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Some properties of the containing spaces and saturated classes of spaces

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. Subjects of this paper are: (a) containing spaces constructed in [2] for an indexed collection $\mathbf{S}$ of subsets, (b) classes consisting of ordered pairs (Q, X), where Q is a subset of a space X, which are called classes of subsets, and (c) the notion of universality in such classes.

We show that if T is a containing space constructed for an indexed collection $\mathbf{S}$ of spaces and for every $X \in \mathbf{S}$, Q^X is a subset of X, then the corresponding containing space $\mathrm{T}|_{\mathbf{Q}}$ constructed for the indexed collection $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ of spaces, under a simple condition, can be considered as a specific subset of T. We prove some "commutative" properties of these specific subsets.

For classes of subsets we introduce the notion of a (properly) universal element and define the notion of a (complete) saturated class of subsets. Such a class is "saturated" by (properly) universal elements. We prove that the intersection of (complete) saturated classes of subsets is also a (complete) saturated class.

We consider the following classes of subsets: (a) $\mathbb{P}(\mathrm{Cl})$, (b) $\mathbb{P}(\mathrm{Op})$, and (c) $\mathbb{P}(\mathrm{n.dense})$ consisting of all pairs (Q, X) such that: (a) Q is a closed subset of X, (b) Q is an open subset of X, and (c) Q is a never dense subset of X, respectively. We prove that the classes $\mathbb{P}(\mathrm{Cl})$ and $\mathbb{P}(\mathrm{Op})$ are complete saturated and the class $\mathbb{P}(\mathrm{n.dense})$ is saturated.

Saturated classes of subsets are convenient to use for the construction of new saturated classes by the given ones.

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1. Introduction.

Agreement concerning notations. We denote by τ a fixed infinite cardinal. The set of all finite subsets of τ is denoted by $\mathcal{F}$. By a space we mean a T_0-space of weight less than or equal to τ.

In this paper we use all notions and notation introduced in [2]. In particular, if an indexed collection of spaces is denoted by the letter "$\mathbf{S}$", a co-mark of $\mathbf{S}$ is denoted by the letter "$\mathbf{M}$", and an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$ is denoted by the letter "R", then we always denote by "T" the containing space $\mathrm{T}(\mathbf{M}, \mathrm{R})$ and by "B^{T}" the standard base for T. If moreover

$$\mathbf{M} = \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\},$$

then the elements of B^{T} are denoted by $U_\delta^{\mathrm{T}}(\mathbf{H})$, $\delta \in \tau$ and $\mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$.

As in [2] we shall be concerned with classes, sets, collections, and families. A class is not necessarily a set. A collection and a family are sets. Any equivalence relation on a set $\mathbf{S}$ (which is considered as a subset of $\mathbf{S} \times \mathbf{S}$ satisfying the well-known conditions) is denoted by the symbol $\sim$ supplied usually with one or more indices. Any ordinal α is identified with the set of all ordinals less than α. For every set X we denote by $\mathcal{P}(X)$ the set of all subsets of X.

For every subset Q of a space X we denote by $\mathrm{Cl}_X(Q)$, $\mathrm{Int}_X(Q)$, and $\mathrm{Bd}_X(Q)$ the closure, the interior, and the boundary of Q in X, respectively.

We shall use the symbol "$\equiv$" in order to introduce new notations without mention to this fact. This will be done as follows. When we introduce an expression A as a notation of an object (a set, an indexed set, a mapping and so on) writing $A \equiv B$ (or $B \equiv A$), where B is another new expression, or when we consider a known object with a known expression A as its notation writing $A \equiv B$ (or $B \equiv A$), where B is a new expression, in both cases we mean that B is considered as another notation of the same object.

In the second section we construct some specific subsets of the containing spaces given in [2]. Suppose that for every space X of an indexed collection $\mathbf{S}$ of spaces a subset Q^X of X is given. Then, the indexed collection $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ is called a restriction of $\mathbf{S}$. Such a restriction can be also treat as an indexed collection of spaces. Any co-mark $\mathbf{M}$ of $\mathbf{S}$ defines by a natural manner a co-mark $\mathbf{M}|_{\mathbf{Q}}$ of $\mathbf{Q}$ and any $\mathbf{M}$-admissible family R of equivalence relations on $\mathbf{S}$ defines an admissible family $\mathrm{R}|_{\mathbf{Q}}$ of equivalence relations on $\mathbf{Q}$. For "almost all" co-marks $\mathbf{M}$ and families R, $\mathrm{R}|_{\mathbf{Q}}$ is $\mathbf{M}|_{\mathbf{Q}}$-admissible. In this case, in parallels with the containing space T, we can also consider the containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}}) \equiv \mathrm{T}|_{\mathbf{Q}}$ for $\mathbf{Q}$ corresponding to the co-mark $\mathbf{M}|_{\mathbf{Q}}$ and the family $\mathrm{R}|_{\mathbf{Q}}$. We show that there exists a natural embedding of $\mathrm{T}|_{\mathbf{Q}}$ into T, which gives us the possibility to consider $\mathrm{T}|_{\mathbf{Q}}$ as a subset of T.

In the third section some "commutative" properties of subsets $\mathrm{T}|_{\mathbf{Q}}$ are given. A restriction $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ is called closed (open) if Q^X is closed (open) in X. Also, the notion of a complete restriction of $\mathbf{S}$ is given. We show that closed and open restrictions are complete. The following restrictions are also considered:

$$\mathbf{Cl}(\mathbf{Q}) \equiv \{\mathrm{Cl}_X(Q^X) : X \in \mathbf{S}\},$$

$$\mathbf{Int}(\mathbf{Q}) \equiv \{\mathrm{Int}_X(Q^X) : X \in \mathbf{S}\},$$

$$\mathbf{Bd}(\mathbf{Q}) \equiv \{\mathrm{Bd}_X(Q^X) : X \in \mathbf{S}\}, \text{ and}$$

$$\mathbf{Co}(\mathbf{Q}) \equiv \{X \setminus Q^X : X \in \mathbf{S}\}.$$

For “almost all” co-marks $\mathbf{M}$ and families R of equivalence relations on $\mathbf{S}$ the following “commutation” relations are proved:

$$\mathrm{T}|_{\mathbf{Cl}(\mathbf{Q})} = \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}),$$

$$\mathrm{T}|_{\mathbf{Int}(\mathbf{Q})} = \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}),$$

$$\mathrm{T}|_{\mathbf{Bd}(\mathbf{Q})} = \mathrm{Bd}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}), \text{ and}$$

$$\mathrm{T}|_{\mathbf{Co}(\mathbf{Q})} = \mathrm{T} \setminus \mathrm{T}|_{\mathbf{Q}}.$$

The first relation is true for any restriction $\mathbf{Q}$ and the others for complete restrictions.

Classes consisting of ordered pairs (Q, X), where Q is a subset of a space X, and called classes of subsets, are considered in the fourth section. An element (Q^T, T) of a class $\mathbb{P}$ of subsets is called universal (respectively, properly universal) if for every element (Q^X, X) of $\mathbb{P}$ there exists a homeomorphism h of X into T such that $h(Q^X) \subset Q^T$ (respectively, $h^{-1}(Q^T) = Q^X$).

Using the above considered properties of subsets $\mathrm{T}|_{\mathbf{Q}}$ of T we define the so-called (complete) saturated classes of subsets. This definition is similar to that of classes of spaces (see [2]). As for the classes of spaces the (complete) saturated classes of subsets not only have (properly) universal elements but they are “saturated” by such elements. We prove that the intersection of (complete) saturated classes is also a (complete) saturated class.

We also show that the classes of subsets

$$\mathbb{P}(\mathrm{Cl}) \equiv \{(Q, X) : Q \text{ is closed in } X\} \text{ and}$$

$$\mathbb{P}(\mathrm{Op}) \equiv \{(Q, X) : Q \text{ is open in } X\}$$

are complete saturated classes and the class

$$\mathbb{P}(\text{n.dense}) \equiv \{(Q, X) : Q \text{ is never dense in } X\}$$

is saturated. For classes $\mathbb{P}(\mathrm{Cl})$ and $\mathbb{P}(\mathrm{Op})$ this follows by the corresponding “commutation” relations.

In the fifth section we introduce the notion of a “commutative operator”. The closure, the interior, and the boundary operators are such operators. Using these operators we construct new (complete) saturated classes of subsets by the given ones.

Finally, in the last section we pose some problems.

2. Specific subspaces of the space $\mathrm{T}(\mathbf{M},\mathrm{R})$.

Definition 2.1. Let $\{V_\delta^X : \delta \in \tau\}$ be a mark of a space X. Then, for every subspace Q of X the indexed set

$$\{V_\delta^Q \equiv Q \cap V_\delta^X : \delta \in \tau\}$$

is a mark of the space Q. This mark is called the *trace on Q of the mark* $\{V_\delta^X : \delta \in \tau\}$ of X.

Lemma 2.2. *Let $\{V_\delta^X : \delta \in \tau\}$ be the mark of a marked space X, Q a subspace of X, and $\{V_\delta^Q : \delta \in \tau\}$ the trace on Q of the mark $\{V_\delta^X : \delta \in \tau\}$. Then,*

$$d_s^X(x) = d_s^Q(x)$$

for every $x \in Q$ and $s \in \mathcal{F} \setminus \{\varnothing\}$. Therefore, $d_s^X(Q) = d_s^Q(Q)$. (We note that the mapping d_s^X is constructed with respect to the mark $\{V_\delta^X : \delta \in \tau\}$ of X and the mapping d_s^Q is constructed with respect to the mark $\{V_\delta^Q : \delta \in \tau\}$ of Q).

Proof. Let $s \in \mathcal{F} \setminus \{\varnothing\}$, $x \in Q$ and $d_s^X(x) = f \in 2^s$. We must prove that $d_s^Q(x) = f$. By the definition of the mapping d_s^X we have

$$x \in X_{(s,f)} = \cap\{X_{(\delta,f(\delta))} : \delta \in s\}.$$

Therefore,

$$x \in Q \cap (\cap\{X_{(\delta,f(\delta))} : \delta \in s\}) =$$

$$\cap\{Q \cap X_{(\delta,f(\delta))} : \delta \in s\} = \cap\{Q_{(\delta,f(\delta))} : \delta \in s\} = Q_{(s,f)}.$$

This means that $d_s^Q(x) = f$. □

Definition 2.3. Suppose that for every element X of an indexed collection $\mathbf{S}$ of spaces a subspace Q^X of X is given. Then, the indexed collection

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$$

is called a *restriction* of $\mathbf{S}$. The element Q^X of $\mathbf{Q}$ will be also denoted by $\mathbf{Q}(X)$.

Definition 2.4. Let $\mathbf{S}$ be an indexed collection of spaces, $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ a restriction of $\mathbf{S}$, $\mathbf{M}$ a co-mark of $\mathbf{S}$, and let $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an indexed family of equivalence relations on $\mathbf{S}$.

The *trace on* $\mathbf{Q}$ *of the co-mark* $\mathbf{M}$ of $\mathbf{S}$ is the co-mark of $\mathbf{Q}$ denoted by $\mathbf{M}|_{\mathbf{Q}}$ and defined as follows: the mark $(\mathbf{M}|_{\mathbf{Q}})(Q^X)$ of an element Q^X of $\mathbf{Q}$ is the trace on Q^X of the mark $\mathbf{M}(X)$ of X.

The *trace on* $\mathbf{Q}$ *of an equivalence relation* $\sim$ on $\mathbf{S}$ is the equivalence relation on $\mathbf{Q}$ denoted by $\sim|_{\mathbf{Q}}$ and defined as follows: two elements Q^X and Q^Y of $\mathbf{Q}$ are $\sim|_{\mathbf{Q}}$-*equivalent* if and only if $X \sim Y$.

The indexed family

$$\mathrm{R}|_{\mathbf{Q}} \equiv \{\sim^s|_{\mathbf{Q}} : s \in \mathcal{F}\}$$

of equivalence relations on $\mathbf{Q}$ is called the *trace on* $\mathbf{Q}$ *of the indexed family* R.

The *trace on* $\mathbf{Q}$ *of an element* $\mathbf{H}$ of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$, denoted by $\mathbf{H}|_{\mathbf{Q}}$, is the set of all elements Q^X of $\mathbf{Q}$ for which $X \in \mathbf{H}$. It is easy to see that $\mathbf{H}|_{\mathbf{Q}}$ is an element of $\mathrm{C}^{\diamondsuit}(\mathrm{R}|_{\mathbf{Q}})$. Obviously, if $\mathbf{H} \in \mathrm{C}(\mathrm{R})$, then $\mathbf{H}|_{\mathbf{Q}} \in \mathrm{C}(\mathrm{R}|_{\mathbf{Q}})$.

By definition, it follows that if $\sim_0$ and $\sim_1$ are two equivalence relations on $\mathbf{S}$ and $\sim_1$ is contained in $\sim_0$, then $\sim_1|_{\mathbf{Q}}$ is contained in $\sim_0|_{\mathbf{Q}}$. Therefore, if R_0 and R_1 are two indexed families of equivalence relations on $\mathbf{S}$ and R_1 is a final refinement of R_0, then $\mathrm{R}_1|_{\mathbf{Q}}$ is a final refinement of $\mathrm{R}_0|_{\mathbf{Q}}$.

Agreement 1. In what follows in this section it is supposed that an indexed collection of spaces denoted by $\mathbf{S}$, a restriction of $\mathbf{S}$ denoted by

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\},$$

a co-mark of $\mathbf{S}$ denoted by

$$\mathbf{M} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\},$$

and an $\mathbf{M}$-admissible family of equivalence relations on $\mathbf{S}$ denoted by

$$\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$$

are fixed.

We note that, in general, the trace on $\mathbf{Q}$ of the $\mathbf{M}$-standard family of equivalence relations on $\mathbf{S}$ is not the $\mathbf{M}|_{\mathbf{Q}}$-standard family of equivalence relations on $\mathbf{Q}$. This justifies the following definition.

Definition 2.5. The $\mathbf{M}$-admissible family R of equivalence relations on $\mathbf{S}$ is said to be $(\mathbf{M}, \mathbf{Q})$-*admissible* if $\mathrm{R}|_{\mathbf{Q}}$ is an $\mathbf{M}|_{\mathbf{Q}}$-admissible family of equivalence relations on $\mathbf{Q}$.

Lemma 2.6. *The family* R *is* $(\mathbf{M}, \mathbf{Q})$*-admissible if and only if for every element* s *of* $\mathcal{F} \setminus \{\varnothing\}$ *there exists an element* t *of* $\mathcal{F} \setminus \{\varnothing\}$ *such that relation* $X \sim^t Y$ *implies relation* $d^X_s(Q^X) = d^Y_s(Q^Y)$ *for every* $X, Y \in \mathbf{S}$.

Proof. Suppose that the condition of the lemma is satisfied. We must prove that the family R is $(\mathbf{M}, \mathbf{Q})$-admissible. Since R is $\mathbf{M}$-admissible it suffices to prove that the family $\mathrm{R}|_{\mathbf{Q}} = \{\sim^s|_{\mathbf{Q}} : s \in \mathcal{F}\}$ is a final refinement of the $\mathbf{M}|_{\mathbf{Q}}$-standard family of equivalence relations on $\mathbf{Q}$. Denote the last family by $\mathrm{R}_{\mathbf{Q}} \equiv \{\sim^s_{\mathbf{Q}}: s \in \mathcal{F}\}$.

Let s and t be elements of $\mathcal{F} \setminus \{\varnothing\}$ satisfying the condition of the lemma. We need to prove that the trace on $\mathbf{Q}$ of the equivalence relation $\sim^t$ on $\mathbf{S}$, that is, the equivalence relation $\sim^t|_{\mathbf{Q}}$ on $\mathbf{Q}$ is contained in the equivalence relation $\sim^s_{\mathbf{Q}}$ on $\mathbf{Q}$. Let $Q^X, Q^Y \in \mathbf{Q}$ and $Q^X \sim^t|_{\mathbf{Q}} Q^Y$. This means that $X \sim^t Y$. By the condition of the lemma, the last relation implies that $d^X_s(Q^X) = d^Y_s(Q^Y)$. By Lemma 2.2, $d^{Q^X}_s(Q^X) = d^{Q^Y}_s(Q^Y)$ and by Lemma 1.4 of [2], $Q^X \sim^s_{\mathbf{Q}} Q^Y$. Thus, the equivalence relation $\sim^t|_{\mathbf{Q}}$ is contained in the equivalence relation $\sim^s_{\mathbf{Q}}$.

Conversely, suppose that the family R is $(\mathbf{M}, \mathbf{Q})$-admissible. Let $s \in \mathcal{F}\setminus\{\varnothing\}$. Since the family $\mathrm{R}|_{\mathbf{Q}}$ is $\mathbf{M}|_{\mathbf{Q}}$-admissible there exists an element t of $\mathcal{F} \setminus \{\varnothing\}$ such that the equivalence relation $\sim^t|_{\mathbf{Q}}$ on $\mathbf{Q}$ is contained in the equivalence relation $\sim^s_{\mathbf{Q}}$ on $\mathbf{Q}$.

Let $X, Y \in \mathbf{S}$ and $X \sim^t Y$, that is, $Q^X \sim^t|_{\mathbf{Q}} Q^Y$. Then, $Q^X \sim^s_{\mathbf{Q}} Q^Y$. By Lemma 1.4 of [2], $d^{Q^X}_s(Q^X) = d^{Q^Y}_s(Q^Y)$ and by Lemma 2.2, $d^X_s(Q^X) = d^Y_s(Q^Y)$. □

Remark 2.7. Lemma 2.6 implies the existence of $(\mathbf{M}, \mathbf{Q})$-admissible families of equivalence relations on $\mathbf{S}$. For example, such a family is the admissible family $\mathrm{R}_0 \equiv \{\sim^s_0: s \in \mathcal{F}\}$ for which $X \sim^s_0 Y$ if and only if $X \sim^s_{\mathbf{M}} Y$ and $d^X_s(Q^X) = d^Y_s(Q^Y)$. (This family is admissible because the set $\mathcal{P}(2^s_X) = \mathcal{P}(2^s_Y)$ is finite).

Notation. Suppose that R is an $(\mathbf{M}, \mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$. Then, besides of the space T we can also consider the containing space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ for the indexed collection $\mathbf{Q}$ corresponding to the co-mark

$$\mathbf{M}|_{\mathbf{Q}} = \{\{U^{Q^X}_\delta = Q \cap U^X_\delta : \delta \in \tau\} : Q^X \in \mathbf{Q}\}$$

and the $\mathbf{M}|_{\mathbf{Q}}$-admissible family $\mathrm{R}|_{\mathbf{Q}}$ of equivalence relations on $\mathbf{Q}$. This containing space is also denoted by $\mathrm{T}|_{\mathbf{Q}}$.

If $\mathbf{H}$ is an element of $\mathrm{C}^\diamondsuit(\mathrm{R})$, then we denote by $\mathrm{T}(\mathbf{H}|_{\mathbf{Q}})$ the subset of T consisting of all points $\mathbf{a}$ for which there exists an element (x, X) of $\mathbf{a}$ such that $X \in \mathbf{H}$ and $x \in Q^X$. It is easy to verify that

$$\mathrm{T}(\mathbf{H}|_{\mathbf{Q}}) = \mathrm{T}|_{\mathbf{Q}} \cap \mathrm{T}(\mathbf{H}).$$

For every $\delta \in \tau$ and $\mathbf{E} \in \mathrm{C}^\diamondsuit(\mathrm{R}|_{\mathbf{Q}})$ the set of all elements $\mathbf{b}$ of $\mathrm{T}|_{\mathbf{Q}}$ for which there exists an element (x, Q^X) of $\mathbf{b}$ such that $x \in U^{Q^X}_\delta$ and $Q^X \in \mathbf{E}$ is denoted by $U^{\mathrm{T}|_{\mathbf{Q}}}_\delta(\mathbf{E})$. Also, we set

$$\mathrm{B}^{\mathrm{T}|_{\mathbf{Q}}} = \{U_\delta^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{E}) : \delta \in \tau,\, \mathbf{E} \in \mathrm{C}^{\diamondsuit}(\mathrm{R}|_{\mathbf{Q}})\}.$$

Lemma 2.8. *For every* $\mathbf{b} \in \mathrm{T}|_{\mathbf{Q}}$ *there exists a unique element* $\mathbf{a}$ *of* T *such that for every* $x \in Q^X$ *the pair* (x, Q^X) *belongs to* $\mathbf{b}$ *if and only if the pair* (x, X) *belongs to* $\mathbf{a}$.

Proof. Let $\mathbf{b} \in \mathrm{T}|_{\mathbf{Q}}$. Consider an element (y, Q^Y) of $\mathbf{b}$ and denote by $\mathbf{a}$ the element of T containing the pair (y, Y).

Now, let $(x, Q^X) \in \mathbf{b}$. Then,

$$X \sim^s|_{\mathbf{Q}} Y \text{ and } d_s^{Q^X}(x) = d_s^{Q^Y}(y) \tag{2.1}$$

for every $s \in \mathcal{F} \setminus \{\varnothing\}$. By Lemma 2.2,

$$d_s^X(x) = d_s^{Q^X}(x) \text{ and } d_s^Y(y) = d_s^{Q^Y}(y). \tag{2.2}$$

Therefore,

$$X \sim^s Y \text{ and } d_s^X(x) = d_s^Y(y) \tag{2.3}$$

for every $s \in \mathcal{F} \setminus \{\varnothing\}$. This means that the pairs (x, X) and (y, Y) belong to the same element of the set T. Thus, $(x, X) \in \mathbf{a}$.

Conversely, let $x \in Q^X$ and $(x, X) \in \mathbf{a}$. Then, relation (2.3) is true for every $s \in \mathcal{F} \setminus \{\varnothing\}$. Lemma 2.2 implies relation (2.2). For every $s \in \mathcal{F} \setminus \{0\}$ relations (2.2) and (2.3) imply relation (2.1), which means that the pairs (x, Q^X) and (y, Q^Y) belong to the same element of the set $\mathrm{T}|_{\mathbf{Q}}$. Thus, $(x, Q^X) \in \mathbf{b}$.

Obviously, the element $\mathbf{a} \in \mathrm{T}$ satisfying the condition of the lemma is uniquely determined. □

Definition 2.9. Let $\mathbf{b}$ be an arbitrary element of $\mathrm{T}|_{\mathbf{Q}}$ and $\mathbf{a}$ the unique element of T satisfying the condition of Lemma 2.8. We define a mapping $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ of $\mathrm{T}|_{\mathbf{Q}}$ into T setting

$$e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b}) = \mathbf{a}.$$

Below, we shall prove that this mapping is an embedding, which will be called the *natural embedding* of the space $\mathrm{T}|_{\mathbf{Q}}$ into the space T.

Lemma 2.10. *Let* $\delta \in \tau$, $\mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$ *and let* $\mathbf{L}$ *be the trace on* $\mathbf{Q}$ *of* $\mathbf{H}$. *Then, an element* $\mathbf{b}$ *of* $\mathrm{T}|_{\mathbf{Q}}$ *belongs to* $U_\delta^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{L})$ *if and only if the element* $\mathbf{a} \equiv e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b})$ *belongs to* $U_\delta^{\mathrm{T}}(\mathbf{H})$.

Proof. For every $X \in \mathbf{S}$ the indexed set

$$\{U_\delta^{Q^X} = Q^X \cap U_\delta^X : \delta \in \tau\}$$

is the trace on Q^X of the mark $\mathbf{M}(X)$. Let $\mathbf{b} \in \mathrm{T}|_{\mathbf{Q}}$ and $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b}) = \mathbf{a} \in \mathrm{T}$. Suppose that $\mathbf{b} \in U_{\delta}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{L})$ and let $(x, Q^X) \in \mathbf{b}$. By Lemma 2.8, $(x, X) \in \mathbf{a}$. We have

$$Q^X \in \mathbf{L} \text{ and } x \in U_{\delta}^{Q^X} = Q^X \cap U_{\delta}^X$$

and, therefore,

$$X \in \mathbf{H} \text{ and } x \in U_{\delta}^X,$$

which means that $\mathbf{a} \in U_{\delta}^{\mathrm{T}}(\mathbf{H})$.

Similarly we prove that if $\mathbf{a} \in U_{\delta}^{\mathrm{T}}(\mathbf{H})$, then $\mathbf{b} \in U_{\delta}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{L})$. □

Proposition 2.11. *The mapping* $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ *is an embedding of* $\mathrm{T}|_{\mathbf{Q}}$ *into* T.

Proof. First, we prove that the mapping $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ is one-to-one. Indeed, let $\mathbf{b}_1$ and $\mathbf{b}_2$ be two distinct elements of the space $\mathrm{T}|_{\mathbf{Q}}$. Since $\mathrm{T}|_{\mathbf{Q}}$ is a T_0-space (see Proposition 2.9 of [2]) there exists $\delta \in \tau$ and $\mathbf{L} \in \mathrm{C}^{\diamondsuit}(\mathrm{R}|_{\mathbf{Q}})$ such that one of the points $\mathbf{b}_1$ and $\mathbf{b}_2$ belongs to the open set $U_{\delta}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{L})$ and the other does not belong. Let $\mathbf{H} \in \mathrm{C}^{\diamondsuit}(\mathrm{R})$ such that $\mathbf{L}$ is the trace on $\mathbf{Q}$ of $\mathbf{H}$. By Lemma 2.10 one of the points $\mathbf{a}_1 \equiv e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b}_1)$ and $\mathbf{a}_2 \equiv e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b}_2)$ belongs to the set $U_{\delta}^{\mathrm{T}}(\mathbf{H})$ and the other does not belong. This means that $\mathbf{a}_1 \neq \mathbf{a}_2$, that is, $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ is one-to-one.

The continuity of the mappings $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ and $(e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}})^{-1}$ follows by Lemma 2.10 and by the fact that every element of the base $\mathrm{B}^{\mathrm{T}|_{\mathbf{Q}}}$ of the space $\mathrm{T}|_{\mathbf{Q}}$ has the form $U_{\delta}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{L})$ and every element of the base B^{T} of the space T has the form $U_{\delta}^{\mathrm{T}}(\mathbf{H})$. Thus, the mapping $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ is an embedding. □

Agreement 2. In what follows in this paper we identify a point $\mathbf{b}$ of $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ with the point $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b}) \equiv \mathbf{a}$ of $\mathrm{T}(\mathbf{M}, \mathrm{R})$ and consider the space $\mathrm{T}(\mathbf{M}|_{\mathbf{Q}}, \mathrm{R}|_{\mathbf{Q}})$ as a subspace of the space $\mathrm{T}(\mathbf{M}, \mathrm{R})$.

The definitions of the natural embeddings e_{T}^X and $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ imply the following consequence.

Corollary 2.12. *The following relations is true:*

$$\mathrm{T}|_{\mathbf{Q}} = \cup\{e_{\mathrm{T}}^X(Q^X) : X \in \mathbf{S}\}.$$

Proposition 2.13. *Let* $X \in \mathbf{S}$ *and let* $e_X^{Q^X}$ *be the identical embedding of* Q^X *into* X. *Then,*

$$e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}} \circ e_{\mathrm{T}|_{\mathbf{Q}}}^{Q^X} = e_{\mathrm{T}}^X \circ e_X^{Q^X}. \tag{2.4}$$

Proof. Let $x \in Q^X$. Let also $(e_{\mathrm{T}}^X \circ e_X^{Q^X})(x) = e_{\mathrm{T}}^X(x) = \mathbf{a}$. Then, $\mathbf{a}$ is the point of T containing the pair (x, X). On the other hand, by the definition of the mapping $e_{\mathrm{T}|_{\mathbf{Q}}}^{Q^X}$, $e_{\mathrm{T}|_{\mathbf{Q}}}^{Q^X}(x) \equiv \mathbf{b}$ is the point of $\mathrm{T}|_{\mathbf{Q}}$ containing the pair (x, Q^X). By the construction of the embedding $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}$ we have $e_{\mathrm{T}}^{\mathrm{T}|_{\mathbf{Q}}}(\mathbf{b}) = \mathbf{a}$. This proves relation (2.4). □

3. Commutative properties of the subspaces $\mathrm{T}|_{\mathbf{Q}}$.

Agreement. In the present section it is supposed that $\mathbf{S}$, $\mathbf{M}$, $\mathbf{Q}$, and R are the same as in the preceding section and they are fixed.

Definition 3.1. The restriction $\mathbf{Q}$ of $\mathbf{S}$ is said to be *closed* (respectively, *open*) if for every $X \in \mathbf{S}$, Q^X is a closed (respectively, an open) subset of X.

The restriction $\mathbf{Q}$ is said to be an $(\mathbf{M}, \mathrm{R})$-*complete restriction* if for every point $\mathbf{a} \in \mathrm{T}|_{\mathbf{Q}}$ and for every element (x, X) of $\mathbf{a}$ we have $x \in Q^X$.

Notation. Below besides of the restriction $\mathbf{Q}$ we also consider the following restrictions connected with $\mathbf{Q}$:

$$\mathbf{Cl}(\mathbf{Q}) \equiv \{\mathrm{Cl}_X(Q^X) : X \in \mathbf{S}\},$$

$$\mathbf{Bd}(\mathbf{Q}) \equiv \{\mathrm{Bd}_X(Q^X) : X \in \mathbf{S}\},$$

$$\mathbf{Int}(\mathbf{Q}) \equiv \{\mathrm{Int}_X(Q^X) : X \in \mathbf{S}\}, \text{ and}$$

$$\mathbf{Co}(\mathbf{Q}) \equiv \{X \setminus Q^X : X \in \mathbf{S}\}.$$

Lemma 3.2. *Suppose that* $\mathbf{Q}$ *is a closed restriction of* $\mathbf{S}$ *and* R *is an* $(\mathbf{M}, \mathbf{Q})$-*admissible family of equivalence relations on* $\mathbf{S}$*. Then, the following statements are true:*

(1) *The subset* $\mathrm{T}|_{\mathbf{Q}}$ *of* T *is closed.*
(2) *If, moreover,* R *is* $(\mathbf{M}, \mathbf{Co}(\mathbf{Q}))$-*admissible, then*

$$\mathrm{T}|_{\mathbf{Co}(\mathbf{Q})} = \mathrm{T} \setminus \mathrm{T}|_{\mathbf{Q}}. \tag{3.1}$$

(3) *The restriction* $\mathbf{Q}$ *is an* $(\mathbf{M}, \mathrm{R})$-*complete restriction.*

Proof. Let $\mathbf{a}$ be a point of T for which there exists an element (x, X) of $\mathbf{a}$ such that $x \notin Q^X$. Since Q^X is closed in X there exists $\delta \in \tau$ such that $x \in U_\delta^X$ and $U_\delta^X \cap Q^X = \varnothing$. Let $s = \{\delta\}$. Since R is $(\mathbf{M}, \mathbf{Q})$-admissible by Lemma 2.6 there exists an element t of $\mathcal{F}$ such that $\sim^t \subset \sim_{\mathbf{M}}^s$ and $d_s^X(Q^X) = d_s^Y(Q^Y)$ for every $Y \in \mathbf{S}$ for which $X \sim^t Y$.

Let $\mathbf{H}$ be the $\sim^t$-equivalence class of X. Then, $U_\delta^{\mathrm{T}}(\mathbf{H})$ is an open neighbourhood of $\mathbf{a}$ in T. We prove that

$$U_\delta^{\mathrm{T}}(\mathbf{H}) \cap \mathrm{T}|_{\mathbf{Q}} = \varnothing. \tag{3.2}$$

Indeed, in the opposite case, there exists a point $\mathbf{b}$ belonging to the set $U_\delta^{\mathrm{T}}(\mathbf{H}) \cap \mathrm{T}|_{\mathbf{Q}}$. Let (y, Q^Y) be an element of $\mathbf{b} \in \mathrm{T}|_{\mathbf{Q}}$. By Lemma 2.8, $\mathbf{b}$ as a point of the space T contains the pair (y, Y). Since $\mathbf{b} \in U_\delta^T(\mathbf{H})$ we have $Y \in \mathbf{H}$ and $y \in U_\delta^Y$. Therefore, $X \sim^t Y$. By the choice of t, $X \sim_{\mathbf{M}}^s Y$ and $d_s^X(Q^X) = d_s^Y(Q^Y)$. Since $y \in Q^Y$ there exists a point $z \in Q^X$ such that $d_s^X(z) = d_s^Y(y)$. This equality and the relation $y \in U_\delta^Y$ imply that $z \in U_\delta^X$. Therefore, $Q^X \cap U_\delta^X \neq \varnothing$, which contradicts to the choice of δ. Thus, the relation (3.2) is proved.

Now we prove the statements of the lemma.

(1). By Corollary 2.12 as the above point $\mathbf{a}$ we can take any point of the set $\mathrm{T} \setminus \mathrm{T}|_{\mathbf{Q}}$. In this case, relation (3.2) implies that the set $\mathrm{T}|_{\mathbf{Q}}$ is closed.

(2). As the point $\mathbf{a}$ we can take any point of the set $\mathrm{T}|_{\mathbf{Co(Q)}}$. In this case, relation (3.2) implies that $\mathbf{a} \notin \mathrm{T}|_{\mathbf{Q}}$, that is, $\mathrm{T}|_{\mathbf{Q}} \cap \mathrm{T}|_{\mathbf{Co(Q)}} = \varnothing$. The Corollary 2.12 implies that $\mathrm{T}|_{\mathbf{Q}} \cup \mathrm{T}|_{\mathbf{Co(Q)}} = \mathrm{T}$. The last two relations are equivalent to the relation (3.1).

(3). If $\mathbf{Q}$ is not an $(\mathbf{M}, \mathrm{R})$-complete restriction, then there exists a point $\mathbf{a}$ of $\mathrm{T}|_{\mathbf{Q}}$ and an element (x, X) of $\mathbf{a}$ such that $x \notin Q^X$. Then, relation (3.2) is true for some open neighbourhood $U_\delta^{\mathrm{T}}(\mathbf{H})$ of $\mathbf{a}$, which is a contradiction. Thus, $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction. □

Lemma 3.3. *Suppose that the restriction* $\mathbf{Q}$ *of* $\mathbf{S}$ *is open and the family* R *of equivalence relations on* $\mathbf{S}$ *is* $(\mathbf{M}, \mathbf{Q})$*-admissible and* $(\mathbf{M}, \mathbf{Co(Q)})$*-admissible. Then:*

(1) *The following relation is true:*

$$\mathrm{T}|_{\mathbf{Co(Q)}} = \mathrm{T} \setminus \mathrm{T}|_{\mathbf{Q}}. \tag{3.3}$$

(2) *The subset* $\mathrm{T}|_{\mathbf{Q}}$ *of* T *is open.*

(3) *The restriction* $\mathbf{Q}$ *is an* $(\mathbf{M}, \mathrm{R})$*-complete restriction.*

Proof. (1). Relation (3.3) follows immediately by the statement (2) of Lemma 3.2 if instead of the restriction $\mathbf{Q}$ of this lemma we consider the closed restriction $\mathbf{Co(Q)}$. (Note that $\mathbf{Co(Co(Q))} = \mathbf{Q}$).

(2). Since the restriction $\mathbf{Co(Q)}$ is closed by Lemma 3.2 the subset $\mathrm{T}|_{\mathbf{Co(Q)}}$ of T is closed. Therefore, by relation (3.3) the subset $\mathrm{T}|_{\mathbf{Q}}$ of T is open.

(3). Let $\mathbf{a} \in \mathrm{T}|_{\mathbf{Q}}$ and $(x, X) \in \mathbf{a}$. If $x \notin Q^X$, then $x \in X \setminus Q^X$. By Corollary 2.12, $\mathbf{a} \in \mathrm{T}|_{\mathbf{Co(Q)}}$, which contradicts to the relation (3.3). □

Proposition 3.4. *Suppose that the family* R *is* $(\mathbf{M}, \mathbf{Q})$*-admissible and* $(\mathbf{M}, \mathbf{Cl(Q)})$*-admissible. Then,*

$$\mathrm{T}|_{\mathbf{Cl(Q)}} = \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}).$$

Proof. Since the restriction $\mathbf{Cl(Q)}$ is closed by Lemma 3.2 the subset $\mathrm{T}|_{\mathbf{Cl(Q)}}$ of T is closed. Therefore, since $\mathrm{T}|_{\mathbf{Q}} \subset \mathrm{T}|_{\mathbf{Cl(Q)}}$, it suffices to prove that $\mathrm{T}|_{\mathbf{Cl(Q)}} \subset \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$.

Let $\mathbf{a} \in \mathrm{T}|_{\mathbf{Cl(Q)}}$. Then, by Corollary 2.12 there exists an element (x, X) of $\mathbf{a}$ such that $x \in \mathrm{Cl}_X(Q^X)$. Suppose that $\mathbf{a} \notin \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$. Then, there exists a neighbourhood $U_\delta^{\mathrm{T}}(\mathbf{H}) \in \mathrm{B}^{\mathrm{T}}$ of $\mathbf{a}$ such that $U_\delta^{\mathrm{T}}(\mathbf{H}) \cap \mathrm{T}|_{\mathbf{Q}} = \varnothing$. Corollary 2.12 implies that $U_\delta^Y \cap Q^Y = \varnothing$ for every $Y \in \mathbf{H}$. Since $X \in \mathbf{H}$, $U_\delta^X \cap Q^X = \varnothing$. This means that $x \notin \mathrm{Cl}_X(Q^X)$, which is a contradiction. Thus, $\mathbf{a} \in \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$ and, therefore, $\mathrm{T}|_{\mathbf{Cl(Q)}} \subset \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$. □

Proposition 3.5. *Suppose that* $\mathbf{Q}$ *is* $(\mathbf{M}, \mathrm{R})$*-complete (in particular, by Lemmas 3.2 and 3.3,* $\mathbf{Q}$ *may be a closed or an open restriction of* $\mathbf{S}$*) and the family* R *of equivalence relations on* $\mathbf{S}$ *is* $(\mathbf{M}, \mathbf{Q})$*-admissible,* $(\mathbf{M}, \mathbf{Int}(\mathbf{Q}))$*-admissible, and* $(\mathbf{M}, \mathbf{Co}(\mathbf{Int}(\mathbf{Q})))$*-admissible. Then,*

$$\mathrm{T}|_{\mathbf{Int}(\mathbf{Q})} = \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}).$$

Proof. Obviously, the restriction $\mathbf{Int}(\mathbf{Q})$ is open. By assumptions of the lemma and Lemma 3.3, the subset $\mathrm{T}|_{\mathbf{Int}(\mathbf{Q})}$ is open in T. Therefore, since $\mathrm{T}|_{\mathbf{Int}(\mathbf{Q})} \subset \mathrm{T}|_{\mathbf{Q}}$, it suffices to prove that $\mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}) \subset \mathrm{T}|_{\mathbf{Int}(\mathbf{Q})}$.

Let $\mathbf{a} \in \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$. There exists an open neighbourhood $U_\delta^{\mathrm{T}}(\mathbf{H}) \in B^{\mathrm{T}}$ of $\mathbf{a}$ such that $U_\delta^{\mathrm{T}}(\mathbf{H}) \subset \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$. Let $(x, X) \in \mathbf{a}$. Then, $x \in U_\delta^X$ and $X \in \mathbf{H}$. We prove that $U_\delta^X \subset Q^X$. Indeed, in the opposite case, there exists a point y belonging to the set $U_\delta^X \setminus Q^X$. Let $\mathbf{b}$ be the point of T containing the pair (y, X). Then, $\mathbf{b} \in U_\delta^{\mathrm{T}}(\mathbf{H})$. Since $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction, $(y, X) \in \mathbf{b}$, and $y \notin Q^X$ we have $\mathbf{b} \notin \mathrm{T}|_{\mathbf{Q}}$, which contradicts of the fact that $U_\delta^{\mathrm{T}}(\mathbf{H}) \subset \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}) \subset \mathrm{T}|_{\mathbf{Q}}$. Thus, $U_\delta^X \subset Q^X$, which means that $x \in \mathrm{Int}_X(Q^X)$ and, therefore, $\mathbf{a} \in \mathrm{T}|_{\mathbf{Int}(\mathbf{Q})}$. □

Proposition 3.6. *Suppose that* $\mathbf{Q}$ *is* $(\mathbf{M}, \mathrm{R})$*-complete and* R *is:*

(a) $(\mathbf{M}, \mathbf{Q})$*-admissible,*
(b) $(\mathbf{M}, \mathbf{Cl}(\mathbf{Q}))$*-admissible,*
(c) $(\mathbf{M}, \mathbf{Int}(\mathbf{Q}))$*-admissible,*
(d) $(\mathbf{M}, \mathbf{Co}(\mathbf{Int}(\mathbf{Q})))$*-admissible, and*
(e) $(\mathbf{M}, \mathbf{Bd}(\mathbf{Q}))$*-admissible.*

Then,

$$\mathrm{T}|_{\mathbf{Bd}(\mathbf{Q})} = \mathrm{Bd}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}). \tag{3.4}$$

Proof. Obviously,

$$\mathrm{Bd}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}) = \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}) \setminus \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}).$$

By Proposition 3.4,

$$\mathrm{T}|_{\mathbf{Cl}(\mathbf{Q})} = \mathrm{Cl}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}})$$

and by Proposition 3.5,

$$\mathrm{T}|_{\mathbf{Int}(\mathbf{Q})} = \mathrm{Int}_{\mathrm{T}}(\mathrm{T}|_{\mathbf{Q}}).$$

Therefore, it suffices to prove that

$$\mathrm{T}|_{\mathbf{Bd}(\mathbf{Q})} = \mathrm{T}|_{\mathbf{Cl}(\mathbf{Q})} \setminus \mathrm{T}|_{\mathbf{Int}(\mathbf{Q})}.$$

Let $\mathbf{a} \in \mathrm{T}|_{\mathbf{Bd}(\mathbf{Q})}$. There exists an element (x, X) of $\mathbf{a}$ such that $x \in \mathrm{Bd}_X(Q^X)$. Then, $x \in \mathrm{Cl}_X(Q^X)$ and, therefore, $\mathbf{a} \in \mathrm{T}|_{\mathbf{Cl}(\mathbf{Q})}$. On the other hand, since $\mathbf{Bd}(\mathbf{Q})$ is a closed restriction, by Lemma 3.2, $\mathbf{Bd}(\mathbf{Q})$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction. This means that for every $(y, Y) \in \mathbf{a}$ we have $y \in$

$\mathrm{Bd}_Y(Q^Y)$, that is, $y \notin \mathrm{Int}_Y(Q^Y)$. By Corollary 2.12, $\mathbf{a} \notin \mathrm{T}|_{\mathbf{Int(Q)}}$, that is, $\mathbf{a} \in \mathrm{T}|_{\mathbf{Cl(Q)}} \setminus \mathrm{T}|_{\mathbf{Int(Q)}}$.

Conversely, let $\mathbf{a} \in \mathrm{T}|_{\mathbf{Cl(Q)}} \setminus \mathrm{T}|_{\mathbf{Int(Q)}}$. Since $\mathbf{Cl(Q)}$ and $\mathbf{Int(Q)}$ are $(\mathbf{M}, \mathrm{R})$-complete restrictions, for every $(x, X) \in \mathbf{a}$ we have $x \in \mathrm{Cl}_X(Q^X) \setminus \mathrm{Int}_X(Q^X)$, that is, $x \in \mathrm{Bd}_X(Q^X)$. This means that $\mathbf{a} \in \mathrm{T}|_{\mathbf{Bd(Q)}}$. Thus, relation (3.4) is proved. □

Proposition 3.7. *Suppose that the restriction* $\mathbf{Q}$ *of* $\mathbf{S}$ *is* $(\mathbf{M}, \mathrm{R})$*-complete and the family* R *of equivalence relations on* $\mathbf{S}$ *is* $(\mathbf{M}, \mathbf{Q})$*-admissible and* $(\mathbf{M}, \mathbf{Co(Q)})$*-admissible. Then,* $\mathbf{Co(Q)}$ *is also* $(\mathbf{M}, \mathrm{R})$*-complete restriction and*

$$\mathrm{T}|_{\mathbf{Co(Q)}} = \mathrm{T} \setminus \mathrm{T}|_{\mathbf{Q}}. \tag{3.5}$$

Proof. Let $\mathbf{a} \in \mathrm{T}|_{\mathbf{Co(Q)}}$. There exists an element (x, X) of $\mathbf{a}$ such that $x \in X \setminus Q^X$. Let $(y, Y) \in \mathbf{a}$. If $y \notin Y \setminus Q^Y$, then $\mathbf{a} \in \mathrm{T}|_{\mathbf{Q}}$ and since $\mathbf{Q}$ is a $(\mathbf{M}, \mathrm{R})$-complete restriction we have $x \in Q^X$, which is a contradiction. Therefore, $y \in Y \setminus Q^Y$, which means that $\mathbf{Co(Q)}$ is a $(\mathbf{M}, \mathrm{R})$-complete restriction. This also means that $\mathbf{a} \notin \mathrm{T}|_{\mathbf{Q}}$.

By the above $\mathrm{T}|_{\mathbf{Co(Q)}} \subset \mathrm{T} \setminus \mathrm{T}|_{\mathbf{Q}}$. On the other hand $\mathrm{T}|_{\mathbf{Co(Q)}} \cup \mathrm{T}|_{\mathbf{Q}} = \mathrm{T}$. The last two relations imply (3.5). □

Definition 3.8. Suppose that for every element λ of a set Λ,

$$\mathbf{F}(\lambda) \equiv \{F^X(\lambda) : X \in \mathbf{S}\}$$

is a restriction of $\mathbf{S}$. The *union* (respectively, the *intersection*) *of the restrictions* $\mathbf{F}(\lambda)$ is the restriction $\{F^X : X \in \mathbf{S}\}$ of $\mathbf{S}$ for which

$$F^X = \cup\{F^X(\lambda) : \lambda \in \Lambda\}$$

(respectively,

$$F^X = \cap\{F^X(\lambda) : \lambda \in \Lambda\})$$

for every $X \in \mathbf{S}$. These restrictions are also denoted by

$$\vee\{\mathbf{F}(\lambda) : \lambda \in \Lambda\} \text{ and}$$

$$\wedge\{\mathbf{F}(\lambda) : \lambda \in \Lambda\},$$

respectively.

Proposition 3.9. *Suppose that for every element* λ *of a set* Λ *of cardinality* $\leq \tau$ *an* $(\mathbf{M}, \mathrm{R})$*-complete restriction* $\mathbf{F}(\lambda)$ *of* $\mathbf{S}$ *is given and let* $\mathbf{F}$ *be either the restriction* $\wedge\{\mathbf{F}(\lambda) : \lambda \in \Lambda\}$ *or the restriction* $\vee\{\mathbf{F}(\lambda) : \lambda \in \Lambda\}$*. If the family* R *is* $(\mathbf{M}, \mathbf{F})$*-admissible and* $(\mathbf{M}, \mathbf{F}(\lambda))$*-admissible for every* $\lambda \in \Lambda$*, then* $\mathbf{F}$ *is an* $(\mathbf{M}, \mathrm{R})$*-complete restriction.*

Proof. Suppose that

$$\mathbf{F}(\lambda) = \{F^X(\lambda) : X \in \mathbf{S}\}.$$

Let

$$\mathbf{F} = \wedge\{\mathbf{F}(\lambda) : \lambda \in \Lambda\} \equiv \{F^X : X \in \mathbf{S}\},$$

$\mathbf{a} \in \mathrm{T}|_{\mathbf{F}} \equiv \mathrm{T}(\mathbf{M}|_{\mathbf{F}}, \mathrm{R}|_{\mathbf{F}})$ and $(x, X) \in \mathbf{a}$. There exists a pair $(y, Y) \in \mathbf{a}$ such that $y \in F^Y$. Therefore, $y \in F^Y(\lambda)$ for every $\lambda \in \Lambda$. By Corollary 2.12, $\mathbf{a} \in \mathrm{T}|_{\mathbf{F}(\lambda)}$. Since by assumption $\mathbf{F}(\lambda)$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction, $x \in F^X(\lambda)$ for every $\lambda \in \Lambda$, which means that $x \in F^X$. Thus, $\mathbf{F}$ is also an $(\mathbf{M}, \mathrm{R})$-complete restriction. The case where $\mathbf{F} = \vee\{\mathbf{F}(\lambda) : \lambda \in \Lambda\}$ is proved similarly. □

4. Saturated classes of subsets.

Definition 4.1. In our considerations by a *class of subsets* we mean a class $I\!\!P$ consisting of ordered pairs (Q, X), where Q is a subset of a space X. Such a class is said to be *topological* if for every homeomorphism h of a space X onto a space Y the condition $(Q, X) \in I\!\!P$ implies that $(h(Q), Y) \in I\!\!P$. In what follows all considered classes of subsets are assumed to be topological.

Definition 4.2. Let $I\!\!P$ be a class of subsets. A restriction $\mathbf{Q}$ of an indexed collection $\mathbf{S}$ of spaces is said to be a *$I\!\!P$-restriction* if $(\mathbf{Q}(X), X) \in I\!\!P$ for every $X \in \mathbf{S}$. (We recall that

$$\mathbf{Q} = \{\mathbf{Q}(X) : X \in \mathbf{S}\}).$$

Definition 4.3. A class $I\!\!P$ of subsets is said to be *saturated* if for every indexed collection $\mathbf{S}$ of spaces and for every $I\!\!P$-restriction $\mathbf{Q}$ of $\mathbf{S}$ there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $(\mathbf{M}, \mathbf{Q})$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every elements $\mathbf{H}$ and $\mathbf{L}$ of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$ for which $\mathbf{H} \subset \mathbf{L}$, we have $(\mathrm{T}(\mathbf{H}_{\mathbf{Q}}), \mathrm{T}(\mathbf{L})) \in I\!\!P$.

The considered co-mark $\mathbf{M}^+$ is said to be an *initial co-mark of* $\mathbf{S}$ (*corresponding to $I\!\!P$-restriction* $\mathbf{Q}$) and the family R^+ is said to be an *initial family of* $\mathbf{S}$ (*corresponding to the co-mark* $\mathbf{M}$ *and $I\!\!P$-restriction* $\mathbf{Q}$).

The proof of the next proposition is similar to the proof of Proposition 3.3 of [2]. (About the "intersection of classes" see the Note to Proposition 3.3 of [2]).

Proposition 4.4. *The intersection of no more than τ many saturated classes of subsets is also a saturated class of subsets.*

Notation. We shall denote by:

$$\text{(a) } \mathbb{P}(\mathrm{Cl}), \text{ (b) } \mathbb{P}(\mathrm{Op}), \text{ and (c) } \mathbb{P}(\mathrm{n.dense})$$

the classes of subsets consisting of all ordered pairs (Q, X) such that: (a) Q is closed, (b) Q is open, and (c) Q is nowhere dense in X, respectively.

Proposition 4.5. *The classes* $\mathbb{P}(\mathrm{Cl})$*,* $\mathbb{P}(\mathrm{Op})$*, and* $\mathbb{P}(\mathrm{n.dence})$ *are saturated classes of subsets.*

Proof. By Lemma 3.2 it follows immediately that $\mathbb{P}(\mathrm{Cl})$ is a saturated class. (For every $\mathbb{P}(\mathrm{Cl})$-restriction $\mathbf{Q}$ of an indexed collection $\mathbf{S}$ of spaces as an initial co-mark $\mathbf{M}^+$ we can take any co-mark of $\mathbf{S}$ and for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ as an initial family we can take any $(\mathbf{M}, \mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$).

Similarly, Lemma 3.3 implies that $\mathbb{P}(\mathrm{Op})$ is a saturated class. (In this case, for every $\mathbb{P}(\mathrm{Op})$-restriction $\mathbf{Q}$ of an indexed collection $\mathbf{S}$ of spaces, as an initial co-mark $\mathbf{M}^+$ we can take any co-mark of $\mathbf{S}$ and for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ as an initial family we can consider any $(\mathbf{M}, \mathbf{Q})$-admissible and $(\mathbf{M}, \mathbf{Co}(\mathbf{Q}))$-admissible family of equivalence relations on $\mathbf{S}$).

Now, we prove that $\mathbb{P}(\mathrm{n.dense})$ is a saturated class. Consider an indexed collection $\mathbf{S}$ of spaces and let

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$$

be a $\mathbb{P}(\mathrm{n.dense})$-restriction of $\mathbf{S}$. Therefore, Q^X is a nowhere dense subset of $X \in \mathbf{S}$. Denote by $\mathbf{M}^+$ an arbitrary co-mark of $\mathbf{S}$. We prove that $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the $\mathbb{P}(\mathrm{n.dense})$-restriction $\mathbf{Q}$.

For this purpose we consider an arbitrary co-mark

$$\mathbf{M} \equiv \{\{U_\delta^X : \delta \in \tau\} : X \in \mathbf{S}\}$$

of $\mathbf{S}$, which is a co-extension of $\mathbf{M}^+$, and let R^+ be any $(\mathbf{M}, \mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$. We show that R^+ is an initial family corresponding to the co-mark $\mathbf{M}$ and the $\mathbb{P}(\mathrm{n.dense})$-restriction $\mathbf{Q}$.

Indeed, let

$$\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$$

be an admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, $\mathbf{H}, \mathbf{L} \in \mathrm{C}^\diamond(\mathrm{R})$, and $\mathbf{H} \subset \mathbf{L}$. Then, R is also $(\mathbf{M}, \mathbf{Q})$-admissible. In order to prove that R^+ is an initial family (and, therefore, $\mathbf{M}^+$ is an initial co-mark) we need to prove that the subset $\mathrm{T}(\mathbf{H}|_\mathbf{Q})$ of $\mathrm{T}(\mathbf{L})$ is nowhere dense. For this purpose it suffices to prove that $\mathrm{T}|_\mathbf{Q}$ is a nowhere dense subset of T.

Let U be an open subset of T. Without loss of generality, we can suppose that U has the form $U_\delta^\mathrm{T}(\mathbf{H})$ for some $\delta \in \tau$ and some $\sim^t$-equivalence class $\mathbf{H}$, $t \in \mathcal{F}$. Let $X \in \mathbf{H}$. Since Q^X is nowhere dense in X there exists an element

$\varepsilon \in \tau$ such that $U^X_\varepsilon \subset U^X_\delta$ and $U^X_\varepsilon \cap Q^X = \varnothing$. Let $\{\delta, \varepsilon\} \cup t = s$. By Lemma 2.6 there exists an element q of $\mathcal{F}$ such that $\sim^q \subset \sim^s_{\mathbf{M}}$ and $d^X_s(Q^X) = d^Y_s(Q^Y)$ if $X \sim^q Y$. Let $\mathbf{E}$ be the $\sim^q$-equivalence class of X. In order to prove that the subset $\mathrm{T}|_{\mathbf{Q}}$ of T is nowhere dense it suffices to prove that

$$U^{\mathrm{T}}_\varepsilon(\mathbf{E}) \subset U^{\mathrm{T}}_\delta(\mathbf{H}) \text{ and } U^{\mathrm{T}}_\varepsilon(\mathbf{E}) \cap \mathrm{T}|_{\mathbf{Q}} = \varnothing. \tag{4.1}$$

Let Y be an arbitrary element of $\mathbf{E}$. By the choice of q, $X \sim^s_{\mathbf{M}} Y$. By Lemma 1.1 of [2], $U^Y_\varepsilon \subset U^Y_\delta$. Therefore, $U^{\mathrm{T}}_\varepsilon(\mathbf{E}) \subset U^{\mathrm{T}}_\delta(\mathbf{E}) \subset U^{\mathrm{T}}_\delta(\mathbf{H})$.

We now prove that the set $U^{\mathrm{T}}_\varepsilon(\mathbf{E}) \cap \mathrm{T}|_{\mathbf{Q}}$ is empty. Indeed, in the opposite case, there exists a point $\mathbf{a} \in \mathrm{T}|_{\mathbf{Q}}$ belonging to the set $U^{\mathrm{T}}_\varepsilon(\mathbf{E})$. By Corollary 2.12 there exists a pair $(y, Y) \in \mathbf{a}$ such that $y \in Q^Y$. Then, $y \in U^Y_\varepsilon$ and $Y \in \mathbf{E}$. By the choice of q, $X \sim^s_{\mathbf{M}} Y$ and $d^X_s(Q^X) = d^Y_s(Q^Y)$. Therefore, there exists a point $x \in Q^X$ such that $d^X_s(x) = d^Y_s(y)$. This means that $x \in U^X_\varepsilon$, that is, $U^X_\varepsilon \cap Q^X \neq \varnothing$, which is a contradiction proving relation (3.1). Thus, the set $\mathrm{T}|_{\mathbf{Q}}$ is a nowhere dense subsets of T and, therefore, $\mathbb{P}$(n.dense) is a saturated class. $\square$

Corollary 4.6. *If $I\!\!P$ is a saturated class of subsets, then the classes*

$$\mathbb{P}(\mathrm{Cl}) \cap I\!\!P,\ \mathbb{P}(\mathrm{Op}) \cap I\!\!P,\ \text{and}\ \mathbb{P}(\mathrm{n.dense}) \cap I\!\!P$$

are also saturated classes.

Definition 4.7. A class $I\!\!P$ of subsets is said to be *closed* (respectively, *open*) if for every element $(Q, X) \in I\!\!P$ the subset Q of X is closed (respectively, open).

A restriction $\mathbf{Q}$ of an indexed collection $\mathbf{S}$ of spaces is said to be *complete* if there exists a co-mark $\mathbf{M}$ of $\mathbf{S}$ and an $(\mathbf{M}, \mathbf{Q})$-admissible family R of equivalence relations on $\mathbf{S}$ such that $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction.

A class $I\!\!P$ of subsets is said to be *complete* if for every indexed collection $\mathbf{S}$ of spaces any $I\!\!P$-restriction $\mathbf{Q}$ of $\mathbf{S}$ is complete.

Lemmas 3.2 and 3.3 imply that any closed or any open restriction of any indexed collection of spaces is complete. Therefore, any closed or open class of subsets is complete. In particular, the classes $\mathbb{P}(\mathrm{Cl})$ and $\mathbb{P}(\mathrm{Op})$ are complete.

Lemma 4.8. *Let $\mathbf{Q}$ be a restriction of a collection $\mathbf{S}$ of spaces, $\mathbf{M}_0$ a co-mark of $\mathbf{S}$ and R_0 a family of equivalence relations on $\mathbf{S}$ such that $\mathbf{Q}$ is $(\mathbf{M}_0, \mathrm{R}_0)$-complete. Then, for every co-extension $\mathbf{M}$ of $\mathbf{M}_0$ and $(\mathbf{M}, \mathbf{Q})$-admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R_0, $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction.*

Proof. Let $\mathbf{M}$ be a co-extension of $\mathbf{M}_0$ and θ an indicial mapping of this co-extension. Let also $\mathrm{R} \equiv \{\sim^s : s \in \mathcal{F}\}$ be an $(\mathbf{M}, \mathbf{Q})$-admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of $\mathrm{R}_0 \equiv \{\sim^s_0 : s \in \mathcal{F}\}$. Consider a point $\mathbf{a} \in \mathrm{T}|_{\mathbf{Q}}$ and let $(x, X) \in \mathbf{a}$. We must prove that $x \in \mathbf{Q}(X)$.

There exists an element (y, Y) of $\mathbf{a}$ such that $y \in \mathbf{Q}(Y)$. Let $\mathbf{b}$ be a point of $\mathrm{T}(\mathbf{M}_0, \mathrm{R}_0)$ such that $(y, Y) \in \mathbf{b}$. Since $y \in \mathbf{Q}(Y)$, $\mathbf{b} \in \mathrm{T}(\mathbf{M}_0|_{\mathbf{Q}}, \mathrm{R}_0|_{\mathbf{Q}})$. We prove that $(x, X) \in \mathbf{b}$. Indeed, since R is a final refinement of R_0 and since $X \sim^s Y$ for every $s \in \mathcal{F}$ we have $X \sim_0^t Y$ for every $t \in \mathcal{F}$.

Let t be an arbitrary element of $\mathcal{F} \setminus \{\varnothing\}$ and $s = \theta(t)$. For every $Z \in \mathbf{S}$ denote by $\widetilde{\mathrm{A}}_t^Z$ the t-algebra of Z related to the mark $\mathbf{M}_0(Z)$ and by A_s^Z the s-algebra of Z related to the mark $\mathbf{M}(Z)$. Also denote by $\widetilde{d}_t^Z$ the mapping of Z into 2^t constructed for the algebra $\widetilde{\mathrm{A}}_t^Z$ and by d_s^Z the corresponding mapping constructed for the algebra A_s^Z. It is easy to verify that if for some point $z \in Z$, $d_s^Z(z) = f \in 2^s$, then $\widetilde{d}_t^Z(z) = f \circ \theta|_t \in 2^t$, where $\theta|_t$ is the restriction of θ to $t \subset \tau$.

Since $(x, X), (y, Y) \in \mathbf{a}$ we have $d_s^X(x) = d_s^Y(y)$ for every $s \in \mathcal{F}$. By the above, $\widetilde{d}_t^X(x) = \widetilde{d}_t^Y(y)$ for every $t \in \mathcal{F}$. Therefore, the pairs (x, X) and (y, Y) belong to the same point of $\mathrm{T}(\mathbf{M}_0, \mathrm{R}_0)$. Since $(y, Y) \in \mathbf{b}$ we have $(x, X) \in \mathbf{b}$. Since $\mathbf{Q}$ is an $(\mathbf{M}_0, \mathrm{R}_0)$-complete restriction and $y \in \mathbf{Q}(Y)$ we have $x \in \mathbf{Q}(X)$, which proves the lemma. □

Definition 4.9. Let $I\!P$ be a class of subsets. An element $(Q^T, T) \in I\!P$ is said to be *universal* (respectively, *properly universal*) in $I\!P$ if for every element (Q^Z, Z) of $I\!P$ there exists a homeomorphism h of Z into T such that $h(Q^Z) \subset Q^T$ (respectively, $h^{-1}(Q^T) = Q^Z$).

Proposition 4.10. *In any non-empty complete saturated class of subsets there exist properly universal elements.*

Proof. Let $I\!P$ be a non-empty complete saturated class of subsets. Since $I\!P$ is a topological class there exists an indexed collection $\mathbf{S}$ of spaces and a $I\!P$-restriction $\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$ of $\mathbf{S}$ such that for every element (Q^Z, Z) of $I\!P$ there exists an element X of $\mathbf{S}$ and a homeomorphism f of Z onto X for which $f(Q^Z) = Q^X$.

Since $I\!P$ is complete, $\mathbf{Q}$ is a complete restriction of $\mathbf{S}$. Therefore, there exists a co-mark $\mathbf{M}_0$ of $\mathbf{S}$ and an $(\mathbf{M}_0, \mathbf{Q})$-admissible family R_0 of equivalence relations on $\mathbf{S}$ such that $\mathbf{Q}$ is an $(\mathbf{M}_0, \mathrm{R}_0)$-complete restriction.

On the other hand, since $I\!P$ is a saturated class there exists an initial co-mark $\mathbf{M}^+$ of $\mathbf{S}$ corresponding to the $I\!P$-restriction $\mathbf{Q}$. Let $\mathbf{M}$ be a co-mark of $\mathbf{S}$, which is simultaneously a co-extension of $\mathbf{M}^+$ and $\mathbf{M}_0$.

There exists a family R^+ of equivalence relations on $\mathbf{S}$, which is an initial family corresponding to the co-mark $\mathbf{M}$ and the $I\!P$-restriction $\mathbf{Q}$. Denote by R an admissible family of equivalence relations on $\mathbf{S}$, which is simultaneously a final refinement of R^+ and R_0. By Lemma 4.8, $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction.

Now, we consider the containing space T and its subset $\mathrm{T}|_{\mathbf{Q}}$. By construction, the ordered pair $(\mathrm{T}|_{\mathbf{Q}}, \mathrm{T})$ is an element of $I\!P$. We prove that this element is properly universal in $I\!P$.

Indeed, let (Q^Z, Z) be an element of $I\!P$. There exists an element $X \in \mathbf{S}$ and a homeomorphism f of Z onto X such that $f(Q^Z) = Q^X$. Let e_{T}^X be the natural embedding of X into T and $h = e_{\mathrm{T}}^X \circ f$. Then, h is a homeomorphism of Z into T. We prove that $h^{-1}(\mathrm{T}|_{\mathbf{Q}}) = Q^Z$. It suffices to prove that

$$(e_{\mathrm{T}}^X)^{-1}(\mathrm{T}|_{\mathbf{Q}}) \subset Q^X. \tag{4.2}$$

Let $\mathbf{a} \in \mathrm{T}|_{\mathbf{Q}}$ and $x \in (e_{\mathrm{T}}^X)^{-1}(\mathbf{a})$. By the definition of the mapping e_{T}^X, $(x, X) \in \mathbf{a}$. Since $\mathbf{Q}$ is an $(\mathbf{M}, \mathrm{R})$-complete restriction we have $x \in Q^X$ proving relation (3.2). This completes the proof of the proposition. □

Similarly we can prove the following proposition.

Proposition 4.11. *In any non-empty saturated class of subsets there exist universal elements.*

Corollary 4.12. *In the classes* $\mathbb{P}(\mathrm{Cl})$*,* $\mathbb{P}(\mathrm{Op})$*, and* $\mathbb{P}(\mathrm{Cl}) \cap \mathbb{P}(\mathrm{n.dense})$ *there exist properly universal elements.*

5. Commutative operators.

Definition 5.1. Suppose that for every space X a mapping O_X of the set $\mathcal{P}(X)$ into itself is given. Then, the class of all such mappings is said to be an *operator*. Such an operator is said to be *topological* if for every homeomorphism h of a space X onto a space Y we have

$$h(\mathrm{O}_X(Q)) = \mathrm{O}_Y(h(Q))$$

for every $Q \in \mathcal{P}(X)$.

In what follows, all considered operators are assumed to be topological.

Notation. The class of all spaces will be denoted by $\mathcal{S}$. Let

$$\mathbf{O} \equiv \{\mathrm{O}_X : X \in \mathcal{S}\}$$

be an operator. For every indexed collection $\mathbf{S}$ of spaces and for every restriction

$$\mathbf{Q} \equiv \{Q^X : X \in \mathbf{S}\}$$

of $\mathbf{S}$ we set

$$\mathbf{O}(\mathbf{Q}) = \{\mathrm{O}_X(Q^X) : X \in \mathbf{S}\}.$$

Obviously, $\mathbf{O}(\mathbf{Q})$ is a restriction of $\mathbf{S}$.

Let $I\!P$ be a class of subsets. Denote by $\mathbf{O}(I\!P)$ and $\mathbf{O}^{-1}(I\!P)$ the classes of subsets, which are defined as follows:

$$\mathbf{O}(I\!P) = \{(\mathrm{O}_X(Q), X) : (Q, X) \in I\!P\} \text{ and}$$

$$\mathbf{O}^{-1}(I\!P) = \{(Q, X) : (\mathrm{O}_X(Q), X) \in I\!P\}.$$

It is clear that $\mathbf{O}(I\!P)$ and $\mathbf{O}^{-1}(I\!P)$ are topological classes of subsets.

Below, the following operators will be considered:

$$\mathbf{Bd} = \{\mathrm{Bd}_X : X \in \mathcal{S}\},$$

$$\mathbf{Cl} = \{\mathrm{Cl}_X : X \in \mathcal{S}\}, \text{ and}$$

$$\mathbf{Int} = \{\mathrm{Int}_X : X \in \mathcal{S}\},$$

where Bd_X, Cl_X, and Int_X are the boundary, the closure, and the interior operators in a space X, respectively.

Definition 5.2. Let $\mathbf{O} \equiv \{\mathrm{O}_X : X \in \mathcal{S}\}$ be an operator. This operator is said to be *commutative with respect to a restriction* $\mathbf{Q}$ *of an indexed collection* $\mathbf{S}$ *of spaces* if there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $(\mathbf{M}, \mathbf{Q})$-admissible and $(\mathbf{M}, \mathbf{O}(\mathbf{Q}))$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every elements $\mathbf{H}$ and $\mathbf{L}$ of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$ for which $\mathbf{H} \subset \mathbf{L}$, we have

$$\mathrm{O}_{\mathrm{T}(\mathbf{L})}(\mathrm{T}(\mathbf{H}|_{\mathbf{Q}})) = \mathrm{T}(\mathbf{H}|_{\mathbf{O}(\mathbf{Q})}).$$

The considered co-mark $\mathbf{M}^+$ is said to be an $\mathbf{O}$-*commutative co-mark* (*corresponding to the restriction* $\mathbf{Q}$) and the family R^+ is said to be an $\mathbf{O}$-*commutative family* (*corresponding to the co-mark* $\mathbf{M}$ *and the restriction* $\mathbf{Q}$). (We note that any co-extension of $\mathbf{M}^+$ is also an $\mathbf{O}$-commutative co-mark and any admissible family of equivalence relation on $\mathbf{S}$, which is a final refinement of R^+, is also an $\mathbf{O}$-commutative family).

The operator $\mathbf{O}$ is said to be *commutative with respect to a class* $I\!\!P$ *of subsets* if it is commutative with respect to any $I\!\!P$-restriction of any indexed collection of spaces.

The operator $\mathbf{O}$ is said to be (*completely*) *commutative* if it is commutative with respect to any (complete) restriction of any indexed collection of spaces.

It is clear that a (completely) commutative operator is commutative with respect to any (complete) class of subsets.

Lemmas 3.2 and 3.3 and Propositions [3.4-3.6] imply the following two consequences.

Corollary 5.3. *The operator* $\mathbf{Cl}$ *is commutative.*

Corollary 5.4. *The operators* $\mathbf{Bd}$ *and* $\mathbf{Int}$ *are completely commutative.*

The proofs of the following two proposition are similar. We prove only the second.

Proposition 5.5. *Let* $I\!\!P$ *be a saturated class of subsets and* $\mathbf{O}$ *an operator, which is commutative with respect to* $I\!\!P$. *Then,* $\mathbf{O}(I\!\!P)$ *is a saturated class of subsets.*

Proposition 5.6. *Let $\mathbb{P}$ and $\mathbb{F}$ be saturated classes of subsets and* $\mathbf{O}$ *an operator, which is commutative with respect to $\mathbb{P}$. Then, $\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F})$ is a saturated class of subsets.*

Proof. Suppose that $\mathbf{O} = \{\mathrm{O}_X : X \in \mathcal{S}\}$. Let $\mathbf{S}$ be an indexed collection of spaces and $\mathbf{Q}$ a $(\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F}))$-restriction of $\mathbf{S}$. Therefore, the restriction $\mathbf{Q}$ is a $\mathbb{P}$-restriction, the restriction $\mathbf{G} \equiv \mathbf{O}(\mathbf{Q})$ of $\mathbf{S}$ is an $\mathbb{F}$-restriction, and $\mathbf{O}$ is commutative with respect to $\mathbf{Q}$.

Since $\mathbb{P}$ and $\mathbb{F}$ are saturated classes and $\mathbf{O}$ is commutative with respect to $\mathbb{P}$ there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$, which is simultaneously an initial co-mark corresponding to the $\mathbb{P}$-restriction $\mathbf{Q}$, an initial co-mark corresponding to the $\mathbb{F}$-restriction $\mathbf{G}$, and an $\mathbf{O}$-commutative co-mark corresponding to the restriction $\mathbf{Q}$.

We prove that $\mathbf{M}^+$ is also an initial co-mark of $\mathbf{S}$ corresponding to the $(\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F}))$-restriction $\mathbf{Q}$. Consider a co-extension $\mathbf{M}$ of the co-mark $\mathbf{M}^+$. There exists a family R^+ of equivalence relations on $\mathbf{S}$, which is simultaneously an initial family corresponding to the co-mark $\mathbf{M}$ and the $\mathbb{P}$-restriction $\mathbf{Q}$, an initial family corresponding to the co-mark $\mathbf{M}$ and the $\mathbb{F}$-restriction $\mathbf{G}$, and an $\mathbf{O}$-commutative family corresponding to the co-mark $\mathbf{M}$ and the restriction $\mathbf{Q}$. We show that this family is also an initial family corresponding to the co-mark $\mathbf{M}$ and the $(\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F}))$-restriction $\mathbf{Q}$.

Indeed, let R be an admissible family of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and $\mathbf{H}$ and $\mathbf{L}$ two elements of $\mathrm{C}^{\diamond}(\mathrm{R})$ such that $\mathbf{H} \subset \mathbf{L}$. Then, we can consider the space $\mathrm{T}(\mathbf{L})$ and its subsets $\mathrm{T}(\mathbf{H}|_{\mathbf{G}})$ and $\mathrm{T}(\mathbf{H}|_{\mathbf{Q}})$.

By construction, $(\mathrm{T}(\mathbf{H}|_{\mathbf{Q}}), \mathrm{T}(\mathbf{L})) \in \mathbb{P}$ and $(\mathrm{T}(\mathbf{H}|_{\mathbf{G}}), \mathrm{T}(\mathbf{L})) \in \mathbb{F}$. Since $\mathbf{O}$ is commutative with respect to $\mathbb{P}$ we have

$$\mathrm{T}(\mathbf{H}|_{\mathbf{G}}) = \mathrm{T}(\mathbf{H}|_{\mathbf{O}(\mathbf{Q})}) = \mathrm{O}_{\mathrm{T}(\mathbf{L})}(\mathrm{T}(\mathbf{H}|_{\mathbf{Q}})).$$

This relation shows that $(\mathrm{T}(\mathbf{H}|_{\mathbf{Q}}), \mathrm{T}(\mathbf{L})) \in \mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F})$, which means that the family R^+ is an initial family corresponding to the co-mark $\mathbf{M}$ and the $(\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F}))$-restriction $\mathbf{Q}$ and, therefore, the co-mark $\mathbf{M}^+$ is an initial co-mark of $\mathbf{S}$ corresponding to the $(\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F}))$-restriction $\mathbf{Q}$. Thus, $\mathbb{P} \cap \mathbf{O}^{-1}(\mathbb{F})$ is a saturated class. □

Corollary 5.7. *If $\mathbb{F}$ is a saturated class of subsets, then $\mathbf{Cl}(\mathbb{F})$ and $\mathbf{Cl}^{-1}(\mathbb{F})$ are also saturated classes of subsets.*

Corollary 5.8. *If $\mathbb{P}$ is a complete saturated class of subsets, then $\mathbf{Bd}(\mathbb{P})$ and $\mathbf{Int}(\mathbb{P})$ are also complete saturated classes of subsets.*

Corollary 5.9. *If $\mathbb{F}$ is a saturated class of subsets and $\mathbb{P}$ is a complete saturated class of subsets, then $\mathbb{P} \cap \mathbf{Bd}^{-1}(\mathbb{F})$ and $\mathbb{P} \cap \mathbf{Int}^{-1}(\mathbb{F})$ are also complete saturated classes of subsets.*

In the next proposition, which is easy to prove, we use the following definition of a saturated class of spaces.

Definition 5.10. A class $I\!P$ of spaces is said to be *saturated* if for every indexed collection $\mathbf{S}$ of elements of $I\!P$ there exists a co-mark $\mathbf{M}^+$ of $\mathbf{S}$ satisfying the following condition: for every co-extension $\mathbf{M}$ of $\mathbf{M}^+$ there exists an $\mathbf{M}$-admissible family R^+ of equivalence relations on $\mathbf{S}$ such that for every admissible family R of equivalence relations on $\mathbf{S}$, which is a final refinement of R^+, and for every element $\mathbf{L}$ of $\mathrm{C}^{\diamondsuit}(\mathrm{R})$, the space $\mathrm{T}(\mathbf{L})$ belongs to $I\!P$.

Remark 5.11. The above definition of a saturated class of spaces is slightly different from that of [2]. In [2] instead of the space $\mathrm{T}(\mathbf{L})$ we consider only the space T. However, for the new notion of a saturated class of spaces all results of [2] concerning saturated classes of spaces are hold.

Proposition 5.12. *The following statements are true:*

(1) *If $I\!P$ is a (complete) saturated class of subsets and $I\!E$ is a saturated class of spaces, then the classes*

$$\{(Q,X) \in I\!P : X \in I\!E\} \text{ and } \{(Q,X) \in I\!P : Q \in I\!E\}$$

are (complete) saturated classes of subsets.

(2) *If $I\!F$ is a saturated class of subsets, then the classes*

$$\{X \in \mathcal{S} : (Q,X) \in I\!F \text{ for some subset } Q \text{ of } X\} \text{ and}$$

$$\{Q \in \mathcal{S} : (Q,X) \in I\!F \text{ for some space } X \in \mathcal{S}\}$$

are saturated classes of spaces.

(3) *If $I\!P$ and $I\!E$ are saturated classes of spaces, then the class*

$$\{(Q,X) : Q \in I\!P,\ X \in I\!E, \text{ and } Q \subset X\}$$

is a saturated class of subsets.

6. Concluding remarks and some problems.

1. In [3] (see also [1]) for a given countable ordinal α "very simple examples of Borel sets M_α and A_α (lying in the Hilbert cube H) which are exactly of the multiplicative and additive class α respectively" are constructed. These sets satisfy the following property:

"If X is a metric space and $B \subset X$ is a Borel set of the multiplicative (additive) class α in X, then there exists a continuous mapping φ of X into H such that $\varphi^{-1}(M_\alpha) = B$ (such that $\varphi^{-1}(A_\alpha) = B$)."

Actually in [3] the class of subsets consisting of all pairs (B, X), where B is a Borel set of multiplicative (additive) class α in a metric space X, is considered. The proving property of the elements (M_α, H) and (A_α, H) shows that these elements in some sense can be considered as a "properly universal element" in this class.

By our method we can prove the following result:

For a given countable ordinal α the class $I\!P_m$ (respectively, the class $I\!P_a$) of all pairs (Q, X), where Q is a Borel set of the multiplicative (respectively, additive) class α in a separable metrizable space X, is a complete saturated class of subsets. Therefore, in these classes there exist properly universal elements.

In connection with the above we put the following problem.

(1) Are the elements (M_α, H) and (A_α, H) properly universal elements in the classes $\mathbb{P}_m$ and $\mathbb{P}_a$, respectively?

2. The problem of the existence of (properly) universal elements for different classes of subsets is arisen. Below, we set two questions concerning the class of all subsets, that is, the class of subsets consisting of all pairs (Q, X), where Q is a subset of a space X, whose answers it seems to be negative.

(2) Is there a properly universal element in the class of all subsets?

(3) Is the class of all subsets complete saturated?

The second question is equivalent to the following:

(3a) Is any saturated class of subsets complete saturated?

(Obviously, the class of all subsets is saturated and, therefore, in this class there are universal elements).

Another general problem is the following:

(4) Construct (completely) commutative operators, which are distinct from that of Section 5 and its compositions. (The composition of two operators

$$\mathbf{O}^1 \equiv \{\mathrm{O}^1_X : X \in \mathcal{S}\} \text{ and } \mathbf{O}^2 \equiv \{\mathrm{O}^2_X : X \in \mathcal{S}\}$$

is the operator

$$\mathbf{O} \equiv \{\mathrm{O}_X \equiv \mathrm{O}^1_X \circ \mathrm{O}^2_X : X \in \mathcal{S}\}).$$

References

[1] R. Engelking, W. Holsztyński and R. Sikorski, *Some examples of Borel sets*, Colloq. Math. **15** (1966), 271–274.

[2] S. D. Iliadis, *A construction of containing spaces*, Topology Appl. **107** (2000), 97–116.

[3] R. Sikorski, *Some examples of Borel sets*, Colloq. Math. **5** (1958), 170–171.

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