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This paper must be cited as:

Bru García, R.; Cantó Colomina, R.; Urbano Salvador, AM. (2015). Eigenstructure of rank one updated matrices. *Linear Algebra and its Applications*. 485:372-391.  
doi:10.1016/j.laa.2015.07.036.



The final publication is available at

<http://dx.doi.org/10.1016/j.laa.2015.07.036>

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Additional Information

# Eigenstructure of rank one updated matrices <sup>☆</sup>

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## Abstract

The relationship among eigenvalues of a given square matrix  $A$  and the rank one updated matrix  $A + v_k q^*$ , where  $v_k$  is an eigenvector of  $A$  associated with the eigenvalue  $\lambda_k$  and  $q$  is an arbitrary vector, was described by Brauer in 1952. In this work we study the relations between the Jordan structures of  $A$  and  $A + v_k q^*$ . More precisely, we analyze the generalized eigenvectors of the updated matrix in terms of the generalized eigenvectors of  $A$ , as well as the Jordan chains of the updated matrix. Further, we obtain similar results when we use a generalized eigenvector of  $A$  instead of the eigenvector  $v_k$ .

*Keywords:* One rank perturbation, Jordan form, Jordan chains

*AMS classification:*, 15A03, 15A18, 15A21

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## 1. Introduction

The relationship between the Jordan structure of two matrices sufficiently close one to each other have been largely studied in the matrix literature. Boer and Thijssse [2] obtained relations for matrices with distinct eigenvalues. Furthermore, Marcus and Parilis [5] gave a complete description of the changes in the Jordan structure (counting the length of the Jordan chains) of a perturbed matrix under small perturbations. Moro and Dopico [7] have researched the Jordan blocks of the complex matrix  $A + B$  associated with an eigenvalue  $\lambda$  of  $A$  with geometric multiplicity  $g$ , when  $\text{rank}(B) \leq g$ . Beitia, Hoyos and Zaballa [1] studied the change of the Jordan structure of a perturbed matrix under small perturbations in one row of the given matrix. Recently Mehl, Mehrmann, Ran and Rodman [6] considered the perturbation theory of structured matrices under generic structured rank one perturbations, identifying generic Jordan structures of perturbed matrices.

In [4] the authors presented the relationships between the right and left eigenvectors of matrices  $A$  and  $A + v_k q^T$ , where  $v_k$  is an eigenvector of  $A$ . The same authors applied this result to deflation problems and to pole assignment

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<sup>☆</sup>Supported by the Spanish DGI grant MTM2013-43678-P  
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for single-input single-output (SISO) and multi-input multi-output (MIMO) systems. In the MIMO systems defined by pairs  $(A, B)$  where  $A$  and  $B$  are  $n \times n$  and  $n \times m$  matrices, [4, Proposition 4.5] showed that the solution of the pole assignment is not unique in general. Moreover, it was possible to allocate poles even in the case of uncontrollable and unstable systems given by  $(A, B)$  and the closed-loop system  $A + BF^T$  could be asymptotically stable with the feedback matrix  $F$  computed from the eigenvectors of  $A^T$ . Some connections between pole assignment and assignment of invariant factors on matrices with some prescribed submatrices were studied by Zaballa [11]. Concretely, by using the Hardy-Littlewood-Polya majorization relations and the interlacing inequalities for invariant factors, it was explained in [11, Theorem 2.6] when there exists an  $m \times n$  matrix  $C$  with elements in a field such that  $A + BC$  has  $n$  given monic polynomials as its invariant factors. This kind of applications motivate the study of problems involving Jordan structure as this work.

Thompson [10, Theorem 2] showed how similarity invariants of matrices with elements in a field change under a rank one update. In fact, Thompson obtained the interlacing inequalities for invariant factors of two square matrices  $A$  and  $A + xy^T$  with elements in a field. This result does not say how the spectrum of  $A + xy^T$  is obtained from the spectrum of  $A$  and the generalized eigenvectors of  $A + xy^T$  from the given generalized eigenvectors of  $A$ . In general, this is a very difficult problem that we study in a specific case by using the well known Brauer Theorem [3]. This theorem gives the spectrum of the updated matrix  $A + xy^T$  from the spectrum of the initial matrix  $A$ , where  $x = v_k$  is an eigenvector of  $A$  associated with the eigenvalue  $\lambda_k$  and  $y$  is an arbitrary vector with elements in a field.

In this work we study the relations between the Jordan structures of the matrices  $A \in \mathbb{C}^{n \times n}$  and  $A + v_k q^* \in \mathbb{C}^{n \times n}$  using Brauer's Theorem. Concretely, we obtain the Jordan chains of  $A + v_k q^*$  in terms of the given Jordan chains of  $A$ . Further, similar results when we use a generalized eigenvector of  $A$  instead of the eigenvector  $v_k$  are given.

Throughout this paper, we work with matrices  $A \in \mathbb{C}^{n \times n}$  and denote by  $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$  the spectrum of  $A$  with algebraic multiplicities  $r_1, r_2, \dots, r_s$ . If  $\{v_{it_i}, v_{i,t_i-1}, \dots, v_{i2}, v_{i1}\}$  is a Jordan chain of  $A$  associated with  $\lambda_i$  of length  $t_i$ , then

$$\begin{cases} Av_{i1} = \lambda_i v_{i1} \\ Av_{ij} = \lambda_i v_{ij} + v_{i,j-1}, & j = 2, 3, \dots, t_i, \end{cases}$$

where

$$v_{ij} \in \ker(A - \lambda_i I)^j \setminus \ker(A - \lambda_i I)^{j-1}, \quad j = t_i, t_i - 1, \dots, 1.$$

Suppose that  $A$  has  $k_i$  Jordan chains of length  $t_{i1}, t_{i2}, \dots, t_{ik_i}$ , for each eigenvalue  $\lambda_i$ ,  $i = 1, 2, \dots, s$ . Then the Jordan structure of  $A$  is

$$J_A = J(\lambda_1) \oplus J(\lambda_2) \oplus \dots \oplus J(\lambda_s)$$

where

$$J(\lambda_i) = J_{t_{i_1}}(\lambda_i) \oplus J_{t_{i_2}}(\lambda_i) \oplus \cdots \oplus J_{t_{i_{k_i}}}(\lambda_i)$$

and

$$J_{t_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{t_i \times t_i}$$

Without loss of generality, we apply Brauer's Theorem with the eigenvalue  $\lambda_1$  of  $A$  and the associated eigenvector  $v_1$ . Then the updated matrix  $A + v_1 q^*$  has  $v_1$  as an eigenvector associated with the new eigenvalue  $\mu = \lambda_1 + q^* v_1$ . Moreover,  $\sigma(A + v_1 q^*) = \{\mu, \lambda_1, \lambda_2, \dots, \lambda_s\}$  with algebraic multiplicities 1,  $r_1 - 1$ ,  $r_2$ ,  $\dots$ ,  $r_s$  when  $\mu \notin \sigma(A)$ , or with the corresponding multiplicities if  $\mu \in \sigma(A)$ .

We study the changes of the Jordan structure when  $\mu \notin \sigma(A)$  in Section 2 and when  $\mu \in \sigma(A)$  in Section 3. To do this, we construct the Jordan chains of each eigenvalue of the updated matrix from the Jordan chains of the corresponding eigenvalue of the initial matrix, and we call the Jordan chain  $\{w_{it_i}, w_{i,t_i-1}, \dots, w_{i2}, w_{i1}\}$  of  $A + v_1 q^*$  the *updated Jordan chain* of  $\{v_{it_i}, v_{i,t_i-1}, \dots, v_{i2}, v_{i1}\}$  of  $A$ .

Finally, in Section 4 we study how the Jordan chains of  $A$  are changing when we apply the rank one perturbation  $v_{1j} q^*$ , where  $v_{1j}$  is a generalized eigenvector of  $A$  associated with  $\lambda_1$  instead of its eigenvector  $v_1$  and  $q$  is an  $n$ -dimensional vector such that  $q^* v_{1h} = 0$ ,  $h \leq j$ .

## 2. Jordan structure when $\mu \notin \sigma(A)$

In this section we study the changes of the Jordan structure of a matrix  $A$  when the new eigenvalue  $\mu = \lambda_1 + q^* v_1 \notin \sigma(A)$ . By Brauer's Theorem (see [4]),  $v_1$  is the eigenvector associated with  $\mu$  of the updated matrix  $A + v_1 q^*$ . The updated Jordan chains are obtained in the following results. In Theorem 1 we prove that the Jordan chains of  $A$  and  $A + v_1 q^*$  associated with  $\lambda_i$  have the same length, for  $i = 2, 3, \dots, s$ . In Theorem 2, the Jordan structure of  $A + v_1 q^*$  associated with  $\lambda_1$  is given.

**Theorem 1 (Structure associated with  $\lambda_i \neq \lambda_1$ ).** *Let  $A \in \mathbb{C}^{n \times n}$ . Consider an eigenvalue  $\lambda_i \neq \lambda_1$  and  $v_1$  an eigenvector of  $A$  associated with  $\lambda_1$ . Let  $\{v_{it_i}, v_{i,t_i-1}, \dots, v_{i2}, v_{i1}\}$  be a Jordan chain of  $A$  associated with  $\lambda_i$ . Let  $q$  be an  $n$ -dimensional vector such that  $\mu = \lambda_1 + q^* v_1 \notin \sigma(A)$ . Then  $A + v_1 q^*$  has the updated Jordan chain  $\{w_{it_i}, w_{i,t_i-1}, \dots, w_{i2}, w_{i1}\}$ , associated with the eigenvalue  $\lambda_i$ , where*

$$w_{ij} = v_{ij} - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} v_1, \quad j = 1, 2, \dots, t_i.$$

*Proof.* Since  $\{v_{it_i}, v_{i,t_i-1}, \dots, v_{i2}, v_{i1}\}$  is a Jordan chain associated with the eigenvalue  $\lambda_i$  we have

$$\begin{aligned} Av_{ij} &= \lambda_i v_{ij} + v_{i,j-1}, & j = t_i, t_i - 1, \dots, 2, \\ Av_{i1} &= \lambda_i v_{i1}. \end{aligned}$$

It remains to prove

$$\begin{aligned} (A + v_1 q^*) w_{ij} &= \lambda_i w_{ij} + w_{i,j-1}, & j = t_i, t_i - 1, \dots, 2, \\ (A + v_1 q^*) w_{i1} &= \lambda_i w_{i1}. \end{aligned}$$

For  $j = t_i, t_i - 1, \dots, 2$ , we have

$$\begin{aligned} (A + v_1 q^*) w_{ij} &= (A + v_1 q^*) \left( v_{ij} - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} v_1 \right) \\ &= Av_{ij} + (q^* v_{ij}) v_1 - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} (A + v_1 q^*) v_1 \\ &= \lambda_i v_{ij} + v_{i,j-1} + (q^* v_{ij}) v_1 - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} \mu v_1 \\ &\quad + \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} \lambda_i v_1 - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} \lambda_i v_1 \\ &= \lambda_i \left( v_{ij} - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} v_1 \right) + v_{i,j-1} + (q^* v_{ij}) v_1 \\ &\quad - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} (\mu - \lambda_i) v_1 \\ &= \lambda_i \left( v_{ij} - \sum_{s=1}^j \frac{q^* v_{is}}{(\mu - \lambda_i)^{j+1-s}} v_1 \right) + \left( v_{i,j-1} - \sum_{s=1}^{j-1} \frac{q^* v_{is}}{(\mu - \lambda_i)^{j-s}} v_1 \right) \\ &= \lambda_i w_{ij} + w_{i,j-1}. \end{aligned}$$

Finally, for  $j = 1$ , the expression of the eigenvector associated with  $\lambda_i$

$$w_{i1} = v_{i1} - \frac{q^* v_{i1}}{\mu - \lambda_i} v_1,$$

is proved in [8] (see also [4, Proposition 1.1.]).

As a consequence, we conclude that  $\{w_{it_i}, w_{i,t_i-1}, \dots, w_{i2}, w_{i1}\}$  is the updated Jordan chain of the matrix  $A + v_1 q^*$  associated with  $\lambda_i$ .  $\square$

**Theorem 2 (Structure associated with  $\lambda_1$ ).** Let  $A \in \mathbb{C}^{n \times n}$  and let  $v_1$  be an eigenvector of  $A$  associated with  $\lambda_1$ . Let  $\{v_{1t_1}, v_{1,t_1-1}, \dots, v_{12}, v_{11}\}$  be a Jordan chain of  $A$  associated with  $\lambda_1$  of length  $t_1$ . Let  $q$  be an  $n$ -dimensional vector such that  $\mu = \lambda_1 + q^*v_1 \notin \sigma(A)$ . Then  $A + v_1q^*$  has the updated Jordan chain  $\{w_{1d_1}, w_{1,d_1-1}, \dots, w_{12}, w_{11}\}$  of length  $d_1$ , associated with  $\lambda_1$ , as follows

(1) If  $v_{11} \neq v_1$ , then  $d_1 = t_1$  and

$$w_{1j} = v_{1j} - \sum_{s=1}^j \frac{q^*v_{1s}}{(\mu - \lambda_1)^{j+1-s}} v_1, \quad j = 1, 2, \dots, t_1.$$

(2) If  $v_{11} = v_1$  then  $d_1 = t_1 - 1$ . We distinguish two cases:

(2.1) If  $t_1 = 1$ , there is not any updated Jordan chain of  $\{v_1\}$  associated with  $\lambda_1$  in  $A + v_1q^*$ . Furthermore,  $v_1$  becomes an eigenvector of  $A + v_1q^*$  associated with  $\mu$ .

(2.2) If  $t_1 > 1$ , then

$$\begin{aligned} w_{11} &= v_{12} - \frac{1 + q^*v_{12}}{\mu - \lambda_1} v_1, \\ w_{1j} &= v_{1,j+1} - \sum_{s=3}^{j+1} \frac{q^*v_{1s}}{(\mu - \lambda_1)^{j+2-s}} v_1 - \frac{1 + q^*v_{12}}{(\mu - \lambda_1)^j} v_1, \end{aligned}$$

for  $j = 2, 3, \dots, t_1 - 1$ .

*Proof.*

(1) This case can be proved analogously to Theorem 1.

(2) For  $v_{11} = v_1$ , we consider two cases:

(2.1) The first part of the assertion is clear. The second part is included in Brauer's Theorem.

(2.2) For  $j = t_1 - 1, t_1 - 2, \dots, 2$ , we have

$$\begin{aligned} (A + v_1q^*)(w_{1j}) &= \\ &= (A + v_1q^*) \left( v_{1,j+1} - \sum_{s=3}^{j+1} \frac{q^*v_{1s}}{(\mu - \lambda_1)^{j+2-s}} v_1 - \frac{1 + q^*v_{12}}{(\mu - \lambda_1)^j} v_1 \right) \\ &= Av_{1,j+1} + (q^*v_{1,j+1})v_1 - \sum_{s=3}^{j+1} \frac{q^*v_{1s}}{(\mu - \lambda_1)^{j+2-s}} \mu v_1 - \frac{1 + q^*v_{12}}{(\mu - \lambda_1)^j} \mu v_1 \\ &= \left( \lambda_1 v_{1,j+1} - \lambda_1 \sum_{s=3}^{j+1} \frac{q^*v_{1s}}{(\mu - \lambda_1)^{j+2-s}} v_1 - \lambda_1 \frac{1 + q^*v_{12}}{(\mu - \lambda_1)^j} v_1 \right) \\ &+ v_{1j} + (q^*v_{1,j+1})v_1 - \sum_{s=3}^{j+1} \frac{q^*v_{1s}}{(\mu - \lambda_1)^{j+2-s}} (\mu - \lambda_1)v_1 - \frac{1 + q^*v_{12}}{(\mu - \lambda_1)^j} (\mu - \lambda_1)v_1 \end{aligned}$$

$$\begin{aligned}
&= \lambda_1 \left( v_{1,j+1} - \sum_{s=3}^{j+1} \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+2-s}} v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^j} v_1 \right) \\
&+ \left( v_{1j} - \sum_{s=3}^j \frac{q^* v_{1s}}{(\mu - \lambda_1)^{j+1-s}} v_1 - \frac{1 + q^* v_{12}}{(\mu - \lambda_1)^{j-1}} v_1 \right) = \lambda_1 w_{1j} + w_{i,j-1}.
\end{aligned}$$

As  $v_1$  is the eigenvector of the matrix  $A + v_1 q^*$ , associated with  $\mu$ , then

$$\begin{aligned}
(A + v_1 q^*) w_{11} &= (A + v_1 q^*) \left( v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1} v_1 \right) \\
&= A v_{12} + (q^* v_{12}) v_1 - \frac{1 + q^* v_{12}}{\mu - \lambda_1} \mu v_1 \\
&= \lambda_1 v_{12} + v_1 + (q^* v_{12}) v_1 - \frac{1 + q^* v_{12}}{\mu - \lambda_1} \mu v_1 \\
&= \lambda_1 v_{12} + \left\{ 1 + q^* v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1} \mu \right\} v_1 \\
&= \lambda_1 v_{12} - \lambda_1 \frac{1 + q^* v_{12}}{\mu - \lambda_1} v_1 = \lambda_1 \left( v_{12} - \frac{1 + q^* v_{12}}{\mu - \lambda_1} v_1 \right) = \lambda_1 w_{11},
\end{aligned}$$

that is,  $w_{11}$  is the eigenvector of  $A + v_1 q^*$  associated with  $\lambda_1$  in this chain. Therefore,  $\{w_{1,t_1-1}, w_{1,t_1-2}, \dots, w_{11}\}$  is the updated Jordan chain of length  $t_1 - 1$ , of the matrix  $A + v_1 q^*$ , associated with  $\lambda_1$ .  $\square$

**Example 1.** Consider the following matrix in Jordan form

$$A = J_3(1) \oplus J_2(1) \oplus J_2(2) \oplus J_1(3)$$

whose eigenvalues are  $\lambda_1 = 1$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 3$ . The Jordan chains are:  $\{e_3, e_2, e_1\}$  and  $\{e_5, e_4\}$  associate with  $\lambda_1 = 1$ ,  $\{e_7, e_6\}$  associate with  $\lambda_2 = 2$  and  $\{e_8\}$  associate with  $\lambda_3 = 3$ .

Consider the eigenvector  $e_1$  and the vector  $q^* = [q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7 \ q_8]$ , such that  $q_1 \notin \{0, 1, 2\}$ . Then  $1 + q^* e_1 = 1 + q_1 \notin \sigma(A)$  and the eigenvalues of  $A + e_1 q^*$  are  $\mu = 1 + q_1$ ,  $\lambda_1 = 1$ ,  $\lambda_2 = 2$  and  $\lambda_3 = 3$ . The Jordan chains of this matrix are:

Jordan chains of $A$	Updated Jordan chains of $A + e_1 q^*$
Eigenvalue $\lambda_2 = 2$ $\{e_7, e_6\}$	Eigenvalue $\lambda_2 = 2$ $\left\{ e_7 - \left( \frac{q_6}{(q_1 - 1)^2} + \frac{q_7}{q_1 - 1} \right) e_1, e_6 - \frac{q_6}{q_1 - 1} e_1 \right\}$
Eigenvalue $\lambda_3 = 3$ $\{e_8\}$	Eigenvalue $\lambda_3 = 3$ $\left\{ e_8 - \frac{q_8}{q_1 - 1} e_1 \right\}$

<i>Jordan chains of A</i>	<i>Updated Jordan chains of A + e<sub>1</sub>q*</i>
<i>Eigenvalue</i> $\lambda_1 = 1$ $\{e_5, e_4\}$	<i>Eigenvalue</i> $\lambda_1 = 1$ $\left\{ e_5 - \left( \frac{q_4}{(q_1 - 1)^2} + \frac{q_5}{q_1 - 1} \right) e_1, e_4 - \frac{q_4}{q_1 - 1} e_1 \right\}$
<i>Eigenvalue</i> $\lambda_1 = 1$ $\{e_3, e_2, e_1\}$	<i>Eigenvalue</i> $\lambda_1 = 1$ $\left\{ e_3 - \left( \frac{q_3}{q_1} + \frac{1 + q_2}{q_1^2} \right) e_1, e_2 - \frac{1 + q_2}{q_1} e_1 \right\}$  <i>Eigenvalue</i> $\mu = 1 + q_1$ $\{e_1\}$

The invariant factors of matrices  $A$  and  $A + e_1q^*$  are

$$S_8(A) = (\lambda - 1)^3(\lambda - 2)^2(\lambda - 3) \quad S_7(A) = (\lambda - 1)^2, \\ S_6(A) = S_5(A) = \dots = S_1(A) = 1$$

and

$$S_8(A + e_1q^*) = (\lambda - 1)^2(\lambda - 2)^2(\lambda - 3)(\lambda - (1 + q_1)) \\ S_7(A + e_1q^*) = (\lambda - 1)^2 \\ S_6(A + e_1q^*) = S_5(A + e_1q^*) = \dots = S_1(A + e_1q^*) = 1.$$

Note that Thompson [10, Theorem 2] holds.

### 3. Jordan structure when $\mu \in \sigma(A)$

Now, we study the changes of the Jordan structure of a matrix  $A$  when the new eigenvalue  $\mu = \lambda_1 + q^*v_1$  belongs to  $\sigma(A)$ . We consider two possibilities: first when  $\mu$  is equal to some eigenvalue  $\lambda_i$ , for  $i = 2, 3, \dots, s$ . In this case, without loss of generality, we suppose that  $\mu = \lambda_2$ . Then to get the updated Jordan chains of  $A + v_1q^*$ , we work with Jordan chains associated with the  $\lambda_i$ , for  $i = 3, 4, \dots, s$ , with  $\lambda_1$  and with  $\lambda_2$ . The second possibility is when  $\mu = \lambda_1$ .

#### 3.1. Jordan structure when $\mu = \lambda_2$

We can obtain the updated Jordan chains of  $A + v_1q^*$  associated with  $\lambda_i$ ,  $i = 3, 4, \dots, s$ , applying Theorem 1 and associated with  $\lambda_1$  applying Theorem 2. To study the updated Jordan chains associated with  $\lambda_2$  we consider two cases:

1. All Jordan chains of  $A$  associated with  $\lambda_2$  satisfy that the inner product of the vector  $q$  with the eigenvector of the corresponding chain is zero (case (1) in the next theorem).
2. Otherwise, there exists Jordan chains of  $A$  associated with  $\lambda_2$  such that the inner product of the vector  $q$  with the corresponding eigenvector is nonzero (case (2) in the next theorem).



**Theorem 3 (Structure associated with  $\lambda_2$ ).** Let  $A \in \mathbb{C}^{n \times n}$ . Consider the eigenvalue  $\lambda_2 \neq \lambda_1$  and  $v_1$  an eigenvector of  $A$  associated with  $\lambda_1$ . Suppose that  $A$  has  $k_2$  Jordan chains of length  $t_{2_1} \leq t_{2_2} \leq \dots \leq t_{2_{k_2}}$ , denoted by  $\{v_{2t_{2_g}}^{(g)}, v_{2,t_{2_g}-1}^{(g)}, \dots, v_{22}^{(g)}, v_{21}^{(g)}\}$ , with  $g = 1, 2, \dots, k_2$ . Let  $q$  be an  $n$ -dimensional vector such that  $\mu = \lambda_1 + q^*v_1 = \lambda_2$ . Then the length and number of Jordan chains associated with  $\lambda_2$  of the updated matrix  $A + v_1q^*$  are:

- (1) If  $q^*v_{21}^{(g)} = 0$ , for  $g = 1, 2, 3, \dots, k_2$ , the updated matrix  $A + v_1q^*$  has  $k_2 + 1$  Jordan chains of length  $t_{2_1}, t_{2_2}, \dots, t_{2_{k_2}}, 1$ , defined as follows

- (1.1) For  $g = 1, 2, \dots, k_2$ ,  $\{w_{2t_{2_g}}^{(g)}, w_{2,t_{2_g}-1}^{(g)}, \dots, w_{22}^{(g)}, w_{21}^{(g)}\}$  is the updated Jordan chain of  $\{v_{2t_{2_g}}^{(g)}, v_{2,t_{2_g}-1}^{(g)}, \dots, v_{22}^{(g)}, v_{21}^{(g)}\}$ , where

$$\begin{aligned} w_{2t_{2_g}}^{(g)} &= v_{2t_{2_g}}^{(g)} \\ w_{2l}^{(g)} &= v_{2l}^{(g)} + (q^*v_{2,l+1}^{(g)})v_1, \quad l = t_{2_g} - 1, t_{2_g} - 2, \dots, 2, 1 \end{aligned}$$

- (1.2)  $\{v_1\}$  is the new Jordan chain of length 1.

- (2) There exists  $j_1, j_2, \dots, j_p$ , such that  $t_{2_1} \leq t_{2_{j_1}} \leq t_{2_{j_2}} \leq \dots \leq t_{2_{j_p}} \leq t_{2_{k_2}}$  and  $q^*v_{21}^{(j_i)} \neq 0$ , for  $i = 1, 2, \dots, p$ . Then  $A + v_1q^*$  has  $k_2$  Jordan chains of length  $t_{2_1}, \dots, t_{2_{j_p-1}}, t_{2_{j_p}} + 1, t_{2_{j_p+1}}, \dots, t_{2_{k_2}}$ , defined as follows

- (2.1)  $\{w_{2,t_{2_{j_p}}+1}^{(j_p)}, w_{2t_{2_{j_p}}}^{(j_p)}, \dots, w_{22}^{(j_p)}, w_{21}^{(j_p)}\}$ , of length  $t_{2_{j_p}} + 1$ , is the updated Jordan chain of  $\{v_{2t_{2_{j_p}}}^{(j_p)}, v_{2,t_{2_{j_p}}-1}^{(j_p)}, \dots, v_{22}^{(j_p)}, v_{21}^{(j_p)}\}$ , where

$$\begin{aligned} w_{2,t_{2_{j_p}}+1}^{(j_p)} &= \frac{1}{q^*v_{21}^{(j_p)}} v_{2t_{2_{j_p}}}^{(j_p)} \\ w_{2,l+1}^{(j_p)} &= \frac{1}{q^*v_{21}^{(j_p)}} v_{2l}^{(j_p)} + \frac{q^*v_{2,l+1}^{(j_p)}}{q^*v_{21}^{(j_p)}} v_1, \quad l = t_{2_{j_p}} - 1, \dots, 2, 1 \\ w_{21}^{(j_p)} &= v_1 \end{aligned}$$

- (2.2) For  $i = 1, 2, \dots, p - 1$ ,  $\{w_{2,t_{2_{j_i}}}^{(j_i)}, w_{2,t_{2_{j_i}}-1}^{(j_i)}, \dots, w_{22}^{(j_i)}, w_{21}^{(j_i)}\}$  is the up-

dated Jordan chain of  $\{v_{2t_{2j_i}}^{(j_i)}, v_{2,t_{2j_i}-1}^{(j_i)}, \dots, v_{22}^{(j_i)}, v_{21}^{(j_i)}\}$ , where

$$\begin{aligned} w_{2,t_{2j_i}}^{(j_i)} &= v_{2,t_{2j_i}}^{(j_i)} - \frac{q^* v_{21}^{(j_i)}}{q^* v_{21}^{(j_p)}} v_{2t_{2j_p}}^{(j_p)} \\ w_{2l}^{(j_i)} &= v_{2l}^{(j_i)} - \frac{q^* v_{21}^{(j_i)}}{q^* v_{21}^{(j_p)}} v_{2l}^{(j_p)} + \left( q^* v_{2,l+1}^{(j_i)} - \frac{q^* v_{21}^{(j_i)}}{q^* v_{21}^{(j_p)}} q^* v_{2,j+1}^{(j_p)} \right) v_1, \\ l &= t_{2j_i} - 1, \dots, 2, 1 \end{aligned}$$

(2.3) For  $1 \leq g \leq k_2$ , such that  $g \notin \{j_1, j_2, \dots, j_p\}$ , it is satisfies that  $q^* v_{21}^{(j_i)} \neq 0$ . Then  $\{w_{2t_{2g}}^{(g)}, w_{2,t_{2g}-1}^{(g)}, \dots, w_{22}^{(g)}, w_{21}^{(g)}\}$ , that it is the updated Jordan chain of  $\{v_{2t_{2g}}^{(g)}, v_{2,t_{2g}-1}^{(g)}, \dots, v_{22}^{(g)}, v_{21}^{(g)}\}$ , is defined in case (1.1)

*Proof.*

(1) If  $q^* v_{21}^{(g)} = 0$ , for  $g = 1, 2, 3, \dots, k_2$ , we have

(1.1) For  $g = 1, 2, 3, \dots, k_2$ ,

$$\begin{aligned} (A + v_1 q^*) w_{2t_{2g}}^{(g)} &= (A + v_1 q^*) v_{2t_{2g}}^{(g)} = \lambda_2 v_{2t_{2g}}^{(g)} + v_{2t_{2g}-1}^{(g)} + (q^* v_{2t_{2g}}^{(g)}) v_1 \\ &= \lambda_2 w_{2t_{2g}}^{(g)} + w_{2t_{2g}-1}^{(g)}. \end{aligned}$$

Further, for  $l = t_{2g} - 1, t_{2g} - 2, \dots, 2$ ,

$$\begin{aligned} (A + v_1 q^*) w_{2l}^{(g)} &= (A + v_1 q^*) (v_{2l}^{(g)} + (q^* v_{2,l+1}^{(g)}) v_1) \\ &= \lambda_2 v_{2l}^{(g)} + v_{2,l-1}^{(g)} + (q^* v_{2l}^{(g)}) v_1 + \lambda_2 (q^* v_{2,l+1}^{(g)}) v_1 \\ &= \lambda_2 (v_{2l}^{(g)} + (q^* v_{2,l+1}^{(g)}) v_1) + (v_{2,l-1}^{(g)} + (q^* v_{2l}^{(g)}) v_1) \\ &= \lambda_2 w_{2l}^{(g)} + w_{2,l-1}^{(g)} \end{aligned}$$

and finally

$$\begin{aligned} (A + v_1 q^*) w_{21}^{(g)} &= (A + v_1 q^*) (v_{21}^{(g)} + (q^* v_{22}^{(g)}) v_1) \\ &= \lambda_2 (v_{21}^{(g)} + (q^* v_{22}^{(g)}) v_1) = \lambda_2 w_{21}^{(g)}. \end{aligned}$$

Therefore,  $\{w_{2t_{2g}}^{(g)}, w_{2,t_{2g}-1}^{(g)}, \dots, w_{22}^{(g)}, w_{21}^{(g)}\}$  is the updated Jordan chain of  $\{v_{2t_{2g}}^{(g)}, v_{2,t_{2g}-1}^{(g)}, \dots, v_{22}^{(g)}, v_{21}^{(g)}\}$ , of length  $t_{2g}$ , associated with  $\lambda_2$ .

(1.2) Since

$$(A + v_1 q^*) v_1 = A v_1 + (q^* v_1) v_1 = (\lambda_1 + q^* v_1) v_1 = \lambda_2 v_1$$

$\{v_1\}$  is the new Jordan chain of length 1 associated with  $\lambda_2$ .

(2) The results can be proved with the techniques used before.  $\square$

**Example 2.** Consider the matrix  $A = J_3(1) \oplus J_2(2) \oplus J_2(2) \oplus J_2(2)$ , whose Jordan chains are:  $\{v_{13}^{(1)}, v_{12}^{(1)}, v_{11}^{(1)}\} = \{e_3, e_2, e_1\}$  associated with  $\lambda_1 = 1$ ,  $\{v_{22}^{(1)}, v_{21}^{(1)}\} = \{e_5, e_4\}$ ,  $\{v_{22}^{(2)}, v_{21}^{(2)}\} = \{e_7, e_6\}$  and  $\{v_{22}^{(3)}, v_{21}^{(3)}\} = \{e_9, e_8\}$  associated with  $\lambda_2 = 2$ . If we take the vector  $q^* = [1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7 \ q_8 \ q_9]$  then  $\mu = 1 + q^*e_1 = 2 \in \sigma(A)$  and the eigenvalues of  $A + e_1q^*$  are  $\lambda_1 = 1$  and  $\lambda_2 = 2$ . The length of the Jordan chains of  $A + e_1q^*$  depend on the values of  $q_4, q_6$  and  $q_8$ . If  $q_4$  and  $q_6$  are not zero and  $q_8 = 0$ , and the Jordan chains of  $A + e_1q^*$  are:

- For  $\lambda_1 = 1$  by Theorem 2 and taking into account that  $q_1 = 1$ , the updated Jordan chain of  $\{e_3, e_2, e_1\}$  is  $\{e_3 - (q_3 + (1 + q_2))e_1, e_2 - (1 + q_2)e_1\}$ .
- For  $\lambda_2 = 2$  by Theorem 3 the updated Jordan chains are:

Jordan chains of $A$	Updated Jordan chains of $A + e_1q^*$
$\{e_9, e_8\}$ $q^*e_8 = q_8 = 0$	$\{e_9, e_8 + q_9e_1\}$
$\{e_5, e_4\}$ $q^*e_4 = q_4 \neq 0$ chain of maximal length	$\left\{ \frac{1}{q_4} e_5, \frac{1}{q_4} e_4 + \frac{q_5}{q_4} e_1, e_1 \right\}$
$\{e_7, e_6\}$ $q^*e_6 = q_6 \neq 0$	$\left\{ e_7 - \frac{q_6}{q_4} e_5, e_6 - \frac{q_6}{q_4} e_4 + \left( q_7 - \frac{q_6}{q_4} q_5 \right) e_1 \right\}$

### 3.2. Jordan structure when $\mu = \lambda_1$

Now, we study the case  $\mu = \lambda_1$ , that is when no eigenvalue is changed. Note that,  $q^*v_1 = 0$ . Since we do not change any eigenvalue, it seems nothing is done. However, we can have a new matrix, with some special property, with the same spectrum as the first one but with a new Jordan structure. In this case we obtain the updated Jordan chains of  $A + v_1q^*$  associated with  $\lambda_i, i = 2, 3, \dots, s$ , applying Theorem 1 with  $\mu = \lambda_1$ . The Jordan structure of  $A + v_1q^*$  associated with  $\lambda_1$  is given in the following theorem.

**Theorem 4 (Structure associated with  $\lambda_1$ ).** Let  $A \in \mathbb{C}^{n \times n}$ . Consider the eigenvalue  $\lambda_1$  and the associated eigenvector  $v_1$ . Suppose that  $A$  has  $k_1$  Jordan chains of length  $t_{1,1}, t_{1,2}, \dots, t_{1,k_1}$  associated with  $\lambda_1$ , denoted by  $\{v_{1t_{1,g}}^{(g)}, v_{1,t_{1,g}-1}^{(g)}, \dots, v_{12}^{(g)}, v_{11}^{(g)}\}$ , with  $g = 1, 2, \dots, k_1$  and  $v_{11}^{(1)} = v_1$ . Let  $q$  be an  $n$ -dimensional vector such that  $q^*v_1 = 0$ . Then the length and number of Jordan chains associated with  $\lambda_1$  of the updated matrix  $A + v_1q^*$  are:

(1) If  $q^*v_{11}^{(g)} = 0$ , for  $g = 1, 2, 3, \dots, k_1$ , the updated matrix  $A + v_1q^*$  has  $k_1$  Jordan chains of length  $t_{1_1}, t_{1_2}, \dots, t_{1_{k_1}}$ , defined as follows

(1.1)  $\{w_{1_{t_{1_1}}}^{(1)}, w_{1, t_{1_1}-1}^{(1)}, \dots, w_{1_2}^{(1)}, w_{11}^{(1)}\}$  of length  $t_{1_1}$ , is the updated Jordan chain of  $\{v_{1_{t_{1_1}}}^{(1)}, v_{1, t_{1_1}-1}^{(1)}, \dots, v_{1_2}^{(1)}, v_{11}^{(1)} = v_1\}$ , where

$$\begin{aligned} w_{11}^{(1)} &= v_1 \\ w_{1_{t_{1_1}}}^{(1)} &= \frac{1}{1 + q^*v_{1_2}^{(1)}} v_{1_{t_{1_1}}}^{(1)}, \\ w_{1l}^{(1)} &= \frac{1}{1 + q^*v_{1_2}^{(1)}} v_{1l}^{(1)} + \frac{q^*v_{1, l+1}^{(1)}}{1 + q^*v_{1_2}^{(1)}} v_1, \quad l = t_{1_1} - 1, \dots, 3, 2, \end{aligned}$$

(1.2) For  $g = 2, \dots, k_1$ ,  $\{w_{1_{t_{1_g}}}^{(g)}, w_{1, t_{1_g}-1}^{(g)}, \dots, w_{1_2}^{(g)}, w_{11}^{(g)}\}$  is the updated Jordan chain of  $\{v_{1_{t_{1_g}}}^{(g)}, v_{1, t_{1_g}-1}^{(g)}, \dots, v_{1_2}^{(g)}, v_{11}^{(g)}\}$  where

$$\begin{aligned} w_{1_{t_{1_g}}}^{(g)} &= v_{1_{t_{1_g}}}^{(g)}, \\ w_{1l}^{(g)} &= v_{1l}^{(g)} + (q^*v_{1, l+1}^{(g)})v_1, \quad l = t_{1_g} - 1, t_{1_g} - 2, \dots, 2, 1 \end{aligned}$$

Note that, in this case  $A$  and  $A + v_1q^*$  are similar because they have the same Jordan structure with different generalized eigenvectors.

(2) If there exists a unique index  $j$ ,  $2 \leq j \leq k_1$ , such that  $q^*v_{11}^{(j)} \neq 0$ , we have

(2.1) If  $t_{1_1} = 1$  the updated matrix  $A + v_1q^*$  has the following  $k_1 - 1$  Jordan chains of length  $t_{1_2}, t_{1_3}, \dots, t_{1_j} + 1, t_{1_{j+1}}, \dots, t_{1_{k_1}}$ ,

(2.1.1)  $\{w_{1, t_{1_j}+1}^{(j)}, w_{1_{t_{1_j}}}^{(j)}, \dots, w_{1_2}^{(j)}, w_{11}^{(j)}\}$ , of length  $t_{1_j} + 1$ , is the updated Jordan chain of  $\{v_{1_{t_{1_j}}}^{(j)}, v_{1, t_{1_j}-1}^{(j)}, \dots, v_{1_2}^{(j)}, v_{11}^{(j)}\}$  where

$$\begin{aligned} w_{1, t_{1_j}+1}^{(j)} &= \frac{1}{q^*v_{11}^{(j)}} v_{1_{t_{1_j}}}^{(j)} \\ w_{1, l+1}^{(j)} &= \frac{1}{q^*v_{11}^{(j)}} v_{1l}^{(j)} + \frac{q^*v_{1, l+1}^{(j)}}{q^*v_{11}^{(j)}} v_1, \quad l = t_{1_j} - 1, \dots, 2, 1, \\ w_{11}^{(j)} &= v_1 \end{aligned}$$

(2.1.2) For  $g = 2, 3, \dots, k_1$ ,  $g \neq j$ ,  $\{w_{1, t_{1_g}}^{(g)}, w_{1_{t_{1_g}}}^{(g)}, \dots, w_{1_2}^{(g)}, w_{11}^{(g)}\}$  is the updated Jordan chain of  $\{v_{1_{t_{1_g}}}^{(g)}, v_{1, t_{1_g}-1}^{(g)}, \dots, v_{1_2}^{(g)}, v_{11}^{(g)}\}$  and it is defined as in case (1.2).

(2.2) If  $t_{1_1} > 1$ , we have two cases:

(2.2.1)  $t_{1_1} > t_{1_j}$ , then the updated matrix  $A + v_1 q^*$  has the following  $k_1$  Jordan chains of length  $t_{1_1}, t_{1_2}, \dots, t_{1_j}, t_{1_{j+1}}, \dots, t_{1_{k_1}}$ ,

(2.2.1.1)  $\{w_{1,t_{1_1}}^{(1)}, w_{1,t_{1_1}-1}^{(1)}, \dots, w_{12}^{(1)}, w_{11}^{(1)}\}$  is the updated Jordan chain of  $\{v_{1,t_{1_1}}^{(1)}, v_{1,t_{1_1}-1}^{(1)}, \dots, v_{12}^{(1)}, v_{11}^{(1)} = v_1\}$  and it is defined as in case (1.1).

(2.2.1.2) For  $g = 2, 3, \dots, k_1, g \neq j$ ,  $\{w_{1,t_{1_g}}^{(g)}, w_{1,t_{1_g}-1}^{(g)}, \dots, w_{12}^{(g)}, w_{11}^{(g)}\}$  is the updated Jordan chain of  $\{v_{1,t_{1_g}}^{(g)}, v_{1,t_{1_g}-1}^{(g)}, \dots, v_{12}^{(g)}, v_{11}^{(g)}\}$  and it is defined as in case (1.2).

(2.2.1.3)  $\{w_{1,t_{1_j}}^{(j)}, w_{1,t_{1_j}-1}^{(j)}, \dots, w_{12}^{(j)}, w_{11}^{(j)}\}$  is the updated Jordan chain of  $\{v_{1,t_{1_j}}^{(j)}, v_{1,t_{1_j}-1}^{(j)}, \dots, v_{12}^{(j)}, v_{11}^{(j)}\}$ , where

$$w_{1,t_{1_j}}^{(j)} = v_{1,t_{1_j}}^{(j)} - \frac{q^* v_{11}^{(j)}}{1 + q^* v_{12}^{(1)}} v_{1,t_{1_j}+1}^{(1)}$$

$$w_{1l}^{(j)} = v_{1l}^{(j)} - \frac{q^* v_{11}^{(j)}}{1 + q^* v_{12}^{(1)}} v_{1,l+1}^{(1)} + \left( q^* v_{1,l+1}^{(j)} - \frac{q^* v_{11}^{(j)}}{1 + q^* v_{12}^{(1)}} q^* v_{1,l+2}^{(1)} \right) v_1$$

$$l = t_{1_j} - 1, t_{1_j} - 2, \dots, 2, 1,$$

(2.2.2)  $t_{1_1} \leq t_{1_j}$ , then the updated matrix  $A + v_1 q^*$  has the following  $k_1$  Jordan chains of length  $t_{1_1} - 1, t_{1_2}, \dots, t_{1_j} + 1, t_{1_{j+1}}, \dots, t_{1_{k_1}}$ ,

(2.2.2.1)  $\{w_{1,t_{1_1}-1}^{(1)}, w_{1,t_{1_1}-2}^{(1)}, \dots, w_{12}^{(1)}, w_{11}^{(1)}\}$ , of length  $t_{1_1} - 1$ , is the updated Jordan chain of  $\{v_{1,t_{1_1}}^{(1)}, v_{1,t_{1_1}-1}^{(1)}, \dots, v_{12}^{(1)}, v_{11}^{(1)} = v_1\}$ , where

$$w_{1,t_{1_1}-1}^{(1)} = v_{1,t_{1_1}}^{(1)} - \frac{1 + q^* v_{12}^{(1)}}{q^* v_{11}^{(j)}} v_{1,t_{1_1}-1}^{(j)}$$

$$w_{1l}^{(1)} = v_{1,l+1}^{(1)} - \frac{1 + q^* v_{12}^{(1)}}{q^* v_{11}^{(j)}} v_{1l}^{(j)} + \left( q^* v_{1,j+2}^{(1)} - \frac{1 + q^* v_{12}^{(1)}}{q^* v_{11}^{(j)}} q^* v_{1,l+1}^{(j)} \right) v_1$$

$$l = t_{1_j} - 2, t_{1_j} - 3, \dots, 2, 1,$$

(2.2.2.2) For  $g = 2, 3, \dots, k_1, g \neq j$ ,  $\{w_{1,t_{1_g}}^{(g)}, w_{1,t_{1_g}-1}^{(g)}, \dots, w_{12}^{(g)}, w_{11}^{(g)}\}$  is the updated Jordan chain of  $\{v_{1,t_{1_g}}^{(g)}, v_{1,t_{1_g}-1}^{(g)}, \dots, v_{12}^{(g)}, v_{11}^{(g)}\}$  and it is defined as in case (1.2).

(2.2.2.3)  $\{w_{1,t_{1_j}+1}^{(j)}, w_{1,t_{1_j}}^{(j)}, \dots, w_{12}^{(j)}, w_{11}^{(j)}\}$ , of length  $t_{1_j} + 1$ , is the up-

dated Jordan chain of  $\{v_{1t_{1_j}}^{(j)}, v_{1,t_{1_j}-1}^{(j)}, \dots, v_{12}^{(j)}, v_{11}^{(j)}\}$ , where

$$\begin{aligned} w_{1,t_{1_j}+1}^{(j)} &= \frac{1}{q^* v_{11}^{(j)}} v_{1t_{1_j}}^{(j)} \\ w_{1l}^{(j)} &= \frac{1}{q^* v_{11}^{(j)}} v_{1,l-1}^{(j)} + \frac{q^* v_{1l}^{(j)}}{q^* v_{11}^{(j)}} v_1, \quad l = t_{1_j}, \dots, 3, 2, \\ w_{11}^{(j)} &= v_1 \end{aligned}$$

(3) If there exist indices  $j_1, j_2, \dots, j_p$ , such that  $t_{1_{j_1}} \leq t_{1_{j_2}} \leq \dots \leq t_{1_{j_p}}$ , with  $2 \leq j_i \leq k_1$  and  $q^* v_{11}^{(j_i)} \neq 0$ ,  $i = 1, 2, \dots, p$ , we have:

(3.1) If  $t_{1_1} > t_{1_{j_p}}$  then the updated matrix  $A + v_1 q^*$  has the following  $k_1$  Jordan chains of length  $t_{1_1}, t_{1_2}, \dots, t_{1_{k_1}}$ ,

(3.1.1)  $\{w_{1t_{1_1}}^{(1)}, w_{1,t_{1_1}-1}^{(1)}, \dots, w_{12}^{(1)}, w_{11}^{(1)}\}$  of length  $t_{1_1}$ , is the updated Jordan chain of  $\{v_{1t_{1_1}}^{(1)}, v_{1,t_{1_1}-1}^{(1)}, \dots, v_{12}^{(1)}, v_{11}^{(1)} = v_1\}$  and it is defined as in case (1.1).

(3.1.2) For  $2 \leq g \leq k_1$ ,  $g \notin \{j_1, j_2, \dots, j_p\}$ ,  $\{w_{1,t_{1_g}}^{(g)}, w_{1,t_{1_g}-1}^{(g)}, \dots, w_{12}^{(g)}, w_{11}^{(g)}\}$  is the updated Jordan chain of  $\{v_{1t_{1_g}}^{(g)}, v_{1,t_{1_g}-1}^{(g)}, \dots, v_{12}^{(g)}, v_{11}^{(g)}\}$  and it is defined as in case (1.2).

(3.1.3) For  $i = 1, 2, \dots, p$ ,  $\{w_{1,t_{1_{j_i}}}^{(j_i)}, w_{1,t_{1_{j_i}}-1}^{(j_i)}, \dots, w_{12}^{(j_i)}, w_{11}^{(j_i)}\}$  is the updated Jordan chain of  $\{v_{1t_{1_{j_i}}}^{(j_i)}, v_{1,t_{1_{j_i}}-1}^{(j_i)}, \dots, v_{12}^{(j_i)}, v_{11}^{(j_i)}\}$  and it is defined as in case (2.2.1.3).

(3.2) If  $1 < t_{1_1} \leq t_{1_{j_p}}$  then the updated matrix  $A + v_1 q^*$  has the following  $k_1$  Jordan chains of length  $t_{1_1} - 1, t_{1_2}, \dots, t_{1_{j-1}}, t_{1_j} + 1, \dots, t_{1_{k_1}}$ ,

(3.2.1)  $\{w_{1,t_{1_1}-1}^{(1)}, w_{1,t_{1_1}-2}^{(1)}, \dots, w_{12}^{(1)}, w_{11}^{(1)}\}$ , of length  $t_{1_1} - 1$ , is the updated Jordan chain of  $\{v_{1t_{1_1}-1}^{(1)}, v_{1,t_{1_1}-2}^{(1)}, \dots, v_{12}^{(1)}, v_{11}^{(1)} = v_1\}$  and defined as in case (2.2.2.1).

(3.2.2) For  $2 \leq g \leq k_1$ ,  $g \notin \{j_1, j_2, \dots, j_p\}$ ,  $\{w_{1,t_{1_g}}^{(g)}, w_{1,t_{1_g}-1}^{(g)}, \dots, w_{12}^{(g)}, w_{11}^{(g)}\}$  is the updated Jordan chain of  $\{v_{1t_{1_g}}^{(g)}, v_{1,t_{1_g}-1}^{(g)}, \dots, v_{12}^{(g)}, v_{11}^{(g)}\}$  and it is defined as in case (1.2).

(3.2.3)  $\{w_{1,t_{1_j}+1}^{(j)}, w_{1,t_{1_j}}^{(j)}, \dots, w_{12}^{(j)}, w_{11}^{(j)}\}$ , of length  $t_{1_j} + 1$ , is the updated Jordan chain of  $\{v_{1t_{1_j}+1}^{(j)}, v_{1,t_{1_j}}^{(j)}, \dots, v_{12}^{(j)}, v_{11}^{(j)}\}$  and it is defined as in case (2.2.2.3).

(3.2.4) For  $i = 1, 2, \dots, p - 1$ ,  $\{w_{1,t_{1_{j_i}}}^{(j_i)}, w_{1,t_{1_{j_i}}-1}^{(j_i)}, \dots, w_{12}^{(j_i)}, w_{11}^{(j_i)}\}$  is the

updated Jordan chain of  $\{v_{1t_{1j_i}}^{(j_i)}, v_{1,t_{1j_i}-1}^{(j_i)}, \dots, v_{12}^{(j_i)}, v_{11}^{(j_i)}\}$ , where

$$\begin{aligned} w_{1t_{1j_i}}^{(j_i)} &= v_{1t_{1j_i}}^{(j_i)} - \frac{q^* v_{11}^{(j_i)}}{q^* v_{11}^{(j_p)}} v_{1t_{1j_i}}^{(j_p)} \\ w_{1l}^{(j_i)} &= v_{1l}^{(j_i)} - \frac{q^* v_{11}^{(j_i)}}{q^* v_{11}^{(j_p)}} v_{1l}^{(j_p)} + \left( q^* v_{1,l+1}^{(j_i)} - \frac{q^* v_{11}^{(j_i)}}{q^* v_{11}^{(j_p)}} q^* v_{1,l+1}^{(j_p)} \right) v_1 \\ l &= t_{1j_i} - 1, t_{1j_i} - 2, \dots, 2, 1, \end{aligned}$$

(3.3) If  $t_1 = 1$  then the updated matrix  $A + v_1 q^*$  has  $k_1 - 1$  Jordan chains of length  $t_{1_2}, t_{1_3}, \dots, t_{1_{j-1}}, t_{1_j} + 1, \dots, t_{1_{k_1}}$  defined as in cases (3.2.2), (3.2.3) and (3.2.4).

*Proof.* It follows with the same techniques of the previous theorems.  $\square$

**Example 3.** Consider the matrix  $A = J_3(\lambda_1) \oplus J_3(\lambda_1) \oplus J_2(\lambda_1)$  whose eigenvalue is  $\lambda_1$ , with 3 associated Jordan chains ( $k_1 = 3$ ) of length  $t_{1_1} = 3$ ,  $t_{1_2} = 3$  and  $t_{1_3} = 2$ . These chains are

$$\begin{aligned} \{v_{13}^{(1)}, v_{12}^{(1)}, v_{11}^{(1)}\} &= \{e_3, e_2, e_1\}, & \{v_{13}^{(2)}, v_{12}^{(2)}, v_{11}^{(2)}\} &= \{e_6, e_5, e_4\}, \\ \{v_{12}^{(3)}, v_{11}^{(3)}\} &= \{e_8, e_7\} \end{aligned}$$

where  $e_i$  denotes the  $i$ th unit vector,  $i = 1, 2, \dots, 8$ . We suppose that  $v_{11}^{(1)} = e_1$  and  $q^* = [q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6 \ q_7 \ q_8]$ . If  $q^* e_1 = 0$  the matrix  $A + v_1 q^*$  has the only eigenvalue  $\lambda_1$ . The generalized eigenvectors and the length of the Jordan chains depend on the values  $q_i$ ,  $i = 2, 3, \dots, 8$ . For example, if  $q^* v_{11}^{(j)} = 0$ , for  $j = 2, 3$ , we have that  $q_4 = q_7 = 0$ . We are in case (1), then the number of chains and their lengths do not change. The updated Jordan chains of  $A + v_1 q^*$  are:

Jordan chains of $A$	Updated Jordan chains of $A + v_1 q^*$
$\{e_3, e_2, e_1\}$	$\left\{ \frac{1}{1+q_2} e_3, \frac{1}{1+q_2} e_2 + \frac{q_3}{1+q_2} e_1, e_1 \right\}$
$\{e_6, e_5, e_4\}$	$\{e_6, e_5 + q_6 e_1, e_4 + q_5 e_1\}$
$\{e_8, e_7\}$	$\{e_8, e_7 + q_8 e_1\}$

Note that in this case  $A + v_1 q^*$  is similar to  $A$ .

#### 4. Updating with a generalized eigenvector

In this section we give results when we replace an eigenvector of a matrix  $A$  by a generalized eigenvector of  $A$  in the rank one perturbation matrix. The

following theorem can be considered as an extension of Brauer's Theorem [3] to a generalized eigenvector associated with the corresponding eigenvalue of  $A$ . For simplicity in the proof we consider the eigenvalue  $\lambda_1$ .

**Theorem 5.** *Let  $A \in \mathbb{C}^{n \times n}$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ , and algebraic multiplicities  $r_1, r_2, \dots, r_s$ , respectively. Let  $\{v_{1t_1}, v_{1,t_1-1}, \dots, v_{12}, v_{11}\}$  be a Jordan chain of  $A$  of length  $t_1$ , associated with  $\lambda_1$ . Let  $q$  be an  $n$ -dimensional vector such that  $q^*v_{1h} = 0$ ,  $h = 1, 2, \dots, k-1$ , with  $k \leq t_1$ . Then the matrix  $A + v_{1k}q^*$  has eigenvalues  $\lambda_1 + q^*v_{1k}, \lambda_1, \lambda_2, \dots, \lambda_s$  with algebraic multiplicities*

1.  $r_1, r_2, \dots, r_s$ , if  $q^*v_{1k} = 0$ .
2.  $1, r_1 - 1, r_2, \dots, r_s$ , if  $q^*v_{1k} \neq 0$  and  $\lambda_1 + q^*v_{1k} \notin \sigma(A)$ .
3.  $r_1 - 1, r_2, \dots, r_{i-1}, r_i + 1, r_{i+1}, \dots, r_s$ , if  $q^*v_{1k} \neq 0$ ,  $\lambda_1 + q^*v_{1k} \in \sigma(A)$  and  $\lambda_1 + q^*v_{1k} = \lambda_i$ ,  $1 < i \leq s$ .

*Proof.* The idea of the proof is similar to that of Brauer's Theorem given in [9]. Consider the matrix  $V_1 = [v_{11} \ v_{12} \ \dots \ v_{1,k-1} \ v_{1k} \ v_{1,k+1} \ \dots \ v_{1t_1}]$ . Let  $V = [V_1 \ V_2]$  be a nonsingular matrix and  $V^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ , such that

$$V^{-1}AV = \left[ \begin{array}{c|c} J_{t_1}(\lambda_1) & U_1AV_2 \\ \hline & U_2AV_2 \end{array} \right],$$

where  $\sigma(U_2AV_2) = \{\lambda_1, \lambda_2, \dots, \lambda_s\}$ , with algebraic multiplicities  $r_1 - t_1, r_2, \dots, r_s$ , respectively. Then

$$V^{-1}(A + v_{1k}q^*)V = V^{-1}AV + V^{-1}(v_{1k}q^*)V$$

$$= \left[ \begin{array}{cccccc|c} J_{k-1}(\lambda_1) & E_1 & & & & & \\ & \lambda_1 + q^*v_{1k} & 1 + q^*v_{1,k+1} & \cdots & q^*v_{1,t_1-1} & q^*v_{1t_1} & \\ & & \lambda_1 & & & & \\ & & & \ddots & \vdots & \vdots & U_1AV_2 + q^*V_2 \\ & & & & \lambda_1 & 1 & \\ \hline & & & & & \lambda_1 & U_2AV_2 \end{array} \right],$$

where

$$E_1^T = [0 \ 0 \ \dots \ 1]_{1 \times (k-1)}$$

Therefore

$$\sigma(A + v_{1k}q^*) = \{\lambda_1 + q^*v_{1k}, \lambda_1\} \cup \sigma(U_2AV_2),$$

and the result follows.  $\square$

Now, we study the relationship among the Jordan chains of a matrix  $A$  and the rank one updated matrix  $A + v_{i_l}^{(j)}q^*$ , where  $v_{i_l}^{(j)}$  is a generalized eigenvector of  $A$  associated with  $\lambda_i$ , with the additional condition  $q^*v_{i_h}^{(j)} = 0$ , for  $h = 1, 2, \dots, l$  (note that in this case  $\sigma(A) = \sigma(A + v_{i_l}^{(j)}q^*)$ ). Suppose that  $A$  has the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ , with algebraic multiplicities  $r_1, r_2, \dots, r_s$ , respectively,



and for each eigenvalue  $\lambda_i$ ,  $i = 1, 2, \dots, s$ , there exist  $k_i$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{k_i}}$ . Then  $A$  has the Jordan structure

$$J_A = J(\lambda_1) \oplus J(\lambda_2) \oplus \dots \oplus J(\lambda_s)$$

where

$$J(\lambda_i) = J_{t_{i_1}}(\lambda_i) \oplus J_{t_{i_2}}(\lambda_i) \oplus \dots \oplus J_{t_{i_{k_i}}}(\lambda_i)$$

For  $j = 1, 2, \dots, k_i$ , we denote by  $\{v_{it_{i_j}}^{(j)}, v_{i,t_{i_j}-1}^{(j)}, \dots, v_{i2}^{(j)}, v_{i1}^{(j)}\}$  the  $j$ th right Jordan chain associated with  $\lambda_i$  of length  $t_{i_j}$ , where

$$v_{ih}^{(j)} \in (\ker(A - \lambda_i I)^h) \setminus (\ker(A - \lambda_i I)^{h-1}), \quad h = 1, 2, \dots, t_{i_j},$$

and by  $\{l_{i1}^{(j)*}, l_{i2}^{(j)*}, \dots, l_{i,t_{i_j}-1}^{(j)*}, l_{it_{i_j}}^{(j)*}\}$  the  $j$ th left Jordan chain associated with  $\lambda_i$  of length  $t_{i_j}$ , where

$$l_{ih}^{(j)} \in (\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}+1-h}) \setminus (\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}-h}) \quad h = 1, 2, \dots, t_{i_j}.$$

The next results study how the Jordan chains of the rank one updated matrix  $A + v_{i_1}^{(j)} q^*$  have changed from the Jordan chains of  $A$ . Different possibilities can be obtained in terms of the election of the generalized eigenvector  $v_{i_1}^{(j)}$  and the vector  $q$ . Concretely, in Theorem 6 from two Jordan chains of length  $p$  and  $q$  of  $A$  associated with the eigenvalue  $\lambda_i$  we construct only one Jordan chain of  $A + v_{i_1}^{(j)} q^*$  of length  $p + q$  associated with the same eigenvalue. Theorem 7 obtains, for  $\lambda_i$ , two Jordan chains of length  $p$  and  $q$  of  $A + v_{i_1}^{(j)} q^*$  from only one Jordan chain of length  $p + q$  of  $A$ . Finally, in Theorem 8 we construct two Jordan chains of length  $r$  and  $s$  of  $A + v_{i_1}^{(j)} q^*$  from two Jordan chains of length  $p$  and  $q$  of  $A$ , such that,  $r + s = p + q$ .

**Theorem 6.** *Let  $A \in \mathbb{C}^{n \times n}$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ , and algebraic multiplicities  $r_1, r_2, \dots, r_s$ , respectively. Suppose that for each eigenvalue  $\lambda_i$ ,  $i = 1, 2, \dots, s$ , the matrix  $A$  has  $k_i$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{k_i}}$ . Let  $q = l_{i1}^{(j)*} \in (\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}}) \setminus (\ker(A^* - \overline{\lambda_i} I)^{t_{i_j}-1})$  be a left generalized eigenvector of the  $j$ th Jordan chain associated with the eigenvalue  $\lambda_i$  and let  $v_{it_{j-1}}^{(j-1)} \in (\ker(A - \lambda_i I)^{t_{i_{j-1}}}) \setminus (\ker(A - \lambda_i I)^{t_{i_{j-1}}-1})$  be a right generalized eigenvector of the  $(j-1)$ th Jordan chain associated with the same eigenvalue.*

*Then the matrix  $A + v_{it_{j-1}}^{(j-1)} q^*$  has, associated with  $\lambda_i$ ,  $k_i - 1$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{j-2}}, t_{i_{j-1}} + t_{i_j}, t_{i_{j+2}}, \dots, t_{i_{k_i}}$ . Moreover, the Jordan structure of  $A$  and  $A + v_{it_{j-1}}^{(j-1)} q^*$ , associated with the eigenvalues  $\lambda_q$ ,  $1 \leq q \leq s$ ,  $q \neq i$ , is the same.*

*Proof.* Since

$$q^* v_{ih}^{(j-1)} = l_{i1}^{(j)*} v_{ih}^{(j-1)} = 0, \quad h = 1, 2, \dots, t_{j-1},$$

the matrix  $A + v_{it_{j-1}}^{(j-1)} q^*$  has the eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ , and algebraic multiplicities  $r_1, r_2, \dots, r_s$ , respectively, by Theorem 5.

Now, to study the Jordan structure of this matrix consider the nonsingular matrix  $V$  such that

$$V^{-1}AV = J_A = J(\lambda_1) \oplus J(\lambda_2) \oplus \dots \oplus J(\lambda_i) \oplus \dots \oplus J(\lambda_{s-1}) \oplus J(\lambda_s)$$

and let  $g$  be the integer number

$$g = \sum_{l=1}^{i-1} \sum_{h=1}^{k_l} t_{lh} + (t_{i_1} + t_{i_2} + \dots + t_{i_{j-1}}).$$

Since

$$V^{-1} \left( A + v_{it_{j-1}}^{(j-1)} q^* \right) V = V^{-1}AV + \left( V^{-1}v_{it_{j-1}}^{(j-1)} \right) \left( l_{i1}^{(j)*} V \right) = V^{-1}AV + e_g e_{g+1}^T,$$

where  $e_i$  is the  $i$ th unit vector, then

$$V^{-1}AV + e_g e_{g+1}^T = J_{A+v_{it_{j-1}}^{(j-1)} q^*} = J(\lambda_1) \oplus J(\lambda_2) \oplus \dots \oplus \tilde{J}(\lambda_i) \oplus \dots \oplus J(\lambda_{s-1}) \oplus J(\lambda_s)$$

with

$$\tilde{J}_{\lambda_i} = \begin{bmatrix} J_{t_{i_1}}(\lambda_i) & \dots & O & O & \dots & O \\ \vdots & & \vdots & \vdots & & \vdots \\ O & \dots & J_{t_{i_{j-1}}}(\lambda_i) & E & \dots & O \\ O & \dots & O & J_{t_{i_j}}(\lambda_i) & \dots & O \\ \vdots & & \vdots & \vdots & & \vdots \\ O & \dots & O & O & \dots & J_{t_{i_{k_i}}}(\lambda_i) \end{bmatrix},$$

and

$$E = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{t_{i_{j-1}} \times t_{i_j}}$$

□

**Example 4.** Consider a matrix  $A$  with eigenvalues  $\lambda_1$  and  $\lambda_2$ , such that its the Jordan form is

$$J_A = V^{-1}AV = J_3(\lambda_1) \oplus J_2(\lambda_1) \oplus J_2(\lambda_2)$$

where

$$\begin{aligned} V &= [v_{11}^{(1)} \ v_{12}^{(1)} \ v_{13}^{(1)} \ v_{11}^{(2)} \ v_{12}^{(2)} \ v_{21}^{(1)} \ v_{22}^{(1)}] \\ (V^{-1})^* &= \begin{bmatrix} l_{11}^{(1)} & l_{12}^{(1)} & l_{13}^{(1)} & l_{11}^{(2)} & l_{12}^{(2)} & l_{21}^{(1)} & l_{22}^{(1)} \end{bmatrix} \end{aligned}$$

are the matrices of the right and left generalized eigenvectors of  $A$ , respectively. Taking  $q^* = l_{11}^{(2)*}$  and the vector  $v_{13}^{(1)}$ , the updated matrix  $A + v_{13}^{(1)} l_{11}^{(2)*}$  has the Jordan structure

$$V^{-1} \left( A + v_{13}^{(1)} l_{11}^{(2)*} \right) V = V^1 A V + V^{-1} \left( v_{13}^{(1)} l_{11}^{(2)*} \right) V = J_5(\lambda_1) + J_2(\lambda_2).$$

**Theorem 7.** Let  $A \in \mathbb{C}^{n \times n}$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ , and algebraic multiplicities  $r_1, r_2, \dots, r_s$ , respectively. Suppose that for each eigenvalue  $\lambda_i$ ,  $i = 1, 2, \dots, s$ , the matrix  $A$  has  $k_i$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{k_i}}$ . Let  $q = -l_{ip}^{(j)} \in \left( \ker(A^* - \bar{\lambda}_i I)^{t_{i_j} + 1 - p} \right) \setminus \left( \ker(A^* - \bar{\lambda}_i I)^{t_{i_j} - p} \right)$ ,  $1 < p \leq t_{i_j}$ , be a left generalized eigenvector of the  $j$ th Jordan chain associated with the eigenvalue  $\lambda_i$  and let  $v_{ip-1}^{(j)} \in \left( \ker(A - \lambda_i I)^{p-1} \right) \setminus \left( \ker(A - \lambda_i I)^{p-2} \right)$  be a right generalized eigenvector of the  $j$ th Jordan chain associated with the same eigenvalue.

Then the matrix  $A + v_{ip-1}^{(j)} q^*$  has, associated with  $\lambda_i$ ,  $k_i + 1$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{j-2}}, t_{i_{j-1}}, t_{i_j} - p + 1, p - 1, t_{i_{j+1}}, \dots, t_{i_{k_i}}$ . Moreover, the Jordan structure of  $A$  and  $A + v_{ip-1}^{(j)} q^*$ , associated with the eigenvalues  $\lambda_q$ ,  $1 \leq q \leq s$ ,  $q \neq i$ , is the same.

*Proof.* The proof is analogous to that of Theorem 6.  $\square$

**Example 5.** Consider the matrix  $A$  given in Example 4. If we take the vector  $q^* = -l_{12}^{(1)*}$  the updated matrix  $A + v_{11}^{(1)} (-l_{12}^{(1)*})$  has the Jordan structure

$$\begin{aligned} V^{-1} \left( A + v_{11}^{(1)} (-l_{12}^{(1)*}) \right) V &= V^1 A V + V^{-1} \left( v_{11}^{(1)} (-l_{12}^{(1)*}) \right) V \\ &= J_1(\lambda_1) \oplus J_2(\lambda_1) \oplus J_2(\lambda_1) \oplus J_2(\lambda_2) \end{aligned}$$

**Theorem 8.** Let  $A \in \mathbb{C}^{n \times n}$  with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ , and algebraic multiplicities  $r_1, r_2, \dots, r_s$ , respectively. Suppose that for each eigenvalue  $\lambda_i$ ,  $i = 1, 2, \dots, s$ , the matrix  $A$  has  $k_i$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{k_i}}$ . Let  $q = l_{ip}^{(j)} \in \left( \ker(A^* - \bar{\lambda}_i I)^{t_{i_j} + 1 - p} \right) \setminus \left( \ker(A^* - \bar{\lambda}_i I)^{t_{i_j} - p} \right)$ ,  $1 < p \leq t_{i_j}$ , be a left generalized eigenvector of the  $j$ th Jordan chain associated with the eigenvalue  $\lambda_i$  and let  $v_{it_{j-1}}^{(j-1)} \in \left( \ker(A - \lambda_i I)^{t_{j-1}} \right) \setminus \left( \ker(A - \lambda_i I)^{t_{j-1} - 1} \right)$  be a right generalized eigenvector of the  $(j-1)$ th Jordan chain associated with the same eigenvalue.

Then the matrix  $A + v_{it_{j-1}}^{(j-1)} q^*$  has, associated with  $\lambda_i$ ,  $k_i$  Jordan chains of length  $t_{i_1}, t_{i_2}, \dots, t_{i_{j-2}}, t_{i_{j-1}} + t_{i_j} - p + 1, p - 1, t_{i_{j+2}}, \dots, t_{i_{k_i}}$ . Moreover, the Jordan structure of  $A$  and  $A + v_{it_{j-1}}^{(j-1)} q^*$ , associated with the eigenvalues  $\lambda_q$ ,  $1 \leq q \leq s$ ,  $q \neq i$ , is the same.

*Proof.* The proof is analogous to that of Theorem 6.  $\square$

**Example 6.** Again, consider the matrix  $A$  in Example 4. Now, taking  $q^* = l_{12}^{(2)*}$  and the vector  $v_{13}^{(1)}$ , we have

$$V^{-1} \left( A + v_{13}^{(1)} l_{12}^{(2)*} \right) V = \left[ \begin{array}{ccccc|cc} \lambda_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{array} \right],$$

which is similar to

$$J = J_4(\lambda_1) \oplus J_1(\lambda_1) \oplus J_2(\lambda_2).$$

**Remark 1.** We can obtain the same result as in Example 6 in two steps. First obtaining a Jordan chain of length 1 as in Example 5 and then applying Theorem 6 to matrix  $A + v_{11}^{(1)} (-l_{12}^{(1)*})$  to obtain a Jordan chain of length 4. That is

$$\begin{aligned} & V^{-1} \left( A + v_{11}^{(1)} (-l_{12}^{(1)*}) + v_{13}^{(1)} l_{11}^{(2)*} \right) V \\ &= V^{-1} \left( A + v_{11}^{(1)} (-l_{12}^{(1)*}) \right) V + V^{-1} \left( v_{13}^{(1)} l_{11}^{(2)*} \right) V \\ &= J_1(\lambda_1) \oplus J_4(\lambda_1) \oplus J_2(\lambda_2) = V^{-1} \left( A + [v_{11}^{(1)} \ v_{13}^{(1)}] \begin{bmatrix} -l_{12}^{(1)*} \\ l_{11}^{(2)*} \end{bmatrix} \right) V. \end{aligned}$$

Note that in this case we apply a rank two perturbation.

## 5. Conclusions and open problems

We have studied the Jordan structure of the one rank updated matrix  $A + v_k q^*$ , where  $v_k$  is an eigenvector of  $A$  and  $q$  is an  $n$ -dimensional vector, in terms of the Jordan structure of  $A$ . Moreover, we have given the expressions of the generalized eigenvectors of the updated matrix. In particular, we have obtained the changes of the Jordan chains when the new eigenvalue  $\mu \notin \sigma(A)$  in Section 2 and when  $\mu \in \sigma(A)$  in Section 3. In both sections all Jordan chains associated with the eigenvalues of  $A + v_k q^*$  are given.

When the updated matrix is constructed with a generalized eigenvector of  $A$ , instead of  $v_k$ , many different new possibilities appear as open problems. We have studied in Section 4 some of these possibilities when the spectrum of the updated matrix is the same as that of  $A$ .

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