Dynamics of the solutions of the water hammer equations☆

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Abstract

A water hammer is a pressure wave that occurs, accidentally or intentionally, in a filled liquid pipeline when a tap is suddenly closed, or a pump starts or stops, or when a valve closes or opens. A water hammer wave propagates through pipes reflecting on features and boundaries. This phenomenon is governed by a pair of coupled quasi-linear partial differential equations of first order, that are usually solved using the method of characteristics.

In this note we provide a representation of the solution using an operator theoretical approach based on the theory of C_0-semigroups and cosine operator functions, when considering this phenomenon on a compressible fluid along an infinite pipe. We provide an integro-differential equation that represents this phenomenon and it only involves the discharge. In addition, the representation of the solution in terms of a specific C_0-semigroup lets us show that hypercyclicity and the topologically mixing property can occur when considering this phenomenon on certain weighted spaces of integrable and continuous functions on the real line.

Keywords: Hydraulics, C_0-semigroups, Cosine operator function, Sine operator function, Devaney chaos, hypercyclicity, Water Hammer equations, wave propagation.

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1. Introduction

Water hammer (or *hydraulic transient* or *hydraulic shock*) is a pressure flow wave that occurs in the liquid through a pipe when, for example, a pump starts or stops or when a valve closes or opens, and it propagates through the pipe network at a high speed. In some cases the effect may result in damages in the pipe like leaks, intrusion of foreign substances, cavitation, etc.

Water hammer can be study as the propagation of pressure waves in compressible media (water, oil,etc) in rigid pipelines. They are also valuable as they carry information through the pipeline network, and they can also be used to locate features, defects and leaks [1]. It is important from the point of view of engineering, that a careful and accurate design of pipeline networks includes the modelling of possible water hammers taking into account the system boundary interactions.

The water hammer phenomenon in closed conduits has been extensively studied over the years. It was firstly studied by many researches at the end of the 19th century and the early 1900's. Menabrea, Joukowsky, and Allievi, among others, worked on transient flow theory and, as a result, Joukowsky developed the fundamental equations of this phenomenon that have been used until nowadays (see the book of Chaudry [2] for a review). The study of water hammers (or transients) is a complex field in fluid dynamics. Nevertheless, after some simplifications, water hammer wave propagation modelling can be summarized by mass and momentum conservation equations.

An essential component of a water hammer is the reflection of its wave at the end (boundaries and elements like pumps or valves) of a pipe. For instance, when a pump shuts down, the wave that it originates will be reflected at the end of the pipe and will return while interfering with the initial wave [1]. Wave reflections can be produced from pipe features such as valves, orifices, and junctions, which must be treated as boundary conditions in numerical computations. Wave superposition can also be considered when several water hammer waves, originated from multiple sources and their reflection at boundaries, meet at some point. Even, in some cases, standing waves are also possible.

A flow is said to be *steady* if the flow conditions, such as pressure, velocity, and discharge, at a point do not change with respect to the time. Disturbances in a steady state flow produce a transient flow. Such a disturbance may be caused by starting or stopping a pump, opening or closing a valve, starting-up a hydraulic turbine, vibrations, etc. Flow disturbances induce spatial and temporal changes in the velocity and pressure along the pipe system. It is usually assumed that such transient flows are on the axial direction, neglecting the radial fluxes. With this assumption, the one-dimensional classical water hammer equations governing the axial and temporal variations of the cross-sectional average of the field variables in transient pipe flows are derived by applying the conservation laws of mass and momentum. A detailed derivation of the governing water hammer equations can be found in Chaudhry [2], where the following assumptions were made: *flow in the conduit is one-dimensional, the velocity distribution is uniform over the cross section of the conduit, the conduit walls and the fluid are linearly elastic and formulas for computing the steady-state friction losses in conduits are valid during the transient state.*

The governing equations for one-dimensional water hammer flows through closed conduits are derived from the classical *mass and momentum conservation* equations and with the previously mentioned assumptions. These are given by the next pair of coupled partial differential equations:

$$Q_t + g\mathcal{A}H_x + \frac{f}{2\mathcal{D}\mathcal{A}}Q|Q| = 0, \tag{1}$$

$$\frac{v^2}{g\mathcal{A}}Q_x + H_t = 0 \tag{2}$$

where $Q(x,t)$ and $H(x,t)$ represent the discharge and the piezometric head at the centerline of the conduit above the specified datum, f is the friction factor (which is assumed to be constant), g is the acceleration due to gravity, v is the fluid wave velocity and $\mathcal{A}$ and $\mathcal{D}$ are the the cross-sectional area and the diameter of a conduit, respectively. The parameters $\mathcal{A}$ and $\mathcal{D}$, are characteristics of the conduit system and are time invariant, but may be functions of x. The wave velocity v depends upon the characteristics of the system.

Equation (1) is known as the *dynamic equation* and equation (2) as the *continuity equation*. Equations (1) and (2) constitute the fundamental equations for 1D water hammer problems and contain all the physics necessary to model wave propagations in pipe systems. This pair of coupled nonlinear partial differential equations can be represented as

$$\begin{cases} \begin{pmatrix} Q(x,t) \\ H(x,t) \end{pmatrix}_t & = \begin{pmatrix} 0 & g\mathcal{A}\frac{d}{dx} \\ \frac{v^2}{g\mathcal{A}}\frac{d}{dx} & 0 \end{pmatrix} \begin{pmatrix} Q(x,t) \\ H(x,t) \end{pmatrix} + \begin{pmatrix} F(Q(x,t),t) \\ 0 \end{pmatrix}, \\ \begin{pmatrix} Q(x,0) \\ H(x,0) \end{pmatrix} & = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \quad x \in \mathbb{R}. \end{cases} \tag{3}$$

where $F(y,t) = -\frac{fy|y|}{2\mathcal{D}\mathcal{A}}$.

The set of partial differential equations (1) and (2) (or equivalently the formulation in (3)) are nonlinear and hyperbolic, and there is not a closed form of the solution of these equations. However, by neglecting or linearizing the nonlinear terms, several numerical methods have been developed to solve these equations for different sets of boundary conditions. The most popular one is the method of characteristics. Further information regarding the description of this method can be found in [2] (see also [1]). Nevertheless, one can also represent the solution of (3) in terms of C_0-semigroups and cosine operator functions. We observe that this treatment is possible because the coefficient $\mathcal{A}$ in (3) is function only of the space parameter x.

The paper is organized as follows: In Section 2 we recall some basic definitions on C_0-semigroups and cosine operator functions. We also introduce some notions on the linear dynamics of C_0-semigroups, such as transitivity, hypercyclicity, and the topologically mixing property. Some characterizations of these properties for the translation C_0-semigroup on certain weighted spaces of continuous and integrable functions are provided. In Section 3 we study the solution of the abstract Cauchy problem originated from the water hammer equations from the point of view of operator theory, providing some general results that can be applied to other similar equations. In Section 4 we apply these results to the particular formulation of the water hammer equations on some function spaces. By simplicity of the treatment, in this section we consider only the case that the coefficient $\mathcal{A}$ is constant. It is remarkable that one can obtain the representation of the solution in terms of C_0-semigroups and cosine operator functions, showing the hyperbolic character of the equation. In particular, we are able to characterize the solution solely in terms of an integro-differential equation that only involves $Q(t)$. This is an interesting feature that seems to be new in the literature on the subject. Finally, in Section 5 we present some results on the dynamics of these solutions when considering the problem on some weighted spaces of continuous and integrable functions.

2. Preliminaries

2.1. C_0-semigroups

Let Δ be denote either the set $\mathbb{R}_+ \cup \{0\}$ or the set $\mathbb{R}$. A one-parameter family $\{T(t)\}_{t\in\Delta}$ of bounded linear operators on a Banach space X is a C_0-semigroup of bounded linear operators if it satisfies

(a) $T(0) = I$, where I stands for the identity operator on X.

(b) $T(t + s) = T(t)T(s)$ for every $t, s \in \Delta$.

(c) $\lim_{t\in\Delta, t\to 0} T(t)x = x$ for every $x \in X$, that is the family $\{T(t)\}_{t\in\Delta}$ is strongly continuous.

The generator B of a C_0-semigroup $\{T(t)\}_{t\in\Delta}$ is defined as

$$Bx := \lim_{t\in\Delta, t\to 0} \frac{T(t)x - x}{t} = T'(0)x \tag{4}$$

whenever this limit exists; the domain of B is the set of all elements $x \in X$ for which this limit exists. The term C_0-semigroup is usually used for the case $\Delta = \mathbb{R}_+ \cup \{0\}$, and the term C_0-group is used when $\Delta = \mathbb{R}$.

Given a C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ with B as its generator, it is clear that its restriction to the family of operators with $t \geq 0$, namely $\{T_+(t)\}_{t\geq 0}$, is a C_0-semigroup of bounded linear operators whose generator is also B. Moreover, if $T_-(t) := T(-t)$ for every $t \geq 0$, then the restriction of the former C_0-group to the family $\{T_-(t)\}_{t\geq 0}$ is also a C_0-semigroup of bounded linear operators with generator $-B$.

Linear dynamics of C_0-semigroups has been widely studied in the last years starting with the seminal paper by Desch, Schappacher and Webb [3]. A C_0-semigroup $\{T(t)\}_{t\in\Delta}$ is said to be *hypercyclic* if there exists some $x \in X$ such that its orbit by the C_0-semigroup is dense in X, i.e. the closure of the set $\{T(t)x : t \in \Delta\}$ coincides with X. We also say that a C_0-semigroup $\{T(t)\}_{t\in\Delta}$ is *topologically transitive* if for every pair of non-empty open sets $U, V \subset X$ we have that $T(t)U \cap V \neq \emptyset$ for some $t \in \Delta \setminus \{0\}$. In the linear setting both notions coincide, see [4, 5]. A stronger notion is the one of topologically mixing. A C_0-semigroup $\{T(t)\}_{t\in\Delta}$ is *topologically mixing* if for every pair of non-empty open sets $U, V \subset X$ we have that there exists some $t_0 > 0$ such that $T(t)U \cap V \neq \emptyset$ for every $|t| \geq t_0$. These definitions can be considered for C_0-groups, too.

A measurable function, $\rho : J \to \mathbb{R}_+$, with $J = \mathbb{R}_+$ or $\mathbb{R}$, is said to be an *admissible weight function* if the following conditions hold:

1. $\rho(\tau) > 0$ for all $\tau \in J$, and

2. there exists constants $M \geq 1$ and $w \in \mathbb{R}$ such that $\rho(\tau) \leq Me^{w|t|}\rho(t+\tau)$ for all $\tau, t \in J$.

Given $L > 0$, there exist some $0 < m \leq M$, depending on ρ and L only, such that for every $t \in \mathbb{R}$ $m\rho(t) \leq \rho(\tau) \leq M\rho(t+L)$ for every $t \leq \tau \leq t+L$, see [3, 4.2].

For $J = \mathbb{R}_+$ or $\mathbb{R}$, we define the weighted spaces $L^p_\rho(J)$, with $1 \leq p < \infty$, and $C_{0,\rho}(J)$ as

$$L^p_\rho(J) := \left\{ u : J \to \mathbb{K} \text{ measurable} : ||u||_p := \left(\int_J |u(\tau)|^p \rho(\tau) d\tau \right)^{1/p} < \infty \right\}. \tag{5}$$

$$C_{0,\rho}(J) := \left\{ u : J \to \mathbb{K} \text{ continuous} : ||u||_\infty := \sup_{\tau \in J} |u(\tau)|\rho(\tau) < \infty \text{ and } \lim_{\tau \to \infty} |u(\tau)|\rho(\tau) = 0 \right\}. \tag{6}$$

If $\rho \equiv 1$, then we will simply denote them as $L^p(J)$ and $C_0(J)$. The translation C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on any of the function spaces $L^p_\rho(J)$ or $C_{0,\rho}(J)$ is given by

$$T(t)f(x) := f(x+t) \quad \text{for all } x \in J, t \geq 0. \tag{7}$$

Characterizations of the hypercyclity of the translation C_0-semigroup on these spaces in terms of the admissible weight function were provided in [3]:

Theorem 2.1. *[3, Th. 4.7] Let X be one of the spaces $L^p_\rho(\mathbb{R}_+)$ or $C_{0,\rho}(\mathbb{R}_+)$ with an admissible weight function ρ. The translation C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X is hypercyclic if, and only if, $\liminf_{t\to\infty} \rho(t) = 0$.*

Theorem 2.2. *[3, Th. 4.8] Let X be one of the spaces $L^p_\rho(\mathbb{R})$ or $C_{0,\rho}(\mathbb{R})$ with an admissible weight function ρ. The translation C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X is hypercyclic if, and only if, for every $\theta \in \mathbb{R}$ there exists a sequence of positive real numbers $\{t_j\}_{j\in\mathbb{N}}$ numbers such that*

$$\lim_{j\to\infty} \rho(\theta + t_j) = \lim_{j\to\infty} \rho(\theta - t_j) = 0. \tag{8}$$

Furthermore, condition (8) also works for the hypercyclicity of the translation C_0-group $\{T(t)\}_{t\in\mathbb{R}}$, and it is equivalent to the hypercyclicity of the C_0-semigroups $\{T_+(t)\}_{t\geq 0}$ and $\{T_-(t)\}_{t\geq 0}$. Similar characterizations for the topologically mixing property were provided by Bermúdez et al in [6] showing that the translation C_0-semigroup $\{T(t)\}_{t\geq 0}$ is topologically mixing if, and only if, $\lim_{t\to\infty} \rho(t) = 0$ for the case $J = \mathbb{R}_+$ (resp. $\lim_{t\to\infty} \rho(t) = \lim_{t\to\infty} \rho(t) = 0$ for the case $J = \mathbb{R}$).

Other dynamical properties were also studied for the translation C_0-semigroup in terms of the admissible weight function, see for instance [7] for the chaos in the sense of Devaney, [8] for the distributional chaos, and [9, 10] for frequent hypercyclicity.

2.2. Cosine operator functions

A *cosine operator function* $\{C(t)\}_{t\in\mathbb{R}}$ is a family of bounded linear operators defined on a Banach space X that satisfies the following properties:

(a) $C(0) = I$

(b) $2C(t)C(s) = C(t+s) + C(t-s)$ for every $t, s \in \mathbb{R}$.

(c) $\lim_{t\to 0} C(t)x = x$ for every $x \in X$.

In this case, the generator A of a cosine operator fucntion $\{C(t)\}_{t\in\mathbb{R}}$ is defined by

$$Ax := \lim_{t\to 0} \frac{2(C(t)x - x)}{t^2} = C''(0)x \tag{9}$$

whenever this limit exists. The sine operator function is defined by

$$S(t)x := \int_0^t C(s)xds, \qquad \text{for every } x \in X,\ t \geq 0, \tag{10}$$

and $S(t) = -S(-t)$ for $t < 0$. Note that, $S(0) = 0$, $S'(t) = C(t)$ and $C'(t) = AS(t)$. For more properties on cosine and sine operator functions, we refer to the reader to the seminal paper of Travis and Webb [11] and the monograph of Arendt et al [12].

In particular, if B is the generator of a C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ on X, then B^2 generates a cosine operator function $\{C(t)\}_{t\in\mathbb{R}}$ given by

$$C(t) := \frac{T_+(t) + T_-(t)}{2}, \tag{11}$$

for every $t \in \mathbb{R}$, whereas the sine operator function satisfies

$$BS(t) := \frac{T_+(t) - T_-(t)}{2}. \tag{12}$$

for every $t \in \mathbb{R}$.

The study of hypercyclicity and the topologically mixing property for cosine operator functions was initially started by Bonilla and Miana in [13], where, among other results, they provide sufficient conditions for the hypercyclicity of the translation cosine operator function generated by the translation C_0-group on the spaces $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$, and they also characterize when the topologically mixing property holds for this cosine operator function on these spaces. These results were obtained from the aformentioned characterization of these properties for the translation C_0-semigroup on these spaces. Kalmes characterized hypercyclicity and the topologically mixing property for some cosine operator functions generated by certain second order partial differential operators [14]. Similar results for the case of cosine operator functions generated from shifts have been given by Chang and Chen in [15]. In addition, Chen also considered the chaos in the sense of Devaney [16], giving a characterization of chaotic cosine operator functions generated by weighted translations on the Lebesgue space $L^p(G)$, with G a locally compact group.

3. An operator theoretical approach to the solutions of the water hammer equations

We notice that the linear water hammer equations can be reformulated using an operator theoretical approach. Let X be a complex Banach space, and let $F : X \times \mathbb{R}_+ \to X$ be a continuous function. In this section we consider the abstract Cauchy problem of the type originated from the nonlinear water hammer equations, which will be formulated as follows:

$$\begin{cases} \begin{pmatrix} Q(t) \\ H(t) \end{pmatrix}_t &= \begin{pmatrix} 0 & \alpha B \\ \frac{1}{\alpha} B & 0 \end{pmatrix} \begin{pmatrix} Q(t) \\ H(t) \end{pmatrix} + \begin{pmatrix} F(Q(t),t) \\ 0 \end{pmatrix}, \\ \begin{pmatrix} Q(0) \\ H(0) \end{pmatrix} &= \begin{pmatrix} \phi \\ \varphi \end{pmatrix}. \end{cases} \tag{13}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$; $\phi, \varphi \in X$ and B is a linear closed operator with domain $\mathrm{Dom}(B)$ in X. Concerning (3) we remark that later on (Section 4), we will consider $\mathcal{A}$ as a constant parameter; $\alpha = \frac{g\mathcal{A}}{v}$ and $Bf = v\frac{d}{dx}$ on an appropriate Banach space X.

Definition 3.1. Let X be a complex Banach space. We say that a pair (Q, H) is a classical solution of (13) if $Q : [0, \infty) \to X$ and $H : [0, \infty) \to X$ are continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$, $Q(t), H(t) \in Dom(B)$ for each $t \in (0, \infty)$, and the pair (Q, H) satisfies the abstract Cauchy problem formulated in (13).

We consider the operator-valued matrix

$$A := \begin{pmatrix} 0 & \alpha B \\ \frac{1}{\alpha} B & 0 \end{pmatrix} \tag{14}$$

with domain $\mathrm{Dom}(A) := \mathrm{Dom}(B) \times \mathrm{Dom}(B)$ defined on the product space $X \times X$ equipped with the usual norm. Our first result provides sufficient conditions for the well-posedness of the abstract linear water hammer problem.

Theorem 3.2. *Suppose that B is the generator of a C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ on X. Then A is the generator of a C_0-group $\{\mathcal{T}(t)\}_{t\geq 0}$ on $X \times X$ given by*

$$\mathcal{T}(t) := \frac{1}{2}\mathcal{T}_+(t) \begin{pmatrix} I & \alpha I \\ \frac{1}{\alpha} I & I \end{pmatrix} + \frac{1}{2}\mathcal{T}_-(t) \begin{pmatrix} I & -\alpha I \\ \frac{-1}{\alpha} I & I \end{pmatrix} \quad \textit{for every } t \geq 0, \tag{15}$$

where

$$\mathcal{T}_\pm(t) := \begin{pmatrix} T_\pm(t) & 0 \\ 0 & T_\pm(t) \end{pmatrix},$$

respectively.

Proof. Clearly, $\mathcal{T}(0) = I \times I$ on $X \times X$. The strong continuity of $\{\mathcal{T}(t)\}_{t\geq 0}$ on $X \times X$ is clear from the strong continuity of $\{T(t)\}_{t\in\mathbb{R}}$ on X. Hence, it is enough to prove that the operators in $\{\mathcal{T}(t)\}_{t\geq 0}$ satisfy the functional equation

$$\mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s) \quad \text{for all } t, s \in \mathbb{R}. \tag{16}$$

Indeed, note that

$$4\mathcal{T}(t)\mathcal{T}(s) = (T(t)\mathcal{P} + T(-t)\mathcal{Q})(T(s)\mathcal{P} + T(-s)\mathcal{Q}) \tag{17}$$

where $\mathcal{P} = \begin{pmatrix} I & \alpha I \\ \frac{1}{\alpha} I & I \end{pmatrix}$ and $\mathcal{Q} = \begin{pmatrix} I & -\alpha I \\ \frac{-1}{\alpha} I & I \end{pmatrix}$ verify the properties $\mathcal{P}^2 = 2\mathcal{P}$, $\mathcal{Q}^2 = 2\mathcal{Q}$ and $\mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} = 0$. Therefore

$$\begin{aligned} 4\mathcal{T}(t)\mathcal{T}(s) &= (T(t)\mathcal{P} + T(-t)\mathcal{Q})(T(s)\mathcal{P} + T(-s)\mathcal{Q}) \\ &= T(t+s)\mathcal{P}^2 + T(t-s)\mathcal{P}\mathcal{Q} + T(-t+s)\mathcal{Q}\mathcal{P} + T(-t-s)\mathcal{Q}^2 \\ &= 2T(t+s)\mathcal{P} + 2T(-t-s)\mathcal{Q} \\ &= 4\mathcal{T}(t+s). \end{aligned} \tag{18}$$

Finally, we observe that the generator of the C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ can be computed as

$$\begin{aligned} \mathcal{T}'(t)|_{t=0} &= \frac{1}{2}\mathcal{T}'_+(t)|_{t=0} \begin{pmatrix} I & \alpha I \\ \frac{1}{\alpha} I & I \end{pmatrix} + \frac{1}{2}\mathcal{T}'_-(t)|_{t=0} \begin{pmatrix} I & -\alpha I \\ \frac{-1}{\alpha} I & I \end{pmatrix} \\ &= \frac{1}{2} B \begin{pmatrix} I & \alpha I \\ \frac{1}{\alpha} I & I \end{pmatrix} - \frac{1}{2} B \begin{pmatrix} I & -\alpha I \\ \frac{-1}{\alpha} I & I \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha B \\ \frac{1}{\alpha} B & 0 \end{pmatrix} = A, \end{aligned} \tag{19}$$

that is defined on $\mathrm{Dom}(B) \times \mathrm{Dom}(B)$. $\square$

Remark 3.3. The above theorem states that if the first order abstract Cauchy problem

$$u'(t) = Bu(t), \quad u(0) = u_0 \tag{20}$$

is well-posed on X, then the homogeneous water hammer equations are also well-posed on $X \times X$ in the sense that A is the generator of a C_0-group there.

Remark 3.4. It is interesting to observe that it is by no means inmediate to see that a family of bounded and linear operators defined by a formula like (15) has the semigroup property. This is a consequence of the very special behavior of the operators $\mathcal{P}$ and $\mathcal{Q}$ involved in the case of the water hammer equations.

Remark 3.5. An explicit description of the C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ on $X \times X$ is the following

$$\mathcal{T}(t)(\phi, \varphi) = \left(T_+(t)\left(\frac{\phi}{2} + \frac{\alpha\varphi}{2}\right) + T_-(t)\left(\frac{\phi}{2} - \frac{\alpha\varphi}{2}\right), T_+(t)\left(\frac{\phi}{2\alpha} + \frac{\varphi}{2}\right) + T_-(t)\left(\frac{-\phi}{2\alpha} + \frac{\varphi}{2}\right)\right), \tag{21}$$

for every $t \geq 0$ and $(\phi, \varphi) \in X \times X$.

According to the C_0-semigroup theory, we can deduce that there is a mild solution for the non-homogeneous water hammer equations. These mild solutions can be defined as follows.

Definition 3.6. Let X be a complex Banach space. We say that a pair (Q, H) is a mild solution of the problem (13) if $Q : [0, \infty) \to X$ and $H : [0, \infty) \to X$ are continuous and satisfy

$$\begin{pmatrix} Q(t) \\ H(t) \end{pmatrix} = \mathcal{T}(t) \begin{pmatrix} \phi \\ \varphi \end{pmatrix} + \int_0^t \mathcal{T}(t-s) \begin{pmatrix} F(Q(s), s) \\ 0 \end{pmatrix} ds. \tag{22}$$

for any $(\phi, \varphi) \in X \times X$.

More precisely, from Theorem 3.2 we deduce the components of the mild solution of problem (13) when $(\phi, \varphi) \in X \times X$, which are given by

$$
\begin{aligned}
Q(t) &= T_+(t)\left(\frac{\phi}{2}+\frac{\alpha\varphi}{2}\right)+T_-(t)\left(\frac{\phi}{2}-\frac{\alpha\varphi}{2}\right)+\frac{1}{2}\int_0^t (T_+(t-s)+T_-(t-s))F(Q(s),s)ds \\
&= C(t)\phi+\alpha BS(t)\varphi+\int_0^t C(t-s)F(Q(s),s)ds
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
H(t) &= T_+(t)\left(\frac{\phi}{2\alpha}+\frac{\varphi}{2}\right)+T_-(t)\left(\frac{-\phi}{2\alpha}+\frac{\varphi}{2}\right)+\frac{1}{2\alpha}\int_0^t (T_+(t-s)-T_-(t-s))F(Q(s),s)ds, \\
&= \frac{1}{\alpha}BS(t)\phi+C(t)\varphi+\frac{1}{\alpha}\int_0^t BS(t-s)F(Q(s),s)ds.
\end{aligned}
\tag{24}
$$

An equivalent formulation of the solution of the nonlinear water hammer equations (13) is given in the following result. It shows the remarkable property that the dynamics of the solution is completely determined by an integro-differential equation satisfied by $Q(t)$. In particular, we explicitly deduce the way in that $H(t)$ is determined by $Q(t)$ and the initial conditions.

Theorem 3.7. *Suppose that B is the generator of a C_0-group $\{T(t)\}_{t\in\mathbb{R}}$ on X and let $F : X \times \mathbb{R}_+ \to \mathrm{Dom}(B)$ be given. A pair (Q, H) is a mild solution of the nonlinear water hammer equation (13) if and only if, for all $(\phi, \varphi) \in \mathrm{Dom}(B) \times \mathrm{Dom}(B)$, Q satisfies the integro-differential equation*

$$
Q'(t) = B^2\int_0^t Q(s)ds + F(Q(t),t) + \alpha B\varphi \tag{25}
$$

and

$$
H(t) = \frac{1}{\alpha}B\int_0^t Q(s)ds + \varphi
$$

with initial conditions $(Q(0), H(0)) = (\phi, \varphi)$.

Proof. Let us denote $G(s) := F(Q(s), s)$ for all $s \geq 0$. Suppose that (Q, H) is a mild solution obtained from some initial conditions $(\phi, \varphi) \in \mathrm{Dom}(B)\times\mathrm{Dom}(B)$. By definition, we have

$$
Q(t) = C(t)\phi + \alpha BS(t)\varphi + \int_0^t C(t-s)G(s)ds \tag{26}
$$

and

$$
H(t) = \frac{1}{\alpha}BS(t)\phi + C(t)\varphi + \frac{1}{\alpha}\int_0^t BS(t-s)G(s)ds. \tag{27}
$$

Since $\phi, \varphi \in \mathrm{Dom}(B)$ we obtain that Q is differentiable and satisfies

$$
Q'(t) = B^2S(t)\phi + \alpha BC(t)\varphi + G(t) + \int_0^t B^2S(t-s)G(s)ds. \tag{28}
$$

On the other hand, since B is a closed operator, we obtain by (26)

$$\begin{aligned} B^2\int_0^t Q(s)ds &= B^2S(t)\phi + \alpha B^2\int_0^t BS(s)\varphi ds + \int_0^t\int_0^s B^2C(s-\tau)G(\tau)d\tau ds \\ &= B^2S(t)\phi + \alpha B^2\int_0^t BS(s)\varphi ds + \int_0^t B^2\int_0^{t-s} C(\tau)d\tau G(s)ds \\ &= B^2S(t)\phi + \alpha B^2\int_0^t BS(s)\varphi ds + \int_0^t B^2S(t-s)G(s)ds \end{aligned} \tag{29}$$

Hence, inserting (28) in (29) we get

$$\begin{aligned} B^2\int_0^t Q(s)ds &= B^2S(t)\phi + \alpha B^2\int_0^t BS(s)\varphi ds + Q_t(t) - B^2S(t)\phi - \alpha BC(t)\varphi - G(t) \\ &= \alpha B^2\int_0^t BS(s)\varphi ds + Q'(t) - \alpha BC(t)\varphi - G(t). \end{aligned} \tag{30}$$

Note that

$$B^2\int_0^t BS(s)\varphi ds = B\int_0^t B^2S(s)\varphi ds = B\int_0^t C'(s)\varphi ds = BC(t)\varphi - B\varphi.$$

Therefore

$$B^2\int_0^t Q(s)ds = \alpha BC(t)\varphi - \alpha B\varphi + Q'(t) - \alpha BC(t)\varphi - G(t) = -\alpha B\varphi + Q'(t) - G(t), \tag{31}$$

proving (25). Moreover from (27) we have

$$H'(t) = \frac{1}{\alpha}B[C(t)\phi + \alpha BS(t)\varphi + \int_0^t C(t-s)G(s)ds] = \frac{1}{\alpha}BQ(t),$$

showing the first implication.

Conversely, suppose that $Q(t)$ satisfies (25) with initial conditions $(Q(0), H(0)) = (\phi, \varphi) \in \mathrm{Dom}(B) \times \mathrm{Dom}(B)$. Then first multiplying by $C(t-s)$ and then integrating over $[0, t]$ we obtain

$$\begin{aligned} \int_0^t C(t-s)Q_t(s)ds &= B^2\int_0^t C(t-s)\int_0^s Q(\tau)d\tau ds + \int_0^t C(t-s)G(s)ds + \alpha B\int_0^t C(t-s)\varphi ds \\ &= B^2\int_0^t \left(\int_0^{t-s} C(\tau)d\tau\right) Q(s)ds + \int_0^t C(t-s)G(s)ds + \alpha B\int_0^t C(s)\varphi ds \\ &= B^2\int_0^t S(t-s)Q(s)ds + \int_0^t C(t-s)G(s)ds + \alpha BS(t)\varphi. \end{aligned} \tag{32}$$

Note that

$$\frac{d}{dt}\int_0^t C(t-s)Q(s)ds = \frac{d}{dt}\int_0^t C(s)Q(t-s)ds = C(t)Q(0) + \int_0^t Q'(t-s)C(s)ds, \tag{33}$$

but also

$$\frac{d}{dt}\int_0^t C(t-s)Q(s)ds = C(0)Q(t) + \int_0^t C'(t-s)Q(s)ds = Q(t) + \int_0^t B^2 S(t-s)Q(s)ds. \tag{34}$$

It follows that

$$\int_0^t C(t-s)Q'(s)ds = \int_0^t C(s)Q'(t-s)ds = Q(t) + B^2\int_0^t S(t-s)Q(s)ds - C(t)\phi, \tag{35}$$

because $Q(0) = \phi$. Therefore, from (32) and (35) we obtain that

$$Q(t)+B^2\int_0^t S(t-s)Q(s)ds - C(t)\phi = B^2\int_0^t S(t-s)Q(s)ds + \int_0^t C(t-s)G(s)ds + \alpha BS(t)\varphi \tag{36}$$

equivalently

$$Q(t) - C(t)\phi = \int_0^t C(t-s)G(s)ds + \alpha BS(t)\varphi \tag{37}$$

which proves that $Q(t)$ satisfies (26). Now, define

$$H(t) := \frac{1}{\alpha}B\int_0^t Q(s)ds + \varphi. \tag{38}$$

Then from (26), integrating over $[0,t]$ and then multiplying by B we obtain

$$\begin{aligned} B\int_0^t Q(s)ds &= B\int_0^t C(s)\phi ds + \alpha\int_0^t B^2 S(s)\varphi ds + B\int_0^t\int_0^s C(s-\tau)G(\tau)d\tau \\ &= BS(t)\phi + \alpha\int_0^t C'(s)\varphi ds + B\int_0^t\int_0^{t-s} C(\tau)d\tau G(s)ds \\ &= BS(t)\phi + \alpha(C(t)-I)\varphi + B\int_0^t S(t-s)G(s)ds \\ &= BS(t)\phi + \alpha C(t)\varphi - \alpha\varphi + B\int_0^t S(t-s)G(s)ds \end{aligned} \tag{39}$$

Hence

$$H(t) = \frac{1}{\alpha}B\int_0^t Q(s)ds + \varphi = \frac{1}{\alpha}BS(t)\phi + C(t)\varphi + \frac{1}{\alpha}B\int_0^t S(t-s)G(s)ds \tag{40}$$

which proves (27) and finish the proof.

□

Remark 3.8. As a consequence of the proof of Theorem 3.7, we have that $H(t)$ is determined by $Q(t)$ by means of the formula

$$H(t) = \frac{1}{\alpha}B\int_0^t Q(s)ds + \varphi.$$

4. Representations of the solutions to the water hammer equations

Let X be either $L^p_\rho(\mathbb{R})$, with $1 \le p < \infty$, or $C_{0,\rho}(\mathbb{R})$. Let $v \in \mathbb{R} \setminus \{0\}$, for every $f \in X$ we define

$$(T(t)f)(x) = f(x - vt) \tag{41}$$

for every $t, x \in \mathbb{R}$. It is well known and easy to check that $\{T(t)\}_{t\in\mathbb{R}}$ is a C_0-group of bounded linear operators on X. The generator of $\{T(t)\}_{t\in\mathbb{R}}$ is defined on $\mathrm{Dom}(D) := \{f : f \in X, f'\text{exists}, f' \in X\}$ as

$$(Df)(x) = -vf'(x) \text{ for } f \in \mathrm{Dom}(D). \tag{42}$$

Remark 4.1. Note that we can analogously treat with the case that v is not constant but depend only on x. Indeed, under appropriate conditions on $v(x)$ it is enough to consider the C_0-semigroup

$$(T(t)f)(x) = f(x - v(x)t)$$

having as generator $(Df)(x) = -v(x)f'(x)$. For more information on the C_0-semigroups generated by D see [17, Ch. VI, Sec. 4].

Note that the cosine operator function induced by $\{T(t)\}_{t\in\mathbb{R}}$ is given as

$$(C(t)f)(x) = \frac{1}{2}\left(f(x - vt) + f(x + vt)\right) \qquad \text{for all } t \in \mathbb{R}, \tag{43}$$

that has $(D^2 f)(x) = v^2 f''(x)$ as its generator. Observe that

$$D(S(t)f)(x) = \frac{-v}{2}\left(f(x - vt) - f(x + vt)\right). \tag{44}$$

Let us define the operator A on $\mathrm{Dom}(D) \times \mathrm{Dom}(D)$ as

$$A := \begin{pmatrix} 0 & \frac{g\mathcal{A}}{v} D \\ \frac{v}{g\mathcal{A}} D & 0 \end{pmatrix}. \tag{45}$$

Rescaling equations (1) and (2) by $1/v$, and according to Theorem 3.2, the solution C_0-semigroup $\{\mathcal{T}(t)\}_{t\ge 0}$ to the linear water hammer equations in (3), having A as generator, is given by:

$$\mathcal{T}(t) := \frac{1}{2}\mathcal{T}_+(t)\begin{pmatrix} I & \frac{g\mathcal{A}}{v} I \\ \frac{v}{g\mathcal{A}} I & I \end{pmatrix} + \frac{1}{2}\mathcal{T}_-(t)\begin{pmatrix} I & -\frac{g\mathcal{A}}{v} I \\ -\frac{v}{g\mathcal{A}} I & I \end{pmatrix} \tag{46}$$

that can be rewritten in terms of the cosine operator function as follows

$$\mathcal{T}(t) := \frac{1}{2}\mathcal{C}(t)\begin{pmatrix} I & \frac{g\mathcal{A}}{v} I \\ \frac{v}{g\mathcal{A}} I & I \end{pmatrix} - \frac{1}{2}\mathcal{T}_-(t)\begin{pmatrix} 0 & \frac{g\mathcal{A}}{v} I \\ \frac{v}{g\mathcal{A}} I & 0 \end{pmatrix}, \tag{47}$$

where

$$\mathcal{C}(t) := \begin{pmatrix} C(t) & 0 \\ 0 & C(t) \end{pmatrix}.$$

In particular, it shows that the solution of the linear part can be decomposed into two parts, where one of them cannot be stable, namely the first part dominated by the cosine operator function, because of the identity

$$I = 2C(t)^2 - C(2t), \quad t \in \mathbb{R}, \tag{48}$$

which follows from the part (b) of the definition of a cosine operator function.

Using the explicit representation of the C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$, we deduce from Theorem 3.2 (see also Remark 3.5) the following result, which resembles Euler's formula for the solutions of the wave equation.

Corollary 4.2. A classical solution of the linear water hammer equation with initial conditions $(\phi, \varphi) \in X \times X$ is given by

$$Q(x,t) = \frac{1}{2}\varphi(x-vt) + \frac{g\mathcal{A}}{2v}\phi(x-vt) + \frac{1}{2}\varphi(x+vt) - \frac{g\mathcal{A}}{2v}\phi(x+vt), \tag{49}$$

and

$$H(x,t) = \frac{v}{2g\mathcal{A}}\varphi(x-vt) + \frac{1}{2}\phi(x-vt) - \frac{v}{2g\mathcal{A}}\varphi(x+vt) + \frac{1}{2}\phi(x+vt). \tag{50}$$

for every $x \in \mathbb{R}$ and $t \geq 0$.

Now we return to the nonlinear water hammer equations. Directly from Theorem 3.7 and the definitions of D and D^2 we obtain the following equivalent formulation to the water hammer equations given in terms of an integro-differential equation that only involves $Q(t)$. This seems to be new in the existing literature on the subject.

Theorem 4.3. *A pair (Q, H) is a solution of the nonlinear water hammer equation if and only if, Q satisfies the integral equation*

$$Q_t(x,t) = v^2 \int_0^t Q_{xx}(x,s)ds + F(Q(x,t),t) + g\mathcal{A}H_x(x,0), \tag{51}$$

and

$$H(x,t) = \frac{v}{g\mathcal{A}} \int_0^t Q_x(x,s)ds + H(x,0).$$

Remark 4.4. In the light of the above theorem, we observe that with regular initial conditions we obtain

$$Q_{tt}(x,t) = v^2 Q_{xx}(x,t) + F_t(Q(x,t),t) \tag{52}$$

which is the inhomogeneous wave equation with source term given by $F_t(Q(x,t),t)$.

5. Dynamics for the homogeneous water hammer equation

In this section we will study when the hypercyclicity and the topologically mixing property hold for the solution C_0-semigroup of the frictionless water hammer equations on $X \times X$, where X can be either $L^p_\rho(\mathbb{R})$, $1 \leq p < \infty$ or $C_{0,\rho}(\mathbb{R})$, with ρ an admissible weight function. These results will be obtained by an application of the Hypercyclicity Criterion for C_0-semigroups, which is a generalization of the version given by Bès and Peris in [18].

Theorem 5.1. *[19, 20]**Hypercyclicity Criterion.** Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on X, two dense subsets $Y, Z \subseteq X$, an increasing sequence of real positive numbers $(t_k)_k$ tending to ∞, and a sequence of mappings $S(t_k) : Z \to X$,$k \in \mathbb{N}$ such that*

(a) $\lim_{k\to\infty} T(t_k)y = 0$ *for all* $y \in Y$,
(b) $\lim_{k\to\infty} S(t_k)z = 0$ *for all* $z \in Z$, *and*
(c) $\lim_{k\to\infty} T(t_k)S(t_k)z = z$ *for all* $z \in Z$.

Then, the C_0-semigroup is hypercyclic (in fact it is weakly mixing).

When replacing in the statement of this result the increasing sequence of real positive numbers $(t_k)_k$ by the whole set $\mathbb{R}_+$, one gets the topologically mixing property.

Theorem 5.2. *[6, 20]**Mixing Criterion.** Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on X, two dense subsets $Y, Z \subseteq X$, and a family of mappings $S(t) : Z \to X$, $t \geq 0$, such that*

(a) $\lim_{t\to\infty} T(t)y = 0$ *for all* $y \in Y$,
(b) $\lim_{t\to\infty} S(t)z = 0$ *for all* $z \in Z$, *and*
(c) $\lim_{t\to\infty} T(t)S(t)z = z$ *for all* $z \in Z$.

Then the C_0-semigroup is topologically mixing.

For the case of the solution C_0-semigroup to the water hammer equations one gets a similar characterization of the hypercyclicity on the spaces $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$, to the one obtained of the hypercyclicity of the translation C_0-semigroup on these spaces. This prove can be compared with the ones of [3, Th. 4.8] and [6, Th. 4.3].

Theorem 5.3. *Let $X = L^p_\rho(\mathbb{R})$, with $1 \leq p < \infty$, or $X = C_{0,\rho}(\mathbb{R})$ with ρ an admissible function. There exists a increasing sequence of positive real numbers $\{t_k\}_{k\in\mathbb{N}}$ tending to ∞ satisfying*

$$\lim_{k\to\infty} \rho(t_k) = \lim_{k\to\infty} \rho(-t_k) = 0, \tag{53}$$

if, and only if, the solution C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ to the water hammer equations is hypercyclic.

Proof. Let us define the norm $||\cdot||$ on $X\times X$ as $||(\phi,\varphi)|| := \left(||\phi||^p_\rho + ||\varphi||^p_\rho\right)^{1/p}$. Set $\{t_k\}_{k\in\mathbb{N}}$ an increasing sequence of real positive numbers satisfying (53). In order to apply the Hypercylicity Criterion, Theorem 5.1, we consider the dense sets $Y \times Y, Z \times Z \subset X \times X$, where Y, Z stand for the set continuous functions with compact support, the sequence $\{t'_k\}_k$, with $t'_k := t_k/v$ for all $k \in \mathbb{N}$, and the mappings $\mathcal{S}(t'_k) := \mathcal{T}(-t'_k)$ for all $k \in \mathbb{N}$.

To prove (a), take $(\phi,\varphi) \in Y \times Y$ with $\text{supp}(\phi), \text{supp}(\varphi) \subseteq [-L, L]$ for some $L > 0$. Fix $\varepsilon > 0$. For $t'_k \geq L$ we have

$$\left|\left|T_+(t'_k)\left(\frac{\phi}{2} + \frac{\alpha\varphi}{2}\right)\right|\right|^p_\rho = \frac{1}{2^p}\int_{-L+vt'_k}^{L+vt'_k} |\phi(s - vt'_k) + \alpha\varphi(s - vt'_k)|^p \rho(s)ds. \tag{54}$$

By the admissibility of the weight we have that there exists $M > 0$ and $w \in \mathbb{R}$ such that $\rho(s+t_k) \leq Me^{w|-s|}\rho(t_k) \leq Me^{wL}\rho(t_k)$ for all $-L \leq s \leq L$. Then

$$\begin{aligned} \left|\left|T_+(t'_k)\left(\frac{\phi}{2}+\frac{\alpha\varphi}{2}\right)\right|\right|_\rho^p &= \frac{1}{2^p}\int_{-L}^{L}|\phi(s)+\alpha\varphi(s)|^p\rho(s+t_k)ds \\ &\leq \frac{Me^{wL}\rho(t_k)}{2^p}\int_{-L}^{L}|\phi(s)+\alpha\varphi(s)|^p ds. \end{aligned} \tag{55}$$

On the other hand, and also by the admissibility of the weight function ρ, we have $\rho(0) \leq Me^{w|s|}\rho(s) \leq Me^{wL}\rho(s)$ for all $-L \leq s \leq L$. Then

$$||\phi+\alpha\varphi||_\rho^p = \int_{-L}^{L}|\phi(s)+\alpha\varphi(s)|^p\rho(s)ds \geq \frac{\rho(0)}{Me^{wL}}\int_{-L}^{L}|\phi(s)+\alpha\varphi(s)|^p ds. \tag{56}$$

Combining estimations (55) and (56), we have

$$\left|\left|T_+(t'_k)\left(\frac{\phi}{2}+\frac{\alpha\varphi}{2}\right)\right|\right|_\rho^p \leq \frac{M^2e^{w2L}\rho(t_k)}{\rho(0)2^p}||\phi+\alpha\varphi||_\rho^p. \tag{57}$$

Then, there exists $k_1 \in \mathbb{N}$ such that for all $k \geq k_1$ we have $\left|\left|T_+(t'_k)\left(\frac{\phi}{2}+\frac{\alpha\varphi}{2}\right)\right|\right|_\rho^p \leq \frac{\varepsilon^p}{4}$. In a similar way, there is $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$ we have $\left|\left|T_+(t'_k)\left(\frac{\phi}{2\alpha}+\frac{\varphi}{2}\right)\right|\right|_\rho^p \leq \frac{\varepsilon^p}{4}$. Besides, considering $T_-(t'_k)$ instead of $T_+(t'_k)$ one can obtain similar estimations to (57) but with $\rho(-t_k)$ instead of $\rho(t_k)$, and then we also obtain that there exists $k_3 \in \mathbb{N}$ such that for all $k \geq k_3$ we have $\left|\left|T_-(t'_k)\left(\frac{\phi}{2}-\frac{\alpha\varphi}{2}\right)\right|\right|_\rho^p \leq \frac{\varepsilon^p}{4}$, and $k_4 \in \mathbb{N}$ such that for all $k \geq k_4$ we have $\left|\left|T_-(t'_k)\left(\frac{-\phi}{2\alpha}+\frac{\varphi}{2}\right)\right|\right|_\rho^p \leq \frac{\varepsilon^p}{4}$. From the explicit representation of $\mathcal{T}(t)(\phi,\varphi)$ given in (21), the definition of the norm $||\cdot||$ on $X\times X$, the triangle inequality, the aforementioned estimations, and taking $k_0 := \max\{k_1,k_2,k_3,k_4\}$, we have $||\mathcal{T}(t'_k)(\phi,\varphi)|| \leq \varepsilon$ for all $k \geq k_0$, which gives the proof of (a) in the Hypercyclicity Criterion.

By the structure of group, see Theorem 3.2, and since the operators $\mathcal{S}(t'_k)$ coincide with $\mathcal{T}(-t'_k)$ for all $k \in \mathbb{N}$, one can obtain similar estimations to the former ones and prove (b) in the statement of the Hypercyclicity Criterion. Finally, condition (c) holds just by the group law.

Conversely, there exists $t'_n := t_n/v > L/v$ and $(\phi_n,\varphi_n) \in X\times X$ such that

$$||(\phi_n,\varphi_n)|| < \frac{1}{n} \text{ and } \left|\left|\mathcal{T}(t'_k)(\phi_n,\varphi_n)-(\chi_{[0,L]},0)\right|\right| < \frac{1}{n}. \tag{58}$$

where $\chi_{[0,L]}$ stands for the characteristic function of the interval $[0,L]$. Let us define

$$\left(\tilde{\phi}_n,\tilde{\varphi}_n\right) = \left(\phi|_{[-t_n,-t_n+L]},\varphi|_{[-t_n,-t_n+L]}\right). \tag{59}$$

On the one hand, taking into account the expression of $\mathcal{T}(t'_n)(\tilde{\phi}_n, \tilde{\varphi}_n)$ in (21), we have that

$$\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right) + T_-(t'_n)\left(\frac{\tilde{\phi}_n}{2} - \frac{\alpha\tilde{\varphi}_n}{2}\right) - \chi_{[0,L]}\right|\right|_\rho^p < \frac{1}{n}, \tag{60}$$

which yields

$$\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right) - \chi_{[0,L]}\right|\right|_\rho^p < \frac{1}{n}, \tag{61}$$

and then

$$\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p \geq \left|\left|\chi_{[0,L]}\right|\right|_\rho^p - \frac{1}{n}. \tag{62}$$

Therefore, there exists some $n_0 \in \mathbb{N}$ such that $\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p$ must be strictly positive for every $n \geq n_0$.

On the other hand, using the admissibility of the weight function ρ we first get

$$\begin{aligned}\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p &= \int_0^L |\tilde{\phi}_n(s - vt'_n) + \alpha\tilde{\varphi}_n(s - vt'_n)|^p \rho(s) ds \\ &\leq Me^{wL}\rho(L) \int_{-t_n}^{-t_n+L} |\tilde{\phi}_n(s) + \alpha\tilde{\varphi}_n(s)|^p \rho(s) ds.\end{aligned} \tag{63}$$

and later

$$\begin{aligned}\left|\left|\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p &= \int_{-t_n}^{-t_n+L} |\tilde{\phi}_n(s) + \alpha\tilde{\varphi}_n(s)|^p \rho(s) ds \\ &\geq \frac{1}{Me^{wL}}\rho(-t_n) \int_{-t_n}^{-t_n+L} |\tilde{\phi}_n(s) + \alpha\tilde{\varphi}_n(s)|^p ds.\end{aligned} \tag{64}$$

Finally, combining the estimations (63) and (64), we get

$$\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p \leq \frac{Me^{w2L}\rho(L)}{\rho(-t_n)}\left|\left|\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p \leq \frac{Me^{w2L}\rho(L)}{\rho(-t_n)}\frac{1}{n^p} \tag{65}$$

Therefore, $\left|\left|T_+(t'_n)\left(\frac{\tilde{\phi}_n}{2} + \frac{\alpha\tilde{\varphi}_n}{2}\right)\right|\right|_\rho^p$ tends to 0 which leads to a contradiction with (62). □

Theorem 5.4. *Let $X = L^p_\rho(\mathbb{R})$, with $1 \leq p < \infty$ and ρ an admissible weight function. The following condition holds*

$$\lim_{t\to\infty} \rho(t) = \lim_{t\to-\infty} \rho(t) = 0. \tag{66}$$

if, and only if, the solution C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ of the water hammer equations is topologically mixing.

Proof. Take an arbitrary increasing sequence $\{t_j\} \subseteq \mathbb{R}_+$. By condition (66), then it satisfies that $\lim_{j\to\infty} \rho(t_j) = \lim_{j\to\infty} \rho(-t_j) = 0$. So that, following the guidelines in the proof of Theorem 5.3 we have that the discretization $\{\mathcal{T}(t_j)\}_j$ of the C_0-semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ is topologically mixing. Then, the C_0-semigroup $\{\mathcal{T}(t)\}_{t\in\mathbb{R}}$ is topologically mixing by [5, Prop. 7.21], see also [6].

The proof of the converse implication is analogous to the case of hypercyclicity but taking the sequence of all real positive numbers $t \geq 0$. □

Remark 5.5. Following the same approach as in [21, 5] for showing the hypercyclicity, topologically mixing, and Devaney chaos of the solution C_0-semigroup associated to the Hyperbolic Heat equation on the Herzog's spaces of analytic functions with certain growth control at ∞, as in [22], one can prove that the solution C_0-semigroup corresponding to water hammer equations also presents such behaviors. This results complements similar results shown for the wave equation [21], the Bioheat equation [23], and the Moore-Gibson-Thompson equation in acoustics [24].

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