Solving linear and quadratic random matrix differential equations: A mean square approach

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Abstract

In this paper linear and Riccati random matrix differential equations are solved taking advantage of the so called L_p-random calculus. Uncertainty is assumed in coefficients and initial conditions. Existence of the solution in the L_p-random sense as well as its construction are addressed. Numerical examples illustrate the computation of the expectation and variance functions of the solution stochastic process.

Keywords: random models, random matrix bilateral differential equation, mean square random calculus, L_p-random matrix calculus.

1. Introduction

The main target of control theory is to develop mathematical models and procedures for the design of complex dynamic systems. The necessity for control appears because operating and designing a dynamical system is usually subject to uncertainties that cannot be exactly predicted. The uncertainty may be due to errors, inherent difficulties (physical or economical) to measure quantities, the appearance of unexpected events, breakdowns, etc. Therefore, it is appropriate to investigate control processes with the aid of models incorporating randomness [1].

Dynamic systems are frequently modelled by differential equations whose unknown is the state of the system. In the ordinary differential equations framework the randomness can be incorporated in different ways, depending on the way the uncertainty appears in the model and the meaning of the derivatives, i.e., the operational calculus used. When one considers stochastic differential equations and uncertainty appears modelled in terms of Gaussian white noise, the proper operational rules are based on Itô calculus. This approach was initiated by Langevin [2] in the study of Brownian motion, Pontryagin et al. [3] and many other authors later. Since the seminal papers by Wonham [4, 5], a number of recent contributions have addressed the study of the Riccati differential equation appearing in stochastic control of linear problems [6, 7, 8, 9]. In these cases, randomness is handled taking advantage of the so called Itô calculus [10, 11].

Otherwise, linear filtering models with stationary coefficients occur, for instance, in the study of the position of a satellite which cannot be observed at some unexpected random times. It is natural to consider these kind of problems where the uncertainty is not modelled in terms of Brownian motion and Itô calculus, allowing other types of randomness. Additionally to Itô calculus approach, the mean square calculus provides a different manner to consider uncertainty in differential equations. This approach has two suitable properties. The first one is that our solution, say X, coincides with the one of the deterministic case, i.e., when random data is deterministic. The second property is that, if $X_n \to X$ as $n \to \infty$ in the mean square sense, then the expectation and the variance of the approximation X_n will converge to the expectation and the variance of the exact solution X, respectively, [12].

The treatment of differential equations where uncertainty is not forced by a process whose sample trajectories are somewhat irregular (nowhere differentiable), such as a Brownian motion or Wiener process, but rather by other mild

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class of randomness, has been developed in recent years taking advantage of the mean square random calculus. It has been done in both scenarios, the scalar and the matrix framework [13, 14, 15, 16, 17, 18]. It is also well-known in population modelling the prominent role played by Riccati differential equation, in both the deterministic and the random cases [19, 20].

In this paper, we deal with the following random matrix Riccati initial value problem (IVP):

$$W'(t) + W(t)\,A + D\,W(t) + W(t)\,B\,W(t) - C = 0\,, \quad W(0) = W_0\,, \tag{1}$$

where coefficients $A \in L_q^{n\times n}(\Omega)$, $D \in L_q^{m\times m}(\Omega)$, $B \in L_q^{n\times m}(\Omega)$, $C \in L_q^{m\times n}(\Omega)$ and initial condition $W_0 \in L_q^{m\times n}(\Omega)$ are random matrices of size $n\times n$, $m\times m$, $n\times m$, $m\times n$ and $m\times n$, respectively, and the unknown $W(t) \in L_q^{m\times n}(\Omega)$ is a matrix stochastic process (s.p.) of size $m\times n$, all of them defined in certain spaces, $L_q^{r\times s}(\Omega)$, that will be defined later. In (1), the meaning of the derivative $W'(t)$ must be understood in the mean square sense which will be specified in Section 2. In that section, some preliminary definitions and results about L_p-random scalar calculus are given. We also include the proof of important results related to the L_p-random matrix operational calculus that will play an important role in the construction of solutions to IVP (1). Section 3 deals with the solution of the random linear matrix differential equation in the L_p-random sense. The results obtained in this section are applied to solve the random matrix bilateral Riccati differential equation (1) in Section 4. The approach used is somewhat inspired in the study of the deterministic Riccati operator equation presented in [21]. Section 5 illustrates the theoretical results through several numerical examples and simulations. Conclusions are drawn in the last section.

2. Random matrix calculus

The aim of this section is to establish the basis of a random matrix calculus allowing the introduction of matrix stochastic processes, operational rules and the definition of the matrix exponential stochastic process. Although the main motivation is finding the solution to the random matrix Riccati IVP (1), the random matrix calculus must be consistent with the so called L_p-random calculus introduced in [12] and [14] for the random scalar calculus, corresponding to $p = 2$ and $p = 4$, respectively.

Throughout this paper, the triplet $(\Omega, \mathcal{F}, \mathbb{P})$ will denote a complete probability space. Let $x : \Omega \longrightarrow \mathbb{R}$ be a random variable (r.v.). It is said to be of order p if $\mathrm{E}\left[\,|x|^p\right] < +\infty$, $p \geq 1$, where $\mathrm{E}\left[\cdot\right]$ denotes the expectation operator. The space $L_p(\Omega)$ of all r.v.'s of order p (assuming we do not distinguish between r.v.'s that are equal with probability one), endowed with the norm

$$\|x\|_p = \left(\mathrm{E}\left[\,|x|^p\right]\right)^{1/p}\,, \tag{2}$$

has a Banach space structure [11, p.9]. It is interesting to recall some important results that will be used later in the matrix operational calculus. If $x \in L_p(\Omega)$ and $0 < q \leq p$, then $x \in L_q(\Omega)$. This is a consequence of Liapunov inequality

$$\left(\mathrm{E}\left[|x|^q\right]\right)^{\frac{1}{q}} \leq \left(\mathrm{E}\left[|x|^p\right]\right)^{\frac{1}{p}}\,, \quad \text{or equivalently} \quad \|x\|_q \leq \|x\|_p, \quad \text{for} \quad 0 < q \leq p, \tag{3}$$

whenever $\mathrm{E}\left[|x|^p\right] < +\infty$. As the norm $\|\cdot\|_p$ is not submultiplicative [22, Sec.3], it is convenient to remember that [15]

$$\|x\,y\|_p \leq \|x\|_{2p}\,\|y\|_{2p}\,, \quad x, y \in L_{2p}(\Omega)\,. \tag{4}$$

For the random scalar calculus, if $a \in L_p(\Omega)$ and $\{x_n : n \geq 0\}$ is a sequence in $(L_p(\Omega), \|\cdot\|_p)$ converging to $x \in L_p(\Omega)$, then the sequence $\{a\,x_n : n \geq 0\}$ does not necessarily converge in the norm $\|\cdot\|_p$ to the r.v. $a\,x$. However, according to [22, Lem. 6], if $\{x_n : n \geq 0\} \subseteq L_{2p}(\Omega)$ and $a \in L_{2p}(\Omega)$ then

$$a\,x_n \xrightarrow[n\to+\infty]{\|\cdot\|_p} a\,x\,. \tag{5}$$

Hereinafter, $\mathcal{T}$ will denote an interval of the real line, $\mathbb{R}$. A stochastic process (s.p.), $\{x(t) : t \in \mathcal{T} \subseteq \mathbb{R}\}$, is said to be of order p if $x(t) \in L_p(\Omega)$ for each $t \in \mathcal{T}$, i.e., $\mathrm{E}\left[|x(t)|^p\right] < +\infty$, $\forall t \in \mathcal{T}$. Let $x_{i,j} \in L_p(\Omega)$, $1 \leq i \leq m$, $1 \leq j \leq n$, and let $X = \left(x_{i,j}\right)_{m\times n}$ be the matrix of the r.v.'s $x_{i,j}$. Then the space of all such random matrices, $L_p^{m\times n}(\Omega)$, endowed with the norm

$$\|X\|_p = \sum_{i=1}^{m}\sum_{j=1}^{n}\left\|x_{i,j}\right\|_p\,, \quad x_{i,j} \in L_p(\Omega)\,, \tag{6}$$

has a Banach space structure. Although we use the same notation for the norms $\|\cdot\|_p$ in (2) and (6), no confusion is possible because lower case letters are used for scalar quantities and capital letters are used for matrix quantities.

The next result is a natural extension of inequality (4) to the random matrix framework.

Proposition 1. *Let $X = (x_{i,k}) \in L_{2p}^{m\times n}(\Omega)$ and $Y = (y_{k,j}) \in L_{2p}^{n\times q}(\Omega)$. Then*

$$\|X\,Y\|_p \leq \|X\|_{2p}\,\|Y\|_{2p}\,. \tag{7}$$

Proof. One one hand, by (4) one gets

$$\|X\,Y\|_p = \sum_{i=1}^{m}\sum_{j=1}^{q}\left\|\sum_{k=1}^{n} x_{i,k}\,y_{k,j}\right\|_p \leq \sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}\left\|x_{i,k}\,y_{k,j}\right\|_p \leq \sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}\left\|x_{i,k}\right\|_{2p}\left\|y_{k,j}\right\|_{2p}\,. \tag{8}$$

On the other hand, manipulating the right-hand side of expression (8) one obtains

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{q}\sum_{k=1}^{n}\left\|x_{i,k}\right\|_{2p}\left\|y_{k,j}\right\|_{2p} &= \sum_{k=1}^{n}\left\{\left(\sum_{i=1}^{m}\left\|x_{i,k}\right\|_{2p}\right)\left(\sum_{j=1}^{q}\left\|y_{k,j}\right\|_{2p}\right)\right\} \leq \left(\sum_{k=1}^{n}\sum_{i=1}^{m}\left\|x_{i,k}\right\|_{2p}\right)\left(\sum_{k=1}^{n}\sum_{j=1}^{q}\left\|y_{k,j}\right\|_{2p}\right)\\ &= \left(\sum_{i=1}^{m}\sum_{k=1}^{n}\left\|x_{i,k}\right\|_{2p}\right)\left(\sum_{k=1}^{n}\sum_{j=1}^{q}\left\|y_{k,j}\right\|_{2p}\right) = \|X\|_{2p}\,\|Y\|_{2p}\,.\end{aligned} \tag{9}$$

From (8) and (9), the result is established. □

Taking into account Proposition 1 and the proof of the scalar result (5), see [22, Lem. 6], it is easy to establish the following lemma that we state without proof.

Lemma 1. *Let $A \in L_{2p}^{m\times n}(\Omega)$, and $\{X_\ell :\ \ell \geq 0\} \subseteq L_{2p}^{n\times q}(\Omega)$ such that $X_\ell \xrightarrow[\ell\to+\infty]{\|\cdot\|_{2p}} X \in L_{2p}^{n\times q}(\Omega)$. Then*

$$A\,X_\ell \xrightarrow[\ell\to+\infty]{\|\cdot\|_p} A\,X\,. \tag{10}$$

We have seen that the concept of scalar s.p. in the space $L_p(\Omega)$ is a collection of r.v.'s, indexed by time, that belong to $L_p(\Omega)$. The definition of matrix s.p. of size $m \times n$, say $\{X(t) :\ t \in \mathcal{T} \subseteq \mathbb{R}\}$ in the space $L_p^{m\times n}(\Omega)$ follows analogously from the definition of random matrix, simply by imposing that $X(t) \in L_p^{m\times n}(\Omega)$ for each $t \in \mathcal{T}$. In accordance with the definition of a scalar differentiable s.p. in $L_p(\Omega)$, we define the concept of differentiability of a matrix s.p. in the space $(L_p^{m\times n}(\Omega), \|\cdot\|_p)$ as follows

Definition 1. Let $\{X(t),\ t \in \mathcal{T}\}$ be a matrix s.p. in $L_p^{m\times n}(\Omega)$. We say that $X(t)$ is p-differentiable or $\|\cdot\|_p$-differentiable at $t_0 \in \mathcal{T}$, being $X'(t_0)$ its p-derivative or $\|\cdot\|_p$-derivative, indistinctly, if there exists a random matrix $X'(t_0) \in L_p^{m\times n}(\Omega)$ such that

$$\left\|\frac{X(t_0+h)-X(t_0)}{h} - X'(t_0)\right\|_p \xrightarrow[h\to 0]{} 0\,, \quad t_0, t_0+h \in \mathcal{T}.$$

It is easy to prove that if all the entries $x_{i,j}(t) \in L_p(\Omega)$ of the matrix s.p. $X(t) = (x_{i,j}(t)) \in L_p^{m\times n}(\Omega)$ are p-differentiable scalar s.p.'s with p-derivative $x'_{i,j}(t_0)$, $t_0 \in \mathcal{T}$, then $X(t)$ is a p-differentiable matrix s.p. at t_0 and its p-derivative is the random matrix $X'(t_0) = (x'_{i,j}(t_0)) \in L_p^{m\times n}(\Omega)$. Reciprocally, if the matrix s.p. $X(t)$ is p-differentiable with p-derivative $X'(t)$, then its entries $x_{i,j}(t)$ are all p-differentiable and the p-derivative $x'_{i,j}(t)$ of entry $x_{i,j}(t)$ is the (i, j)-entry of the $X'(t)$ matrix.

Lemma 2. *Let $G \in L_p^{m\times n}(\Omega)$ and $g(t)$ be a deterministic differentiable function. Then, the matrix s.p. $G(t) = Gg(t)$ is p-differentiable and its p-derivative is given by $G'(t) = Gg'(t)$.*

Proof. It follows directly from the definition of the derivative in the p–norm:

$$\left\|\frac{G(t+h)-G(t)}{h}-G'(t)\right\|_p=\left\|\frac{Gg(t+h)-Gg(t)}{h}-Gg'(t)\right\|_p=\|G\|_p\left|\frac{g(t+h)-g(t)}{h}-g'(t)\right|\xrightarrow[h\to 0]{}0,$$

where in the last step we have used that $\|G\|_p<+\infty$ and the differentiability (in the classical or deterministic sense) of $g(t)$. □

The next result is a rule for p-differentiability of the product of two $2p$-differentiable matrix s.p.'s. It constitutes a generalization of [14, Lemma 3.14] to the matrix scenario.

Proposition 2. *Let $F(t)\in L_{2p}^{m\times n}(\Omega)$ and $G(t)\in L_{2p}^{n\times q}(\Omega)$ be $2p$-differentiable matrix s.p.'s at $\mathcal{T}\subseteq\mathbb{R}$, being $F'(t)$ and $G'(t)$ its $2p$-derivatives, respectively. Then, $H(t)=F(t)G(t)\in L_p^{m\times q}(\Omega)$ and is a p-differentiable matrix s.p. with its p-derivative is given by*

$$H'(t)=F'(t)G(t)+F(t)G'(t)\,.$$

Proof. Let us consider

$$\left\|\frac{F(t+h)G(t+h)-F(t)G(t)}{h}-\{F'(t)G(t)+F(t)G'(t)\}\right\|_p=\left\|\frac{F(t+h)G(t+h)-F(t)G(t)-hF'(t)G(t)-hF(t)G'(t)}{h}\right\|_p$$

and add and subtract $F(t+h)G(t)$, then applying triangular inequality to obtain

$$\leq\left\|F(t+h)\frac{G(t+h)-G(t)}{h}-F(t)G'(t)\right\|_p+\left\|\frac{F(t+h)-F(t)}{h}G(t)-F'(t)G(t)\right\|_p$$

next, we add and subtract $F(t+h)G'(t)$, then applying again the triangular inequality together with (7) one gets

$$\leq\|F(t+h)\|_{2p}\left\|\frac{G(t+h)-G(t)}{h}-G'(t)\right\|_{2p}+\|F(t+h)-F(t)\|_{2p}\left\|G'(t)\right\|_{2p}+\left\|\frac{F(t+h)-F(t)}{h}-F'(t)\right\|_{2p}\|G(t)\|_{2p}\,. \tag{11}$$

Since $F(t)\in L_{2p}^{m\times n}(\Omega)$ and $G(t),G'(t)\in L_{2p}^{n\times q}(\Omega)$, then $\|F(t+h)\|_{2p}$, $\|G(t)\|_{2p}$ and $\|G'(t)\|_{2p}$ are finite $\forall t,\ t+h\in\mathcal{T}$. Moreover, because of $\|\cdot\|_{2p}$-differentiability, and hence $\|\cdot\|_{2p}$-continuity, of $F(t)$ and $G(t)$, one gets

$$\|F(t+h)-F(t)\|_{2p}\xrightarrow[h\to 0]{}0,\quad\left\|\frac{F(t+h)-F(t)}{h}-F'(t)\right\|_{2p}\xrightarrow[h\to 0]{}0,\quad\left\|\frac{G(t+h)-G(t)}{h}-G'(t)\right\|_{2p}\xrightarrow[h\to 0]{}0\,.$$

This implies that all the terms in (11) tend to zero as $h\to 0$. Thereby, the result is established. □

The following result constitutes a generalization of inequality (17) of [22]:

$$\left\|\prod_{i=1}^{s}Y_i\right\|_q\leq\prod_{i=1}^{s}\left(\left\|(Y_i)^{2^{s-1}}\right\|_q\right)^{\frac{1}{2^{s-1}}},\qquad E\left[(Y_i)^{2^{s-1}q}\right]<+\infty,\ \ 1\leq i\leq s,\ \ q>0. \tag{12}$$

It is obtained by applying [22, Prop. 12] to $X_i=(Y_i)^q$. Hence, inequality (17) of [22] is a particular case of (12) when $q=4$.

As shall be seen later, the solution of the Riccati random matrix differential equation (1) will be expressed in terms of the inverse of a random matrix involving some random inputs. Then, we will need to guarantee the existence of an ordinary neighbourhood where that random inverse matrix is well-defined. Next, we introduce some definitions and results addressed to tackle this issue through the determinant of a random matrix. Although the random matrix differential equation (1) is autonomous, i.e., its matrix of coefficients does not depend upon time t, in order to provide more generality both conditions and results will be given for s.p.'s instead of r.v.'s.

Definition 2. Let $\{a_{i,j}(t),\ 1\leq i,j\leq n\}$ be s.p.'s defined for $t\in\mathcal{T}\subset\mathbb{R}$. The determinant of the matrix s.p. of size $n\times n$, $A_n(t)=(a_{i,j}(t))_{n\times n}$, is defined by

$$\det(A_n(t))=\sum_{\sigma_n=(j_1,\ldots,j_n)\in S_n}\operatorname{sgn}(\sigma_n)\,a_{1,j_1}(t)\cdots a_{n,j_n}(t), \tag{13}$$

where, as usual, S_n denotes the set of all permutations of $(1,2,\ldots,n)$ and $\operatorname{sgn}(\sigma_n)$ stands for the signature of the permutation σ_n.

Notice that the determinant of a random matrix is a r.v. Since $A_n(t)$ is a matrix s.p., in the context of Definition 2, $\det(A_n(t))$ is a scalar s.p. As an extension of its scalar counterpart, we introduce the following.

Definition 3. A stochastic process $\{U(t) : t \in \mathcal{T}\}$ is said to be invertible if its determinant $\det(U(t))$ is different from zero with probability one for every $t \in \mathcal{T}$.

In the context of the above definition, let $p \geq 1$ be fixed, and assume that the following statistical moments exist and are finite

$$\mathrm{E}\left[(a_{i,j}(t))^{2^{n-1}p}\right] < \infty, \quad \forall i, j : \ 1 \leq i, j \leq n, \ n \geq 1, \ \forall t \in \mathcal{T}. \tag{14}$$

Then, using inequality (12) one gets that the determinant of the matrix s.p. $A_n(t)$ is well-defined in the p-norm:

$$\|\det(A_n(t))\|_p \leq \sum_{\sigma_n=(j_1,\ldots,j_n)\in S_n} \left\|a_{1,j_1}(t)\cdots a_{n,j_n}(t)\right\|_p \leq \sum_{\sigma_n=(j_1,\ldots,j_n)\in S_n} \prod_{k=1}^{n} \left(\left\|\left(a_{i,j_k}(t)\right)^{2^{n-1}}\right\|_p\right)^{\frac{1}{2^{n-1}}} < \infty. \tag{15}$$

Notice that in the last step, hypothesis (14) has been applied. Inequality (15) can be straightforwardly generalized to matrix stochastic processes of size $n-r$, $A_{n-r}(t)$, $0 \leq r \leq n-1$ considering the $(2^r p)$-norm

$$\|\det(A_{n-r}(t))\|_{2^r p} \leq \sum_{\sigma_{n-r}=(j_1,\ldots,j_{n-r})\in S_{n-r}} \left\|a_{1,j_1}(t)\cdots a_{n-r,j_{n-r}}(t)\right\|_{2^r p} \leq \sum_{\sigma_{n-r}=(j_1,\ldots,j_{n-r})\in S_{n-r}} \prod_{l=1}^{n-r} \left(\left\|\left(a_{l,j_l}(t)\right)^{2^{n-r-1}}\right\|_{2^r p}\right)^{\frac{1}{2^{n-r-1}}} < \infty. \tag{16}$$

Notice that if $r = 0$ in (16) one obtains inequality (15).

Proposition 3. *Let $\{a_{i,j}(t),\ 1 \leq i, j \leq n\}$ be s.p.'s defined for $t \in \mathcal{T} \subset \mathbb{R}$ satisfying condition* (14) *in an ordinary neighbourhood of t:*

$$\exists\ \epsilon > 0 \quad \text{such that} \quad \mathrm{E}\left[(a_{i,j}(s))^{2^{n-1}p}\right] < +\infty, \quad \forall s \in (t-\epsilon, t+\epsilon),\ \epsilon > 0, \quad i, j :\ 1 \leq i, j \leq n,\ n, p \geq 1,\ \forall t \in \mathcal{T}. \tag{17}$$

Assume that $a_{i,j}(t)$, $1 \leq i, j \leq n$ are continuous in the $(2^{n-1}p)$–norm. Then, the determinant of the matrix s.p. of size $n \times n$, $A_n(t) = (a_{i,j}(t))_{n\times n}$, defined by (13)*, is continuous in the p–norm.*

Proof. Throughout the proof, we will assume that $n \geq 2$, otherwise the result is trivial. Let $0 < |h| < \epsilon$, $t, t+h \in \mathcal{T}$ and consider the following development based on the Laplace's formula to compute the determinant of matrix $A_n(t)$ in terms of the cofactors $(-1)^{1+j}A_{n-1}^{(1,j)}(t)$ of elements $a_{1,j}(t)$, $1 \leq j \leq n$, of the first row

$$\begin{aligned}\|\det(A_n(t+h)) - \det(A_n(t))\|_p \;=\; & \Big\|\Big\{a_{1,1}(t+h)(-1)^{1+1}\det\left(A_{n-1}^{(1,1)}(t+h)\right) + \cdots + a_{1,n}(t+h)(-1)^{1+n}\det\left(A_{n-1}^{(1,n)}(t+h)\right)\Big\} \\ -\; & \Big\{a_{1,1}(t)(-1)^{1+1}\det\left(A_{n-1}^{(1,1)}(t)\right) + \cdots + a_{1,n}(t)(-1)^{1+n}\det\left(A_{n-1}^{(1,n)}(t)\right)\Big\}\Big\|_p .\end{aligned} \tag{18}$$

Now, we add and subtract $\pm\det\left(A_{n-1}^{(1,1)}(t)\right)a_{1,1}(t+h)(-1)^{1+1}$, ..., $\pm\det\left(A_{n-1}^{(1,n)}(t)\right)a_{1,n}(t+h)(-1)^{1+n}$ in the sum of the right-hand side of (18) and then we apply triangular inequality together with inequality (4). This yields

$$\begin{aligned}\|\det(A_n(t+h)) - \det(A_n(t))\|_p \;=\; & \Big\|\Big\{\det\left(A_{n-1}^{(1,1)}(t+h)\right) - \det\left(A_{n-1}^{(1,1)}(t)\right)\Big\}a_{1,1}(t+h)(-1)^{1+1} \\ +\; & \{a_{1,1}(t+h) - a_{1,1}(t)\}\det\left(A_{n-1}^{(1,1)}(t)\right)(-1)^{1+1} \\ & \vdots \\ +\; & \Big\{\det\left(A_{n-1}^{(1,n)}(t+h)\right) - \det\left(A_{n-1}^{(1,n)}(t)\right)\Big\}a_{1,n}(t+h)(-1)^{1+n} \\ +\; & \{a_{1,n}(t+h) - a_{1,n}(t)\}\det\left(A_{n-1}^{(1,n)}(t)\right)(-1)^{1+n}\Big\|_p \\ \leq\; & \left\|\det\left(A_{n-1}^{(1,1)}(t+h)\right) - \det\left(A_{n-1}^{(1,1)}(t)\right)\right\|_{2p}\left\|a_{1,1}(t+h)\right\|_{2p} \\ +\; & \left\|a_{1,1}(t+h) - a_{1,1}(t)\right\|_{2p}\left\|\det\left(A_{n-1}^{(1,1)}(t)\right)\right\|_{2p} \\ & \vdots \\ +\; & \left\|\det\left(A_{n-1}^{(1,n)}(t+h)\right) - \det\left(A_{n-1}^{(1,n)}(t)\right)\right\|_{2p}\left\|a_{1,n}(t+h)\right\|_{2p} \\ +\; & \left\|a_{1,n}(t+h) - a_{1,n}(t)\right\|_{2p}\left\|\det\left(A_{n-1}^{(1,n)}(t)\right)\right\|_{2p} .\end{aligned} \tag{19}$$

By Liapunov inequality (3) and hypothesis (17), one obtains

$$\left\|a_{1,j_1}(t+h)-a_{1,j_1}(t)\right\|_{2p} \le \left\|a_{1,j_1}(t+h)-a_{1,j_1}(t)\right\|_{2^{n-1}p}, \quad 1\le j_1\le n, \quad n\ge 2. \tag{20}$$

Hence, taking into account that by hypothesis $a_{1,j_1}(t)$, $1\le j_1\le n$, are $\|\cdot\|_{2^{n-1}p}$-continuous, one gets

$$\left\|a_{1,j_1}(t+h)-a_{1,j_1}(t)\right\|_{2p} \xrightarrow[h\to 0]{} 0, \quad 1\le j_1\le n. \tag{21}$$

Since $A_{n-1}^{(1,j_1)}(t)$ has size $(n-1)\times(n-1)$, under hypothesis (17) and applying (16) with $r=1$ one gets $\left\|\det\left(A_{n-1}^{(1,j_1)}(t)\right)\right\|_{2p} < +\infty$, $1\le j_1\le n$.

Therefore,

$$\left\|a_{1,j_1}(t+h)-a_{1,j_1}(t)\right\|_{2p}\left\|\det\left(A_{n-1}^{(1,j_1)}(t)\right)\right\|_{2p} \xrightarrow[h\to 0]{} 0, \quad 1\le j_1\le n. \tag{22}$$

To conclude the proof, we now need to show that

$$\left\|\det\left(A_{n-1}^{(1,j_1)}(t+h)\right)-\det\left(A_{n-1}^{(1,j_1)}(t)\right)\right\|_{2p}\left\|a_{1,j_1}(t+h)\right\|_{2p} \xrightarrow[h\to 0]{} 0, \quad 1\le j_1\le n. \tag{23}$$

With this goal, we now adapt the reasoning exhibited previously in (18)–(19) developing the determinants of size $(n-1)\times(n-1)$ that appear in (23) using the Laplace's formula in terms of the cofactors $(-1)^{2+j_2}A_{n-2}^{(2,j_2)}(t)$, $1\le j_2\le n$, $j_2\ne j_1$, which correspond to the elements of the second row of the original matrix $A_n(t)$, except the element a_{2,j_1}. This yields

$$\begin{aligned}
\left\|\det\left(A_{n-1}^{(1,j_1)}(t+h)\right)-\det\left(A_{n-1}^{(1,j_1)}(t)\right)\right\|_{2p}\left\|a_{1,j_1}(t+h)\right\|_{2p} &= \Big\{\Big\|\left\{\det\left(A_{n-2}^{(2,1)}(t+h)\right)-\det\left(A_{n-2}^{(2,1)}(t)\right)\right\}a_{2,1}(t+h)(-1)^{1+1}\\
&+ \left\{a_{2,1}(t+h)-a_{2,1}(t)\right\}\det\left(A_{n-2}^{(2,1)}(t)\right)(-1)^{1+1}\\
&\ \ \vdots\\
&+ \left\{\det\left(A_{n-2}^{(2,j_1-1)}(t+h)\right)-\det\left(A_{n-2}^{(2,j_1-1)}(t)\right)\right\}a_{2,j_1-1}(t+h)(-1)^{1+(j_1-1)}\\
&+ \left\{a_{2,j_1-1}(t+h)-a_{2,j_1-1}(t)\right\}\det\left(A_{n-2}^{(2,j_1-1)}(t)\right)(-1)^{1+(j_1-1)}\\
&+ \left\{\det\left(A_{n-2}^{(2,j_1+1)}(t+h)\right)-\det\left(A_{n-2}^{(2,j_1+1)}(t)\right)\right\}a_{2,j_1+1}(t+h)(-1)^{1+j_1}\\
&+ \left\{a_{2,j_1+1}(t+h)-a_{2,j_1+1}(t)\right\}\det\left(A_{n-2}^{(2,j_1+1)}(t)\right)(-1)^{1+j_1}\\
&\ \ \vdots\\
&+ \left\{\det\left(A_{n-2}^{(2,n)}(t+h)\right)-\det\left(A_{n-2}^{(2,n)}(t)\right)\right\}a_{2,n}(t+h)(-1)^{1+(n-1)}\\
&+ \left\{a_{2,n}(t+h)-a_{2,n}(t)\right\}\det\left(A_{n-2}^{(2,n)}(t)\right)(-1)^{1+(n-1)}\Big\|_{2p}\Big\}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&\le \left\|\det\left(A_{n-2}^{(2,1)}(t+h)\right)-\det\left(A_{n-2}^{(2,1)}(t)\right)\right\|_{2^2p}\left\|a_{2,1}(t+h)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&+ \left\|a_{2,1}(t+h)-a_{2,1}(t)\right\|_{2^2p}\left\|\det\left(A_{n-2}^{(2,1)}(t)\right)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&\ \ \vdots\\
&+ \left\|\det\left(A_{n-2}^{(2,j_1-1)}(t+h)\right)-\det\left(A_{n-2}^{(2,j_1-1)}(t)\right)\right\|_{2^2p}\left\|a_{2,j_1-1}(t+h)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&+ \left\|a_{2,j_1-1}(t+h)-a_{2,j_1-1}(t)\right\|_{2^2p}\left\|\det\left(A_{n-2}^{(2,j_1-1)}(t)\right)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&+ \left\|\det\left(A_{n-2}^{(2,j_1+1)}(t+h)\right)-\det\left(A_{n-2}^{(2,j_1+1)}(t)\right)\right\|_{2^2p}\left\|a_{2,j_1+1}(t+h)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&+ \left\|a_{2,j_1+1}(t+h)-a_{2,j_1+1}(t)\right\|_{2^2p}\left\|\det\left(A_{n-2}^{(2,j_1+1)}(t)\right)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&+ \left\|\det\left(A_{n-2}^{(2,n)}(t+h)\right)-\det\left(A_{n-2}^{(2,n)}(t)\right)\right\|_{2^2p}\left\|a_{2,n}(t+h)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\\
&+ \left\|a_{2,n}(t+h)-a_{2,n}(t)\right\|_{2^2p}\left\|\det\left(A_{n-2}^{(2,n)}(t)\right)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}.
\end{aligned} \tag{24}$$

In the above expression, all the summands of the form

$$\left\|a_{2,j_2}(t+h)-a_{2j_2}(t)\right\|_{2^2p}\left\|\det\left(A_{n-2}^{(2,j_2)}(t)\right)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p},\quad 1\le j_1,j_2\le n,\ j_2\neq j_1,$$

tend to zero as $h\to 0$ because the $\|\cdot\|_{2^{n-1}p}$-continuity of $\{a_{2,j_2}(t)\}$ (and hence, using the Liapunov's inequality, the $\|\cdot\|_{2^2p}$-continuity of $\{a_{2j_2}(t)\}$) and the finiteness of $\left\|\det\left(A_{n-2}^{(2,j_2)}(t)\right)\right\|_{2^2p}$ (by applying inequality (16) for $r=2$) and $\left\|a_{1j_1}(t+h)\right\|_{2p}$ (by the Liapunov's inequality and hypothesis (17)). Thereby, to conclude the proof it must be proven that

$$\left\|\det\left(A_{n-2}^{(2,j_2)}(t+h)\right)-\det\left(A_{n-2}^{(2,j_2)}(t)\right)\right\|_{2^2p}\left\|a_{2,j_2}(t+h)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\xrightarrow[h\to 0]{}0,\quad 1\le j_1,j_2\le n,\quad j_2\neq j_1.$$

Again, we can repeat the previous reasoning in $n-3$ additional steps. This leads to show that is enough to prove

$$\begin{gathered}\left\|\det\left(a_{n,n}(t+h)\right)-\det\left(a_{n,n}(t)\right)\right\|_{2^{n-1}p}\left\|a_{n-1,j_{n-1}}(t+h)\right\|_{2^{n-1}p}\cdots\left\|a_{2,j_2}(t+h)\right\|_{2^2p}\left\|a_{1,j_1}(t+h)\right\|_{2p}\xrightarrow[h\to 0]{}0,\\ 1\le j_1,\cdots,j_{n-1}\le n,\quad j_k\neq j_l\ \text{ if }\ k\neq l,\ k,l\in\{1,\ldots,n-1\}\end{gathered}\tag{25}$$

to conclude the proof. Notice that all the terms of the form $\left\|a_{k,j_k}(t+h)\right\|_{2^kp}$, $1\le k\le n-1$, are finite (by Liapunov's inequality and hypothesis (17)) and

$$\left\|\det\left(a_{n,n}(t+h)\right)-\det\left(a_{n,n}(t)\right)\right\|_{2^{n-1}p}\xrightarrow[h\to 0]{}0,$$

because the $\|\cdot\|_{2^{n-1}p}$-continuity of $a_{n,n}(t)$. Thus (25) holds and the proof is completed. □

Let us assume that $U(t)\in L_{2p}^{n\times n}(\Omega)$ is invertible and $2p$-differentiable and that its inverse, $(U(t))^{-1}\in L_{2p}^{n\times n}(\Omega)$ is a $2p$-differentiable matrix s.p. Then there exists an ordinary neighbourhood $\mathcal{I}=]t_0-\delta,t_0+\delta[$, $\delta>0$ such that $U(t)\in L_{2p}^{n\times n}(\Omega)$ is invertible for all $t\in\mathcal{I}$. Moreover, notice that by Proposition 2

$$\left(U(t)(U(t))^{-1}\right)'=(I_n)'=0_n\Rightarrow U'(t)(U(t))^{-1}+U(t)\left((U(t))^{-1}\right)'=0_n\Rightarrow\left((U(t))^{-1}\right)'=-(U(t))^{-1}U'(t)(U(t))^{-1},$$

where 0_n and I_n denote the null and identity random matrix of size n in $L_{2p}^{n\times n}(\Omega)$, respectively. Therefore in the interval $\mathcal{I}$, one gets

Corollary 1. Let $U(t)\in L_{2p}^{n\times n}(\Omega)$ be an invertible matrix s.p. on the interval $t\in\mathcal{I}=]t_0-\delta,t_0+\delta[\subseteq\mathbb{R}$, $\delta>0$. Let us assume that its inverse $(U(t))^{-1}$ is in $L_{2p}^{n\times n}(\Omega)$ and is 2p-differentiable. Then, its p-derivative is given by

$$\left((U(t))^{-1}\right)'=-(U(t))^{-1}U'(t)(U(t))^{-1}\quad\forall t\in\mathcal{I}.\tag{26}$$

3. Random linear matrix differential systems

This section deals with the solution of random linear matrix differential systems of the form

$$\left.\begin{array}{rcl}Y'(t)&=&L\,Y(t),\quad t>0,\\ Y(0)&=&Y_0,\end{array}\right\}\tag{27}$$

where $L\in L_p^{m\times m}(\Omega)$, $Y(t),Y_0\in L_p^{m\times n}(\Omega)$. Apart from the fact that system (27) is the natural extension to the random framework of the classical linear homogeneous matrix deterministic systems, here they have a particular relevance because the solution of the random matrix Riccati differential equation (1) will be constructed in terms of the solution of a random rectangular linear differential system of the form (27).

The fact that the solutions of deterministic linear systems of type (27), as well as the solution of random scalar linear differential equations, are given in terms of the exponentials of its coefficient L, [14, 13], suggest that under appropriate conditions, to be specified later, the random matrix exponential $\exp(L\,t)$ will play a relevant role justifying that a natural candidate solution of (27) is

$$Y(t)=\exp(L\,t)Y_0\,.\tag{28}$$

Let us assume that the random matrix coefficient $L=(l_{i,j})$ has entries $l_{i,j}:\Omega\to\mathbb{R}$ such that there exist positive constants $m_{i,j}$, $h_{i,j}$ satisfying

$$E\left[|l_{i,j}|^r\right]\le m_{i,j}\left(h_{i,j}\right)^r<+\infty,\quad\forall r\ge 0,\forall i,j:\ 1\le i,j\le m.\tag{29}$$

Note that condition (29) guarantees that $L = (l_{i,j}) \in L_p^{m\times m}(\Omega)$, $p \geq 1$ because,

$$\|l_{i,j}\|_p = \left(E\left[|l_{i,j}|^p\right]\right)^{1/p} < +\infty\,, \quad \forall i,j:\ 1 \leq i,j \leq m\,. \tag{30}$$

Next, we will show that under condition (29) the random matrix series

$$\sum_{k\geq 0} \frac{L^k t^k}{k!}\,, \tag{31}$$

is absolutely convergent in the space $(L_p^{m\times m}(\Omega), \|\cdot\|_p)$ for all $t \in \mathbb{R}$.

Let us denote the (i,j)-th component of matrix L^k by $l_{i,j}^{(k)}$, i.e.,

$$L^k = \left(l_{i,j}^{(k)}\right)_{m\times m}\,, \quad l_{i,j}^{(k)} = \sum_{s_1,s_2,\ldots,s_{k-1}=1}^{m} l_{i,s_1} l_{s_1,s_2} \cdots l_{s_{k-1},j}\,, \tag{32}$$

and note that

$$\|L^k\|_p = \sum_{i=1}^{m}\sum_{j=1}^{m} \|l_{i,j}^{(k)}\|_p \leq \sum_{i=1}^{m}\sum_{j=1}^{m} \sum_{s_1,s_2,\ldots,s_{k-1}=1}^{m} \|l_{i,s_1} l_{s_1,s_2} \cdots l_{s_{k-1},j}\|_p\,. \tag{33}$$

By applying (12) and hypothesis (29), it follows that

$$\begin{aligned}
\|l_{i,s_1} l_{s_1,s_2} \cdots l_{s_{k-1},j}\|_p &\leq \left(\|\left(l_{i,s_1}\right)^{2^{k-1}}\|_p\right)^{\frac{1}{2^{k-1}}} \left(\|\left(l_{s_1,s_2}\right)^{2^{k-1}}\|_p\right)^{\frac{1}{2^{k-1}}} \cdots \left(\|\left(l_{s_{k-1},j}\right)^{2^{k-1}}\|_p\right)^{\frac{1}{2^{k-1}}} \\
&= \left(E\left[\left(l_{i,s_1}\right)^{2^{k-1}p}\right]\right)^{\frac{1}{2^{k-1}p}} \left(E\left[\left(l_{s_1,s_2}\right)^{2^{k-1}p}\right]\right)^{\frac{1}{2^{k-1}p}} \cdots \left(E\left[\left(l_{s_{k-1},j}\right)^{2^{k-1}p}\right]\right)^{\frac{1}{2^{k-1}p}} \\
&\leq \left(m_{i,s_1}\left(h_{i,s_1}\right)^{2^{k-1}p}\right)^{\frac{1}{2^{k-1}p}} \left(m_{s_1,s_2}\left(h_{s_1,s_2}\right)^{2^{k-1}p}\right)^{\frac{1}{2^{k-1}p}} \cdots \left(m_{s_{k-1},j}\left(h_{s_{k-1},j}\right)^{2^{k-1}p}\right)^{\frac{1}{2^{k-1}p}} \\
&= \left(m_{i,s_1} m_{s_1,s_2} \cdots m_{s_{k-1},j}\right)^{\frac{1}{2^{k-1}p}} h_{i,s_1} h_{s_1,s_2} \cdots h_{s_{k-1},j}\,.
\end{aligned} \tag{34}$$

Let us denote

$$\hat{m} = \max\{m_{i,j}:\ 1 \leq i,j \leq m\} < +\infty\,, \quad \hat{h} = \max\{h_{i,j}:\ 1 \leq i,j \leq m\} < +\infty\,. \tag{35}$$

Then, from (34) one gets

$$\|l_{i,s_1} l_{s_1,s_2} \cdots l_{s_{k-1},j}\|_p \leq (\hat{m})^{\frac{k}{2^{k-1}p}} \left(\hat{h}\right)^k\,. \tag{36}$$

Taking into account (36), expression (33) implies

$$\|L^k\|_p \leq \sum_{i=1}^{m}\sum_{j=1}^{m} \sum_{s_1,s_2,\ldots,s_{k-1}=1}^{m} (\hat{m})^{\frac{k}{2^{k-1}p}} \left(\hat{h}\right)^k = m^{k+1} (\hat{m})^{\frac{k}{2^{k-1}p}} \left(\hat{h}\right)^k\,. \tag{37}$$

Let us denote

$$\alpha_k(t) = \frac{m^{k+1} (\hat{m})^{\frac{k}{2^{k-1}p}} \left(\hat{h}\right)^k |t|^k}{k!}\,, \quad k \geq 0\,, \tag{38}$$

and note that

$$\frac{\|L^k\|_p |t|^k}{k!} \leq \alpha_k(t)\,, \quad \frac{\alpha_{k+1}(t)}{\alpha_k(t)} = (\hat{m})^{\frac{1-k}{2^k p}} \frac{m\hat{h}|t|}{k+1} \xrightarrow[k\to+\infty]{} 0\,, \quad \forall t \in \mathbb{R}\,. \tag{39}$$

Thus series (31) is absolutely convergent in the space $(L_p^{m\times m}(\Omega), \|\,\|_p)$ and thereby we can define

$$\exp(Lt) = \sum_{k\geq 0} \frac{L^k t^k}{k!}\,, \quad \forall t \in \mathbb{R}\,. \tag{40}$$

The next result is to check that series function $\exp(Lt)$ defined by (40) is termwise differentiable in the norm $\|\cdot\|_p$. This can be justified by applying the Lemma 3 stated below. This result is an extension of [23, Th.3.1] to the matrix framework for the q-norm. Indeed, this latter result corresponds to Lemma 3 in the particular case $q = 2$ (mean square convergence). The case $q = 4$ (mean fourth convergence) was already used in reference [18]. The proof of Lemma 3 would just require an adaptation of [23, Th.3.1] as well as the involved intermediate results developed in [23] that includes understanding that the integral of a matrix function $M(t) = (m_{i,j}(t))_{m\times n} \in L_p^{m\times n}(\Omega)$ is the matrix of the integrals of its components, i.e.,

$$\int_a^b M(t)\,\mathrm{d}t = \left(\int_a^b m_{i,j}(t)\,\mathrm{d}t\right)_{m\times n}.$$

Thus, we state without proof the next result.

Lemma 3. *Assume that, for each $k \geq 0$, the s.p. $\{U_k(t) : t \in \mathcal{T}\} \in L_q^{m\times n}(\Omega)$ is $\|\cdot\|_q$-differentiable for all $t \in \mathcal{T}$, $U_k'(t)$ is $\|\cdot\|_q$-continuous for all $t \in \mathcal{T}$,*

$$\sum_{k\geq 0} U_k(t) \ \text{ is } \ \|\cdot\|_q - \text{convergent and } \ \sum_{k\geq 0} U_k'(t) \ \text{ is } \ \|\cdot\|_q - \text{uniformly convergent for all } \ t \in \mathcal{T}.$$

Then, for each $t \in \mathcal{T}$, $U(t)$ is $\|\cdot\|_q$-differentiable and

$$\left(\sum_{k\geq 0} U_k(t)\right)' = \sum_{k\geq 0} U_k'(t).$$

Under condition (29) imposed on $L \in L_{2p}^{m\times m}(\Omega) \subset L_p^{m\times m}(\Omega)$, assuming that $Y_0 \in L_{2p}^{m\times n}$, hence $Y_0 \in L_p^{m\times m}(\Omega)$, by (40), Proposition 2, Lemmas 2 and 3, it follows that

$$(\exp(Lt)Y_0)' = \left[\left(\sum_{k\geq 0} \frac{L^k t^k}{k!}\right) Y_0\right]' = \left(\sum_{k\geq 0} \frac{L^k t^k}{k!}\right)' Y_0 = \left[\sum_{k\geq 0} \left(\frac{L^k t^k}{k!}\right)'\right] Y_0 = \left(\sum_{k\geq 1} \frac{L^k t^{k-1}}{(k-1)!}\right) Y_0 = L\exp(Lt)Y_0. \tag{41}$$

Remark 1. Notice that, in order to reach the above conclusion in the $L_p(\Omega)$ sense, we need to apply Proposition 2 and so we require that $(\exp(Lt))'$ be in the $L_{2p}(\Omega)$ sense. Then, we need to apply Lemma 3 with $q = 2p$. For that we must prove that the series

$$\sum_{k\geq 1} \frac{L^k t^{k-1}}{(k-1)!} \tag{42}$$

is $2p$-uniformly convergent for all real t. It can be proved, with a slight modification of arguments used previously to prove that series (31) is $\|\cdot\|_p$-convergent. Observe that all expressions from (33) to (37) are still valid for the $2p$-norm just changing p by $2p$. This leads to the following majorizing series of (42)

$$\sum_{k\geq 1} \gamma_k(t), \quad \gamma_k(t) = \frac{m^{k+1}\,(\hat{m})^{\frac{k}{2^k p}}\left(\hat{h}\right)^k |t|^{k-1}}{(k-1)!}.$$

Let $R > 0$ arbitrary but fixed and take $|t| < R$. Then using radio test one gets

$$\gamma_k(t) < \frac{m^{k+1}\,(\hat{m})^{\frac{k}{2^k p}}\left(\hat{h}\right)^k R^{k-1}}{(k-1)!} := \hat{\gamma}_k(t),$$

and

$$\frac{\hat{\gamma}_{k+1}(t)}{\hat{\gamma}_k(t)} = (\hat{m})^{\frac{1-k}{2^{k+1}p}}\,\frac{m\hat{h}R}{k} \xrightarrow[k\to+\infty]{} 0, \quad \forall R > 0.$$

Based on the so-called Weierstrass test, this proves that series (42) is $\|\cdot\|_{2p}$-uniformly convergent on the interval $|t| \leq R$.

Therefore, $Y(t) = \exp(Lt)Y_0$ is a solution of problem (27) on that interval and, since this is true for all $R > 0$, it is the solution for all t. The following result has been established:

Theorem 1. *Let $L \in L_{2p}^{m\times m}(\Omega)$ and $Y_0 \in L_{2p}^{m\times n}(\Omega)$ and assume that L satisfies condition* (29)*. Then, $Y(t) = \exp(Lt)Y_0$ is a solution of the random initial value problem* (27) *in $L_p^{m\times n}(\Omega)$ for all $t \in \mathbb{R}$.*

Remark 2. Notice that if random variable L satisfies condition (29), then it is guaranteed that $L \in L_{2p}^{m\times m}(\Omega)$.

Remark 3. It is important to point out that condition (29) is quite strong. There are standard r.v.'s that do not satisfy it. In fact, if x is an exponential r.v. $x \sim \text{Exp}(\lambda)$, $\lambda > 0$, then

$$\mathrm{E}[|x|^r] = \mathrm{E}[x^r] = \frac{r!}{\lambda^r}.$$

Notice that using the Stirling's approximation $r! \approx \sqrt{2\pi r}\left(\frac{r}{\exp(1)}\right)^r$, being $\exp(1) \approx 2.718281\ldots$ the Euler's constant, one gets

$$\lim_{r\to\infty} \frac{r!}{(\lambda H)^r} = \sqrt{2\pi} \lim_{r\to\infty} \sqrt{r}\left(\frac{r}{\lambda H \exp(1)}\right)^r = +\infty.$$

As a consequence, condition (29) is not fulfilled. Nevertheless, this condition is useful in applications because it is easy to check that bounded r.v.'s do satisfy it. Moreover, unbounded r.v.'s, like exponential, can be approximated by truncating them. This approach is supported by Chebyshev's inequality

$$\mathbb{P}[\{\omega \in \Omega : |x(\omega) - \mu_x| \geq k\sigma_x\}] \leq \frac{1}{k^2}, \quad k > 0,$$

which holds for any r.v. x with finite expected value μ_x and finite variance $\sigma_x^2 > 0$. In particular, the interval $[\mu_x - 10\sigma_x, \mu_x + 10\sigma_x]$ contains at least 99% of probability mass of x independently of the probability distribution of r.v. x. Of course, this lower bound can be improved if the probability distribution of x is known.

4. Random Riccati matrix differential equation

In this section we take advantage of the well-known linear hamiltonian matrix approach, see [24, p.11] developed to the study of the Riccati deterministic matrix problem, in order to generate a solution to the random matrix differential problem (1). An excellent study of Riccati matrix equations in the context of control systems can be found in [25].

Given the random IVP (1) where $A \in L_q^{n\times n}(\Omega)$, $B \in L_q^{n\times m}(\Omega)$, $C \in L_q^{m\times n}(\Omega)$, $D \in L_q^{m\times m}(\Omega)$ and $W_0 \in L_q^{m\times n}(\Omega)$, let us consider the random linear matrix problem (27) where

$$L = \left[\begin{array}{c|c} A & B \\ \hline C & -D \end{array}\right], \qquad Y_0 = \begin{bmatrix} I_n \\ W_0 \end{bmatrix}, \tag{43}$$

where I_n is the identity matrix of size n. Note that, if L satisfies condition (29), then by Theorem 1, $Y(t)$ given by (28) is a local $L_{2p}^{(n+m)\times n}(\Omega)$ solution of (27) in an ordinary neighbourhood $\mathcal{N}_Y(0)$ about $t = 0$.

Let us consider the block-decomposition

$$Y(t) = \begin{bmatrix} U(t) \\ V(t) \end{bmatrix}; \quad U(t) \in L_{2p}^{n\times n}(\Omega), \quad V(t) \in L_{2p}^{m\times n}(\Omega), \tag{44}$$

and let us write problem (27) in the form

$$\begin{bmatrix} U(t) \\ V(t) \end{bmatrix}' = \left[\begin{array}{c|c} A & B \\ \hline C & -D \end{array}\right] \begin{bmatrix} U(t) \\ V(t) \end{bmatrix}; \qquad \begin{bmatrix} U(0) \\ V(0) \end{bmatrix} = \begin{bmatrix} I_n \\ W_0 \end{bmatrix}. \tag{45}$$

Note that $U(0) = [I_n, 0]\, Y(0) = [I_n, 0]\, \exp(L\,0)\, Y_0 = [I_n, 0] \begin{bmatrix} I_n \\ W_0 \end{bmatrix} = I_n$, and that if $U(t)$ is invertible in $L_{2p}^{n\times n}(\Omega)$ in an ordinary neighbourhood $\mathcal{N}_U(0)$ of $t = 0$ and $(U(t))^{-1}$ lies in $L_{2p}^{n\times n}(\Omega)$, then the stochastic process

$$W(t) = V(t)\,(U(t))^{-1}, \qquad t \in \mathcal{N}_U(0), \tag{46}$$

is well-defined and lies in $L_p^{m\times n}(\Omega)$.

Let us consider the block-decomposition

$$\exp(L\,t) = \left[\begin{array}{c|c} Z_{1,1}(t) & Z_{1,2}(t) \\ \hline Z_{2,1}(t) & Z_{2,2}(t) \end{array}\right] \in L_{2p}^{(n+m)\times(n+m)}(\Omega)\,, \tag{47}$$

with

$$Z_{1,1}(t) \in L_q^{n\times n}(\Omega), \quad Z_{1,2}(t) \in L_q^{n\times m}(\Omega), \quad Z_{2,1}(t) \in L_q^{m\times n}(\Omega), \quad Z_{2,2}(t) \in L_{2p}^{m\times m}(\Omega). \tag{48}$$

Then, from (28), (44), (45) and (47) we can write

$$U(t) = Z_{1,1}(t) + Z_{1,2}(t)W_0\,; \qquad V(t) = Z_{2,1}(t) + Z_{2,2}(t)W_0\,, \;\; t \in \mathcal{N}_U(0)\,, \tag{49}$$

and from Theorem 1, both s.p.'s $U(t) \in L_{2p}^{n\times n}(\Omega)$ and $V(t) \in L_{2p}^{m\times n}(\Omega)$, defined by (49), are p-differentiable. Hence, we can write $W(t)$, defined by (46), as

$$W(t) = V(t)\,(U(t))^{-1} = \left(Z_{2,1}(t) + Z_{2,2}(t)W_0\right)\left(Z_{1,1}(t) + Z_{1,2}(t)W_0\right)^{-1}\,, \qquad t \in \mathcal{N}_U(0)\,. \tag{50}$$

By Proposition 2, Corollary 1, (45), (46) and, assuming that $(U(t))^{-1} = (Z_{11}(t) + Z_{12}(t)W_0)^{-1} \in L_{2p}^{n\times n}(\Omega)$ and is $2p$-differentiable, it follows that

$$\begin{aligned} W'(t) &= V'(t)\,(U(t))^{-1} + V(t)\left[-\,(U(t))^{-1}\;U'(t)\;(U(t))^{-1}\right] \\ &= [C\,U(t) - D\,V(t)]\;(U(t))^{-1} - V(t)\;(U(t))^{-1}\;U'(t)\;(U(t))^{-1} \\ &= C - D\,W(t) - W(t)\,[A\,U(t) + B\,V(t)]\;(U(t))^{-1} \\ &= C - D\,W(t) - W(t)\,A - W(t)\,B\,W(t)\,, \end{aligned}$$

and $W(0) = V(0)\;(U(0))^{-1} = W_0$.

Summarizing the following result has been established

Theorem 2. *Let us assume that random matrices L and Y_0 defined by* (43) *lie in $L_{4p}^{(n+m)\times(n+m)}(\Omega)$ and $L_{4p}^{(n+m)\times n}(\Omega)$, respectively, and L satisfies condition* (29)*. Let $Z_{i,j}(t)$ be the block-entries of* $\exp(L\,t)$ *defined by* (47)–(48) *and let $U(t)$, $V(t)$ be defined by* (49) *with $U(0) = I_n$, $V(0) = W_0 \in L_{4p}^{m\times n}(\Omega)$. If $\mathcal{N}_U(0)$ is an ordinary neighbourhood of $t = 0$ where $U(t) \in L_{2p}^{n\times n}(\Omega)$ is $2p$-differentiable, invertible and $(U(t))^{-1} \in L_{2p}^{n\times n}(\Omega)$ is $2p$-differentiable, then $W(t)$ defined by* (50) *is a solution of random IVP* (1) *in $L_p^{m\times n}(\Omega)$.*

Thinking of applications, it is also interesting the study of the linear bilateral random problem

$$W'(t) + W(t)\,A + D\,W(t) = 0\,, \qquad W(0) = W_0\,, \tag{51}$$

that is a particular case of (1) where $B = O_{n\times m}$, $C = O_{m\times n}$. With the notation of Theorem 2, observe that L is the block-diagonal matrix

$$L = \mathrm{diag}(A, -D) = \left[\begin{array}{c|c} A & O \\ \hline O & -D \end{array}\right] \tag{52}$$

and

$$\exp(L\,t) = \left[\begin{array}{c|c} \exp(t\,A) & O \\ \hline O & \exp(-t\,D) \end{array}\right], \tag{53}$$

$$U(t) = Z_{1,1}(t) = \exp(t\,A)\,; \qquad V(t) = Z_{2,2}(t)\,W_0 = \exp(-t\,D)\,W_0\,. \tag{54}$$

Note that $\mathcal{N}_U(0)$ is the whole real line because $U(t) = \exp(t\,A)$ is invertible for all $t \in \mathbb{R}$, with $(U(t))^{-1} = \exp(-t\,A)$. Using hypotheses of Theorem 2, the solution of (51) in all the real line is given by

$$W(t) = \exp(-t\,D)\,W_0\,\exp(-t\,A)\,. \tag{55}$$

In this case, condition (29) upon random matrix L can be expressed directly in terms of the same property for random matrices A and D. Hence, the following result has been established:

Corollary 2. Assume that random matrices $A \in L_{2p}^{n\times n}(\Omega)$, $D \in L_{2p}^{m\times n}(\Omega)$ satisfy condition (29) and $W_0 \in L_{2p}^{m\times n}(\Omega)$. Then $W(t)$ defined by (55) is a $L_p^{m\times n}(\Omega)$ solution of problem (51).

5. Numerical examples

This section is devoted to present three examples where the theoretical results previously established are illustrated. In order to show the capability of the proposed method in different scenarios, the first and second examples consider, respectively, two particular cases of that random IVP where $m = n = 1$, thus corresponding to the scalar case. Specifically, the first is a numerical example whereas the second shows an application to the recent random SI-type epidemiological model [26] in order to model the early stages of the AIDS epidemic. Finally, the last example deals with a random matrix Riccati IVP of the form (1).

We point out that the uncertainty assigned to each one of the involved random input parameters in all examples is considered through a wide range of probability distributions such as beta, exponential, Gaussian, etc. In the examples, we will compute the main statistical moments of the solution s.p., namely, the mean and the variance functions.

Example 1. *Let us consider the following scalar Riccati random differential equation*

$$w'(t) + a\,w(t) + b\,(w(t))^2 - c = 0\,, \qquad w(0) = w_0, \tag{56}$$

which is obtained as a particular case of (1) *taking*

$$m = n = 1\,, \quad W(t) = w(t)\,, \quad W(0) = w_0\,, \quad A = D = \frac{a}{2}\,, \quad B = b\,, \quad C = c\,. \tag{57}$$

We will assume that r.v. a has a Gaussian distribution of mean $\mu = 2$ and standard deviation $\sigma = 0.1$ truncated at the interval $[1.5, 2.5]$, $a \sim N_{[1.5,2.5]}(2; 0.1)$; b has an exponential distribution of parameter $\lambda = 1/3$ truncated at the interval $[1, 6]$, $b \sim Exp_{[1,6]}(1/3)$; c has a uniform distribution on the interval $[0.5, 1.5]$, $c \sim U(0.5, 1.5)$; and finally w_0 has a Gaussian distribution of mean $\mu = 1$ and standard deviation $\sigma = 0.1$ truncated at the interval $[0.5, 1.5]$, $w_0 \sim N_{[0.5,1.5]}(1; 0.1)$. To simplify subsequent expressions involved in computations, we consider that these four r.v.'s are defined in a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as well as they are independent. In order to compute the expectation, the following steps have been performed.

Step 1. Representation of the solution s.p. of (56) *in terms of the random data.*
Compute the random matrix exponential

$$\exp(Lt) = \left[\begin{array}{c|c} Z_{1,1}(t) & Z_{1,2}(t) \\ \hline Z_{2,1}(t) & Z_{2,2}(t) \end{array}\right], \qquad L = \left[\begin{array}{c|c} \frac{a}{2} & b \\ \hline c & -\frac{a}{2} \end{array}\right],$$

using, for example, Mathematica software. This yields

$$\left.\begin{aligned}
Z_{1,1}(t) &= -\frac{(a - \sqrt{a^2+4bc})\exp\left(-\frac{1}{2}t\sqrt{a^2+4bc}\right)}{2\sqrt{a^2+4bc}} + \frac{(a + \sqrt{a^2+4bc})\exp\left(\frac{1}{2}t\sqrt{a^2+4bc}\right)}{2\sqrt{a^2+4bc}}\,,\\
Z_{1,2}(t) &= -\frac{b\exp\left(-\frac{1}{2}t\sqrt{a^2+4bc}\right)}{\sqrt{a^2+4bc}} + \frac{b\exp\left(\frac{1}{2}t\sqrt{a^2+4bc}\right)}{\sqrt{a^2+4bc}}\,,\\
Z_{2,1}(t) &= -\frac{c\exp\left(-\frac{1}{2}t\sqrt{a^2+4bc}\right)}{\sqrt{a^2+4bc}} + \frac{c\exp\left(\frac{1}{2}t\sqrt{a^2+4bc}\right)}{\sqrt{a^2+4bc}}\,,\\
Z_{2,2}(t) &= -\frac{(-a - \sqrt{a^2+4bc})\exp\left(-\frac{1}{2}t\sqrt{a^2+4bc}\right)}{2\sqrt{a^2+4bc}} + \frac{(-a + \sqrt{a^2+4bc})\exp\left(\frac{1}{2}t\sqrt{a^2+4bc}\right)}{2\sqrt{a^2+4bc}}\,.
\end{aligned}\right\} \tag{58}$$

Observe that entries $\pm a/2$, b and c of matrix L satisfy condition (29) *since a, b and c are bounded r.v.'s. According to* (50) *and* (58)*, represent explicitly the solution s.p. of scalar random Riccati IVP* (56)*, $w(t)$, in terms of the random parameters as follows*

$$w(t) = V(t)(U(t))^{-1} = \frac{Z_{2,1}(t) + Z_{2,2}(t)\,w_0}{Z_{1,1}(t) + Z_{1,2}(t)\,w_0}\,. \tag{59}$$

Step 2. Computation of the expectation of the solution s.p. $w(t)$ *given by* (59).

Denote by $f_{w_0}(w_0)$, $f_a(a)$, $f_b(b)$ *and* $f_c(c)$ *the probability density functions of* w_0, a, b *and* c, *respectively. Compute the expectation of* $w(t)$ *as follows*

$$E\left[w(t)\right] = \int_{\mathbb{R}^4} w(t) f_{w_0}(w_0) f_a(a) f_b(b) f_c(c)\, \mathrm{d}w_0\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c\,. \tag{60}$$

Step 3. Computation of the standard deviation of the solution s.p. $w(t)$ *given by* (59).

Compute

$$E\left[(w(t))^2\right] = \int_{\mathbb{R}^4} (w(t))^2 f_{w_0}(w_0) f_a(a) f_b(b) f_c(c)\, \mathrm{d}w_0\, \mathrm{d}a\, \mathrm{d}b\, \mathrm{d}c\,, \tag{61}$$

and then, determine the standard deviation by

$$\sigma\left[w(t)\right] = +\sqrt{E\left[(w(t))^2\right] - \left(E\left[w(t)\right]\right)^2}\,, \tag{62}$$

using (60) *and* (61).

Figure 1 shows $E[w(t)]$ *and* $E[w(t)] \pm \sigma[w(t)]$ *on the time interval* $[0, 5]$. *We observe that, in this particular case, the expectation and standard deviation of the solution stabilize as time goes on.*

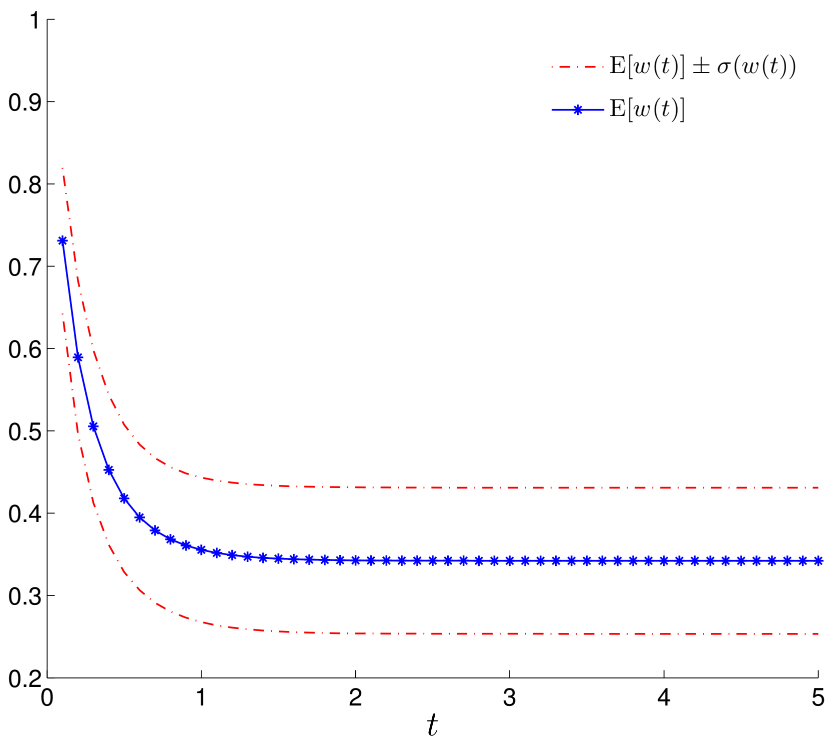


Figure 1: Evolution of the expectation, $\mathrm{E}[w(t)]$, and plus/minus the standard deviation, $\sigma[w(t)]$, of the solution s.p. $w(t)$ of the scalar random Riccati IVP (56) on the temporal domain $t \in [0, 5]$ in the context of Example 1.

Finally, in order to legitimate the earlier application of Theorem 2, notice that it remains to check that $U(t) \in L_{2p}^{1\times 1}(\Omega)$ *is* $2p$*-differentiable and invertible and that* $(U(t))^{-1} \in L_{2p}^{1\times 1}(\Omega)$ *is* $2p$*-differentiable. According to* (58), *let us first observe that* $U(t)$ *has the following form*

$$U(t) = \alpha_1 \exp(\beta_1 t) + \alpha_2 \exp(\beta_2 t), \tag{63}$$

where $\alpha_i = \alpha_i(\omega)$ and $\beta_i = \beta_i(\omega)$, $i = 1, 2$, $\omega \in \Omega$ are defined by

$$\alpha_1 = \frac{-a - 2bw_0 + \sqrt{a^2 + 4bc}}{2\sqrt{a^2 + 4bc}}, \quad \alpha_2 = \frac{a + 2bw_0 + \sqrt{a^2 + 4bc}}{2\sqrt{a^2 + 4bc}}, \quad \beta_1 = -\frac{\sqrt{a^2 + 4bc}}{2} < 0, \quad \beta_2 = \frac{\sqrt{a^2 + 4bc}}{2} > 0.$$

Moreover, taking into account the domains of bounded absolutely continuous r.v.'s a, b, c and w_0, it is clear that $a^2 + 4bc = (a(\omega))^2 + 4b(\omega)c(\omega) > 0$ for all $\omega \in \Omega$, thus $\alpha_i = \alpha_i(\omega)$ and $\beta_i = \beta_i(\omega)$, $i = 1, 2$, are well-defined and, for each $t \geq 0$ and $p \geq 1$ fixed, one gets

$$M_{t,p} := \max_{\omega \in \Omega}\{(\alpha_1 \exp(\beta_1 t) + \alpha_2 \exp(\beta_2 t))^{2p}\} < +\infty.$$

Then, one gets

$$\begin{aligned} E[(U(t))^{2p}] &= \int_{1.5}^{2.5}\int_{1}^{6}\int_{0.5}^{1.5}\int_{0.5}^{1.5} (\alpha_1 \exp(\beta_1 t) + \alpha_2 \exp(\beta_2 t))^{2p} f_{w_0}(w_0) f_c(c) f_b(b) f_a(a)\, \mathrm{d}w_0\, \mathrm{d}c\, \mathrm{d}b\, \mathrm{d}a \\ &\leq M_{t,p}\left(\int_{0.5}^{1.5} f_{w_0}(w_0)\, \mathrm{d}w_0\right)\left(\int_{0.5}^{1.5} f_c(c)\, \mathrm{d}c\right)\left(\int_{1}^{6} f_{b0}(b)\, \mathrm{d}b\right)\left(\int_{1.5}^{2.5} f_a(a)\, \mathrm{d}a\right) = M_{t,p} < +\infty. \end{aligned}$$

Notice that in the last step we have used that every integral is 1. This shows that $U(t) \in L_{2p}^{1\times 1}(\Omega)$ for each $t \geq 0$. Taking into account that $\alpha_i = \alpha_i(\omega)$ and $\beta_i = \beta_i(\omega)$, $i = 1, 2$ lie in closed finite intervals, one can check that $U(t) = U(t)(\omega) > 0$ for all $\omega \in \Omega$ and defining

$$m_{t,p} := \min_{\omega \in \Omega}\{(\alpha_1 \exp(\beta_1 t) + \alpha_2 \exp(\beta_2 t))^{2p}\} > 0,$$

it is straightforward to show, using an analogous argument, that, for each $t \geq 0$ and $p \geq 1$ fixed, one gets

$$E\left[((U(t))^{-1})^{2p}\right] \leq \frac{1}{m_{t,p}} < +\infty.$$

Bearing in mind that, by (63)*, $U(t)$ is a linear combination of two exponential processes, to prove that $U(t)$ is $2p$-differentiable about $t = 0$ it is enough to observe that, for a s.p., $g(t) = \exp(\beta t)$, one gets*

$$\left(\left\|\frac{g(h) - g(0)}{h} - g'(0)\right\|_{2p}\right)^{2p} = E\left[\left(\frac{\exp(\beta h) - 1}{h} - \beta\right)^{2p}\right] = E\left[\left(\frac{\exp(\beta h) - (1 + \beta h)}{h}\right)^{2p}\right] = O(h^{2p}) \longrightarrow 0 \quad as \quad h \to 0.$$

A similar argument justifies that $(U(t))^{-1}$ is $2p$-differentiable about $t = 0$ since $U(t) = U(t)(\omega) > 0$ for all $\omega \in \Omega$.

Example 2. *SI-type models are useful to study simple epidemics where the only transition in the population is from susceptible (S) to infected (I). It is assumed that the total population size, say $\hat{n}$, is constant for all time t because this hypothesis is credible during certain time-intervals, particularly in developed countries as well as for populations under control. SI-models can be described by the following IVP*

$$s'(t) = -\frac{\beta}{\hat{n}}\, s(t)[\hat{n} - s(t)]\,, \quad s(0) = m, \tag{64}$$

where $s(t)$ is the number of susceptibles at the time instant t, m represents the initial number of susceptibles and $\beta > 0$ denotes the transmission rate of decline in the number of susceptibles. In [27], authors rewritten equation (64) *in terms of the proportion of susceptibles at time t, $w(t) = s(t)/\hat{n}$, obtaining the following scalar Riccati random differential equation*

$$w'(t) = -\beta\, w(t)[1 - w(t)]\,, \quad w(0) = w_0, \tag{65}$$

where $w_0 = m/\hat{n}$ is the initial proportion of susceptibles verifying $w_0 \in [0, 1]$. In this manner, the authors assume that the initial condition w_0 is a r.v., following a beta distribution, that is $w_0 \sim Be(a; b)$, whose domain is the interval $[0, 1]$. And for simplicity, they consider that the transmission rate β in (65) *is deterministic. However, using our theoretical results previously developed, we can introduce uncertainty in both parameters w_0 and β and compute the prevalence* *of people with HIV antibodies in a representative sample of homosexual men. Identifying all the elements of the scalar Riccati random differential equation* (65) *as we did in Example 1, we obtain*

$$m = n = 1\,, \quad W(t) = w(t)\,, \quad W(0) = w_0\,, \quad A = D = \frac{\beta}{2}\,, \quad B = -\beta\,, \quad C = 0\,. \tag{66}$$

According to [27], we assume $w_0 \sim Be(a = 3.4998; b = 0.2168)$ *and consider parameter* β *as a r.v. following a Gaussian distribution of mean* $\mu = 1.18$ *and standard deviation* $\sigma = 0.11$ *truncated at the interval* $[\mu - 3\sigma, \mu + 3\sigma] = [0.85, 1.51]$*, that is* $\beta \sim N_{[0.85,1.51]}(1.18; 0.11)$*, instead of taking the deterministic estimation,* $\hat{\beta} = 1.18(\pm 0.11)$*, used in [27].*

Following similar steps as the ones described in the Example 1, we obtain the expressions

$$\left.\begin{array}{l} Z_{1,1} = \exp\left(\frac{t\beta}{2}\right)\,, \\ Z_{1,2} = -\exp\left(-\frac{t\beta}{2}\right)(-1 + \exp(t\beta))\,, \\ Z_{2,1} = 0 \\ Z_{2,2} = \exp\left(-\frac{t\beta}{2}\right)\,, \end{array}\right\}$$

and the solution s.p. of scalar random Riccati IVP (65)*,* $w(t)$*, in terms of the random parameters is*

$$w(t) = V(t)(U(t))^{-1} = \frac{Z_{2,1}(t) + Z_{2,2}(t)\,w_0}{Z_{1,1}(t) + Z_{1,2}(t)\,w_0} = \frac{\exp\left(-\frac{t\beta}{2}\right) w_0}{\exp\left(\frac{t\beta}{2}\right) - \exp\left(-\frac{t\beta}{2}\right)(-1 + \exp(t\beta))\; w_0}\,. \tag{67}$$

Observe that entries $\pm\beta/2$ *and* $-\beta$ *of the matrix* $L = \left[\begin{array}{c|c} \frac{\beta}{2} & -\beta \\ \hline 0 & -\frac{\beta}{2} \end{array}\right]$ *satisfy condition* (29) *since* β *is a bounded r.v. The expectation function,* $E[w(t)]$*, can be computed with Mathematica software from* (67) *using expression*

$$E\left[w(t)\right] = \int_0^1 w(t) f_{w_0}(w_0) f_\beta(\beta)\, \mathrm{d}w_0\, \mathrm{d}\beta\,.$$

In Figure 2 we have plotted $E\left[w(t)\right]$ *together with the four observed data points of the prevalence of HIV antibodies in a representative sample of homosexual men (San Francisco City Clinic cohort, 1978–1984), see [27].*

Finally, it must be checked that $U(t) \in L_{2p}^{1\times 1}(\Omega)$ *is* $2p$*-differentiable and* $(U(t))^{-1} \in L_{2p}^{1\times 1}(\Omega)$ *is* $2p$*-differentiable, being*

$$U(t) = \exp\left(\frac{t\beta}{2}\right) - \exp\left(-\frac{t\beta}{2}\right)(-1 + \exp(t\beta))\; w_0.$$

We omit this proof since it can be proved following a similar reasoning we used in Example 1.

Example 3. *Let us consider the random Riccati IVP* (1) *where*

$$W(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}, \quad W_0 = \begin{bmatrix} w_{1,0} \\ w_{2,0} \end{bmatrix}, \quad A = a\,, \quad B = \begin{bmatrix} b_{1,1} & b_{1,2} \end{bmatrix}, \quad C = \begin{bmatrix} c_{1,1} \\ c_{2,1} \end{bmatrix}, \quad D = \begin{bmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{bmatrix}. \tag{68}$$

We will assume that $w_{2,0} = 1$ *and* $b_{1,2} = c_{2,1} = d_{1,2} = d_{2,1} = d_{2,2} = 0$*. The rest of the parameters are assumed to be r.v.'s with the following distributions:* $w_{1,0}$ *has a beta distribution of parameters* $\alpha = 3$ *and* $\beta = 2$*,* $w_{1,0} \sim Be(3; 2)$*;* a *has a beta distribution of parameters* $\alpha = 2$ *and* $\beta = 1$*,* $a \sim Be(2; 1)$*;* $b_{1,1}$ *has an exponential distribution of parameter* $\lambda = 1$ *truncated at the interval* $[2, 3]$*,* $b_{1,1} \sim Exp_{[2,3]}(1)$*;* $c_{1,1}$ *has a Gaussian distribution of mean* $\mu = 1$ *and standard deviation* $\sigma = 0.1$ *truncated at the interval* $[0.5, 1.5]$*,* $c_{1,1} \sim N_{[0.5,1.5]}(1; 0.1)$ *and, finally* $d_{1,1}$ *has a uniform distribution on the interval* $[1, 2]$*,* $d_{1,1} \sim U(1, 2)$*. We will assume that all the input parameters are independent r.v.'s.*

In order to compute the expectation, the following steps have been performed.

Step 1 . Representation of the matrix solution s.p. in terms of the random data.

Compute the solution (28) *of random IVP* (27) *where*

$$L = \left[\begin{array}{c|cc} a & b_{1,1} & 0 \\ \hline c_{1,1} & -d_{1,1} & 0 \\ 0 & 0 & 0 \end{array}\right], \qquad Y_0 = \left[\begin{array}{c} 1 \\ \hline w_{1,0} \\ 1 \end{array}\right]. \tag{69}$$

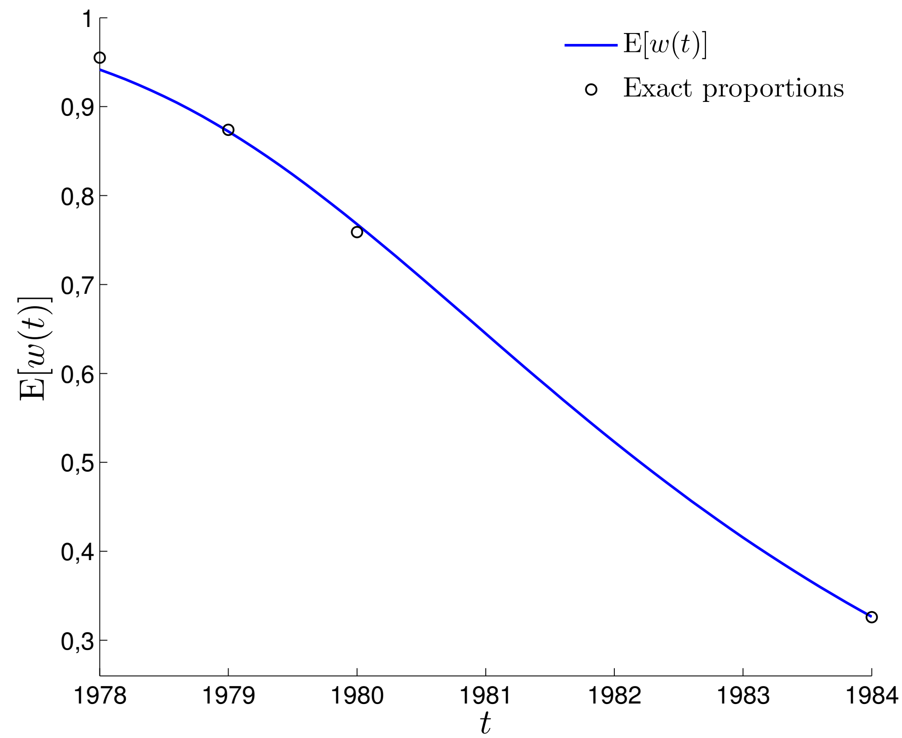


Figure 2: Expectation of the percentage of non-HIV+ from year 1978 until 1984, $E[w(t)]$, in a sample of homosexual men and the four exact percentages (0.955, 0.874, 0.759 and 0.326) at time points 0, 1, 2 and 6 corresponding to the years 1978, 1979, 1980 and 1984, respectively.

Note that entries a, $b_{1,1}$, $c_{1,1}$, and $-d_{1,1}$ of matrix L satisfy condition (29) *since a, $b_{1,1}$, $c_{1,1}$ and $d_{1,1}$ are bounded r.v.'s. Define a column vector of size* 3×1

$$Y(t) = \exp(L\,t)\;Y_0 = \left[\begin{array}{c|c} Z_{1,1}(t) & Z_{1,2}(t) \\ \hline Z_{2,1}(t) & Z_{2,2}(t) \end{array}\right] \left[\begin{array}{c} 1 \\ \hline W_0 \end{array}\right] = \left[\begin{array}{c|cc} z_{1,1}(t) & z_{1,2}(t) & z_{1,3}(t) \\ \hline z_{2,1}(t) & z_{2,2}(t) & z_{2,3}(t) \\ z_{3,1}(t) & z_{3,2}(t) & z_{3,3}(t) \end{array}\right] \left[\begin{array}{c} 1 \\ \hline w_{1,0} \\ 1 \end{array}\right]. \tag{70}$$

According to (50) *and* (70)*, represent explicitly the solution s.p. of random Riccati IVP* (1)*, $W(t) = [w_1(t)\;w_2(t)]^{\mathsf{T}}$, in terms of the random parameters as follows*

$$\begin{aligned} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} &= \left(Z_{2,1}(t) + Z_{2,2}(t)W_0\right)\left(Z_{1,1}(t) + Z_{1,2}(t)W_0\right)^{-1} \\ &= \left\{ \begin{bmatrix} z_{2,1}(t) \\ z_{3,1}(t) \end{bmatrix} + \begin{bmatrix} z_{2,2}(t) & z_{2,3}(t) \\ z_{3,2}(t) & z_{3,3}(t) \end{bmatrix} \begin{bmatrix} w_{1,0} \\ 1 \end{bmatrix} \right\} \left\{ z_{1,1}(t) + \begin{bmatrix} z_{1,2}(t) & z_{1,3}(t) \end{bmatrix} \begin{bmatrix} w_{1,0} \\ 1 \end{bmatrix} \right\}^{-1}. \end{aligned} \tag{71}$$

Step 2. Computation of the expectation.

Expression (71) *gives a representation of components $w_i(t)$, $i = 1, 2$, of $W(t)$ in terms of the random input parameters $w_{1,0}$, a, $b_{1,1}$, $c_{1,1}$ and $d_{1,1}$. Denote by $f_{w_{1,0}}(w_{1,0})$, $f_a(a)$, $f_{b_{1,1}}(b_{1,1})$, $f_{c_{1,1}}(c_{1,1})$ and $f_{d_{1,1}}(d_{1,1})$ their probability density functions (p.d.f.'s), respectively. Compute the expectation of the solution s.p. $W(t)$ as follows*

$$E\left[w_i(t)\right] = \int_{\mathbb{R}^5} w_i(t) f_{w_{1,0}}(w_{1,0}) f_a(a) f_{b_{1,1}}(b_{1,1}) f_{c_{1,1}}(c_{1,1}) f_{d_{1,1}}(d_{1,1})\, \mathrm{d}w_{1,0}\, \mathrm{d}a\, \mathrm{d}b_{1,1}\, \mathrm{d}c_{1,1}\, \mathrm{d}d_{1,1}, \quad i = 1, 2. \tag{72}$$

Step 3. Computation of the standard deviation.

Compute

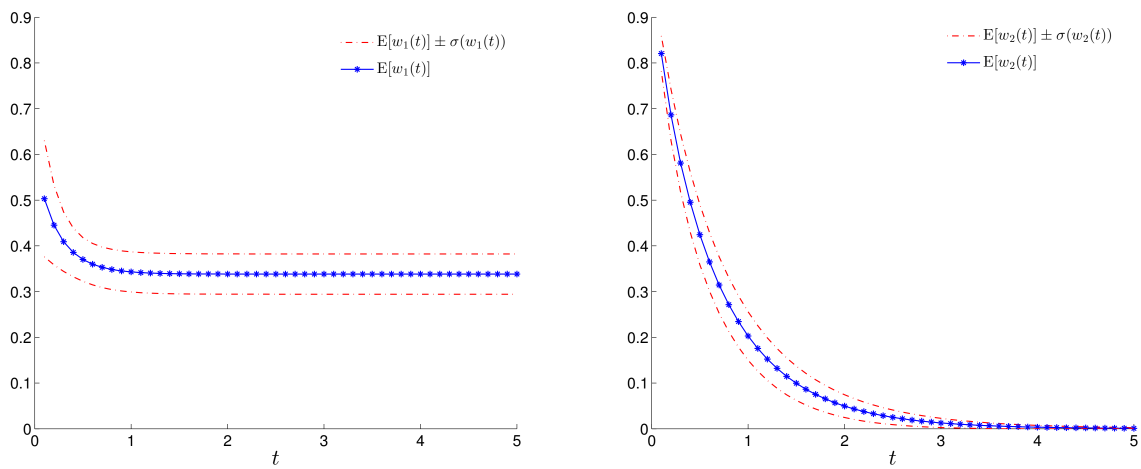
$$E\left[(w_i(t))^2\right] = \int_{\mathbb{R}^5} (w_i(t))^2 \, f_{w_{1,0}}(w_{1,0}) f_a(a) f_{b_{1,1}}(b_{1,1}) f_{c_{1,1}}(c_{1,1}) f_{d_{1,1}}(d_{1,1}) \, \mathrm{d}w_{1,0} \, \mathrm{d}a \, \mathrm{d}b_{1,1} \, \mathrm{d}c_{1,1} \, \mathrm{d}d_{1,1}, \quad i = 1, 2, \quad (73)$$

and then, determine the standard deviation by

$$\sigma[w_i(t)] = +\sqrt{E\left[(w_i(t))^2\right] - (E[w_i(t)])^2}, \quad i = 1, 2, \quad (74)$$

where $E[w_i(t)]$ is given by (72).

Figure 3 shows the expectation plus/minus the standard deviation for each one of the two components, $w_1(t)$ (plot (a)) and $w_2(t)$ (plot (b)), of the solution s.p. $W(t)$ of the Riccati random differential equation (1)*. In this particular example, we observe that the expectations and standard deviations of both components tend to stabilization.*



(a) (b)

Figure 3: Expectations $\mathrm{E}[w_i(t)]$ and plus/minus the standard deviations $\mathrm{E}[w_i(t)] \pm \sigma[w_i(t)]$, $i = 1, 2$, of the two components of the solution $W(t)$ of the random Riccati IVP (1) on the time domain $t \in [0, 5]$ in the context of Example 3.

We finally point out that it must be checked that $U(t) \in L_{2p}^{1\times 1}(\Omega)$ is $2p$-differentiable and $(U(t))^{-1} \in L_{2p}^{1\times 1}(\Omega)$ is $2p$-differentiable, being

$$U(t) = z_{1,1}(t) + \begin{bmatrix} z_{1,2}(t) & z_{1,3}(t) \end{bmatrix} \begin{bmatrix} w_{1,0} \\ 1 \end{bmatrix}.$$

This can be done following a similar reasoning we used in Example 1.

6. Conclusions

Riccati matrix differential equations with uncertainty play a relevant role in many different type of real problems such as population dynamics and control theory, for instance [28]. When uncertainty is driven by Brownian motion, the differentiability is considered in the Itô calculus sense and models are formulated by Itô type stochastic differential equations. In this paper, we consider an alternative type of randomness and we then apply the so called L_p-random calculus to solve random differential equations. Throughout this paper we have established some results belonging to the L_p-random matrix calculus to extend methods of deterministic calculus to the random framework. This has been done assuming certain conditions involving statistical moments of coefficients, forcing term and initial condition of the random differential equation. Although these conditions are, from a mathematical point of view, somewhat strong, they are met in many practical situations. Several numerical examples illustrate the applicability of the results established through this paper.

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