Low-complexity root-finding iteration functions with no derivatives of any order of convergence ☆

Alicia Cordero[a], Juan R. Torregrosa[a,*]

[a] *Instituto Universitario de Matemática Multidisciplinar*
Universitat Politècnica de València
Camino de Vera s/n, 46022 València, Spain

Abstract

In this paper, a procedure to design Steffensen-type methods of different orders for solving nonlinear equations is suggested. By using a particular divided difference of first order we can transform many iterative methods into derivative-free iterative schemes, holding the order of convergence of the departure original method. Numerical examples and the study of the dynamics are made to show the performance of the presented schemes and to compare them with another ones.

Keywords: Nonlinear equation; iterative method; derivative-free scheme; divided difference; efficiency index; optimal order

1. Introduction

Solving nonlinear equations is a classical problem which has interesting applications in various branches of science and engineering. In this study, we describe new iterative methods to find a simple root α of a nonlinear equation $f(x) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval I.

In the last years, a lot of papers have developed the idea of removing derivatives from the iteration function in order to avoid defining new functions, and calculate iterates only by using the function that describes the problem, trying to preserve the convergence order. The interest of these methods is to be applied on nonlinear equations when there are many problems for obtaining and evaluating the derivatives involved, or when there is no analytical function to derive.

The known Newton's method for finding α uses the iterative expression

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$

which converges quadratically in some neighborhood of α. If the derivative $f'(x_k)$ is replaced by the forward-difference approximation

$$f'(x_k) \approx f[z_k, x_k] = \frac{f(z_k) - f(x_k)}{z_k - x_k}, \tag{1}$$

where $z_k = x_k + f(x_k)$, the Newton's method becomes

$$x_{k+1} = x_k - \frac{f(x_k)}{f[z_k, x_k]}, \quad k = 0, 1, \ldots$$

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*Corresponding author
Email addresses: acordero@mat.upv.es (Alicia Cordero), jrtorre@mat.upv.es (Juan R. Torregrosa)

which is the known Steffensen's method, (see [6]). This scheme is a touch competitor of Newton's method. Both are of second order, both require two functional evaluations per step, but in contrast to Newton's method, Steffensen's scheme is derivative-free.

Commonly, the efficiency of an iterative method is measured by the efficiency index defined as $p^{1/d}$, where p is the order of convergence and d is the number of functional evaluations per step. Kung and Traub conjectured in [4] that the order of convergence of any multipoint method can not exceed the bound 2^{d-1}. The schemes that reach this bound are called optimal methods.

In this paper, by using the idea of Steffensen's scheme, we are going to design a procedure that allows to transform many iterative schemes for solving nonlinear equations into derivative-free method, preserving the order of convergence.

The rest of the paper is organized as follows: in Section 2 we describe the mentioned procedure, design some optimal derivative-free iterative schemes and establish the convergence order of these methods. In Section 3 different numerical tests, using smooth and non-smooth functions, and functions with zeros of multiplicity greater than one, confirm the theoretical results and allow us to compare the new methods with the starting ones. In Section 4, some dynamical aspects associated to the presented methods are studied. We finish this manuscript with some conclusions and remarks.

2. Development of the procedure

Traub in [10] presented an iterative method with third order of convergence, which needs three functional evaluations per step, two of the function f and one of the derivative f'. It is known that if we replace f' by the forward-difference approximation, the resulting method has the iterative expression

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, \\ x_{k+1} &= y_k - \frac{f(y_k)}{f[z_k,x_k]}, \end{aligned}$$

where $z_k = x_k + f(x_k)$, and it has order of convergence three. So, we ask the following question:
Every time that f' is replaced by the forward-difference approximation, do you always preserve the order of convergence?
The answer is negative as we can see now. It is known that Ostrowski's method, given by the iterative expression

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ x_{k+1} &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)}, \end{aligned}$$

has order of convergence four and uses three functional evaluations per iteration, so it is an optimal method in the sense of Kung-Traub conjecture. However, if we use the mentioned approximation of f', the resulting scheme

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, \\ x_{k+1} &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f[z_k,x_k]}, \end{aligned} \qquad (2)$$

where $z_k = x_k + f(x_k)$, has only order of convergence three, being its error equation

$$e_{k+1} = -c_2^2(1+f'(\alpha))\left[3+f'(\alpha)+2(1+f'(\alpha))^2\right]e_k^3 + O(e_k^4),$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \ldots$ and $e_k = x_k - \alpha$.

Nevertheless, if we use $z_k = x_k + f(x_k)^2$ then the iterative method (2) has order of convergence four, preserving the order and the optimality of Ostrowski's scheme. As a more general result, if we apply this idea to the King's family schemes, which contains the Ostrowski's method for a particular value of the parameter (see [3]), we obtain an uniparametric family of optimal derivative-free methods of order four.

Theorem 1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is close enough to α, then the iterative method described by*

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, \\ x_{k+1} &= y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f[z_k,x_k]}, \end{aligned} \tag{3}$$

where $z_k = x_k + f(x_k)^2$ and β is a real parameter, has optimal fourth convergence order and its error equation is

$$e_{k+1} = c_2\left(-c_2 f'(\alpha)^2 + (1+2\beta)c_2^2 - c_3\right)e_k^4 + O(e_k^5),$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$ for $j = 2, 3, \ldots$ and $e_k = x_k - \alpha$.

Proof. By using Taylor's expansion about α, we have

$$f(x_k) = f'(\alpha)\left[e_k + c_2e_k^2 + c_3e_k^3 + c_4e_k^4\right] + O(e_k^5) \tag{4}$$

and

$$z_k - \alpha = e_k + f'(\alpha)^2\left[e_k^2 + 2c_2e_k^3 + (2c_3 + c_2^2)e_k^4\right] + O(e_k^5). \tag{5}$$

Then

$$\begin{aligned} f(z_k) &= f'(\alpha)e_k + f'(\alpha)(c_2 + f'(\alpha)^2)e_k^2 + f'(\alpha)(c_3 + 4c_2f'(\alpha)^2)e_k^3 \\ &\quad + f'(\alpha)(c_4 + 5(c_3 + c_2^2)f'(\alpha)^2 + c_2f'(\alpha)^4)e_k^4 + O(e_k^5). \end{aligned}$$

By substituting these expressions in the first step of (3), we obtain

$$\begin{aligned} y_k - \alpha &= c_2e_k^2 + (c_2f'(\alpha)^2 - 2c_2^2 + 2c_3)e_k^3 + (-c_2^2f'(\alpha)^2 + 4c_2^3 + 3c_3f'(\alpha)^2 - 7c_2c_3 + 3c_4)e_k^4 \\ &\quad + O(e_k^5) \end{aligned}$$

and using again Taylor's expansion

$$\begin{aligned} f(y_k) &= c_2f'(\alpha)e_k^2 + f'(\alpha)(c_2f'(\alpha)^2 - 2c_2^2 + 2c_3)e_k^3 \\ &\quad + f'(\alpha)(-c_2^2f'(\alpha)^2 + 5c_2^3 + 3c_3f'(\alpha)^2 - 7c_2c_3 + 3c_4)e_k^4 + O(e_k^5). \end{aligned}$$

Finally, by replacing these expressions in the second step of (3), we obtain the error equation of the method

$$e_{k+1} = c_2\left(-c_2f'(\alpha)^2 + (1+2\beta)c_2^2 - c_3\right)e_k^4 + O(e_k^5)$$

and this completes the proof. ■

We can extend the previous results in the following way.

Theorem 2. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is close enough to α, then the iterative method described by*

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, \\ x_{k+1} &= y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f[z_k,x_k]}, \end{aligned}$$

where $z_k = x_k + \gamma f(x_k)^n$ and β, γ are real parameters, has optimal fourth convergence order for all $n \geq 2$ and for all β, γ, $\gamma \neq 0$. Its error equation is

$$e_{k+1} = c_2\left(-\gamma c_2f'(\alpha)^2 + (1+2\beta)c_2^2 - c_3\right)e_k^4 + O(e_k^5), \text{ for } n = 2$$

and

$$e_{k+1} = \left((1+2\beta)c_2^3 - c_2c_3\right)e_k^4 + O(e_k^5), \text{ for } n \geq 3.$$

Can we extend this technique for higher order schemes? For methods of order greater than four we consider, for example, the three-point iterative method introduced by Sharma et al. in [8], with optimal order of convergence eight, derived from King's family followed by a new step obtained by a particular rational approximation:

$$\begin{array}{rcl} y_k & = & x_k - \frac{f(x_k)}{f'(x_k)}, \\ w_k & = & y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} & = & x_k - \frac{(P+Q+R)f(x_k)}{Pf[w_k,x_k]+Qf'(x_k)+Rf[y_k,x_k]}, \end{array} \tag{6}$$

where $P = (x_k - y_k)f(x_k)f(y_k)$, $Q = (y_k - w_k)f(y_k)f(w_k)$ and $R = (w_k - x_k)f(w_k)f(x_k)$.

By using the approximation $f'(x_k) \approx f[z_k, x_k]$, where $z_k = x_k + \gamma f(x_k)^n$, we obtain a two-parametric derivative-free family

$$\begin{array}{rcl} y_k & = & x_k - \frac{f(x_k)}{f[z_k,x_k]}, \\ w_k & = & y_k - \frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}\frac{f(y_k)}{f[z_k,x_k]}, \\ x_{k+1} & = & x_k - \frac{(P+Q+R)f(x_k)}{Pf[w_k,x_k]+Qf[z_k,x_k]+Rf[y_k,x_k]}, \end{array} \tag{7}$$

verifying the following result.

Theorem 3. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ in an open interval I. If x_0 is close enough to α, then the iterative methods described by (7), for all values of parameters β and γ, $\gamma \neq 0$, have the following error equations:*

(i) If $n = 1$, the new methods have order of convergence 5 and they error equation is

$$e_{k+1} = -f'(\alpha)^2\gamma^2(1 + \gamma f'(\alpha))^2 c_2^4 e_k^5 + O(e_k^6).$$

(ii) If $n = 2$, the new scheme reaches seventh convergence order and its error equation is

$$e_{k+1} = f'(\alpha)^2\gamma c_2^3(-f'(\alpha)^2\gamma c_2 + (1 + 2\beta)c_2^2 - c_3)e_k^7 + O(e_k^8).$$

(iii) If $n \geq 3$, we obtain an optimal derivative-free family of order eight (preserving the order of the original scheme (6)). The error equation for $n = 3$ is

$$e_{k+1} = c_2^2((1 + 2\beta)c_2^2 - c_3)(c_2^3 + c_2(f'(\alpha)^3\gamma - 2c_3) + c_4)e_k^8 + O(e_k^9)$$

and for $n \geq 4$

$$e_{k+1} = c_2^2((1 + 2\beta)c_2^2 - c_3)(c_2^3 - 2c_2c_3 + c_4)e_k^8 + O(e_k^9).$$

Let us note that the factor $(1 + \gamma f'(\alpha))^2$, in case $n = 1$, allows us to apply the techniques used for several researchers (see, for example [11]) in order to obtain iterative schemes with memory and with order of convergence greater than 5. These procedures can not be applied in cases $n = 2$ and $n \geq 3$.

We obtain similar results to Theorem 3 by applying this idea to any scheme with optimal order of convergence eight. For example, we consider the three-point iterative method introduced by Sharma et al. in [9], with optimal eighth-order convergence

$$\begin{array}{rcl} y_k & = & x_k - \frac{f(x_k)}{f'(x_k)}, \\ w_k & = & y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} & = & w_k - \left(1 + \frac{f(w_k)}{f(x_k)}\right)\frac{f[x_k,y_k]f(w_k)}{f[x_k,w_k]f[y_k,w_k]}. \end{array} \tag{8}$$

By using the approximation $f'(x_k) \approx f[z_k, x_k]$, where $z_k = x_k + \gamma f(x_k)^n$, we obtain a derivative-free scheme verifying, for every non-zero value of parameter γ:

(i) If $n=1$, the new method has order of convergence 5 and its error equation is

$$e_{k+1} = -f'(\alpha)^2\gamma^2(1+\gamma f'(\alpha))^2 c_2^4 e_k^5 + O(e_k^6).$$

(ii) If $n=2$, the new scheme reaches seventh convergence order and its error equation is

$$e_{k+1} = -f'(\alpha)^2\gamma(f'(\alpha)^2\gamma c_2 - c_2^2 + c_3)e_k^7 + O(e_k^8).$$

(iii) If $n \geq 3$, the convergence order of Sharma's scheme is held and we obtain optimal derivative-free schemes of order eight. The error equation for $n=3$ is

$$e_{k+1} = c_2^2(c_2^2 - c_3)(3c_2^3 + c_2(f'(\alpha)^3 - 4c_3) + c_4)e_k^8 + O(e_k^9)$$

and for $n \geq 4$

$$e_{k+1} = c_2^2(c_2^2 - c_3)(3c_2^3 - 4c_2c_3 + c_2c_4)e_k^8 + O(e_k^9).$$

This idea also provides good results in optimal schemes obtained from weight functions procedure. By using this technique, Liu and Wang presented in [5] the iterative scheme

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ w_k &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f'(x_k)}, \\ x_{k+1} &= w_k - \frac{f(w_k)}{f'(x_k)}\left[\left(\frac{f(x_k)-f(y_k)}{f(x_k)-2f(y_k)}\right)^2 + \frac{f(w_k)}{f(y_k)+f(w_k)} + G(\mu_k)\right], \end{aligned}$$

where $\mu_k = \frac{f(w_k)}{f(x_k)}$. They proved that if x_0 is close enough to the solution and G is a sufficiently differentiable function such that $G(0)=0$ and $G'(0)=4$, then the scheme is convergent with order eight. We modify this method in the following sense:

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k,x_k]}, \\ w_k &= y_k - \frac{f(x_k)}{f(x_k)-2f(y_k)}\frac{f(y_k)}{f[z_k,x_k]}, \\ x_{k+1} &= w_k - \frac{f(w_k)}{f[z_k,x_k]}\left[\left(\frac{f(x_k)-f(y_k)}{f(x_k)-2f(y_k)}\right)^2 + \frac{f(w_k)}{f(y_k)+f(w_k)} + G(\mu_k)\right], \end{aligned} \tag{9}$$

where $\mu_k = \frac{f(w_k)}{f(x_k)}$ and $z_k = x_k + \gamma f(x_k)^n$, and we can establish the following result.

Theorem 4. *Assume that f and G are sufficiently differentiable functions and f has a simple zero $\alpha \in I$. If the initial estimation x_0 is close enough to α, then the methods defined by (9) converge to α with eighth-order for all $n \geq 3$, $\gamma \neq 0$ and under conditions $G(0)=0$ and $G'(0)=4$. The error equation is*

$$e_{k+1} = c_2(c_2^2 - c_3)(14c_2^4 + c_2^2(\gamma f'(\alpha)^3 - 17c_3) + c_3^2 + c_2c_4)e_k^8 + O(e_k^9), \text{ for } n=3$$

and

$$e_{k+1} = c_2(c_2^2 - c_3)(14c_2^4 - 17c_2^2c_3 + 2c_3^2 + c_2c_4)e_k^8 + O(e_k^9), \text{ for } n \geq 4.$$

Remark Every time that we apply the approximation of the derivative $f'(x_k) \approx f[z_k, x_k]$, with $z_k = x_k + \gamma f(x_k)^n$, on an optimal scheme of order four (2^2) or eighth (2^3), we need to use $n \geq 2$ or $n \geq 3$, respectively. So, we conjecture that if the optimal order is 2^q, we will need $n \geq q$ for preserving the order of convergence. We also conjecture that this idea provides analogous results when it is applied to scheme with optimal order of convergence 2^q, $q = 1, 2, 3, \ldots$.

When the original scheme has not optimal order p, the approximation $f'(x_k) \approx f[z_k, x_k]$, with $z_k = x_k + \gamma f(x_k)^n$, allows to hold the order of convergence, but we are not sure about the minimum value of n that gives the order of the original scheme.

3. Numerical results

In this section, we compare the derivative-free schemes described with the methods used to design them. Specifically, we compare Ostrowski's method (OM) and the schemes obtained by using in it the approximation $f'(x_k) \approx f[x_k + f(x_k)^n, x_k]$ for $n = 1$ and $n = 2$, denoted by OM-DF1 and OM-DF2, respectively. For eighth-order we use Sharma (SM) and SGG's method (SGGM), described by equations (8) and (6), with $\beta = 0$, respectively; and the schemes obtained by using in it the previous approximation of $f'(x_k)$ for $n = 1$, $n = 2$ and $n = 3$, denoted by SM-DF1, SM-DF2, SM-DF3, SGGM-DF1, SGGM-DF2 and SGGM-DF3, respectively. These methods are employed to solve some nonlinear equations of two classes: smooth functions

(1) $f_1(x) = \sin^2 x - x^2 + 1, \quad \alpha \approx \pm 1.4044916,$

(2) $f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad \alpha \approx -1.2076478,$

(3) $f_3(x) = e^{\sin x} - 1 - x/5, \quad \alpha = 0,$

(4) $f_4(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3, \quad \alpha \approx 2.331968,$

(5) $f_5(x) = (x-1)^3 - 1, \quad \alpha = 2.$

and non-smooth functions or functions with multiple roots

(1) $g_1(x) = \sqrt{x-1}, \quad \alpha = 1,$

(2) $g_2(x) = \begin{cases} x(x-1), & x < 0, \\ -2x(x+1), & x \geq 0, \end{cases} \quad \alpha = 0,$

(3) $g_3(x) = |x^2 - 9|, \quad \alpha = 3,$

(4) $g_4(x) = \begin{cases} x, & x \leq 0, \\ x^2, & x > 0, \end{cases} \quad \alpha = 0,$

(5) $g_5(x) = (\sin^2 x - 2x + 1)^3, \quad \alpha \approx 0.714835, m(\alpha) = 3.$

Let us note that the derivative of g_1 is not defined in the root $\alpha = 1$. Functions g_2, g_3 and g_4 are non-differentiable in $x = 0$, $x = \pm 3$ and $x = 0$, respectively. Finally, g_5 is a function with a multiple zero.

Numerical computations with smooth functions have been carried out using variable precision arithmetic, with 2000 digits, in Matlab 7.13, whereas that for functions g_i, $i = 1, 2 \ldots, 5$ we have used Mathematica. The stopping criterion used is $|x_{k+1} - x_k| < tol$ or $|f(x_{k+1})| < tol$, with $tol = 10^{-500}$ for smooth functions and $tol = 10^{-50}$ for the other functions. In any case, we show, in the different tables, $incr1 = |x_{k+1} - x_k|$ and $incr2 = |f(x_{k+1})|$ for the last iteration, the number of iterations needed to reach the wished tolerance and the approximate computational order of convergence (ACOC), according to (see [2])

$$p \approx ACOC = \frac{\ln(|x_{k+1} - x_k| / |x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}| / |x_{k-1} - x_{k-2}|)}.$$

The value of ACOC that appears in Tables from 1 to 4 is the last coordinate of vector ACOC when the variation between its values is small. In other case we will denote it by '-'. On the other hand, 'nc' denotes that the method does not converge with, at least, 10^4 iterations. In Tables 1 and 2 we summarize the results obtained by Ostrowski's method and its variants when their are applied to smooth and non-smooth functions, respectively. In Table 1 we can observe that, for initial approximation near to the solution, the behavior of all methods is similar, stable and the ACOC is very near of the theoretical order of convergence. However, derivative-free methods have problems of convergence (see for instance f_2) when the initial estimation is not near of the solution.

In Table 2 we can observe the erratic behavior of the methods on non-smooth functions. For g_1, g_4 and g_5 Ostrowski's scheme has only linear convergence which coincides with that of the other schemes when they

Table 1: Ostrowski-type methods for smooth functions

	OM	OM-DF1	OM-DF2
$f_1, x_0 = 1$			
incr1	0.125e-439	0.973e-418	0.116e-361
incr2	0.250e-1753	0.516e-1249	0.150e-1440
iter	6	8	6
ACOC	4.0000	3.0000	4.0000
$f_2, x_0 = 3$			
incr1	0.114e-192	nc	nc
incr2	0.130e-764	nc	nc
iter	14	nc	nc
ACOC	4.0000	-	-
$f_3, x_0 = 0.5$			
incr1	0.127e-155	0.785e-188	0.360e-196
incr2	0.514e-618	0.217e-560	0.788e-782
iter	5	6	5
ACOC	4.0000	3.0000	4.0090
$f_4, x_0 = 2$			
incr1	0.337e-324	0.283e-291	0.302e-237
incr2	0.273e-1293	0.189e-871	0.137e-944
iter	5	6	5
ACOC	3.9998	3.0000	4.0000
$f_5, x_0 = 1.7$			
incr1	0.139e-130	nc	0.116e-285
incr2	0.736e-517	nc	0.449e-1136
iter	5	nc	6
ACOC	4.0000	-	4.0000

Table 2: Ostrowski-type methods for other functions

	OM	OM-DF1	OM-DF2
$g_1, x_0 = 0.7$			
incr1	0.896e-50	nc	0.617e-50
incr2	0.563e-25	nc	0.496e-25
iter	143	nc	212
ACOC	1.0000	-	1.0000
$g_2, x_0 = 0.5$			
incr1	0.291e-30	0.225e-15	nc
incr2	0.144e-121	0.774e-62	nc
iter	4	4	nc
ACOC	4.0000	3.9220	-
$g_3, x_0 = 0.5$			
incr1	0.402e-42	nc	nc
incr2	0.726e-171	nc	nc
iter	5	nc	nc
ACOC	4.0000	-	-
$g_4, x_0 = 0.5$			
incr1	0.776e-25	0.103e-24	0.284e-24
incr2	0.668e-51	0.118e-50	0.898e-50
iter	42	41	41
ACOC	1.0000	1.0000	1.0000
$g_5, x_0 = 1$			
incr1	0.161e-16	0.173e-16	0.161e-16
incr2	0.173e-50	0.213e-50	0.171e-50
iter	44	44	44
ACOC	1.0000	1.0000	1.0000

converge. In particular, derivative-free methods have a bad behavior for g_3, with independence of the initial approximation.

We show in Tables 3 and 4 the results obtained by Sharma and SGG's methods (with $\beta = 0$ in case of SGGM) and their variants when they are applied to smooth and non-smooth functions, respectively. Again, theoretical results are confirmed by numerical ones, as Sharma and SGG's method and their derivative-free variants have a very similar behavior. Indeed, if an initial estimation far away from the solution is chosen, the derivative-free variants can not converge, although SGG-type methods seem to be more stable and robust than Sharma-type schemes, for smooth functions.

Table 3: Sharma and SGG-type methods for smooth functions

	Sharma	Sharma-DF1	Sharma-DF2	Sharma-DF3	SGGM	SGGM-DF1	SGGM-DF2	SGGM-DF3
$f_1, x_0 = 1$								
incr1	0.502e-182	0.861e-448	0.924e-116	0.243e-227	0.738e-244	0.829e-150	0.123e-129	0.702e-240
incr2	0.336e-1444	0	0.182e-798	0.107e-1805	0.153e-1939	0.497e-741	0.135e-895	0.559e-1906
iter	4	6	4	4	4	5	4	4
ACOC	8.3628	5.0000	6.5748	7.6851	8.0000	5.0000	7.0005	8.0001
$f_2, x_0 = 3$								
incr1	0.158e-128	nc	nc	nc	0.384e-308	nc	nc	nc
incr2	0.316e-1016	nc	nc	nc	0.108e-2008	nc	nc	nc
iter	4	nc	nc	nc	21	nc	nc	nc
ACOC	8.0000	-	-	-	8.0000	-	-	-
$f_3, x_0 = 0.5$								
incr1	0.184e-292	0.287e-116	0.168e-236	0.116e-286	0.251e-235	0.155e-136	0.146e-247	0.998e-308
incr2	0.240e-2009	0.491e-575	0.447e-1648	0.106e-2009	0.167e-1872	0.226e-676	0.165e-1725	0.166e-2010
iter	4	4	4	4	4	4	4	4
ACOC	7.7729	-	7.1520	7.7375	-	5.0000	7.0000	8.0000
$f_4, x_0 = 2$								
incr1	0.119e-86	0.143e-150	0.279e-323	0.984e-487	0.178e-82	0.273e-151	0.156e-324	0.165e-488
incr2	0.986e-686	0.860e-749	0.108e-2008	0.108e-2008	0.123e-652	0.433e-752	0.0	0.0
iter	3	4	4	4	3	4	4	4
ACOC	-	5.0141	6.9275	7.9346	7.9666	5.0000	7.0005	-
$f_5, x_0 = 1.7$								
incr1	0.109e-220	0.231e-295	0.179e-90	0.147e-70	0.775e-280	0.425e-237	0.804e-106	0.684e-110
incr2	0.652e-1753	0.285e-1467	0.133e-620	0.128e-550	0.0	0.599e-1176	0.487e-728	0.263e-865
iter	4	8	4	4	4	6	4	4
ACOC	8.2431	4.9998	6.7072	-	8.0000	5.0000	7.0024	7.9593

In Table 4 we observe that, for functions g_1, g_4 and g_5 Sharma and SGG's schemes and their variants have lost the eighth-order convergence. In addition, derivative-free variants have a confuse behavior, as sometimes a better approximation of the derivative deserves a worst result.

In the following section we will try to find answers to the questions arisen in numerical tests. Is the stability worst when the exponent of $f(x_k)$ increases? Does this behavior depend on the original scheme and the problem to be solved?

4. Dynamical Analysis

In the following, a brief discussion will be made on the dynamical aspects associated to the introduced methods. It is important to note that dynamics has been revealed as a very useful tool to deep in the understanding of the rational functions that rise when an iterative scheme is applied to solve a nonlinear equation $f(z) = 0$. The dynamical properties of this rational function give us important information about numerical features of the method as its stability and reliability. In order to get this aim, we will recall some basic concepts that can be found in (see [1]).

Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the orbit of a point $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0,\ R(z_0),\ R^2(z_0), ..., R^n(z_0), ...\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_0 \in \hat{\mathbb{C}}$ is called a fixed point if $R(z_0) = z_0$. A periodic point z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$. A pre-periodic point is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. Moreover, a fixed point z_0 is called attractor if $|R'(z_0)| < 1$, superattractor if $|R'(z_0)| = 0$, repulsor if $|R'(z_0)| > 1$ and parabolic if $|R'(z_0)| = 1$. Then, the basin of

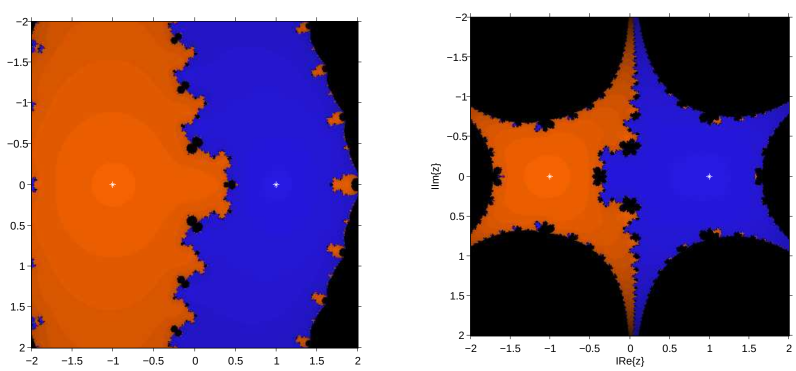
Table 4: Sharma and SGG-type methods for other functions

	Sharma	Sharma-DF1	Sharma-DF2	Sharma-DF3	SGG	SGGM-DF1	SGGM-DF2	SGGM-DF3
$g_1, x_0 = 0.7$								
incr1	0.801e-50	nc	0.524e-50	0.588e-50	0.379e-50	nc	0.412e-50	nc
incr2	0.376e-25	nc	0.442e-25	0.323e-25	0.373e-25	nc	0.381e-25	nc
iter	72	$> 10^4$	96	75	103	nc	113109	nc
ACOC	1.0	-	1.844	1.0	1.0	-	1.0	-
$g_2, x_0 = 0.5$								
incr1	0.591e-27	0.628e-13	nc	0.299e-27	0.2912e-30	0.8871e-12	nc	0.5053e-36
incr2	0.892e-217	0.463e-91	nc	0.130e-219	0.104e-243	0.5189e-83	nc	0.2973e-289
iter	3	4	nc	3	3	4	nc	3
ACOC	-	-	-	-	-	4.860	-	-
$g_3, x_0 = 0.5$								
incr1	0.137e-8	nc	nc	nc	0.791e-21	nc	nc	nc
incr2	0.791e-75	nc	nc	nc	0.328e-173	nc	nc	nc
iter	4	nc	nc	nc	5	nc	nc	nc
ACOC	5.546	-	-	-	-	-	-	-
$g_4, x_0 = 0.5$								
incr1	0.160e-24	0.534e-24	0.154e-24	0.158e-24	0.181e-24	0.102e-24	0.182e-24	0.182e-24
incr2	0.715e-51	0.804e-51	0.664e-51	0.707e-51	0.668e-51	0.289e-51	0.675e-51	0.674e-51
iter	30	29	30	30	28	28	28	28
ACOC	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
$g_5, x_0 = 1$								
incr1	0.208e-16	0.213e-16	0.207e-16	0.208e-16	4.206e-16	3.994e-16	4.209e-16	4.206e-16
incr2	0.780e-51	0.842e-51	0.777e-51	0.780e-51	3.623e-52	3.103e-52	3.632e-52	3.622e-52
iter	32	32	32	32	30	30	30	30
ACOC	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

attraction of an attractor α is defined as:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \;:\; R^n(z_0) \to \alpha,\ n \to \infty\}.$$

The Fatou set of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the Julia set, $\mathcal{J}(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.



(a) $n = 1$ (b) $n = 2$

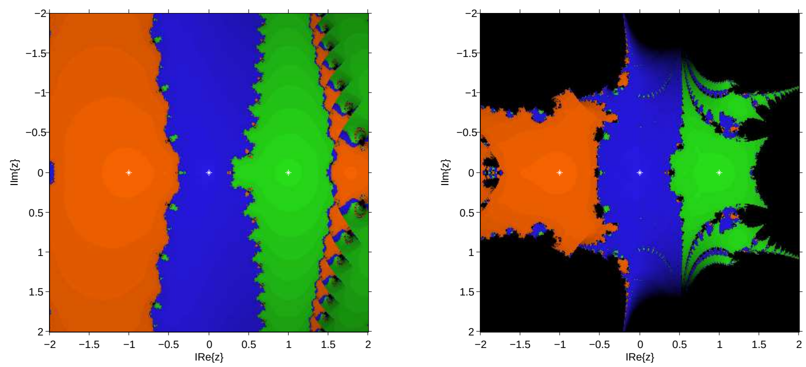
Figure 1: Dynamical planes associated to modified Ostrowski's scheme on $p(z) = z^2 - 1$

Let us start by the simplest case: if Ostrowski's scheme is applied on a quadratic polynomial, it is known that the associated dynamical plane will be composed by two semi-planes as the only connected components of the basins of attraction of the the two roots. When the modified Ostrowski's schemes (for $n = 1$ and $n = 2$) act on the same polynomial (see Figure 1), the basins remain wide for $n = 1$, although the number of connected components has increased to infinity. This makes smaller the region of stable behavior. However, they are greater than the ones obtained for $n = 2$. This seems to be the cost for holding the order of convergence of the original method. Moreover, in both cases back regions appear, that represent starting points with no convergence.

When the non-smooth function $g_2(z)$ is considered, for $z \in \mathbb{C}$,

$$g_2(z) = \begin{cases} z(z+1), & z < 0 \\ -2z(z-1), & z \geq 0. \end{cases}$$

Then, modified Ostrowski's scheme shows a stable behavior, with clean basins of attraction are observed and little black areas, pre-images of the infinity appear (see Figure 2). In addition, three basins of attraction, corresponding to the roots $z = -1$, $z = 0$ and $z = 1$, are showed in different colors.



(a) $n = 1$, order 3

(b) $n = 2$, order 4

Figure 2: Dynamical planes associated to modified Ostrowski's scheme on $g_2(z)$

Let us now consider the behavior of eighth-order schemes: in Figures 3, 4 and 5, the dynamical planes for variants of both methods, SM and SGGM, by using exponents up to $n = 3$ are showed. Analyzing the dynamical planes associated to modified Sharma's scheme on $p(z) = z^2 - 1$ (see Figures 3), it can be noticed that they have better conditions for convergence than modified Ostrowski's scheme for $n = 1$ and $n = 2$, as there are no wide black regions; nevertheless, dynamics becomes 'interesting' when $n = 3$, with the appearance of 2-periodic orbits, whose detail can be observed in Figure 4. Respect to the derivative-free variants of SGGM, it can be noticed that their behavior is more similar to the Ostrowski-like schemes: there exist wide black regions, but the vicinity of the roots is "clean", there are wider balls centered at the roots with full convergence to the solutions. In case of $n = 3$ (Figure 5c)), the wideness of the basins have been reduced respect to the ones of $n = 1$ and $n = 2$.

In Figures 6 and 7, the dynamical planes associated to modified Sharma and SGG's schemes on g_2 are showed. We can observe that the role of the function is capital, as the complexity of the dynamical plane is much higher than in simpler cases.

5. Conclusions

We have introduced a simple technique, based in a particular divided difference of first order, which applied to an iterative scheme for nonlinear equations provides a derivative-free iterative method, holding the order of convergence. Some numerical test are provided on smooth and non-smooth functions to show the performance of the new methods. The analysis of the dynamics shows that, in general, some of the basins of attraction widen when the powers increase, the stability is worse when the original order is recovered and also pre-images of the infinity appear.

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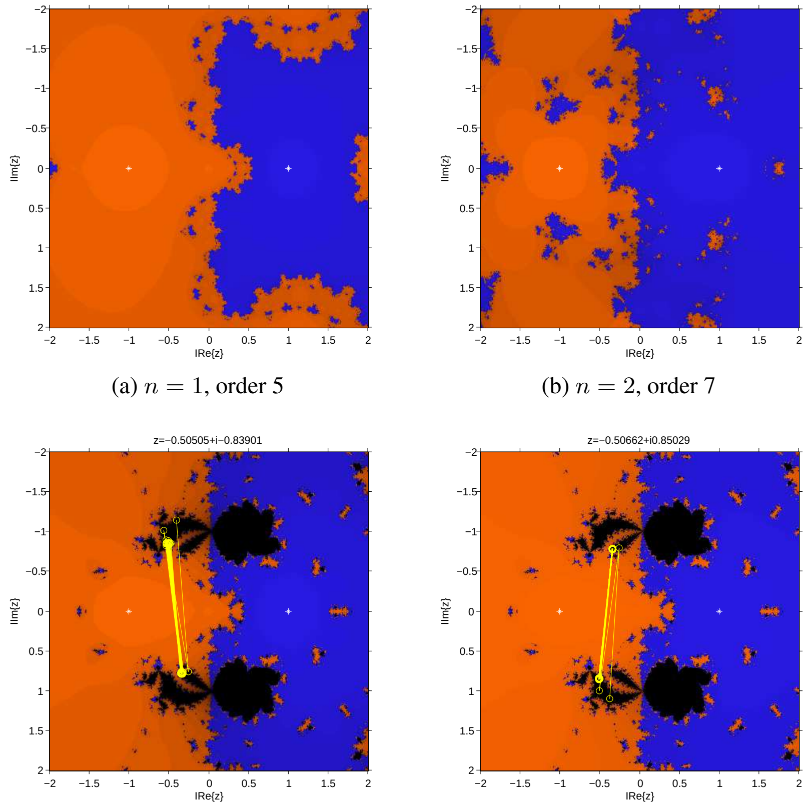


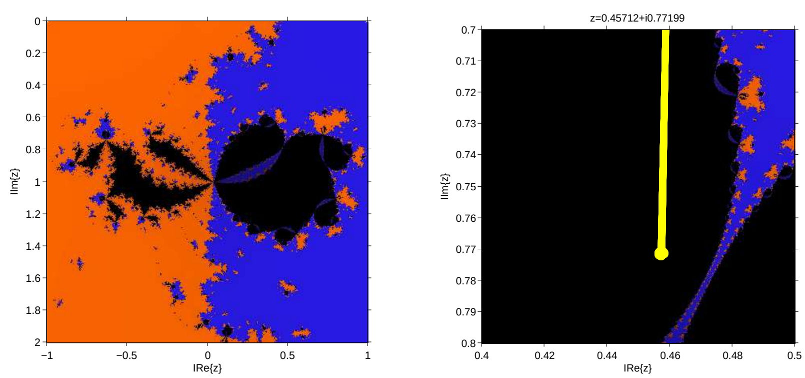
(a) $n = 1$, order 5

(b) $n = 2$, order 7

(c) $n = 3$, order 8

(d) $n = 3$, order 8

Figure 3: Dynamical planes associated to modified Sharma's scheme on $p(z) = z^2 - 1$



(a) $n = 3$, order 8

(b) $n = 3$, order 8

Figure 4: Details about modified Sharma's scheme on $p(z) = z^2 - 1$

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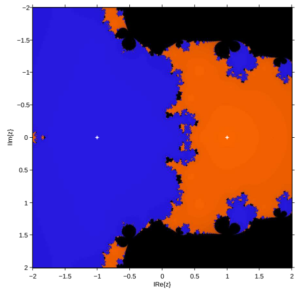
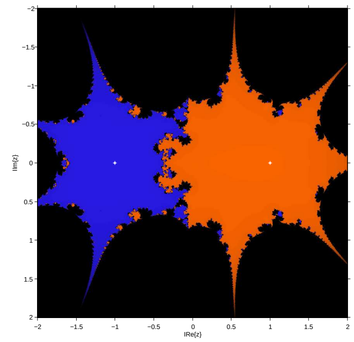
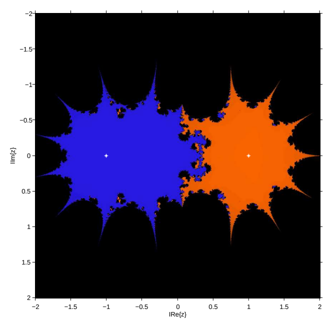




(a) $n = 1$, order 5

(b) $n = 2$, order 7



(c) $n = 3$, order 8

Figure 5: Dynamical planes associated to modified SGG's scheme on $p(z) = z^2 - 1$

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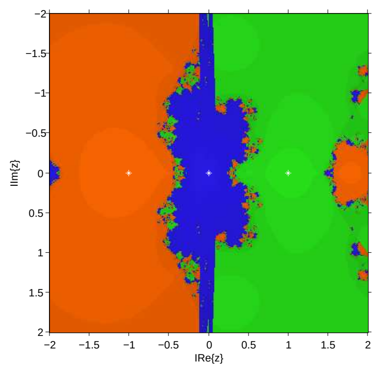
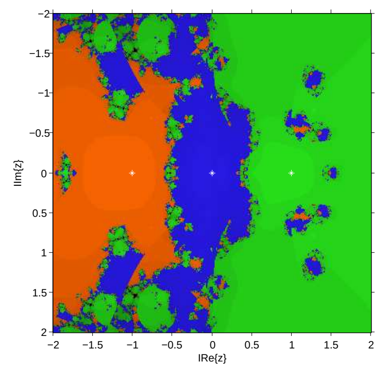
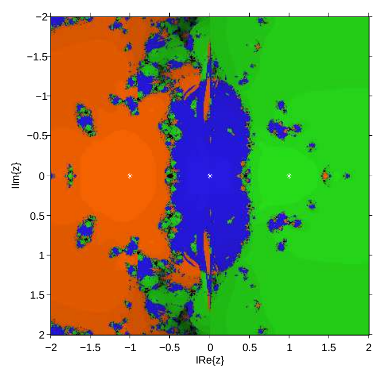


(a) $n = 1$, order 5

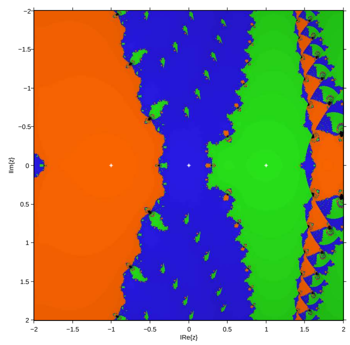


(b) $n = 2$, order 7

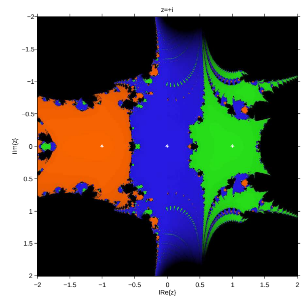


(c) $n = 3$, order 8

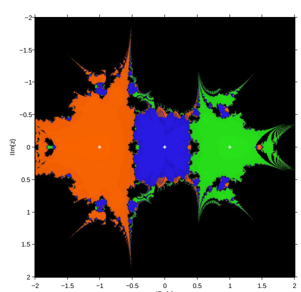
Figure 6: Dynamical planes associated to modified Sharma's scheme on $g_2(z)$



(a) $n = 1$, order 5



(b) $n = 2$, order 7



(c) $n = 3$, order 8

Figure 7: Dynamical planes associated to modified SGG's scheme on $g_2(z)$