A stable family with high order of convergence for solving nonlinear equations *

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Abstract

Recently, Li et al. [1] have published a new family of iterative methods, without memory, with order of convergence five or six, which are not optimal in the sense of Kung and Traub's conjecture. Therefore, we attempt to modify this suggested family in such a way that it becomes optimal. To this end, we consider the same two first steps of the mentioned family, and furthermore, we introduce a better approximation for $f'(z)$ in the third step based on interpolation idea as opposed to the Taylor's series used in the work of Li et al. Theoretical, dynamical and numerical aspects of the new family are described and investigated in details.
Keywords: Nonlinear equations, optimal iterative methods, efficiency index, parameter space, basin of attraction, stability.

1 Introduction

Iterative methods for approximating simple zeros of a real-valued function is an active research area which has progressed thanks to the advances in modern computers both in software and hardware. The principal base for constructing these methods is the significant and substantial works by Traub's [2] and Kung and Traub's [3]. In other words, Traub classified iterative methods in some sense. Here, we are interested in iterative multi-point methods without memory. Regarding the construction of these methods, we need and recall two basic criteria. Traub says for constructing an one-point method, having convergence order p, we require p functional evaluations. Designs and developments of one-point methods is not a considerable task because of its less efficiency. On the other hand, Kung and Traub conjectured that any multi-point iterative method, without memory, with $d+1$ functional evaluations per step, has an order of convergence at most 2^d. When this bound is reached the method is called *optimal*. Indeed, these methods overcome theoretical and practical issue of single point methods regarding computational evaluations and convergence rate.

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As far as we know, Ostrowski's [4], Jarratt's [5], and King's [6] methods are the first optimal two-point methods of fourth-order. Moreover, Neta [7] and Bi et al. [8, 9] are pioneers in developing optimal eighth-order methods. Recently, new families of iterative methods of optimal eighth-order have been published in [10, 11, 12]. A good review of optimal and no optimal iterative schemes of different orders of convergence can be found in [13].

Recently, Li et al. [1] have developed a new family of three-point methods based on modification of Chebyshev-Halley's scheme. It is worth mentioning that the methods of this family are not optimal in the sense of Kung-Traub's conjecture, since it consumes four functional evaluations per iteration having convergence order five, and a particular member of the family has convergence order six. This family, denoted by LLK5, has the following iterative expression (indices are dropped for simplicity).

$$\begin{cases} y = x - \frac{f(x)}{f'(x)}, \\ z = x - \left(1 + \frac{f(y)}{f(x)-2\beta f(y)}\right)\frac{f(x)}{f'(x)}, \\ \hat{x} = z - \frac{f(z)}{f'(x)+\tilde{f}''(x)(z-x)}, \end{cases} \tag{1.1}$$

where β is a real parameter and $\tilde{f}''(x) = \frac{2f(y)f'(x)^2}{f(x)^2}$. Moreover, its error equation is

$$\hat{e} = 2(\beta-1)c_2^2c_3e^5 + \Big((4\beta^2 - 14\beta + 9)c_2^3c_3 + (8\beta-7)c_2c_3^2\Big)e^6 + O(e^7), \tag{1.2}$$

being $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k = 2, 3, \ldots$, $e = x - \alpha$ and α a root of $f(x) = 0$.

It can easily be observed that for $\beta = 1$, the first two steps result Ostrowski's method, and, in addition, the convergence order becomes six. In this work, we attempt to derive an optimal three-point method without memory from (1.1) by changing the denominator of the last step. To this end, we suitably approximate $f'(z)$ in the third step instead of $f'(x) + \tilde{f}''(x)(z-x)$ which has been computed by Li et al. [1]. Our approximate is based on Newton-Hermite interpolation at the given data $f(x)$, $f'(x)$, $f(y)$, and $f(z)$. This optimal eighth-order scheme is a particular case of a sixth-order iterative family, depending on parameter β.

Thereafter, we will analyze the stability of the elements of this class of iterative schemes on quadratic polynomials, in terms of the asymptotic behavior of their fixed points and also by using the associated parameter planes, that will allow us to find the most stable elements of the family, under a numerical point of view.

Now, we are going to recall some dynamical concepts of complex dynamics (see [14]) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), ..., R^n(z_0), ...\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_f \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_f) = z_f$. A *periodic point* z of period $p > 1$ is a point such that $R^p(z) = z$ and $R^k(z) \neq z$, for $k < p$. A *pre-periodic point* is a point z that is not periodic but there exists a $k > 0$ such that $R^k(z)$ is periodic. A *critical point* z^* is a point where the derivative of the rational function vanishes, $R'(z^*) = 0$. Moreover, a fixed point z_f is called *attractor* if $|R'(z_f)| < 1$, *superattractor* if $|R'(z_f)| = 0$, *repulsor* if $|R'(z_f)| > 1$ and *parabolic* if $|R'(z_f)| = 1$. So, a superattracting fixed point is also a critical point.

On the other hand, the *basin of attraction* of an attractor $\alpha \in \hat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to α:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \ : \ R^n(z_0) \to \alpha, \ n \to \infty\}.$$

The *Fatou set* of the rational function R, $\mathcal{F}(R)$ is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complementary set in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That is, the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The rest of the paper is organized as follows. In Section 2 we present our family of iterative methods, prove the rate of convergence and give the asymptotic error. A particular member of the family is an optimal and very stable eighth-order scheme. In Section 3 we analyze the dynamical behavior of the family, studying the fixed and critical points of the rational function associated to the family on quadratic polynomials. This analysis allows us to obtain some elements of the family with good stability properties. The numerical study presented in Section 4 confirm the theoretical results and allows us to compare our methods with other known ones. The paper finishes with some conclusions and the references used in it.

2 Improved methods and convergence analysis

The main object of this section is to modify method (1.1) so that it has optimal convergence order eight: it must use only four function evaluations. We keep on the first two steps of (1.1), which lead a parametric family of iterative schemes of order three for any value of parameter, and we modify the third step. It is sufficient to find a suitable approximate for $f'(z)$ in the denominator of the third step. Although there are some different and effective approaches for this approximation, we prefer to use the Hermite-Newton interpolation method. Suppose $f(x)$, $f'(x)$, $f(y)$, and $f(z)$ are available. Then, the interpolation polynomial is given by

$$H_3(t) = f(z) + (t-z)f[z,y] + (t-z)(t-y)f[z,y,x] + (t-z)(t-y)(t-x)f[z,y,x,x], \quad (2.1)$$

where

$$f[x_0, x_1, \ldots, x_{k-1}, x_k] = \begin{cases} \frac{f[x_1,\ldots,x_k]-f[x_0,\ldots,x_{k-1}]}{x_k-x_0}, & x_0 \neq x_k, \\ \frac{f^{(k)}(x)}{k!}, & x_0 = \ldots = x_k (= x), \end{cases}$$

is the generalized divided differences at $x_0 \leq x_1 \leq \cdots \leq x_{k-1} \leq x_k$. Hence,

$$\begin{aligned} H_3'(t) &= f[z,y] + (z-y)f[z,y,x] + (z-y)(z-x)f[z,y,x,x] \\ &= f[z,y] + 2(z-y)f[z,y,x] - (z-y)f[y,x,x]. \end{aligned} \quad (2.2)$$

Now, we set $f'(z) \approx H_3'(z)$. Consequently, our modified family, denoted by F6, has the following iterative expression:

$$\begin{cases} y = x - \frac{f(x)}{f'(x)}, \\ z = x - \left(1 + \frac{f(y)}{f(x)-2\beta f(y)}\right)\frac{f(x)}{f'(x)}, \\ \hat{x} = z - \frac{f(z)}{f[z,y]+2(z-y)f[z,y,x]-(z-y)f[y,x,x]}. \end{cases} \quad (2.3)$$

We are going to analyze the convergence order of this family by means of the use of Taylor's expansions for the different expressions of the iterative formula. To do this, we show the Mathematica code for obtaining the mentioned Taylor's series.

Taylor's expansion of $f(x)$ and $f'(x)$ about α, taking into account that $f(\alpha) = 0$, is

$$f(x) = f'(\alpha)(e + c_2e^2 + c_3e^3 + c_4e^4 + c_5e^5 + c_6e^6 + c_7e^7 + c_8e^8) + O(e^9) \quad (2.4)$$

and

$$f'(x) = f'(\alpha)(1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + 6c_6e^5 + 7c_7e^6 + 8c_8e^7) + O(e^8), \quad (2.5)$$

where $e = x - \alpha$ and $c_k = \frac{f^{(k)}(\alpha)}{f'(\alpha)k!}$, $k = 2, 3, \ldots$.

$\texttt{In[1]} := \texttt{f[e\_]} = \texttt{f1a}(\texttt{e} + \texttt{c}_2\texttt{e}^2 + \texttt{c}_3\texttt{e}^3 + \texttt{c}_4\texttt{e}^4);$ (*$f1a$ denotes $f'(\alpha)$*)
$\texttt{In[2]} := \texttt{e}_\texttt{y} = \texttt{e} - \text{Series}\left[\frac{\texttt{f[e]}}{\texttt{f}'\texttt{[e]}}, \{\texttt{e}, 0, 8\}\right];$ (*First Step: Newton's Iteration*)
$\texttt{In[3]} := \texttt{e}_\texttt{z} = \texttt{e} - \left(1 + \frac{\texttt{f[e}_\texttt{y}\texttt{]}}{\texttt{f[e]}-2\beta\texttt{f[e}_\texttt{y}\texttt{]}}\right)\frac{\texttt{f[e]}}{\texttt{f}'\texttt{[e]}};$ (*Second Step: Family of order three*)
$\texttt{In[4]} := \texttt{f[x\_, y\_]} = \frac{\texttt{f[x]}-\texttt{f[y]}}{\texttt{x}-\texttt{y}};$
$\texttt{In[5]} := \texttt{f[x\_, y\_, z\_]} = \frac{\texttt{f[x,y]}-\texttt{f[y,z]}}{\texttt{x}-\texttt{z}};$
$\texttt{In[6]} := \texttt{f[x\_, x\_, y\_]} = \frac{\texttt{f}'\texttt{[x]}-\texttt{f[x,y]}}{\texttt{x}-\texttt{y}};$
$\texttt{In[7]} := \hat{\texttt{e}} = \texttt{e}_\texttt{z} - \frac{\texttt{f[z]}}{\texttt{f[e}_\texttt{z}\texttt{,e}_\texttt{y}\texttt{]}+(\texttt{e}_\texttt{z}-\texttt{e}_\texttt{y})\left(2\texttt{f[e}_\texttt{z}\texttt{,e}_\texttt{y}\texttt{,e]}-\texttt{f[e,e,e}_\texttt{y}\texttt{]}\right)}$//FullSimplify (*Third Step*)

$$\begin{aligned}\texttt{Out[7]} = \hat{\texttt{e}} \;=\; & 4(-1+\beta)^2\texttt{c}_2^5\texttt{e}^6 + 2(-1+\beta)\texttt{c}_2^3\left(2\left(9-14\alpha+4\beta^2\right)\texttt{c}_2^3 + 2(-7+8\beta)\texttt{c}_2\texttt{c}_3 - \texttt{c}_4\right)\texttt{e}^7\\ & +\texttt{c}_2^2\left(\left(201-636\beta+692\beta^2-304\beta^3+48\beta^4\right)\texttt{c}_2^5\right.\\ & +2\left(-151+410\beta-340\beta^2+80\beta^3\right)\texttt{c}_2^3\texttt{c}_3 + \left(23-62\beta+40\beta^2\right)\texttt{c}_2^2\texttt{c}_4\\ & +(11-12\beta)\texttt{c}_3\texttt{c}_4 + \texttt{c}_2\left(\left(73-168\beta+96\beta^2\right)\texttt{c}_3^2\right.\\ & \left.\left.-4(-1+\beta)\texttt{c}_5\right)\right)\texttt{e}^8 + \texttt{O}(\texttt{e}^9). \quad \text{(*Final error equation*)}\end{aligned}$$

We can observe that for $\beta = 1$ the final expression of the error equation is

$$\hat{e} = c_2^2\left(c_2^5 - 2c_2^3c_3 + c_2c_3^2 + c_2^2c_4 - c_3c_4\right)e^8 + O(e^9).$$

As we have seen above, the following result can be stated.

Theorem 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a real function sufficiently differentiable in an open interval I and $\alpha \in I$ a simple root of $f(x) = 0$. Assume that x_0 is an initial guess close enough to α. Then, the modified three-point iterative family (2.3) has order of convergence six, being the case of $\beta = 1$ an optimal scheme with convergence order eight.*

3 Dynamical analysis

The application of iterative methods for solving nonlinear equations on an arbitrary polynomial, gives rise to rational functions whose dynamics are not well-known.

From the numerical point of view, the dynamical properties of the rational function associated with an iterative method give us important information about its stability and reliability. In most of studies presented in the literature (see, for example, [15, 16, 17, 18, 19, 20, 21] for multipoint iterative methods), interesting dynamical planes, including periodical behavior and other anomalies, have been obtained. We are interested in the parameter planes associated to a family of iterative methods, which allow us to understand the behavior of the different elements of the family when they are applied to quadratic polynomials, helping us in the election of particular ones with good numerical properties.

The graphical tools used to obtain the parameter planes and the different dynamical planes have been designed by Chicharro et al. in [22] and are implemented in Matlab language.

Let us consider $p(z) = (z-a)(z-b)$, an arbitrary quadratic polynomial with roots a and b and denote by $M6(z, \beta, a, b)$ the fixed point operator corresponding to the family (2.3) applied to $p(z)$. P. Blanchard, in [23], by considering the conjugacy map

$$H(z) = \frac{z-a}{z-b}, \qquad H^{-1}(z) = \frac{zb-a}{z-1}, \tag{3.1}$$

with the following properties:

$$H(\infty) = 1, \quad H(a) = 0, \quad H(b) = \infty,$$

proved that, for quadratic polynomials, Newton's operator is always conjugate to the rational map z^2. In an analogous way, $M6(z, \beta, a, b)$ is conjugated to operator $F(z, \beta)$,

$$F(z, \beta) = (H \circ M6 \circ H^{-1})(z) = z^6 \frac{(2 - 2\beta + z)^2}{(1 - 2(-1+\beta)z)^2}. \tag{3.2}$$

We observe that parameters a and b have been obviated in $F(z, \beta)$, as a result of the Scaling Theorem that is verified by this iterative scheme. The fixed points of rational function $F(z, \beta)$ that are not associated to the roots of the polynomial $p(z)$ are called *strange fixed points*.

We will study the general convergence of methods (2.3) for quadratic polynomials. To be more precise (see [24, 25]), a given method is generally convergent if the scheme converges to a root for almost every starting point and for almost every polynomial of a given degree.

3.1 Study of the strange fixed and critical points

As we have seen, the sixth-order family of iterative methods (2.3), applied on the generic quadratic polynomial $p(z)$, and after Möbius transformation, gives rise to the rational function (3.2), depending on parameter β. It is clear that this rational function has 0 and ∞ as superattracting fixed points, but also different strange fixed points (that do not correspond to the roots of $p(z)$): $s_1(\beta) = 1$ and the roots of the following polynomial, that will be denoted by $s_i(\beta)$, $i = 2, \ldots, 7$:

$$1 + (5-4\beta)z + \left(9 - 12\beta + 4\beta^2\right) z^2 + \left(9 - 12\beta + 4\beta^2\right) z^3 + \left(9 - 12\beta + 4\beta^2\right) z^4 + (5-4\beta)z^5 + z^6.$$

Nevertheless, the complexity of the operator can be lower depending on the value of the parameter, as we can see in the following result.

Theorem 2. *The number of strange fixed points of operator $F(z, \beta)$ is seven (including $s_1(\beta) = 1$), except in the following cases:*

- *For $\beta = 1$, $\beta = \frac{1}{2}$ and $\beta = \frac{3}{2}$, there not exist strange fixed points and the expression of the operator is, respectively,*

$$F(z, 1) = z^8, \quad F\left(z, \frac{1}{2}\right) = z^6 \quad \text{and} \quad F\left(z, \frac{3}{2}\right) = z^6.$$

- *When $\beta = \frac{13}{6}$, there are five different strange fixed points (being the multiplicity of $s_1(\beta) = 1$ three), and*

$$F\left(z, \frac{13}{6}\right) = z^6 \frac{(7-3z)^2}{(3-7z)^2}.$$

Let us remark that the schemes corresponding to $\beta \in \{1, \frac{1}{2}, \frac{3}{2}\}$ satisfy Cayley's Test (see [26]), being the one of $\beta = 1$ the only element of the family (2.3) whose operator is conjugate to the rational map z^8.

In order to analyze the stability of each one of these strange fixed points, we define the stability function of each fixed point as $St_i(\beta) = |F'(s_i(\beta), \beta)|$, for each $i = 1, 2, \ldots, 7$: if $St_i(\beta) < 1$, then the strange fixed point $s_i(\beta)$ is attractive, parabolic in case of $St_i(\beta) = 1$ and repulsive in other case. As far it is possible, we analyze both analytically and graphically the character of all these points in the complex plane, showing for complex values of β which is the value of the corresponding stability function. It is found that $s_1(\beta) = 1$ is attractive for values of β in a circular region around $\beta = 2$; for this value, $St_1(2) = 0$ and $s_1(2) = 1$ is superattractive. This is shown in the following result.

Theorem 3. *The character of the strange fixed point $s_1(\beta) = 1$ is:*

i) If $\left|\beta - \frac{61}{30}\right| < \frac{2}{15}$, then $s_1(\beta) = 1$ is an attractor and it is a superattractor if $\beta = -2$.

ii) When $\left|\beta - \frac{61}{30}\right| = \frac{2}{15}$, $s_1(\beta) = 1$ is a parabolic point.

iii) If $\left|\beta - \frac{61}{30}\right| > \frac{2}{15}$, then $s_1(\beta) = 1$ is a repulsor.

Proof. It can be easily checked that

$$St_1(\beta) = \frac{8(-2+\beta)}{-3+2\beta}.$$

Then,

$$\left|\frac{8(-2+\beta)}{-3+2\beta}\right| \leq 1 \qquad \text{is equivalent to} \qquad 64\,|-2+\beta| \leq |-3+2\beta|\,.$$

Now, let us consider $\beta = a + ib$ an arbitrary complex number. So,

$$64\left(4 - 4a + a^2 + b^2\right) \leq 9 - 12a + 4a^2 + 4b^2.$$

By simplifying

$$60a^2 + 60b^2 - 244a + 247 \leq 0,$$

that is,

$$\left(a - \frac{61}{30}\right)^2 + b^2 \leq \left(\frac{2}{15}\right)^2.$$

Therefore,

$$|St_1(\beta)| \leq 1 \;\text{ if and only if }\; \left|\beta - \frac{61}{30}\right| \leq \frac{2}{15}.$$

It is clear that $\beta = 2$ makes $s_1(\beta)$ superattracting. Finally, if β satisfies $\left|\beta - \frac{61}{30}\right| > \frac{2}{15}$, then $|St_1(\beta)| > 1$ and $s_1(\beta) = 1$ is a repulsive point. □

Therefore, following Theorem 3, it is stated that the stability region of $s_1(\beta)$, that is, the set of values of β that makes the strange fixed point $s_1(\beta) = 1$ attracting is defined by the area of the complex plane where $\left|\beta - \frac{61}{30}\right| < \frac{2}{15}$. This region has also been obtained numerically and can be seen in Figure 1.

On the other hand, we have stated numerically (by using Mathematica 8.0 software) the stability of strange fixed points $s_i(\beta)$ $i = 2, \ldots, 7$ for values of β in the complex plane, by analyzing the value of their associate stability functions. In Figures 2a and 2b we show the stability region of both of them $St_4(\beta)$ and $St_5(\beta)$. Let us remark that they have as a common region of stability a disk centered in the real axis. These results are summarized in the following theorem.

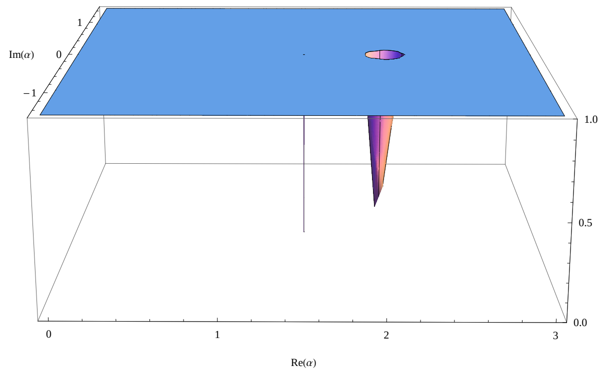
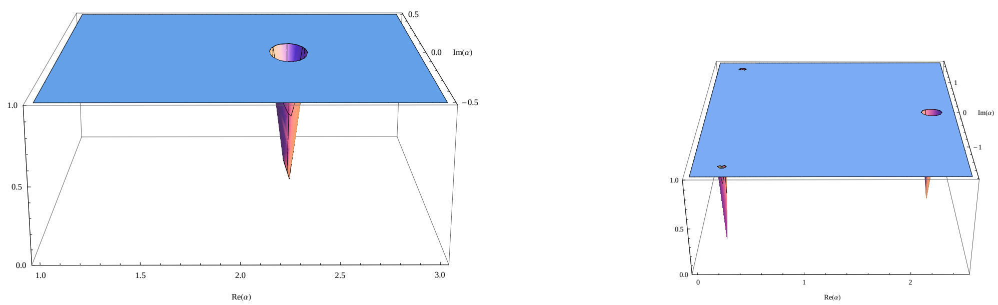


Figure 1: Stability function $St_1(\alpha)$



(a) $St_4(\alpha)$ (b) $St_5(\alpha)$

Figure 2: Stability functions of the strange fixed points $s_4(\beta)$ and $s_5(\beta)$.

Theorem 4. *The stability of the strange fixed points of $F(z,\beta)$ is as follows:*

i) Four of the strange fixed points are repulsive for any value of $\beta \in \mathbb{C}$.

ii) Two strange fixed points, $s_4(\beta)$ and $s_5(\beta)$ are simultaneously attractive for $\beta \in (13/6, 2.2752)$ and simultaneously superattractive for $\beta \approx 0.4304$.

iii) If $\beta = 0.2722+1.4912i$ or $\beta = 0.2722-1.4912i$, then $s_5(\beta)$ is superattracting, meanwhile $s_4(\beta)$ is not superattractive.

By combining the pictures of the stability regions of these fixed points, we obtain an overview of the stability of the family, depending on the value of the parameter. This will be analyzed in deep in the following, by means of the parameter plane.

3.2 Analysis of critical points

A classical result in complex dynamics (see [27]) establishes that there is at least one critical point associated with each invariant Fatou component. It is clear that $z = 0$ and $z = \infty$ (the image by the Möbius transformation of the roots of the polynomial), are critical points and give rise to their respective Fatou components, but there exist in the family some *free critical points*, that is, critical points not associated to the roots, all of them depending on the value of the parameter.

In order to calculate the critical points, we get the first derivative of $F(z,\beta)$,

$$F'(z,\beta) = \frac{4z^5(2+z-2\beta)\left(-3(1+z)^2+(3+z(8+3z))\beta-4z\beta^2\right)}{(-1+2z(-1+\beta))^3}$$

and solving the equation $F'(z,\beta)=0$, we obtain the following result.

Theorem 5. *The set of critical points of operator $F(z,\beta)$ includes 0 and ∞. Moreover, some specific cases must be described:*

a) If $\beta=\frac{1}{2}$, $\beta=1$ or $\beta=\frac{3}{2}$ there not exist free critical points.

b) When $\beta=0$, then -1 is the only free critical point, which is a preimage of the repulsive strange fixed point $s_1(0)=1$.

c) For $\beta=2$, then $1=s_1(2)$ is the only free critical point, which is a superattracting strange fixed point.

In any other case, the number of free critical points is three: $cr_1(\beta)=2(-1+\beta)$, which is a preimage of $z=0$ by $F(z,\beta)$,

$$cr_2(\beta)=\frac{3-4\beta+2\beta^2-\sqrt{-6\beta+19\beta^2-16\beta^3+4\beta^4}}{3(-1+\beta)}$$

and

$$cr_3(\beta)=\frac{3-4\beta+2\beta^2+\sqrt{-6\beta+19\beta^2-16\beta^3+4\beta^4}}{3(-1+\beta)}.$$

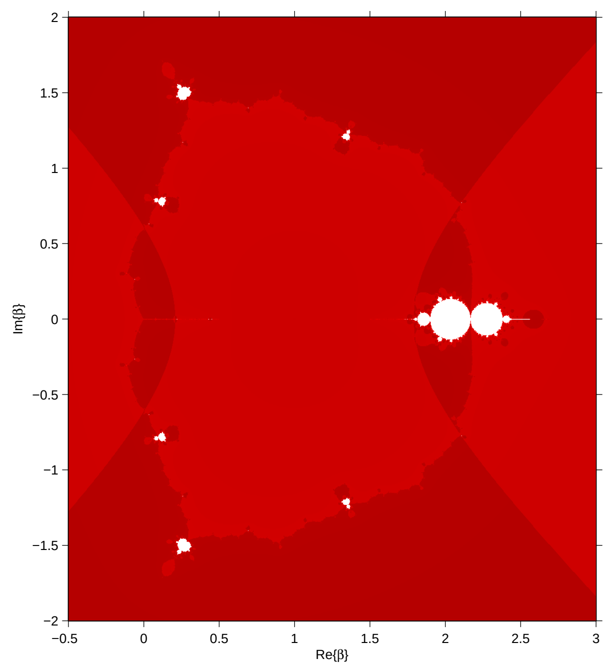
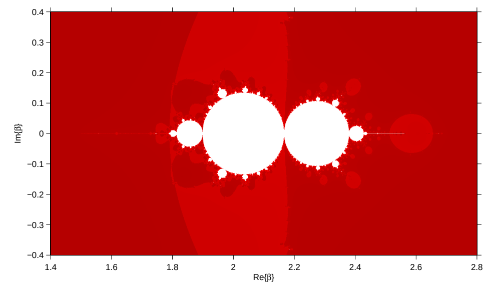


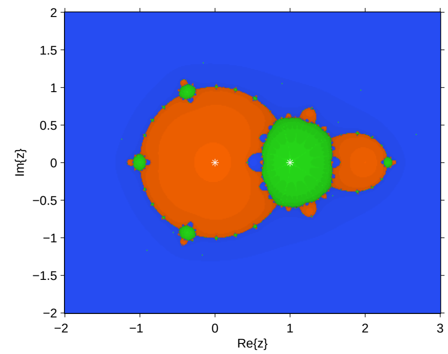
Figure 3: Parameter plane associated to the free critical point $cr_2(\beta)$.

Let us note that $cr_2(\beta)=\frac{1}{cr_3(\beta)}$ so there is only one independent free critical point, that we will use to define the parameter plane. This task is developed by means of associating each point of the parameter plane with a complex value of parameter β, i.e., with an element of family. Specifically, we want to find regions of the parameter plane as much stable as possible, because in that regions we find the best members of the family under numerical point of view.

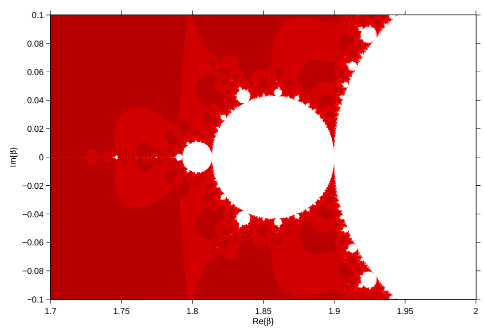
In Figure 3 the parameter plane associated to $cr_2(\beta)$ is shown. Each value of the parameter β is selected in a mesh of 1000×1000 points. Each of them corresponds to one element of the family of iterative schemes, that is executed using $cr_2(\beta)$ as a starting point. Then, we paint a point in red if the iteration of the method converges to the fixed point 0 (related to the root a) or if it converges to ∞ (related to the root b) and in white in any other case, with a maximum of 200 iterations. In Figure 4a we can observe a detail of this parameter plane where the biggest white regions are found.



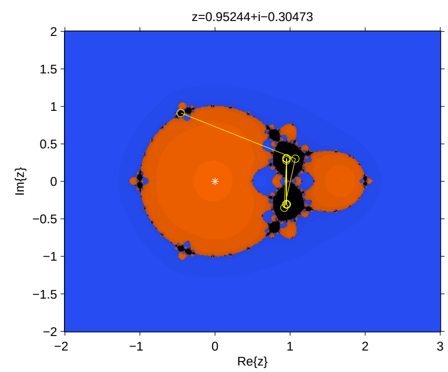
(a) Detail of $F(cr_2(\beta), \beta)$



(b) $\beta = 2$, $s_1(\beta) = 1$ superattractive



(c) Detail of a bulb of D_1



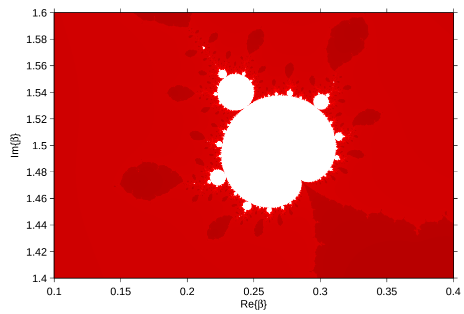
(d) $\beta = 1.85$, 2-periodic orbit

Figure 4: Details of the parameter plane and associated dynamical planes.

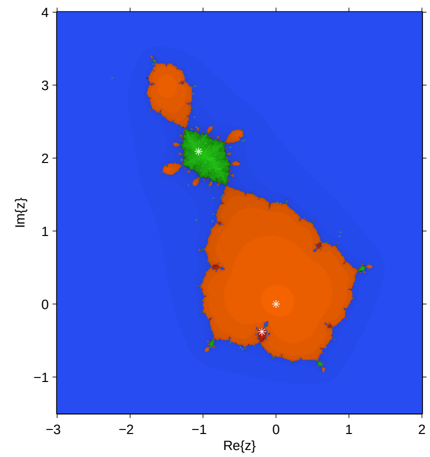
In Figure 3, we can observe that the behavior of the family is stable as there is convergence to 0 or ∞ for the most of complex values of β. The main problem resides in the zone where the parameter is close to 2 (see Figure 4a). In this region (that we will denote by "the cat set", see [28]), the strange fixed points $s_1(\beta) = 1$ (white disk on the left, denoted by D_1), $s_4(\beta)$ and $s_5(\beta)$ (white disk on the right, denoted by D_2) can be attractive or even superattractive (see Theorems 3 and 4). Moreover, the white disks that represent the stability region of these points are surrounded for tangent bulbs, of different sizes; they correspond to values of β for which the iteration of $cr_2(\beta)$ converges to periodic orbits. This kind of behavior can also be observed in the small white Mandelbrot-type sets that surround this region of the parameter plane.

For the representation of the convergence basins of every iterative procedure (dynamical planes) we have also used the software described in [22]. We draw a mesh with eight hundred points per axis; each point of the mesh is a different initial estimation which we introduce in the method. If the scheme reaches one of the attracting fixed points in less than eighty iterations, this point is drawn in different colors, depending on the fixed point that the iterative process converges to (orange and blue for 0 and ∞, respectively, and green, red, ... for strange fixed points). These attracting points are marked in the figures by white stars. The color will be more intense when the number of iterations is lower. Otherwise, if the method arrives at the maximum of iterations, the point will be drawn in black.

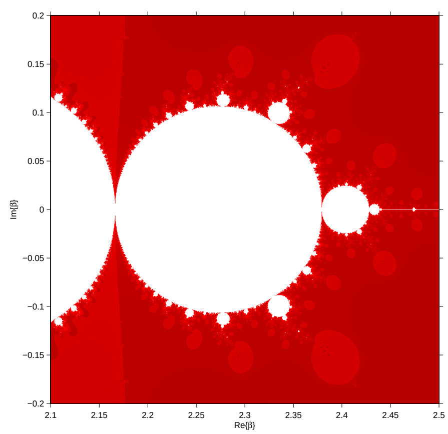
In Figure 4a, a detail of the parameter plane associated to the family (2.3) is shown corresponding to the most unstable region, the cat set. Moreover, in Figure 4b we observe the dynamical plane corresponding to $\beta = 2 \in D_1$, being $s_1(\beta) = 1$ superattracting with a big green basin of attraction. In Figure 4c a detail of D_1 is showed with a big bulb which is part of the loci of periodic orbits of period 2, as can be seen in the associated dynamical plane shown in Figure 4d.



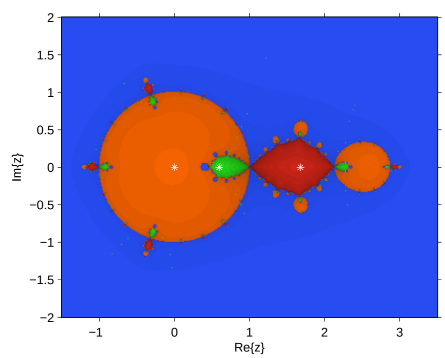
(a) $F(cr_2(\beta), \beta)$, a detail



(b) $\beta = 0.25 + 1.5i$



(c) $F(cr_2(\beta), \beta)$, a detail



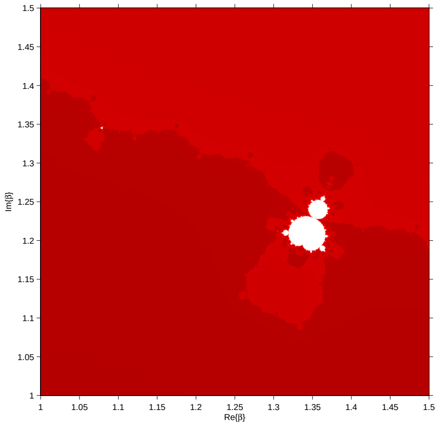
(d) $\beta = 2.3$

Figure 5: Stability of strange fixed points $s_4(\beta)$ and $s_5(\beta)$.

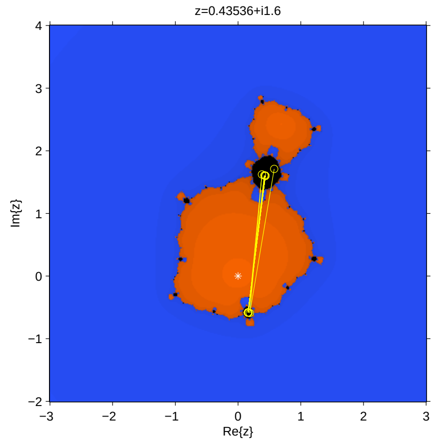
If we represent the dynamical plane associated to a β in the central body of the Mandelbrot set that appears in Figure 5a, $\beta = 0.25 + 1.5i$, or in the white disk D_2 (Figure 5c), $\beta = 2.3$ there exist two attracting strange fixed points, whose respective basins of attraction are presented in green and red, respectively, as can be seen at Figures 5b and 5d.

Moreover, in Figure 6a a detail of Figure 3 can be seen, where a Mandelbrot set in the collar of the cat set appear (see [28]). As it can be seen in Figure 6b, the core (points inside of the cardioid) of this set correspond to values of the parameter whose dynamical planes include 2-periodic orbits. We have plotted in yellow the orbit of a point in this basin of attraction.

Figure 7 corresponds to iterative schemes without stability problems. In it, we found the dynamical plane of the iterative methods for $\beta = 0$ (Figure 7a), $\beta = 1$, $\beta = \frac{1}{2}$, and $\beta = \frac{3}{2}$ (Figure 7b). Also cases where Cayley Test is not verified appear in Figures 7c and 7d, where a very stable behavior can be observed. All of them correspond to values of parameter β inside

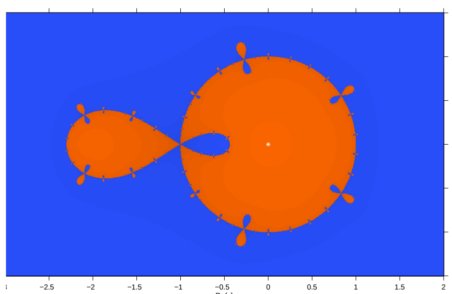


(a) $F(cr_2(\beta), \beta)$, a detail

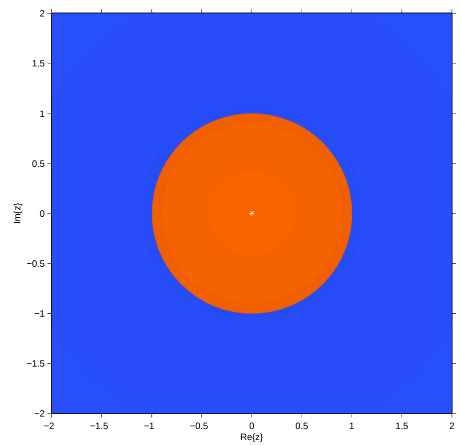


(b) $\beta = 1.3430 + 1.2125i$

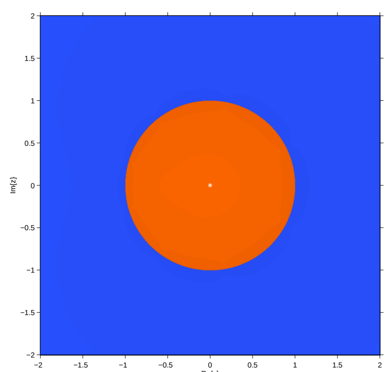
Figure 6: Stability of strange fixed points $s_4(\beta)$ and $s_5(\beta)$.



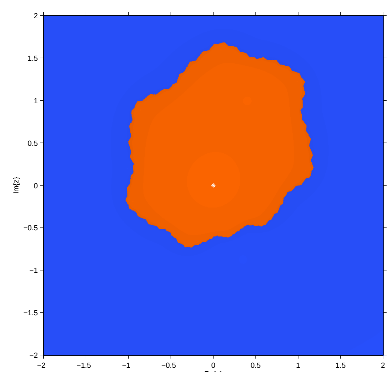
(a) $\beta = 0$



(b) $\beta \in \{\frac{1}{2}, 1, \frac{3}{2}\}$



(c) $\beta = \frac{3}{4}$



(d) $\beta = 1.2 + 0.5i$

Figure 7: Dynamical planes for values of β in which there are not stability problems.

the collar of the cat set and not belonging to any antenna. Schemes corresponding to values of β out of the collar are also stable. Some of this values will be included in the test made in the numerical section, in order to check the behavior of the methods on more complicated problems.

4 Numerical reports

In this section we report some numerical examples to justify the applicability and accuracy of some elements of the developed family (2.3). Specifically, we are going to test the members of our family corresponding to $\beta = 3/2$, $\beta = 3/4$ (order six) and $\beta = 1$ (order eight). We compare our methods of order six with LLK5 for $\beta = 1$ and the scheme of order eighth with the following optimal eighth-order iterative methods without memory, designed by Kung and Traub in [3], denoted by KT8,

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \\ z_k = y_k - \frac{f(x_k)^2}{(f(x_k)-f(y_k))^2} \frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} = z_k - \left(\frac{1}{f(x_k)-f(z_k)} \left(\frac{1}{f[x_k,z_k]} - \frac{1}{f'(x_k)} \right) - \frac{f(y_k)}{(f(x_k)-f(y_k))^2 \, f'(x_k)} \right) \frac{f^2(x_k)f(y_k)}{f(y_k)-f(z_k)}, \end{cases} \tag{4.1}$$

and with the derivative-free optimal eighth-order scheme, also constructed by Kung and Traub in [3], that is denoted by KT8b,

$$\begin{cases} y_k = x_k - \frac{f(x_k)}{f[x_k,w_k]}, \quad w_k = x_k + \gamma f(x_n),\ \gamma \in \mathbb{R},\ k = 0, , 1, 2, \dots, \\ z_k = y_k - \frac{f(w_k)}{f(w_k)-f(y_k)} \frac{f(y_k)}{f[x_k,y_k]}, \\ x_{k+1} = z_k - \frac{f(y_k)f(w_k)\left(y_k - x_k + \frac{f(x_k)}{f[x_k,z_k]}\right)}{(f(y_k)-f(z_k))(f(w_k)-f(z_k))} + \frac{f(y_k)}{f[y_k,z_k]}. \end{cases} \tag{4.2}$$

The errors $|x_k - \alpha|$ denote approximations to the sought zeros, and $a(-b)$ stands for $a \times 10^{-b}$. To show the convergence order in action, modified computational order of convergence (MCOC) introduced in [29] is used:

$$p \approx= MCOC = \frac{\log(|f(x_k)/f(x_{k-1})|)}{\log(|f(x_{k-1})/f(x_{k-2})|)}.$$

All the numerical results have been obtained by using software Mathematica 8.0 and working with 1000 significant digits. Among many test problems, the following four examples are considered [1]

$$\begin{array}{lll} f_1(x) = (x+2)e^x - 1, & \alpha = -0.4428..., & x_0 = -0.5, \\ f_2(x) = x^2 - (2-x)^3, & \alpha = 1, & x_0 = 1.1, \\ f_3(x) = 10xe^{-x^2} - 1, & \alpha = 1.6796..., & x_0 = 1.7, \\ f_4(x) = \sin^2(x) - x^2 + 1, & \alpha = 1.4044..., & x_0 = 1.5. \end{array}$$

Method	$\lvert x_1 - \alpha\rvert$	$\lvert x_2 - \alpha\rvert$	$\lvert x_3 - \alpha\rvert$	MCOC
LLK5, $\beta = 1$	0.1238(-8)	0.1391(-54)	0.2799(-330)	6.00
F6, $\beta = 3/2$	0.5706(-8)	0.5618(-50)	0.5117(-302)	6.00
F6, $\beta = 3/4$	0.1413(-8)	0.3236(-54)	0.4669(-328)	6.00
F6, $\beta = 1$	0.2414(-11)	0.2298(-94)	0.1551(-758)	8.00
KT8b, $\gamma = -1$	0.3166(-11)	0.4117(-93)	0.3369(-748)	8.00
KT8	0.3214(-10)	0.2736(-84)	0.7542(-677)	8.00

Table 1: Numerical results with $f_1(x)$

As we can see in Tables 1-4, the numerical tests confirm the theoretical results of our methods and method F6 ($\beta = 1$) produces slightly better results than optimal eighth-order Kung-Traub's schemes. In relation to the no optimal methods of order six, the numerical results are very similar, obtaining better results with F6 ($\beta = 3/4$) in some cases.

Method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	MCOC
LLK5, $\beta=1$	0.4289(-8)	0.1993(-52)	0.2007(-318)	6.00
F6, $\beta=3/2$	0.7627(-8)	0.2015(-50)	0.6862(-306)	6.00
F6, $\beta=3/4$	0.2343(-8)	0.4238(-54)	0.1484(-328)	6.00
F6, $\beta=1$	0.2154(-11)	0.4737(-97)	0.2597(-782)	8.00
KT8b, $\gamma=-1$	0.2164(-8)	0.1894(-69)	0.6515(-558)	8.00
KT8	0.1569(-10)	0.5646(-89)	0.1585(-716)	8.00

Table 2: Numerical results with $f_2(x)$

Method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	MCOC
LLK5, $\beta=1$	0.5380(-11)	0.1378(-68)	0.3895(-344)	6.00
F6, $\beta=3/2$	0.6221(-10)	0.4630(-61)	0.7865(-368)	6.00
F6, $\beta=3/4$	0.1538(-10)	0.2637(-65)	0.6712(-394)	6.00
F6, $\beta=1$	0.6224(-14)	0.3961(-114)	0.1067(-915)	8.00
KT8b, $\gamma=-1$	0.4546(-10)	0.2030(-79)	0.3206(-634)	8.00
KT8	01932(-12)	0.1077(-100)	0.1000(-806)	8.00

Table 3: Numerical results with $f_3(x)$

Method	$\|x_1-\alpha\|$	$\|x_2-\alpha\|$	$\|x_3-\alpha\|$	MCOC
LLK5, $\beta=1$	0.1752(-7)	0.1044(-47)	0.4695(-289)	6.00
F6, $\beta=3/2$	0.1612(-6)	0.5177(-41)	0.5684(-248)	6.00
F6, $\beta=3/4$	0.4403(-7)	0.5380(-45)	0.1790(-272)	6.00
F6, $\beta=1$	0.2027(-9)	0.6381(-76)	0.2535(-610)	8.00
KT8b, $\gamma=-1$	0.4108(-6)	0.1615(-48)	0.9199(-388)	8.00
KT8	0.5000(-8)	0.5296(-66)	0.8382(-530)	8.00

Table 4: Numerical results with $f_4(x)$

5 Conclusion

In this study, based on using a better approximation for $f'(z)$ in the third step of family (1.1), we transform this fifth-order family (with a particular case of order six), into a new sixth-order family with an optimal eighth-order member, without increasing the number of functional evaluations per step. So the efficiency index is improved.

A dynamical analysis of the sixth-order family on quadratic polynomials has been made, identifying some elements of the family that are specially stable, several of them satisfying Cayley's test. This has been checked in the numerical section for different test functions.

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