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All hypertopologies are hit-and-miss

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This paper is dedicated to my Guru Professor John G. Hocking, who introduced me to the excitement of mathematical research

Abstract. We solve a long standing problem by showing that all known hypertopologies are hit-and-miss. Our solution is not merely of theoretical importance. This representation is useful in the study of comparison of the Hausdorff-Bourbaki or H-B uniform topologies and the Wijsman topologies among themselves and with others. Up to now some of these comparisons needed intricate manipulations. The H-B uniform topologies were the subject of intense activity in the 1960's in connection with the Isbell-Smith problem. We show that they are proximally locally finite topologies from which the solution to the above problem follows easily. It is known that the Wijsman topology on the hyperspace is the proximal ball (hit-and-miss) topology in "nice" metric spaces including the normed linear spaces. With the introduction of a new far-miss topology we show that the Wijsman topology is hit-and-miss for all metric spaces. From this follows a natural generalization of the Wijsman topology to the hyperspace of any T_1 space. Several existing results in the literature are easy consequences of our work.

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1. Introduction

During the early part of the last century there were two main studies on the hyperspace of all non-empty closed subsets of a topological space. Vietoris introduced a topology consisting of two parts (a) the lower finite hit part and (b) the upper miss part. Hausdorff defined a metric on the hyperspace of a metric space and the resulting topology depends on the metric rather than on the topology of the base space. Two equivalent metrics on a metrisable space induce equivalent Hausdorff metric topologies if and only if they are uniformly equivalent. With the discovery of uniform spaces, the Hausdorff metric was generalized to Hausdorff-Bourbaki or H-B uniformity on the hyperspace of a uniformizable (Tychonoff) space. In the 1960's this led to the celebrated Isbell-Smith problem to find necessary and sufficient conditions that two uniformities on the base space are H-equivalent, i.e., they induce topologically equivalent H-B uniformities on the hyperspace. ([9, 17, 18, 19]. For an account see [15, Section 15, Page 87]) The solution presented by Ward is quite intricate. In this paper we show that the H-B uniform topology has two parts (a) the lower locally finite hit part and (b) the upper proximal miss part. From this representation and the knowledge that lower and upper parts act separately in any comparison, Ward's theorem is obvious. We then briefly indicate how the recent work on bounded Hausdorff metric or Attouch-Wets topologies can also be represented in a similar vein and derive results effortlessly.

In 1966, while working on a problem in Convex Analysis, Wijsman introduced a convergence for the closed subsets of a metric space (X, d) viz.

$$A_n \to A \text{ if and only if for each } x \in X, d(x, A_n) \to d(x, A).$$

The resulting topology was found to be quite useful in applications. Moreover, Wijsman topologies are the building blocks of many other topologies e.g.,

(a) the metric proximal topology is the sup of all Wijsman topologies induced by uniformly equivalent metrics, and
(b) the Vietoris topology is the sup of all Wijsman topologies induced by topologically equivalent metrics. ([3])

In addition, the Wijsman topologies are rather intriguing since there are examples of two uniformly equivalent metrics giving non equivalent Wijsman topologies and two non uniformly equivalent metrics, giving equivalent Wijsman topologies! Attempts to topologize the Wijsman topology led to the discovery of (a) the ball topology ([1]) and (b) the proximal ball topology ([4]) which were only partially successful. The lower part of the Wijsman topology coincides with the lower Vietoris topology. It is the upper one which is difficult to handle. In this paper we show that the upper Wijsman topology can be expressed as a *far-miss* topology and this representation makes it easy to compare Wijsman topologies among themselves and with others. We give simple conceptual proofs without any calculations.

2. Preliminaries

(References for uniformities include [9, 10], and for proximities [15].) Suppose $(X, \mathbb{U})$ is a Hausdorff uniform space and suppose $\mathbb{U}$ induces the EF- proximity δ which, in turn, induces the topology T on X. In case X is a metrisable space with a metric d, $\mathbb{U}$ denotes the metric uniformity and δ the metric proximity. The fine uniformity on (X, T) is denoted by $\mathbb{U}^\sharp$ and the coarsest (totally bounded) uniformity, compatible with δ, is denoted by $\mathbb{U}^*$. To keep matters simple, we assume that all entourages are open symmetric. We use the symbol δ_0 to denote the fine LO-proximity on X given by

$$A\delta_0 B \text{ iff } clA \cap clB \neq \varnothing.$$

We denote by $\delta^\sharp$ the fine EF-proximity on X given by

$$A\underline{\delta}^\sharp B \text{ iff } A \text{ and } B \text{ can be separated by a continuous function on } X \text{ to } [0,1].$$

We use the following standard notation :

$CL(X)$ = the family of all non-empty closed subsets of X.

$\mathbb{K}(X)$= the family of all non-empty compact subsets of X.

Δ denotes a non-empty subfamily of $CL(X)$. Without any loss of generality we assume that Δ is closed under finite unions and contains all singletons. We call Δ a **cobase**. ([16]) (In the literature Δ is usually assumed to contain merely all singletons. In our view the above assumptions simplify the results, since we get a *base* for the upper Δ-topologies.) For any set $E \subset X$ and $\mathsf{E} \subset T$ we use the following notation:

$$E^- = \{A \in CL(X) : A \cap E \neq \varnothing\}$$

$$\mathsf{E}^- = \{A \in CL(X) : A \cap E \neq \varnothing \text{ for each } E \in \mathsf{E}\}$$

$$E^{++} = \{A \in CL(X) : A <<_\delta E \text{ i.e. } A\underline{\delta}E^c\}$$

$$E^+ = \{A \in CL(X) : A \subset E \text{ i.e. } A << E \text{ w.r.t. } \delta_0\}$$

The **upper proximal Δ-topology** (w.r.t. δ) $\sigma(\delta\Delta^+)$ is generated by the basis $\{E^{++} : E^c \in \Delta\}$.

The **upper Δ-topology** $\tau(\Delta^+) = \sigma(\delta_0\Delta^+)$.

The **lower Vietoris (or finite) topology** $\tau(V^-)$ has a basis $\{\mathsf{E}^- : \mathsf{E} \subset T \text{ is finite}\}$. The **lower locally finite topology** $\tau(\mathsf{LF}^-)$ has a basis $\{\mathsf{E}^- : \mathsf{E} \subset T \text{ is locally finite}\}$. The **$\mathbb{L}$-lower locally finite topology** $\tau(\mathbb{L}^-)$ has a basis $\{\mathsf{E}^- : \mathsf{E} \in \mathbb{L}\}$ where $\mathbb{L} \subset \{\mathsf{E} \subset T : \mathsf{E} \text{ is locally finite}\}$ satisfies some simple filter condition, i.e., given $E, F \in \mathbb{L}$ there is a $G \in \mathbb{L}$ such that $G^- \subset E^- \cap F^-$ (see [11]).

The **proximal (finite) Δ-topology** (w.r.t. δ) $\sigma(\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(V^-)$. We omit δ if it is obvious from the context and write $\sigma(\Delta)$ for $\sigma(\delta\Delta)$.

The **Δ-topology** $\tau(\Delta) = \tau(\Delta^+) \vee \tau(V^-)$.

The **proximal locally finite Δ-topology** (w.r.t. δ) $\sigma(\mathsf{LF}\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(\mathsf{LF}^-)$.

The **proximal $\mathbb{L}$-locally finite Δ-topology** (w.r.t. δ) $\sigma(\mathbb{L}\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(\mathbb{L}^-)$. We omit the prefix "proximal" and replace σ by τ if $\delta = \delta_0$.

Well known special cases are:

(a) when $\Delta = CL(X)$,
- $\tau(\Delta) = \tau(\mathsf{V})$ the **Vietoris or finite topology** ([12])
- $\sigma(\delta\Delta) = \sigma(\delta)$ the **proximal topology** ([5])
- $\tau(\mathsf{LF}\Delta) = \tau(\mathsf{LF})$ the **locally finite topology** ([2, 11, 14, 20])
- $\sigma(\mathsf{LF}\delta\Delta) = \sigma(\mathsf{LF}\delta)$ the **proximal locally finite topology** ([5])

(b) When $\Delta = \mathbb{K}(X), \tau(\Delta) = \tau(\mathsf{F}) = \sigma(\mathsf{F})$ the **Fell topology** (see [7])

(c) If (X, d) is a metric space, δ is the metric proximity induced by d and Δ denotes the family B of finite unions of proper closed balls of non-negative radii, then

$\tau(\Delta) = \tau(\mathsf{B})$ the **Ball topology** ([1])

$\sigma(\Delta) = \sigma(\mathsf{B})$, the **proximal Ball topology** ([4])

3. Hausdorff-Bourbaki uniformity

Recall that $(X, \mathbb{U})$ is a Hausdorff uniform space, the uniformity $\mathbb{U}$ induces the EF-proximity δ which, in turn, induces the topology T on X. In this section we show that the topology induced on $CL(X)$ by the Hausdorff-Bourbaki or H-B-uniformity $\mathbb{U}_H$ is a hit-and-miss topology.

We recall that a typical nbhd. of $A \in CL(X)$ in the **lower H-B-uniform** topology $\tau(\mathbb{U}_H^-)$ is $\{B \in CL(X) : A \subset U(B)\}$ where $U \in \mathbb{U}$.

In the case of the **upper H-B-uniform** topology $\tau(\mathbb{U}_H^+)$, a typical nbhd. of $A \in CL(X)$ is $\{B \in CL(X) : B \subset U(A)\}$ where $U \in \mathbb{U}$.

It is known and easy to show that $\sigma(\delta^+) = \tau(\mathbb{U}_H^+)$.

The **H-B-uniform** topology $\tau(\mathbb{U}_H) = \tau(\mathbb{U}_H^+) \vee \tau(\mathbb{U}_H^-) = \sigma(\delta^+) \vee \tau(\mathbb{U}_H^-)$.

Lemma 3.1. *For each $A \in CL(X)$ and each $U \in \mathbb{U}$, there is a discrete (and hence a locally finite) family $\{U(x) : x \in Q \subset A\}$ with the properties:*

(a) *if $x, y \in Q$ and $x \neq y$ then $y \notin U(x)$, and*

(b) *$A \subset U(Q)$.*

Proof. By Zorn's Lemma, there is a set $Q \subset A$ which is maximal such that if $x, y \in Q$ and $x \neq y$ then $y \notin U(x)$. Clearly, (b) is satisfied. □

Remark 3.2.

(a) It is obvious that the uniformity $\mathbb{U}$ is totally bounded if and only if for each $A \in CL(X)$ and each $U \in \mathbb{U}$, maximal U-discrete subsets of A are finite.

(b) Let $\mathbb{LU}$ be the collection of families of open sets of the form $\{U(x) : x \in Q \subset A\}$, where $A \in CL(X)$, $U \in \mathbb{U}$ and Q is U-discrete.

The family $\mathbb{LU}$ allows us to define a new lower locally finite "hit" topology.

Definition 3.3. The **lower $\mathbb{LU}$-locally finite topology** $\tau(\mathbb{LU}^-)$ on $CL(X)$ consists of $\{\mathsf{E}^- : \mathsf{E} \in \mathbb{LU}\}$.

If $A \in \mathsf{E}^-$, where $\mathsf{E} \subset T$ is finite, then there is a finite set $Q \subset A$ and a $U \in \mathbb{U}$ such that Q is U-discrete and $A \in \{U(x) : x \in Q\}^- \subset \mathsf{E}^-$. So $\tau(\mathsf{V}^-) \subset \tau(\mathbb{LU}^-)$.

The **proximal $\mathbb{LU}$-locally finite topology** $\sigma(\mathbb{LU}\delta) = \sigma(\delta^+) \vee \tau(\mathbb{LU}^-)$.

Lemma 3.4. $\tau(\mathbb{U}_H^-) = \tau(\mathbb{LU}^-)$.

Proof. Suppose $A \in W = \{B \in CL(X) : A \subset U(B)\} \in \tau(\mathbb{U}_H^-)$ where $U \in \mathbb{U}$. Let $V \in \mathbb{U}$ be such that $V^2 \subset U$. Let Q be a maximal V-discrete subset of A. Then $A \in S = \{V(x)^- : x \in Q\}$ and $S \in \tau(\mathbb{LU}^-)$.

Conversely, suppose $A \in S' = \{U(x)^- : x \in Q \subset A\} \in \tau(\mathbb{LU}^-)$. It is easy to see that if $A \subset U(B)$, then $B \in S'$ and so $A \in \{B \in CL(X) : A \subset U(B)\} \subset S'$. □

Theorem 3.5. $\tau(\mathbb{U}_H) = \sigma(\mathbb{LU}\delta)$.

Therefore, every H-B-uniformity $\mathbb{U}_H$ on $CL(X)$, associated with a uniformity $\mathbb{U}$ on X, induces the proximal $\mathbb{LU}$-locally finite topology on $CL(X)$.

We mention two special cases here. Remark 3.2 (a) gives us

Corollary 3.6 ([5])**.** $\sigma(\delta) = \tau(\mathbb{U}^*_H)$.

i.e., the proximal (finite) topology is induced by the H-B uniformity associated with the coarsest totally bounded uniformity compatible with δ.

Corollary 3.7 ([5])**.** $\sigma(\mathsf{LF}\delta^\sharp) = \tau(\mathbb{U}^\sharp_H)$.

i.e., the proximal locally finite topology w.r.t. the fine EF-proximity $\delta^\sharp$ is induced by the H-B uniformity associated with the fine uniformity $\mathbb{U}^\sharp$.

Proof. It is sufficient to show that if $A \in \mathsf{E}^-$, where E is a locally finite family of nonempty open sets, then there is a subset $Q \subset A$ and a $U \in \mathbb{U}^\sharp$, such that $A \in W = \{U(x)^- : x \in Q\} \in \tau(\mathbb{LU}^{\sharp -})$ and $W \in \mathsf{E}^-$. But this is just a standard argument involving continuous functions and locally finite families of open sets. □

Remark 3.8. The above result includes the following:

(a) X is normal if and only if the H-B uniformity associated with the fine uniformity induces the locally finite topology ([14, 20]).
(b) In a metrisable space, the supremum of all Hausdorff metric topologies, associated with compatible metrics, is the locally finite topology (see [20] for a non-standard treatment and [2] for the usual one).

Remark 3.9. Nachman [13] defined the **strong hyperproximity** on $CL(X)$ as the EF-proximity induced by $\mathbb{U}\widehat{\ }_H$, where $\mathbb{U}\widehat{\ }$is the AN-uniformity (the union of all uniformities compatible with δ). (See [15]). The following result is now obvious. The topology of the strong hyperproximity on $CL(X)$ is the supremum of proximal locally finite hit-and-miss topologies.

We now turn our attention to the celebrated Isbell-Smith problem mentioned in the introduction. Let (X, T) be a Tychonoff space with compatible uniformities $\mathbb{U}, \mathbb{V}$ inducing EF-proximities $\delta(\mathbb{U}), \delta(\mathbb{V})$ and H-B uniformities $\mathbb{U}_H, \mathbb{V}_H$ respectively. Isbell ([9]) first conjectured that if $\mathbb{U} \neq \mathbb{V}$ then $\tau(\mathbb{U}_H) \neq \tau(\mathbb{V}_H)$ suggesting that they induce non-equivalent families of nbhds. of the element $X \in CL(X)$. Smith ([17]) gave a counterexample to show that the latter statement to be false but nevertheless proved several results supporting Isbell's conjecture. Ward ([19]), however, disproved the conjecture and proved the following result:

Theorem 3.10 (see [15], Pages 89–90)**.** *$\tau(\mathbb{V}_H) \subset \tau(\mathbb{U}_H)$ if and only if $\delta(\mathbb{V}) \leq \delta(\mathbb{U})$, and for every $V \in \mathbb{V}$ and every V-discrete set A there is $U \in \mathbb{U}$ such that $U(a) \subset V(a)$ for each $a \in A$.*

Proof. The result follows easily from Theorem 3.5 and the fact that a coarser EF-proximity induces a coarser upper proximal hypertopology. (cf. [5]) □

Three other related results also follow easily:

Theorem 3.11.

(a) *Two uniformities on X that are H-equivalent, are in the same proximity class.*
(b) *Two uniformities on X, at least one of which is totally bounded, are not H-equivalent.*
(c) *Two different metrisable uniformities on X are not H-equivalent.*

4. The Wijsman topology

An unusual topology for the hyperspace of a metric space (X, d) arose from Wijsman's paper [21]. The Wijsman topology is not only useful in applications but also valuable as the building block of many hypertopologies ([3]). It was defined in terms of *convergence* viz.

(4.1) A net $A_n \in CL(X)$ **W-converges** to $A \in CL(X)$ iff for each $x \in X$, $d(x, A_n) \to d(x, A)$, where $d(x, A) = \inf\{d(x, a) : a \in A\}$.

Attempts to characterize the resulting Wijsman topology $\tau(\mathsf{W})$ as a hit-and-miss topology led to the discovery of the ball topology $\tau(\mathsf{B})$ (see [1] for further references) and the proximal ball topology $\sigma(\mathsf{B})$ ([4]). In the latter paper it was shown that $\tau(\mathsf{W}) = \sigma(\mathsf{B})$ in *nice* metric spaces including all normed linear spaces. We now show that the Wijsman topology is a hit-and-far-miss topology in *arbitrary* metric spaces by using the following characterization by Del Prete-Lignola ([6]):

(4.2) A_n W-converges to A if and only if the following conditions are met:
   (1) For every non-empty open set E, $A \in E^-$ implies eventually $A_n \in E^-$;
   (2) Whenever $0 < \varepsilon < \alpha$ and $A \notin S(x, \alpha)^-$ then eventually $A_n \notin S(x, \varepsilon)^-$.

It is well known that (1) is equivalent to convergence in the lower Vietoris topology $\tau(\mathsf{V}^-)$.

And (2) is equivalent to the statement:

If A is *far* from $S(x, \varepsilon)$ (in the metric proximity), then eventually A_n does not intersect $S(x, \varepsilon)$.

Here we find "far" is mixed with "disjoint". We need a new topology mixing the ball and the proximal ball topologies. Suppose B denotes the family of finite unions of all proper closed balls with non-negative radii.

Definition 4.1. The **upper far-miss ball topology** $\tau(\mathsf{FM}^+)$ has a local basis at $A \in CL(X)$, $\{E^+ : E^c \in \mathsf{B}$ and $A \in E^{++}\}$.

The **hit-and-far-miss topology** $\tau(\mathsf{FM}) = \tau(\mathsf{FM}^+) \vee \tau(\mathsf{V}^-)$.

It now follows from the remarks made above that the Wijsman topology is a hit-and-far-miss topology.

Theorem 4.2. $\tau(\mathsf{W}) = \tau(\mathsf{FM})$

Remark 4.3. It is obvious that the Wijsman topology can be generalized to a T_1 space with a compatible proximity δ and a cobase Δ which is a subfamily of $CL(X)$ closed under finite unions. Thus we have a Δ hit-and-far-miss topology defined exactly as in 4.1 with Δ replacing B.

It is a trivial matter to note that the Wijsman topology is coarser than both the ball topology and the proximal topology. (cf. [1, Page 45])

Comparing Wijsman, ball and proximal ball topologies among themselves or with one another is now quite easy and we just give a sample below.

We note that a set E is said to be *weakly totally bounded* or *w-TB* (cf. *strictly d-included*) iff for every $\varepsilon > 0$, there exists a B in B such that $E << B << S(E, \varepsilon)$. (cf. [1, 4])

Theorem 4.4 (cf. [1] Page 45)**.** *The proximal ball topology equals the Wijsman topology if and only if every* $B \in \mathsf{B}$ *is* $w - TB$.

Proof. Suppose $A_0 \neq X$, is a non-empty closed subset of X and $A_0 \in U^{++}$, where $U^c = B \in \mathsf{B}$. The upper proximal ball topology is coarser than the upper Wijsman topology iff there is a V with $V^c = B' \in \mathsf{B}$ with $A_0 \in V^{++}$ and $V^+ \subset U^{++}$. Rewriting the above in terms of the cobase B and the proximity we get the result. □

5. Bounded hypertopologies

Let (X, d) be a metric space. Then using *bounded* sets in the definitions of some upper, some lower, and some both hypertopologies one can get analogues of those defined in Section 2. Thus there are the following: the bounded Vietoris topology, the bounded Hausdorff metric of Attouch-Wets or AW topology, the bounded proximal topology, etc. (see [1]). Many workers in the area do not seem to be aware that *bounded sets* can be defined in any topological space ([8],[15, Page 53]). A *family of bounded subsets* of a topological space is merely a family of non-empty subsets that are closed under finite unions and hereditary. In the present situation we restrict the concept to *closed* subsets. Thus any cobase Δ, which is closed hereditary, serves as a family of bounded sets. So all the results in a metric space are special cases of their analogues in (proximal) Δ-hypertopologies when the cobase consists of all closed metrically bounded subsets. For this reason, we give a short account here about this topic.

(5.1) **The bounded proximal topology:** This is just the **proximal (finite) Δ-topology** (w.r.t. δ) $\sigma(\delta\Delta) = \sigma(\delta\Delta^+) \vee \tau(\mathsf{V}^-)$ where Δ is closed hereditary.

(5.2) **The bounded Vietoris topology:** This is merely the **Δ-topology** $\tau(\Delta) = \tau(\Delta^+) \vee \tau(\mathsf{V}^-)$ where Δ is closed hereditary.

(5.3) **The bounded H-B topology:** Suppose $(X, \mathbb{U})$ is a Hausdorff uniform space, the uniformity $\mathbb{U}$ induces the EF-proximity δ which, in turn, induces the topology T on X. Let Δ be a cobase which is closed hereditary. Let $\mathbb{LB}$ be the collection of all families of open sets of the form $\{U(x) : x \in Q \subset A\}$, where $A \in \Delta$, $U \in \mathbb{U}$ and Q is U-discrete. Then **the bounded H-B topology** on $CL(X)$ is $\tau(\mathbb{LB}\delta) = \tau(\mathbb{LB}\delta^+) \vee \tau(\mathbb{LB}^-) = \sigma(\delta\Delta^+) \vee \tau(\mathbb{LB}^-)$.

If X is a metrisable space with a compatible d, $\mathbb{U}$ is the metric uniformity generated by d, δ is the metric proximity and Δ is the family of all non-empty closed metrically bounded subsets of (X, d), then $\tau(\mathbb{LB}\delta)$ is precisely the **AW-topology** $\tau(AWd)$ on $CL(X)$.

Using these representations it is now an easy task to get results comparing them among themselves and with others. We give a couple of samples below.

Remark 5.1.

(a) It is easy to see that $\sigma(\delta\Delta) \subset \tau(\mathbb{LB}\delta)$ and so $\sigma(\delta\Delta) = \tau(\mathbb{LB}\delta)$ if and only if closed and bounded sets are totally bounded. (cf. [1, Theorem 3.1.4]).

(b) $\sigma(\delta\Delta) = \tau(\mathbb{LB}\delta^+) \vee \tau(\mathsf{V}^-)$. (cf. [1, Theorem 4.2.1]).

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