Let $f \in W,\ g \in S(\Sigma)$. Applying the above lemma and corollary 5.2.6 we have

$$\begin{aligned} \|T(fg)\|_X &= \left\| \int_\Omega fg dm_T \right\|_X \leq \|f\|_{m_T,\Phi} \|g\|_{M_\Phi} \\ &\leq K\|f\|_W \|g\|_{M_\Phi} \leq 2K\|f\|_W\|g\|_{m_T,\Psi}. \end{aligned}$$

To prove the converse we first show that $W \subset L^\Phi(m_T)$ and that the inclusion is continuous. By hypothesis, for every $g \in \mathcal{S}(\Sigma)$ we have $\|T(fg)\|_X \leq K\|f\|_W\|g\|_{m_T,\Psi}$. Hence, from lemma 5.3.1 we get

$$\begin{aligned} \|f\|_{M_\Psi} &= \sup_{g\in B(L^\Psi(m_T))} \left\| \int_\Omega fg dm_T \right\|_X \\ &= \sup_{g\in B(L^\Psi(m_T))} \left\{ \left\| \int_\Omega fg dm_T \right\|_X : g \in \mathcal{S}(\Sigma) \right\} \\ &\leq \|T(f)\|_X \leq K\|f\|_W. \end{aligned}$$

Corollary 5.2.6 yields now $\|f\|_{m_T,\Phi} \leq K\|f\|_W$. Thus, $W \subset L^\Phi_w(m_T)$ and the inclusion is continuous.

To prove the injectivity of i let $A \in \Sigma$, then we have

$$\|f\chi_A\|_{m_T,\Phi} \leq K\|f\chi_A\|_W.$$

Since T is $\mu-$determined, note that $\mu(A) \to 0$ if, and only if, $\lambda(A) \to 0$, where λ is a Rybakov measure for m_T. Since W and $L^\Phi(m_T)$ are order continuous B.f.s., it follows from the inequality above that $\|f\chi_A\|_{m_T,\Phi} \to 0$ when $\mu(A) \to 0$. By (1.1), this implies that $f \in L^\Phi(m_T)$.

It lasts to prove the optimality of $L^\Phi(m_T)$. So suppose that there is a μ-B.f.s. Z and some $K > 0$ such that $W \hookrightarrow Z$ and $\|T(fg)\|_X \leq K\|f\|_Z\|g\|_{m_T,\Psi}$, $\forall f \in Z$ and $g \in \mathcal{S}(\Sigma)$. Similar arguments as those used just before show that Z is continuously included in $L^\Phi(m_T)$. $\square$

Index

Symbols

$B_{\rho,\varepsilon}(\,\cdot\,)$ 23
$\mathcal{B}(K)$ 5

$\mathcal{C}(K)$ 4

$d_r(\,\cdot\,)$ 90
$d_r^m(\,\cdot\,)$ 91

$\varepsilon(\cdot)$ 89
$\mathrm{Ext}(K)$ 19

Γ 13
γ_{g,x^*} 13

$\mathcal{H}$ 9

I_f 14
Id_X 11

$K_{r,s}$ 79

$L^0(\mu)$ 1
$L^1(\mu)$ 1
$L^1(m)$ 6
$L^1_w(m)$ 10
$L^\Phi(\mu)$ 101
$L^\Phi(m)$ 104
$L^p(m)$ 7
Λ 13
$\mathcal{L}(X,Y)$ 11
$\ell_r^m(L^p(m))$ 58
$\ell_r(X)$ 54
$\ell_r^w(X)$ 54
λ 4

$M(W,Y)$ 107
$\mathbf{M}^{(p)}\,[X(\mu)]$ 3
$\mathbf{M}^{(q)}[\cdot]$ 11
$\mathbf{M}_{(q)}[\cdot]$ 11
μ 1
$\mathcal{M}\,(E,F)$ 40

$O^\Phi(m)$ 105
$O^\Phi(\mu)$ 102
$O^\Phi_w(m)$ 105
Ω 1

$\Pi_r(X,Y)$ 54
$\Pi_r^m(L^p(m),Y)$ 57
$\pi(\cdot)$ 89
π_r 54
$\pi_r^m(T)$ 57
$\pi_r^{w-m}(T)$ 62
φ_h 9
p_g 40

ρ 23
$\rho_{\mu,\Phi}$ 102
$r_i(t)$ 79

Σ 1
$\mathcal{S}(\Sigma)$ 1
$\sigma(L^p(m),\Gamma)$ 13
$\sigma(L^p(m),\Lambda)$ 14

$T_p(\cdot)$ 79
τ 40
τ_u 44
τ_{weak^*} 41
$\tilde{\tau}$ 43

$V_{g,\varepsilon}(\,\cdot\,)$ 14
$V_{g,x^*,\varepsilon}(\,\cdot\,)$ 13

$\mathcal{W}\,(E,F)$ 40

X 3
$X(\mu)$ 1
$X(\mu)^+$ 1
$X(\mu)'$ 2
$X(\mu)_{[p]}$ 3
X_a 10
χ_A 1

ζ_g 13

A

absolutely continuous norm 2
admissible function 100

B

Banach function space 1
 B.f.s., 1
Banach lattice 1
Banach space
 injective, 56

C

control measure 4
cotype 79
crossnorm
 finitely generated, 89
 $m-$reasonable, 90
 reasonable, 88
 uniform, 89
 uniform $m-$, 90

D

Δ_2-property 102
dense family of subspaces 69

E

$\varepsilon-$dense 16
extremal set 19

F

Fatou property 2
 $\sigma-$Fatou property, 2

G

generalized tensor norm 91

I

integrable
 $B-$integrable, 6
 $m-$integrable, 4
 $p-$integrable, 7
 $\mathcal{S}^*-$integrable, 5
 scalarly $m-$integrable, 10
integration operator 14

J

James boundary 19

K

Köthe dual 2
Kahane's Inequality 79
Ky Fan's Lemma 45

L

lattice norm 1

M

M–weakly compact 40
$m-$compact 22, 24
$m-$complete 24
$m-$convergent 24
$m-\varepsilon-$norming set 29
$m-$norming set 26
$m-$essentially countably generated 16

$m-$integrable 4
$m-$norm 103
$m-r-\varepsilon-$normed 75
$m-$separation property 23
$m-$topology 13
$m-$weak topology 13
$\mu-$determined 113

N

norm
  $m-$tensor, 90
  Chevet-Saphard, 89
  injective, 89
  Luxemburg, 101
  Orlicz, 101
  projective, 89
  tensor, 89

O

operator 11
  ideal, 11
  $m-r-$summing, 57
  $q-$concave, 11
  $q-$convex, 11
  quotient, 94
  $r-$summing, 53
  $(s, m-r)-$mixing, 86
  $(s, r)-$mixing, 85
  weak $m-r-$summing, 62
order continuity 1
  $\sigma-$order continuity, 2
order ideal 1
Orlicz
  space with respect to m, 104
  class, 102
  functional, 102
  norm, 101
  space, 101
  weak space, 104

P

$p-$convex 3
$p-$integrable 7
$p-$th power 3
positive finite measure space 1
positively balanced 74
property $(r-\star)$ 69

R

$r-\varepsilon-$norming 69
Rademacher functions 78
Rybakov measures 4

S

scalarly $m-$integrable 10
scalarly dominated 4
sequence
  $m-r-$summable, 58
  strongly $r-$summable, 54
  weakly$-r-$summable, 54
$\sigma-$additive 3
strong operator topology 11

T

tensor norm 89
  $m-$right projective, 94
totally bounded 16
trace duality 90
type 78

U

uniform operator topology 11

V

vector duality 8
vector measure 3
  positive, 21
  semivariation, 4
  $\sigma-$additive, 3
  variation, 3

W

weak order unit 1
wealky compactly generated
WGC, 6

Y

Young's function 100
conjugated, 100
Young's inequality 101

Bibliography

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