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Additional Information

## CHAOS FOR THE HYPERBOLIC BIOHEAT EQUATION

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**ABSTRACT.** The Hyperbolic Heat Transfer Equation describes heat processes in which extremely short periods of time or extreme temperature gradients are involved. It is already known that there are solutions of this equation which exhibit a chaotic behaviour, in the sense of Devaney, on certain spaces of analytic functions with certain growth control. We show that this chaotic behaviour still appears when we add a source term to this equation, i.e. in the Hyperbolic Bioheat Equation. These results can also be applied for the Wave Equation and for a higher order version of the Hyperbolic Bioheat Equation.

### 1. Preliminaries.

**1.1. Introduction.** Nowadays, surgery uses high temperature ablative techniques, such as laser, radiofrequency, microwave, or ultrasound energy to heat biological tissues to over 50°C in a localized and safe way. Theoretical modeling can provide information about the biophysics of these techniques quickly and cheaply. Specifically, the thermal problem is modeled using the Bioheat Equation (BE) as the governing equation [25]

$$-u_{xx} + \frac{1}{\alpha} u_t = \frac{1}{k} g, \quad (1)$$

where  $u$  represents the temperature and  $\alpha$  and  $k$  the thermal diffusivity and conductivity of the material. The source term  $g$  refers to internal heat sources and represents different contributions for the heat sources in a biological tissue:

$$g = g_s + g_p + g_m. \quad (2)$$

The subscript  $s$  denotes a surgical heat source (e.g. laser or radiofrequency treatment),  $p$  refers to blood perfusion, and  $m$  to any source related with metabolic activity. The presence of  $g$  in equation (1) differences BE from the classic heat equation.

BE, as the classic heat equation, is based on the Fourier Theory which assumes an infinite thermal energy propagation speed. Although this theory might be suitable for modeling most ablative procedures, there are other surgical procedures in which extremely short periods of time or extreme temperature gradients are involved and

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it is necessary to consider a non-Fourier model: The Hyperbolic Bioheat Equation (HBE) [24],

$$-u_{xx} + \frac{1}{\alpha}(u_t + \tau u_{tt}) = \frac{1}{k}(g + \tau g_t), \quad (3)$$

where  $\tau$  is the thermal relaxation time, which depends on the material and represents the time that goes by since the temperature gradient is imposed until the heat flux is produced. Again, the presence of internal heat sources differences HBE from the hyperbolic heat equation.

The expression of the internal heat sources varies according to the ablative technique employed. Such expression is a function of spatial and temporal variables. The spatial dependence is related with the type of ablative technique employed (radiofrequency, laser, microwaves, or ultrasound). The time-dependence refers to the energy delivering mode (e. g. continuous or pulsed). Therefore, equation (3) can present many different formulations. In [27], the laser heating heat source term in the one-dimensional case for a pulsed protocol was:

$$g_s(t, x) = M e^{-bx} (H(t) - H(t - \Delta t)) \quad x \in \mathbb{R}, t \geq 0 \quad (4)$$

where  $M$  and  $b$  are physical parameters,  $H(t)$  is the Heaviside function and  $\Delta t$  is the time that the laser pulse is applied to the tissue. Figure 1 shows an schematic representation of this kind of source.

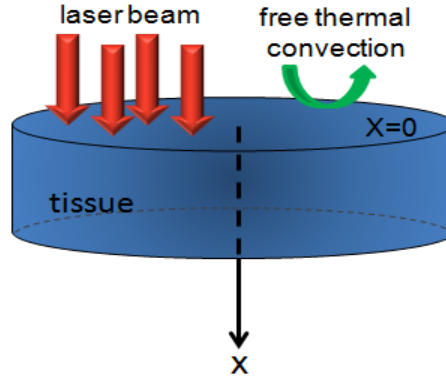


FIGURE 1. Schematic representation of the application of a laser source.

In [28] the authors presented the expression for a spherical laser source

$$g_s(t, R) = \frac{N e^{-dR}}{R} \omega(t), \quad (5)$$

and for an infinitely long cylindrical laser source

$$g_s(t, R) = \frac{P e^{-qR}}{R^{\frac{1}{2}}} \omega(t), \quad (6)$$

being  $N$ ,  $d$ ,  $P$  and  $q$  biophysical parameters and  $R$ , the radial coordinate.  $\omega(t)$  refers to any type of time-dependent protocol for energy delivering. In [21] there

is another example of a source coming from radiofrequency, where the heat of a continuous source in the spherical coordinates was

$$g_s(t, r) = \frac{Q}{R^4} H(t) \quad R > 0, t \geq 0, \quad (7)$$

being  $Q$  a physical parameter.

And in [20] we found the expression for a spherical microwave or ultrasound source

$$g_s(t, R) = \frac{Z e^{-mR}}{R^2} \omega(t) \quad (8)$$

and for an infinitely long cylindrical microwave or ultrasound source

$$g_s(t, R) = \frac{V e^{-wR}}{R} \omega(t) \quad (9)$$

$Z$ ,  $m$ ,  $V$  and  $w$  being biophysical parameters.

The kind of tissue determines the ablation technique and the consideration or not of  $g_p$  and  $g_m$  in equation (2). In this sense,  $g_p = 0$  in non-perfused organs like the cornea, but  $g_p$  is a temperature-dependent expression in well perfused organs like the liver. In the case of  $g_m$  only in organs which can generate metabolic heat is taking into account usually with a constant value.

The dynamical behaviour presented by the solutions of the heat equation, when there is no heat source, has been studied on certain spaces of analytic functions with certain growth control [18]. Similar results were also obtained for the Hyperbolic Heat Transfer Equation (HHTE) in the absence of heat sources [9, 17]. Certainly, it is also interesting to know some aspects of the dynamical behaviour of the HBE. In this sense the aim of this note is to study the chaotic asymptotic behaviour of certain solutions of a Cauchy problem in which the governing equation is the HBE, that is

$$\left\{ \begin{array}{l} \tau u_{tt} + u_t = \alpha u_{xx} + g(t, x) \\ u(0, x) = \varphi_1(x), \quad x \in \mathbb{R} \\ u_t(0, x) = \varphi_2(x), \quad x \in \mathbb{R} \end{array} \right\}, \quad (10)$$

where  $g(t, x)$  represents the corresponding term to the heat source in equation (3),  $\varphi_1(x)$  represents the initial temperature and, and  $\varphi_2(x)$ , the initial variation of temperature. We consider a general case, but with  $g_p = 0$ , since we consider a linear problem. To develop this study, we will represent the solutions of (10) in terms of the solutions of the HHTE which are expressed by a  $C_0$ -semigroup generated by certain first order equation. The solutions of (10), and the associated  $C_0$ -semigroup, will be considered on the product of a certain function space  $X$  of analytic functions with certain growth control with itself, i.e.  $X \oplus X$ . The general treatment of the problem shows that our results could be meaningful for all types of sources used in ablative therapies.

**1.2.  $C_0$ -semigroups.** A family  $\{T_t\}_{t \geq 0}$  of linear and continuous operators on a Banach space  $X$  is said to be a  $C_0$ -semigroup if  $T_0 = Id$ ,  $T_t T_s = T_{t+s}$  for all  $t, s \geq 0$ , and  $\lim_{t \rightarrow s} T_t x = T_s x$  for all  $x \in X$  and  $s \geq 0$ .

Let  $\{T_t\}_{t \geq 0}$  be an arbitrary  $C_0$ -semigroup on  $X$ . It can be shown that an operator defined by  $Ax := \lim_{t \rightarrow 0} \frac{1}{t}(T_t x - x)$  exists on a dense subspace of  $X$ ; denoted by  $D(A)$ . Then  $A$ , or rather  $(A, D(A))$ , is called the (*infinitesimal*) generator of the semigroup. It can also be shown that the infinitesimal generator determines the

semigroup uniquely. If the generator  $A$  is defined on  $X$  ( $D(A) = X$ ), the semigroup is expressed as  $\{T_t\}_{t \geq 0} = \{e^{tA}\}_{t \geq 0}$  [14].

The link between semigroups and differential equations is via the infinitesimal generator. The unique solution of the abstract Cauchy problem

$$\begin{cases} u_t = Au \\ u(0, x) = \varphi(x) \end{cases}, \quad (11)$$

where  $A$  is a linear operator defined on  $X$ , is given by  $u(t, x) = e^{tA}\varphi(x)$ . In that sense,  $u(t, x)$  is called a *classical solution* of the abstract Cauchy problem (11) and the semigroup  $\{T_t\}_{t \geq 0} = \{e^{tA}\}_{t \geq 0}$  is called the *solution semigroup* of (11), whose infinitesimal generator is  $A$ .

In the case of a non-homogeneous Cauchy problem of the form

$$\begin{cases} u_t = Au + g(t, x) \\ u(0, x) = \varphi(x) \end{cases}, \quad (12)$$

where  $g(t, x)$  is a source term, we have the (classical) unique solution given by the following expression:

$$u(t, x) = e^{tA}\varphi(x) + \int_0^t e^{(t-s)A}g(s, x)ds. \quad (13)$$

**1.3. Linear dynamics of  $C_0$ -semigroups.** Given a family of operators  $\{T_t\}_{t \geq 0}$ , we say that this family of operators is *transitive* if for every pair of non-void open sets  $U, V \subset X$  there exists some  $t > 0$  such that  $T_t(U) \cap V \neq \emptyset$ . Furthermore, if there is some  $t_0$  such that the condition  $T_t(U) \cap V \neq \emptyset$  holds for every  $t \geq t_0$  we say that it is *topologically mixing*.

A family of operators  $\{T_t\}_{t \geq 0}$  is said to be *universal* if there exists some  $x \in X$  such that  $\{T_t x : t \geq 0\}$  is dense in  $X$ . When  $\{T_t\}_{t \geq 0}$  is a  $C_0$ -semigroup we refer to it as *hypercyclic* instead of universal. In this setting, transitivity coincides with universality, but it is strictly weaker than topologically mixing [6].

In addition, two notions of chaos are introduced: *Devaney chaos* and *distributional chaos*. First, we recall that an element  $x \in X$  is said to be a *periodic point* of  $\{T_t\}_{t \geq 0}$  if there exists some  $t_0 > 0$  such that  $T_{t_0}x = x$ .

On the one hand, a family of operators  $\{T_t\}_{t \geq 0}$  is said to be *chaotic in the sense of Devaney* if it is hypercyclic (universal) and there exists a dense set of periodic points in  $X$ . On the other hand, it is *distributionally chaotic* if there are an uncountable set  $S \subset X$  and  $\delta > 0$ , so that for each  $\varepsilon > 0$  and each pair  $x, y \in S$  of distinct points we have

$$\begin{aligned} \overline{\text{Dens}}\{s \geq 0 : \|T_s x - T_s y\| \geq \delta\} &= 1 \text{ and} \\ \overline{\text{Dens}}\{s \geq 0 : \|T_s x - T_s y\| < \varepsilon\} &= 1, \end{aligned}$$

where  $\overline{\text{Dens}}(B)$  is the upper density of a Lebesgue measurable subset  $B \subset \mathbb{R}_0^+$  defined as

$$\limsup_{t \rightarrow \infty} \frac{\mu(B \cap [0, t])}{t},$$

with  $\mu$  standing for the Lebesgue measure on  $\mathbb{R}_0^+$ . A vector  $x \in X$  is said to be *distributionally irregular* for the  $C_0$ -semigroup  $\{T_t\}_{t \geq 0}$  if for every  $\delta > 0$  we have

$$\overline{\text{Dens}}\{s \geq 0 : \|T_s x\| \geq \delta\} = 1 \text{ and}$$

$$\overline{\text{Dens}}\{s \geq 0 : \|T_s x\| < \delta\} = 1.$$

Such vectors were considered in [7] so as to get a further insight into the phenomenon of distributional chaos, showing the equivalence between a distributionally chaotic operator and an operator having a distributionally irregular vector. This equivalence has been shown for  $C_0$ -semigroups in [1].

A criterion for Devaney chaos in terms of the abundance of eigenvectors of the infinitesimal generator of a  $C_0$ -semigroup was stated in [13] by Desch, Schappacher, and Webb. Since then, this criterion has been reformulated and applied to several examples of  $C_0$ -semigroups which are solution of certain partial differential equations, see for instance [2, 10, 17]. The following version can be found in [17, Th. 7.30].

**Theorem 1.1. Desch-Schappacher-Webb criterion** *Let  $X$  be a complex separable Banach space and  $\{T_t\}_{t \geq 0}$  a  $C_0$ -semigroup on  $X$  with infinitesimal generator  $(A, D(A))$ , where  $D(A)$  denotes its domain. Assume that there exists a nonempty open connected subset  $U$  of  $\mathbb{C}$  and weakly holomorphic functions  $f_j : U \rightarrow X$ ,  $j \in J$ , such that*

1.  $U \cap i\mathbb{R} \neq \emptyset$ ,
2.  $f_j(\lambda) \in \ker(\lambda I - A)$  for every  $\lambda \in U$ ,  $j \in J$ ,
3. for any  $x^* \in X^*$ , if  $\langle f_j(\lambda), x^* \rangle = 0$  for all  $\lambda \in U$  and  $j \in J$  then  $x^* = 0$ .

*Then  $\{T_t\}_{t \geq 0}$  is topologically mixing and Devaney chaotic.*

The third condition in this result is used in order to prove the density of the span of certain sets of eigenvectors associated to eigenvalues of  $A$  with real part greater, equal, and smaller than 0. A criterion stated in these terms was firstly stated for operators by Godefroy and Shapiro in [15].

**Theorem 1.2. Eigenvalue criterion for chaos.** *Let  $X$  be a complex separable Banach space and  $\{T_t\}_{t \geq 0}$  a  $C_0$ -semigroup on  $X$ . Suppose that the sets*

$$\begin{aligned} X_0 &:= \text{span}\{x \in X : \exists \lambda > 0, T_t x = e^{\lambda t} x, \forall t \geq 0\}, \\ X_1 &:= \text{span}\{x \in X : \exists \lambda < 0, T_t x = e^{\lambda t} x, \forall t \geq 0\}, \\ X_p &:= \text{span}\{x \in X : \exists \lambda \in \mathbb{Q}, T_t x = e^{\pi \lambda i t} x, \forall t \geq 0\} \end{aligned}$$

*are dense in  $X$ , then  $\{T_t\}_{t \geq 0}$  is Devaney chaotic.*

In the proof of Theorem 1.1, Condition (3) of its statement is used to satisfy the hypothesis of Theorem 1.2, and hence to prove that the  $C_0$ -semigroup is Devaney chaotic. As a result, we can replace Condition (3) in Theorem 1.1 with the hypothesis of Theorem 1.2.

In addition, there exist several criteria for distributional chaos [1, 7]. Nevertheless, either the Desch-Schappacher-Webb criterion or the Eigenvalue criterion for chaos imply distributional chaos, [4, Rem. 3.8], see also [5, Cor. 31]. Moreover, in these cases we can affirm that there is a dense distributionally irregular manifold, that is a dense manifold of distributionally irregular vectors.

More information on sufficient conditions for hypercyclicity and chaos for  $C_0$ -semigroups and operators can be found in [2, 6, 12, 13, 17, 19].

## 2. Analysis.

**2.1. The Hyperbolic Heat Transfer Equation.** The chaotic behaviour of the solutions of an abstract Cauchy problem (10) that is given by the Hyperbolic Heat Transfer Equation in the absence of internal heat sources was analyzed on certain spaces of analytic functions with certain growth control in [9], see also [17]. This can be done if we express this second-order PDE as a first-order equation by representing it as a  $C_0$ -semigroup on the product of a certain function space with itself. To do this we set  $u_1 = u$  and  $u_2 = \frac{\partial u}{\partial t}$ . Then the associated first-order equation is

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau} \frac{\partial^2}{\partial x^2} & -\frac{1}{\tau} I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ \begin{pmatrix} u_1(0, x) \\ u_2(0, x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{array} \right. \quad (14)$$

We fix  $\rho > 0$  and consider the space

$$X_\rho = \left\{ f : \mathbb{R} \rightarrow \mathbb{C} ; f(x) = \sum_{n=0}^{\infty} \frac{a_n \rho^n}{n!} x^n, (a_n)_{n \geq 0} \in c_0 \right\}, \quad (15)$$

endowed with the norm  $\|f\| = \sup_{n \geq 0} |a_n|$ , where  $c_0$  is the Banach space of complex sequences tending to 0. Then  $X_\rho$  is a Banach space of analytic functions with a certain growth control. By its definition it is isometrically isomorphic to  $c_0$ . This type of spaces were already used in [18].  $X_\rho$  is a Banach space of analytic functions that is densely embedded in  $\mathcal{C}(\mathbb{R})$  with the topology of uniform convergence on compact sets of  $\mathbb{R}$ , since it contains all polynomials. Essentially,  $X_\rho$  is a space of analytic functions with certain increasing control at infinity. In fact,  $(X_\rho, \|\cdot\|)$  is isometrically isomorphic to  $(c_0(\mathbb{N}_0), \|\cdot\|_\infty)$ .

Since

$$A := \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau} \frac{\partial^2}{\partial x^2} & -\frac{1}{\tau} I \end{pmatrix} \quad (16)$$

is a linear and continuous operator on  $X_\rho \oplus X_\rho$ , we have that  $\{e^{tA}\}_{t \geq 0}$ , with

$$e^{tA} = \sum_{n \geq 0} \frac{(tA)^n}{n!}, \quad (17)$$

is well defined on  $X_\rho \oplus X_\rho$ , and we have that  $(e^{tA})_{t \geq 0}$  is a  $C_0$ -semigroup on  $X_\rho \oplus X_\rho$  (even uniformly continuous), which is the solution semigroup of (14) on  $X_\rho \oplus X_\rho$ , see for instance [14, Ch. 1, Prop. 3.5]. As we have already pointed out, the  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$  is chaotic on  $X_\rho \oplus X_\rho$  [9, Th. 2.1], cf. [17, Prop. 7.35], and distributionally chaotic, too, c.f. [4, Rem. 3.8]. For simplicity we will denote by  $\|\cdot\|_{\rho, \rho}$  the norm in  $X_\rho \oplus X_\rho$  given by the norm  $\|\cdot\|_\rho$  on each copy of the space  $X_\rho$ .

**Theorem 2.1.** [9, Th. 2.1], & [17, Th. 7.35]. *Let  $\rho > 0$  be such that  $\alpha\tau\rho^2 > 2$ . Then the solution semigroup  $\{e^{tA}\}_{t \geq 0}$  of (14) is topologically mixing and Devaney chaotic on  $X_\rho \oplus X_\rho$ .*

The proof of Theorem 2.1 is included here because we will use the same notation in many of the upcoming results. This proof consists on checking the Eigenvalue criterion for chaos stated in Theorem 1.2. For this purpose, we take the holomorphic functions of the form

$$\varphi_{\lambda, z_0, z_1}(x) = z_0 \sum_{n=0}^{\infty} \frac{R_{\lambda}^n x^{2n}}{(2n)!} + z_1 \sum_{n=0}^{\infty} \frac{R_{\lambda}^n x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}, \quad (18)$$

where  $\lambda \in \mathbb{C}$ ,  $z_0, z_1 \in \mathbb{R}$ , and  $R_{\lambda} = (\tau\lambda^2 + \lambda)/\alpha$ . If we take  $\lambda \in V \subset \mathbb{C}$ , with  $V$  the open disk of radius  $r = \sqrt{\alpha\rho^2/2\tau} > 0$  centered at zero, then  $\varphi_{\lambda, z_0, z_1} \in X_{\rho}$ . Then, the functions  $\phi_{z_0, z_1} : V \rightarrow X$ ,  $z_0, z_1 \in \mathbb{R}$ , given by

$$\phi_{\lambda, z_0, z_1} = \begin{pmatrix} \varphi_{\lambda, z_0, z_1} \\ \lambda \varphi_{\lambda, z_0, z_1} \end{pmatrix}. \quad (19)$$

satisfy that  $\phi_{\lambda, z_0, z_1} \in \ker(\lambda I - A)$  for every  $\lambda \in V$ ,  $z_0, z_1 \in \mathbb{R}$ . Finally, the Eigenvalue criterion is applied taking the sets  $X_0, X_1, X_{\rho}$  mentioned in Theorem 1.2 as

$$X_0 := \text{span}\{\phi_{\lambda, z_0, z_1} : \lambda \in V \cap \mathbb{R}^+, z_0, z_1 \in \mathbb{R}\}, \quad (20)$$

$$X_1 := \text{span}\{\phi_{\lambda, z_0, z_1} : \lambda \in V \cap \mathbb{R}^-, z_0, z_1 \in \mathbb{R}\}, \quad (21)$$

$$X_{\rho} := \text{span}\{\phi_{\lambda, z_0, z_1} : \lambda \in V \cap \pi i \mathbb{Q}, z_0, z_1 \in \mathbb{R}\} \quad (22)$$

With a similar approach, one can obtain the chaotic behaviour for the solutions of the Wave Equation:

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \alpha \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ \begin{pmatrix} u_1(0, x) \\ u_2(0, x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{array} \right. \quad (23)$$

where  $\alpha > 0$  is the square of the speed of wave propagation. Again, this equation can be reformulated as a first order equation taking the following operator  $B$  instead of  $A$ .

$$B := \begin{pmatrix} 0 & I \\ \alpha \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \quad (24)$$

In this case, the solution  $C_0$ -semigroup to the abstract Cauchy Problem in (23) presents a Devaney chaotic behaviour on the space  $X_{\rho} \oplus X_{\rho}$  for any  $\rho > 0$  [9, Th. 2.3] and [17, Ex. 7.5.3].

**2.2. The Hyperbolic Bioheat Equation.** Let us consider the HBE given in (10). As in the previous case, we express this second-order equation as a first-order equation by setting  $u_1 = u$  and  $u_2 = \frac{\partial u}{\partial t}$  and taking  $A$  as it has been already defined in (16). In this way, the associated first-order equation is formulated as

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \frac{\alpha}{\tau} \frac{\partial^2}{\partial x^2} & -\frac{1}{\tau} I \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t, x) \end{pmatrix}, \\ \begin{pmatrix} u_1(0, x) \\ u_2(0, x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{array} \right. \quad (25)$$

where the function  $g(t, x)$  represents the internal heat sources. Comparing this expression with the formulation of the HBE in (3), we see that  $g(t, x)$  stands for  $\frac{\tau}{k}(g + \tau g_t)$ .

The unique (classical) solution of the HBE in (25) is given by:



$$u(t, x) = e^{tA}\Phi(x) + \int_0^t e^{(t-s)A}\Psi(s, x)ds \quad (26)$$

where we have used the following notation

$$u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}, \quad \Phi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad \Psi(t, x) = \begin{pmatrix} 0 \\ g(t, x) \end{pmatrix}.$$

**2.3. Dynamics of the Hyperbolic Bioheat Equation.** Since we know that the  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$  is chaotic on the Banach space  $X_\rho \oplus X_\rho$ , in order to study the asymptotic behaviour of the solutions of the HBE we have to analyze the asymptotic properties the second term in (26):

$$h(t, x) := \int_0^t e^{(t-s)A}\Psi(s, x)ds \quad (27)$$

In addition, as the solution  $u(t, x)$  is expected to be an element of  $X_\rho \oplus X_\rho$ , it is reasonable to consider the source term  $\Psi(t, x)$  in this same space. Later, we will see that in fact we are still considering some type of HBE. Moreover, for many applications it is possible to consider the source term  $g(t, x)$  as time-independent. So that, we will firstly consider  $\Psi(t, x)$  as a time-independent function of the form  $\phi_{\lambda, z_0, z_1}(x)$ , for certain values of  $\lambda$  with negative real part.

On the one hand, we will see that in Theorems 2.2 - 2.5 we consider

$$\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x) = \begin{pmatrix} \varphi_{\lambda, z_0, z_1}(x) \\ \lambda \varphi_{\lambda, z_0, z_1}(x) \end{pmatrix} \quad (28)$$

which has introduced a variation in the relationship between  $u_1(t, x)$  and  $u_2(t, x)$  in equation (25): we have  $(u_1)_t = u_2 + \varphi_{\lambda, z_0, z_1}$  instead of just  $(u_1)_t = u_2$ . At first sight, it may seem that the resulting differential equation is no longer the HBE. Fortunately, that is not the case: Let us express the heat source term in the form:

$$\Psi(t, x) = \begin{pmatrix} j_1(t, x) \\ g(t, x) + j_2(t, x) \end{pmatrix} \quad (29)$$

where we assume that  $j_1$ , its time derivative  $(j_1)_t$ , and  $j_2$  are *small*. Then, from the equation (25), it is just an exercise to check that  $u_1(t, x) = u(t, x)$  satisfies the HBE:

$$\tau(u_1)_{tt} + (u_1)_t = \alpha(u_1)_{xx} + g(t, x) + \frac{1}{\tau}j_1(t, x) + j_2(t, x) + (j_1(t, x))_t \quad (30)$$

which corresponds to a small perturbation of the heat source term  $g(t, x)$  in equation (10).

On the other hand, taking a term  $\Psi(t, x)$  as in (28) is not a great restriction. This is due to the fact that we will consider eigenfunctions of  $A$ ,  $\phi_{\lambda, z_0, z_1}$ , in a set whose span is dense in  $X_\rho \oplus X_\rho$ . Therefore, given any source function  $g(t, x)$ , the initial term  $\begin{pmatrix} 0 \\ g(t, x) \end{pmatrix}$  can be approximated by a linear combination of those eigenfunctions multiplied by a certain function depending of  $t$ . Then, for this new source term, we will be able to find an initial condition whose orbit was dense under the action of the operators in the family  $u(t, x)$ .

Before going further, we point out that the notation of the statement of the following theorem and in the rest of the results of this section is the same used in the proof of Theorem 2.1.

**Theorem 2.2.** *Let  $\rho > 0$  be such that  $\alpha\tau\rho^2 > 2$  and let  $\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x)$  with  $\lambda \in V$  and  $\Re(\lambda) < 0$ . The solution family  $\{u(t, \cdot)\}_{t \geq 0}$  of (26) is topologically mixing on  $X_\rho \oplus X_\rho$ .*

*Proof.* Let us take two non-empty open sets  $U_1, U_2 \subseteq X_\rho \oplus X_\rho$ . Let  $W \subset X_\rho \oplus X_\rho$  be an open 0-neighborhood and let  $U'_2$  be an open set in  $X_\rho \oplus X_\rho$  such that  $W + U'_2 \subset U_2$ .

Consider an arbitrary  $\lambda \in V$  with  $\Re(\lambda) < 0$ . On the one hand, since the  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$  is known to be topologically mixing on  $X_\rho \oplus X_\rho$ , then there exists some  $t_0 > 0$  such that for all  $t \geq t_0$  there is some  $\Phi^t \in U_1$  verifying  $e^{tA}(\Phi^t) \in \frac{1}{\lambda}\phi_{\lambda, z_0, z_1} + U'_2$ .

On the other hand, since  $\phi_{\lambda, z_0, z_1}(x)$  does not depend on the time, then we can easily analyze the integral  $h(t, x)$ :

$$\begin{aligned} h(t, x) &= \int_0^t e^{(t-s)A} \phi_{\lambda, z_0, z_1}(x) ds \\ &= \int_0^t e^{\lambda(t-s)} \phi_{\lambda, z_0, z_1}(x) ds = \frac{e^{\lambda t} - 1}{\lambda} \phi_{\lambda, z_0, z_1}(x). \end{aligned}$$

Clearly,  $h(t, x) \in X_\rho \oplus X_\rho$  for every  $t \geq 0$  and, since

$$\lim_{t \rightarrow \infty} h(t, x) = \lim_{t \rightarrow \infty} \frac{e^{\lambda t} - 1}{\lambda} \phi_{\lambda, z_0, z_1}(x) = \frac{-1}{\lambda} \phi_{\lambda, z_0, z_1}(x),$$

and  $\Re(\lambda) < 0$ , we can affirm that there exists  $t_1 > 0$  such that  $h(t, \cdot) + \frac{1}{\lambda}\phi_{\lambda, z_0, z_1} \in W$  for all  $t \geq t_1$ . Therefore, for  $t \geq \max\{t_0, t_1\}$ , we have that  $u(t, \cdot)$  acting on the initial condition  $\Phi^t \in U_1$  yields

$$u(t, \cdot) = (e^{tA}(\Phi^t) + h(t, \cdot)) \in h(t, \cdot) + \frac{1}{\lambda}\phi_{\lambda, z_0, z_1} + U'_2 \subset W + U'_2 \subset U_2,$$

which fulfills the definition of topologically mixing for the solution family  $\{u(t, \cdot)\}_{t \geq 0}$  on  $X_\rho \oplus X_\rho$ .  $\square$

**Remark 1.** The condition that  $\lambda$  belongs to  $V$  is necessary to be sure that  $\phi_{\lambda, z_0, z_1}(x)$  belongs to  $X_\rho \oplus X_\rho$ . This can be replaced by the more general one  $|\tau\lambda^2 + \lambda| < \alpha\rho^2$ .

**Remark 2.** Under the hypothesis of Theorem 2.2 we can get a little more than topologically mixing: since the asymptotic behaviour of the orbits by the  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$  coincides with the asymptotic behaviour of the orbits under the family of operators  $\{e^{tA} + h(t, \cdot)\}_{t \geq 0}$  except by a constant, then we can affirm that every hypercyclic function for the  $C_0$ -semigroup  $\{e^{tA}\}_{t \geq 0}$  is universal for the family of operators  $\{e^{tA} + h(t, \cdot)\}_{t \geq 0}$ . Furthermore, by [11, Th. 2.3] we have that a hypercyclic/universal behaviour is presented on every nontrivial autonomous discretization of  $\{e^{tA} + h(t, \cdot)\}_{t \geq 0}$ , i.e. for the sequence of operators  $\{e^{kt_0A} + h(kt_0, \cdot)\}_{k \in \mathbb{N}}$  for every  $t_0 > 0$ . Furthermore, the set of hypercyclic/universal functions is shared by the family of operators itself and by every nontrivial autonomous discretization. Even more, any nontrivial single operator is also topologically mixing c.f. [6, Th. 3.5].

**Remark 3.** One can also consider the cases  $\Psi(t, x) = f(t)\phi_{\lambda, z_0, z_1}(x)$ , with  $\lambda \in V$  and  $\Re(\lambda) < 0$ , with a similar proof to the one of Theorem 2.2: This can be done by analyzing the asymptotic behaviour of  $h(t, x)$  when  $t$  tends to  $\infty$ .

- i)  $f(t) \in L^1(\mathbb{R}_0^+)$ : We will see that  $\lim_{t \rightarrow \infty} h(t, x) = 0$  for every  $x \in \mathbb{R}$ . Let us fix  $x \in \mathbb{R}$  and  $\varepsilon > 0$ . Since  $f(t) \in L^1(\mathbb{R}_0^+)$ , we can find some  $t' > 0$  such that

$\int_{t'}^{\infty} |f(s)| ds < \frac{\varepsilon}{2|\phi_{\lambda, z_0, z_1}(x)|}$ . Now, as  $\Re(\lambda) < 0$ , we have that there is some  $t_0 > t'$  such that for all  $t \geq t_0$  we have that  $e^{\Re(\lambda)(t-t')} < \frac{\varepsilon}{2\|f\|_1|\phi_{\lambda, z_0, z_1}(x)|}$ . Taking this into account we have

$$\begin{aligned} |h(t, x)| &= |\phi_{\lambda, z_0, z_1}(x)| \left( \int_0^{t'} |e^{\lambda(t-s)} f(s)| ds + \int_{t'}^t |e^{\lambda(t-s)} f(s)| ds \right) \\ &\leq |\phi_{\lambda, z_0, z_1}(x)| \left( e^{\Re(\lambda)(t-t')} \|f\|_1 + \int_{t'}^{\infty} |f(s)| ds \right) < \varepsilon. \end{aligned}$$

for all  $t \geq t_0$ .

- ii)  $f(t) \in L^p(\mathbb{R}_0^+)$  for every  $1 < p < \infty$ : We will also see that  $\lim_{t \rightarrow \infty} h(t, x) = 0$ . Again, let us fix  $x \in \mathbb{R}$  and  $\varepsilon > 0$ . As before, there is some  $t' > 0$  such that  $\int_{t'}^{\infty} |f(s)|^p ds < \frac{\varepsilon}{2|\phi_{\lambda, z_0, z_1}(x)|}$ . Then, there is some  $t_0 > t'$  such that we have

$$e^{\Re(\lambda)t} < \frac{\varepsilon}{2\|f\|_p|\phi_{\lambda, z_0, z_1}(x)|} \left| \frac{\lambda q}{1 - e^{-\lambda q t'}} \right|^{\frac{1}{q}}$$

for all  $t \geq t_0$ . Applying Hlder inequality we get

$$\begin{aligned} |h(t, x)| &= |\phi_{\lambda, z_0, z_1}(x)| \left( \int_0^{t'} |e^{\lambda(t-s)} f(s)| ds + \int_{t'}^t |e^{\lambda(t-s)} f(s)| ds \right) \\ &\leq |\phi_{\lambda, z_0, z_1}(x)| \left( e^{\Re(\lambda)t} \left| \frac{1 - e^{-\lambda q t'}}{\lambda q} \right|^{\frac{1}{q}} \|f\|_p + \int_{t'}^{\infty} |f(s)|^p ds \right) < \varepsilon \end{aligned}$$

for all  $t \geq t_0$ .

- iii)  $f(t)$  a bounded locally integrable function that asymptotically tends to a constant  $L_0$ : Fix  $x \in \mathbb{R}$  and  $\varepsilon > 0$ . We will see that  $\lim_{t \rightarrow \infty} h(t, x) = \frac{-L_0}{\lambda} \phi_{\lambda, z_0, z_1}$ . Since  $\lim_{t \rightarrow \infty} f(t) = L_0$  there exists some  $t' > 0$  such that  $|f(s) - L_0| < \frac{\varepsilon|\lambda|}{8|\phi_{\lambda, z_0, z_1}(x)|}$  for all  $s \geq t'$ . Then there exists some  $t_0 > t'$  such that for all  $t \geq t_0$  we have

$$e^{\Re(\lambda)t} < \min \left\{ \frac{\varepsilon|\lambda|}{2\|f\|_{\infty}|\phi_{\lambda, z_0, z_1}(x)|(1 - e^{-\lambda t'})}, \frac{\varepsilon|\lambda|}{4|L_0 e^{-\lambda t'} \phi_{\lambda, z_0, z_1}(x)|} \right\}.$$

We have to estimate

$$\left| h(t, x) + \frac{L_0}{\lambda} \phi_{\lambda, z_0, z_1} \right|$$

which is smaller or equal than

$$|\phi_{\lambda, z_0, z_1}(x)| \left( \left| \int_0^{t'} e^{\lambda(t-s)} f(s) ds \right| + \left| \int_{t'}^t e^{\lambda(t-s)} f(s) ds + \frac{L_0}{\lambda} \right| \right)$$

The first integral is smaller or equal than

$$e^{\Re(\lambda)t} \|f\|_{\infty} \left| \frac{1 - e^{-\lambda t'}}{\lambda} \right|$$

The second one can be bounded by

$$\left| \int_{t'}^t e^{\lambda(t-s)} (f(s) - L_0) ds \right| + \left| \frac{L_0 e^{\lambda(t-t')}}{\lambda} \right|$$

To sum up we have  $|h(t, x) + \frac{L_0}{\lambda} \phi_{\lambda, z_0, z_1}| < \varepsilon$  for all  $t \geq t_0$ .

**Example 1.** Let us consider a time dependent source term  $\Psi(t, x) = f(t)\phi_{\lambda, z_0, z_1}(x)$  such that the time dependent function  $f(t)$  is obtained from the laser heating source term  $g_s(t, x)$  given in (4) and the corresponding right hand side of (3),  $\frac{\tau}{k}(g + \tau g_t)$ . Thus,

$$\Psi(t, x) = \frac{\tau}{k} (H(t) - H(t - \Delta t) + \tau\delta(t) - \tau\delta(t - \Delta t)) \phi_{\lambda, z_0, z_1}(x) \quad (31)$$

for all  $x \in \mathbb{R}$  and  $t \geq 0$ .

In this case, the integral  $h(t, x)$  involves the distribution  $\delta$ , and

$$h(t, x) = \begin{cases} \frac{\tau}{k} e^{\lambda t} \left( \frac{1 - e^{-\lambda t}}{\lambda} + \tau \right) \phi_{\lambda, z_0, z_1}(x) & \text{if } t < \Delta t \\ \frac{\tau}{k} e^{\lambda t} \left( \frac{1}{\lambda} + \tau \right) (1 - e^{-\lambda \Delta t}) \phi_{\lambda, z_0, z_1}(x) & \text{if } t > \Delta t \end{cases} \quad (32)$$

If  $\Re(\lambda) < 0$ ,  $\lim_{t \rightarrow \infty} h(t, x) = 0$ , and the asymptotic behaviour of the solution family  $\{u(t, \cdot)\}_{t \geq 0}$  of (35) is the same that the solution semigroup  $\{e^{tA}\}_{t \geq 0}$  of (14).

**Example 2.** Analogous calculations can be done for the heat source term coming from radiofrequency given in (7), then

$$\Psi(t, x) = (H(t) + \tau\delta(t)) \phi_{\lambda, z_0, z_1}(x) \quad x \in \mathbb{R}, t \geq 0 \quad (33)$$

and, the  $h(t, x)$  integral is given by:

$$h(t, x) = \left( \frac{-1}{\lambda} + \left( \frac{1}{\lambda} + \tau \right) e^{\lambda t} \right) \phi_{\lambda, z_0, z_1}(x) \quad (34)$$

Thus, the dynamical properties of this example are the same that of the time independent case.

If instead of taking  $\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x)$  with  $\lambda \in V$  and  $\Re(\lambda) < 0$ , we take it with  $\lambda \in V \cap i\mathbb{R}$ , then we can get Devaney chaos.

**Theorem 2.3.** *Let  $\rho > 0$  be such that  $\alpha\tau\rho^2 > 2$  and let  $\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x)$  with  $\lambda \in V \cap i\mathbb{R}$ . The solution family  $\{u(t, \cdot)\}_{t \geq 0}$  of (26) is Devaney chaotic on  $X_\rho \oplus X_\rho$ .*

*Proof.* Take a function  $\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x)$  with  $\lambda \in V \cap i\mathbb{R}$ . There exists some  $\tau \in \mathbb{R}$  such that  $\lambda = \tau\pi i$ . If we check again the computations for  $h(t, x)$  in the proof of Theorem 2.2, we have that for  $t = \frac{2k}{\tau}$ ,  $k \in \mathbb{N}$ ,  $h\left(\frac{2k}{\tau}, x\right) = 0$ .

So that,  $\{u\left(\frac{2k}{\tau}, \cdot\right)\}_k = \{e^{2kA/\tau}\}_k$  which are the iterates of the operator  $e^{2kA/\tau}$ . This operator is Devaney chaotic on  $X_\rho \oplus X_\rho$  by the combination of [9, Th. 2.1] with [19, Th. 2.1]. Then the solution family  $\{u(t, \cdot)\}_{t \geq 0}$  is Devaney chaotic, too.  $\square$

If we consider the abstract Cauchy problem given by the Wave Equation where we have included a source term  $g(t, x)$ ,

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \alpha \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t, x) \end{pmatrix} \\ \begin{pmatrix} u_1(0, x) \\ u_2(0, x) \end{pmatrix} = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{array} \right\} \quad (35)$$

one can also obtain similar results to Theorems 2.2 and 2.3 using the results about the chaotic behaviour of the solutions of the wave equation in (23) on the spaces  $X_\rho \oplus X_\rho$ . One just have to take the operator  $B$  in (24) instead of  $A$  and to consider its corresponding eigenfunctions, which are of the form  $\phi_{\lambda, z_0, z_1}$  with  $R_\lambda = \frac{\lambda^2}{\alpha}$ .

**Theorem 2.4.** *Let  $\rho > 0$ ,  $W$  the open disk of radius  $\alpha\rho^2$  centered at 0, and  $\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x)$  with  $\lambda \in W$  with  $\Re(\lambda) < 0$ . The solution family  $\{u(t, \cdot)\}_{t \geq 0}$  of (35) is topologically mixing on  $X_\rho \oplus X_\rho$ .*

**Theorem 2.5.** *Let  $\rho > 0$ ,  $W$  the open disk of radius  $\alpha\rho^2$  centered at 0, and  $\Psi(t, x) = \phi_{\lambda, z_0, z_1}(x)$  with  $\lambda \in W \cap i\mathbb{R}$ . The solution family  $\{u(t, \cdot)\}_{t \geq 0}$  of (35) is Devaney chaotic on  $X_\rho \oplus X_\rho$ .*

**Remark 4.** As a consequence of [4, Remark 3.8], we can also conclude that under the hypothesis of Theorem 2.3 and Theorem 2.5 the corresponding solution family of operators  $\{u(t, \cdot)\}_{t \geq 0}$  also exhibits a distributionally chaotic behaviour.

The results obtained for the HBE can be also applied to the relativistic heat equation (RHE). The formulation of the RHE only differs from the HBE formulation in the meaning of the coefficients that accompanied second derivatives in equation (10) (see [22]). The RHE is a hyperbolic-like equation, whose theoretical model is based on the theory of relativity and which was designed to overcome the possible conflict between the HHTE and the second law of thermodynamics.

We conclude this section with a comment regarding the stability of the solutions. We recall that a  $C_0$ -semigroup of the form  $\{e^{tA}\}_{t \geq 0}$  defined on a Banach space  $X$  is (uniformly) exponentially stable, [14, p. 296], if there exists  $\varepsilon > 0$  such that

$$\lim_{t \rightarrow \infty} e^{\varepsilon t} \|e^{tA}\| = 0, \quad (36)$$

This situation is not fulfilled in our case. Nevertheless, a weaker version of this condition can also be considered. We say that  $\{e^{tA}\}_{t \geq 0}$  is exponentially stable on a subspace  $Y \subset X$  if there exists  $\varepsilon > 0$  such that for any  $y \in Y$  we have

$$\lim_{t \rightarrow \infty} e^{\varepsilon t} \|e^{tA}y\| = 0, \quad (37)$$

Such study is sometimes considered when analyzing the chaotic behaviour of  $C_0$ -semigroups, see for instance [8, 3]. The HHTE can be seen to be exponentially stable on the subspaces

$$X_{1,\delta} := \text{span}\{\phi_{\lambda, z_0, z_1} : \lambda \in V \text{ with } \Re(\lambda) < \delta, z_0, z_1 \in \mathbb{R}\}, \quad (38)$$

for every  $-\sqrt{\alpha\rho^2/2\tau} < \delta < 0$ . For the case of  $\lambda \in V$  with  $\Re(\lambda) > 0$  we have that the behaviour of the solutions for an initial condition on  $X_1$  escapes to  $\infty$  in norm.

**3. An extension: dynamics of solutions of the higher order HBE.** An extension of the problem considered above is the dynamics of the solutions of higher order linear heat transfer or diffusion equations. Many interesting physical problems are modeled by a fourth, or even higher, diffusion equation. These equations appear in a wide range of areas, including fluid dynamics, electromagnetism and semiconductors, optical tomography, image processing, etc (as an example, see [26, 16] and references therein). In particular, an interesting class of problems described by nonlinear fourth order diffusion equations are thin fluid film flows where surface tension, due to gravity or temperature gradient, is a driving mechanism [23].

The simplest example is the one-dimensional linear fourth order diffusion equation,

$$\left\{ \begin{array}{l} u_t = -\alpha u_{xxxx} \\ u(0, x) = \varphi(x), \quad x \in \mathbb{R} \end{array} \right\}, \quad (39)$$

where  $\alpha$  is the diffusivity. The linear operator  $A = -\alpha\partial_{xxxx}$  is the infinitesimal generator of the solution semigroup  $\{T_t\}_{t \geq 0} = \{e^{tA}\}_{t \geq 0}$ .

To extend a little more our study, let us consider the family of higher order diffusion equations:

$$\left\{ \begin{array}{l} u_t = (-1)^m \alpha_m u_{x \dots 2^{(m+1)} \dots x}, \quad m = 0, 1, 2, \dots \\ u(0, x) = \varphi(x), \quad x \in \mathbb{R} \end{array} \right\}, \quad (40)$$

where  $\alpha_m$  represents the different diffusivities. The factor  $(-1)^m$  in (40) is necessary to ensure sensible physical behavior of its solutions [26], although this is not important from a mathematical point of view for describing the dynamics of the solutions. Family of equations (40) includes for  $m = 0$  the second order heat transfer equation (1) (without source term).

For the Cauchy problems of (40), the infinitesimal generator of each corresponding solution  $C_0$ -semigroup is a linear and continuous operator  $A_m := (-1)^m \alpha_m \partial_x^{2(m+1)}$ ,  $m \in \mathbb{N}^+$ . In order to study the dynamical behavior of the solution semigroup  $\{T_t\}_{t \geq 0} = \{e^{tA_m}\}_{t \geq 0}$  for every  $m \in \mathbb{N}^+$ , we use again the Banach space  $X_\rho$  (15) and the set of holomorphic functions defined as:

$$\varphi_{\lambda, Z}(x) := \sum_{k=0}^{2m+1} z_k \sum_{n=0}^{\infty} \frac{R_\lambda^n x^{2(m+1)n+k}}{(2(m+1)n+k)!}, \quad x \in \mathbb{R}, \quad (41)$$

where  $\lambda \in \mathbb{C}$ ,  $Z = (z_0, z_1, \dots, z_{2m+1}) \in \mathbb{R}^{2(m+1)}$  and  $R_\lambda = (-1)^m \lambda / \alpha_m$ . Functions  $\varphi_{\lambda, Z}(x)$  are eigenvectors of the infinitesimal operator  $A_m$  for every  $m \in \mathbb{N}^+$ . Fixed  $\rho > 0$ , let  $U$  be the open disk of radius  $\alpha_m \rho^{2(m+1)}$  centered at 0, then, for every  $\lambda \in U$  and  $Z \in \mathbb{R}^{2(m+1)}$  the functions  $\varphi_{\lambda, Z}(x)$  belong to the Banach space  $X_\rho$ . Using these holomorphic functions it is not difficult to check that the solution  $C_0$ -semigroup of (40) satisfies the Eigenvalue criterion for chaos and mixing on the Banach space  $X_\rho$  given in Theorem 1.1.

Since this work is devoted to the Hyperbolic Bioheat Equation (10), let us consider the following higher order extension of (10) including an internal source term  $g_m(t, x)$ :

$$\left\{ \begin{array}{l} \tau_m u_{tt} + u_t = (-1)^m \alpha_m u_{x \dots 2^{(m+1)} \dots x} + g_m(t, x), \quad m = 0, 1, 2, \dots \\ u(0, x) = \varphi_1(x), \quad x \in \mathbb{R} \\ u_t(0, x) = \varphi_2(x), \quad x \in \mathbb{R} \end{array} \right\}, \quad (42)$$

where  $\tau_m$  is the relaxation time constant. For  $m = 0$  equation (42) becomes the Hyperbolic Bioheat Equation (10).

Rewriting (42) as a first order Cauchy problem as in (25), the linear operators

$$A_m := \begin{pmatrix} 0 & I \\ (-1)^m \frac{\alpha_m}{\tau_m} \partial_x^{2(m+1)} & -\frac{1}{\tau_m} I \end{pmatrix}, \quad m \in \mathbb{N}^+ \quad (43)$$

are the infinitesimal generators of the solution semigroups  $\{T_t\}_{t \geq 0} = \{e^{tA_m}\}_{t \geq 0}$ .

Therefore, the solutions of this family of higher order equations are of the form:

$$u_m(t, x) = e^{tA_m} \Phi(x) + \int_0^t e^{(t-s)A_m} \Psi_m(s, x) ds \quad (44)$$

where the notation is similar to that used in equation (26).

These solutions have the same dynamical properties on the Banach space  $X_\rho \oplus X_\rho$  that the solution of the second order Hyperbolic Bioheat Equation (Theorem 2.2 and Theorem 2.3). For the homogeneous equation, we check again the Desch-Schappacher-Webb criterion given in Theorem 1.1 using functions  $\varphi_{\lambda, Z}(x)$  of the

form (41). It is just an exercise to check that for  $\lambda \in \mathbb{C}$ ,  $Z = (z_0, z_1, \dots, z_{2m+1}) \in \mathbb{R}^{2(m+1)}$  and

$$R_\lambda = (-1)^m \frac{\tau_m \lambda^2 + \lambda}{\alpha_m}. \quad (45)$$

the set of functions

$$\phi_{\lambda, Z} := \begin{pmatrix} \varphi_{\lambda, Z} \\ \lambda \varphi_{\lambda, Z} \end{pmatrix}. \quad (46)$$

are holomorphic and satisfy that  $\phi_{\lambda, Z} \in \ker(\lambda I - A_m)$  for every  $m \in \mathbb{N}^+$ . Furthermore, for a fixed  $\rho > 0$  such that  $\alpha_m \tau_m \rho^{2(m+1)} > 2$  and for  $\lambda \in U$ , being  $U$  the open disk of radius  $\sqrt{\alpha_m \rho^{2(m+1)}}/2\tau_m > 0$  centered at 0, functions  $\phi_{\lambda, Z}$  belong to the Banach space  $X_\rho \oplus X_\rho$ . Therefore, using the chaos criterion (Theorem 1.1), the  $C_0$ -semigroups  $\{e^{tA_m}\}_{t \geq 0}$  are mixing a chaotic on  $X_\rho \oplus X_\rho$  for all  $m \in \mathbb{N}^+$ .

Discussion for the part of the solution due to the source terms, i.e. the integral  $\int_0^t e^{(t-s)A_m} \Psi_m(s, x) ds$ , is analogous of the discussion of the HBE equation. As a result, the solution family  $\{u_m(t, \cdot)\}_{t \geq 0}$  (44) of the  $2(m+1)$ -order hyperbolic bioheat equation (42) is topologically mixing for  $\lambda$  values with  $\Re(\lambda) < 0$  and Devaney chaotic with  $\lambda \in i\mathbb{R}$  on  $X_\rho \oplus X_\rho$  for each  $m \in \mathbb{N}^+$  (under some additional restrictions analogous to the hypothesis of Theorem 2.2 and Theorem 2.3).

**4. Conclusions.** In this paper we have studied the linear dynamics of the solutions of the HBE in certain spaces of analytic functions that are densely embedded in  $\mathcal{C}(\mathbb{R})$  with the topology of uniform convergence on compact subsets of  $\mathbb{R}$ . Specifically, we have shown the chaotic behaviour for the solutions of a Cauchy problem in which the governing equation is given by the HBE with a time-independent heat source term that is certain eigenfunction for the differential operator  $A$ . Since these eigenfunctions belong to a set whose span is dense in  $X_\rho \oplus X_\rho$ , see Theorem 1.2, given any initial source term  $g(t, x)$ , we can find another source term of the form

$$\Psi(t, x) = \sum_{i=1}^n \phi_{\lambda_i, z_{i,0}, z_{i,1}}(x) \text{ with } \lambda_i \in V \text{ and } \Re(\lambda_i) \leq 0 \text{ for all } 1 \leq i \leq n \quad (47)$$

such that  $\begin{pmatrix} \sum_{i=1}^n \varphi_{\lambda_i, z_{i,0}, z_{i,1}}(x) \\ \sum_{i=1}^n \lambda_i \varphi_{\lambda_i, z_{i,0}, z_{i,1}}(x) \end{pmatrix}$  can be found as close as we want of  $\begin{pmatrix} 0 \\ g(t, x) \end{pmatrix}$  and for this new source term we can find an initial condition with dense orbit under the action of the operators in the family  $u(t, x)$ .

The general treatment of the problem allows us to apply the results to all types of ablative therapies (laser, radiofrequency, microwave, and ultrasound), and with different modes of energy delivering without making any additional computation.

An extension of these results for Hyperbolic Bioheat Equations of higher order is also provided.

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