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NILPOTENT AND ABELIAN HALL SUBGROUPS IN FINITE GROUPS

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ABSTRACT. We give a characterization of the finite groups having nilpotent or abelian Hall π -subgroups which can easily be verified from the character table.

1. INTRODUCTION

One of the main themes in Finite Group Theory is to study the interaction between global and local structure: if p is a prime and G is a finite group, we seek to analyze the relationship between G and its p -local subgroups. Ideally, a local property of G can be read off from the character table of G .

There are not many theorems analyzing local and global structure from the point of view of two different primes, as we do in the main result of this paper.

Theorem A. *Let G be a finite group, and let p and q be different primes. Then some Sylow p -subgroup of G commutes with some Sylow q -subgroup of G if and only if the class sizes of the q -elements of G are not divisible by p and the class sizes of the p -elements of G are not divisible by q .*

Theorem A, of course, gives us a characterization (detectable in the character table) of when a finite group possesses nilpotent Hall $\{p, q\}$ -subgroups. The existence of Hall subgroups (nilpotent or not) is a classical subject in finite group theory, with extensive literature. This characterization in our Theorem A can be seen as a contribution to Richard Brauer's Problem 11. In his celebrated paper [Br], Brauer asks about obtaining information about the existence of (certain) subgroups of a finite group given its character table.

In the course of proving Theorem A we show that for finite simple groups, commuting Sylow subgroups for different primes are actually always abelian, see Theorem 2.1.

Several possible extensions of Theorem A are simply not true. For instance, the fact that p does not divide $|G : \mathbf{C}_G(x)|$ for every q -element $x \in G$ does not guarantee that a

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Sylow p -subgroup of G commutes with some Sylow q -subgroup of G : the semi-affine groups of order $q^p(q^p - 1)p$ provide solvable examples for every choice of p and q . Furthermore, if p does not divide $|G : \mathbf{C}_G(x)|$ for every q -element $x \in G$, it is not even true that P normalizes some Sylow q -subgroup of G ($G = M_{23}$ for $(q, p) = (3, 5)$, or $G = J_4$ with $(q, p) = (3, 7)$ are examples). However, this assertion is true for p -solvable or q -solvable groups, or if $p = 2$, as we will show in Section 4 below.

Recently, the finite groups with nilpotent Hall subgroups have also received attention in [M]. Using Theorem A and the results of [M], we can deduce the following.

Theorem B. *Let G be a finite group, and let π be a set of primes. Then G has nilpotent Hall π -subgroups if and only if for every pair of distinct primes $p, q \in \pi$, the class sizes of the p -elements of G are not divisible by q .*

Theorem B gives an easy algorithm to determine if a group has nilpotent Hall π -subgroups from the character table. (The somewhat weaker result that the property of having nilpotent Hall subgroups is shared by groups with the same character table was proven in [KS].)

As a consequence of Theorem B and the main results of [NT] and [NST] now we have the following explicit way to detect from the character table of a finite group G whether G possesses an abelian Hall π -subgroup, for any set π of primes.

Theorem C. *Let G be a finite group, and let π be a set of primes. Then G has abelian Hall π -subgroups if and only if the two following conditions hold:*

- (i) *For every $p \in \pi$ and every p -element $x \in G$, $|G : \mathbf{C}_G(x)|$ is a π' -integer.*
- (ii) *For every $p \in \pi \cap \{3, 5\}$ and for every irreducible character χ in the principal p -block of G , $\chi(1)$ is not divisible by p .*

As it might be expected, the proofs of Theorems A, B and C use the Classification of Finite Simple Groups.

2. SIMPLE GROUPS AND THEOREM A

The goal of this section is to prove Theorem 2.1, which yields a strong form of Theorem A for simple groups, and another auxiliary result, Theorem 2.2.

Theorem 2.1. *Let S be a finite non-abelian simple group and let p and q be distinct prime divisors of $|S|$. Suppose that all the p -elements of S have q' -conjugacy class sizes and all the q -elements of S have p' -conjugacy class sizes. Then $p, q > 2$ and S has an abelian Hall $\{p, q\}$ -subgroup.*

Since, in contrast to Theorem A, we claim existence of abelian Hall subgroups, Theorem 2.1 obviously does not generalize to arbitrary finite groups.

Theorem 2.2. *Let S be a finite non-abelian simple group of order divisible by an odd prime r . Then S contains a conjugacy class of r -elements of even size.*

We begin with some obvious observations. If $x \in G$, then we denote by x^G the conjugacy class of x in G .

Lemma 2.3. (i) *For each simple group S it suffices to prove Theorem 2.1 and Theorem 2.2 for some quasi-simple group L such that $S \cong L/\mathbf{Z}(L)$.*

(ii) *The case $\min(p, q) = 2$ of Theorem 2.1 follows from Theorem 2.2.*

Proof. (i) Suppose that S satisfies the hypothesis of Theorem 2.1, and that Theorem 2.1 holds for some quasi-simple group L with $S = L/\mathbf{Z}(L)$. Let $g \in L$ be any p -element and let $D/Z := \mathbf{C}_S(gZ)$ for $Z := \mathbf{Z}(L)$. Then for any $x \in D$ we have $xgx^{-1} = f(x)g$ for some $f(x) \in Z$. Moreover, $f \in \text{Hom}(D, Z)$, $\text{Ker}(f) = \mathbf{C}_L(g)$, and $f(x)^{|g|} = 1$ for all $x \in G$, i.e. $\text{Im}(f)$ is a p -group. Thus $|D : C|$ is a p -power. It follows that $|(gZ)^S| = |L : D|$ and $|g^L| = |L : C|$ have the same q -part and so $|g^L|$ is coprime to q . Similarly, $|h^L|$ is coprime to p for all q -elements $h \in L$. Since Theorem 2.1 holds for L , L contains an abelian Hall $\{p, q\}$ -subgroup $P \times Q$, whence $(P \times Q)Z/Z$ is an abelian Hall $\{p, q\}$ -subgroup for S .

A similar argument proves the part of the claim concerning Theorem 2.2.

(ii) Suppose $p = 2 < q$. Since $q \parallel |S|$, by Theorem 2.2 there is a conjugacy class x^S of q -elements in S of even size, a contradiction. \square

Lemma 2.4. *Theorems 2.1 and 2.2 hold in the case S is an alternating group, a sporadic group, or ${}^2F_4(2)'$.*

Proof. The case of 26 sporadic groups and ${}^2F_4(2)'$ can be checked directly using [GAP]. (We remark that the only examples among the sporadic groups are J_1 for $\{p, q\} = \{3, 5\}$ and J_4 for $\{p, q\} = \{5, 7\}$, and in both cases S has cyclic Hall $\{p, q\}$ -subgroups.) Suppose that $S = \mathbf{A}_n$ and $n \geq p > q \geq 2$. Then $\mathbf{C}_{S_n}(g) \cong C_p \times \mathbf{S}_{n-p}$ for a p -cycle $g \in S$. If $p \geq 5$, we have that $|g^{S_n}| = \binom{n}{p}(p-1)!$ is divisible by $2q$ and so q divides $|g^S|$. If $p = 3$, then $n \geq p + 2$ and so $|g^S| = |g^{S_n}|$ is again divisible by q . \square

The rest of the section is devoted to proving Theorems 2.1 and 2.2 for simple groups of Lie type $S \not\cong {}^2F_4(2)'$. For the sake of convenience, we rename p, q in Theorem 2.1 to r, s . We will consider the following setup: $S = G/\mathbf{Z}(G)$, where \mathbf{G} is a simple simply connected algebraic group over the algebraic closure of a finite field of characteristic p and $F : \mathbf{G} \rightarrow \mathbf{G}$ is a Steinberg endomorphism with group of fixed points $G := \mathbf{G}^F$. We let q denote the absolute value of all eigenvalues of F on the character group of an F -stable maximal torus of \mathbf{G} .

Lemma 2.5. *Theorems 2.1 and 2.2 hold in the case S is a simple group of Lie type in characteristic $r = p$.*

Proof. By Lemma 2.4 we may assume that $(S, 2) \neq ({}^2F_4(2)', 2)$. By Lemma 2.3(i) we may replace S by $G = \mathbf{G}^F$. Set $s := 2$ in the case of Theorem 2.2. By [C, Prop. 5.1.7], G contains a regular unipotent p -element $g \in G$. Since $s \nmid |g^G|$, $\mathbf{C}_G(g)$ contains a Sylow s -subgroup Q of G . But every semisimple element in $\mathbf{C}_G(g)$ belongs to $\mathbf{Z}(\mathbf{G})$ by [C, Prop. 5.1.5], whence $Q \leq \mathbf{Z}(G)$. It follows that $s \nmid |S|$ and so we are done. \square

Proof of Theorem 2.2. Assume the contrary: $|S : \mathbf{C}_S(x)|$ is odd for all r -elements $x \in S$. By Lemmas 2.4 and 2.5 we see that S is a simple group of Lie type in characteristic p for some prime $p \neq r$ and $(S, p) \neq ({}^2F_4(2)', 2)$.

Note that if $1 \neq g \in S$ is a **real** r -element, then $|\mathbf{N}_S(\langle g \rangle) : \mathbf{C}_S(g)|$ is even and so $|g^S|$ is even. Thus S cannot contain any real r -element $g \neq 1$. By [TZ, Prop. 3.1], it follows that

$$S \in \{\mathrm{PSL}_n(q), \mathrm{PSU}_n(q) \mid n \geq 3\} \cup \{P\Omega_{4n+2}^\pm(q) \mid n \geq 2\} \cup \{E_6(q), {}^2E_6(q)\}.$$

By Lemma 2.3(i) we may replace S by $G = \mathbf{G}^F$, or by some quotient $L = G/Z$ with $Z \leq \mathbf{Z}(G)$.

Suppose first that $S = \mathrm{SL}_n(q)$ with $n \geq 3$, and set $k := \mathrm{ord}_r(q) \leq n$. If $k \leq n/2$, then $\mathrm{SL}_n(q) \geq \mathrm{Sp}_{2k}(q)$ contains a nontrivial real r -element by [TZ, Prop. 3.1], a contradiction. Similarly, if $2|k$, then again $\mathrm{SL}_n(q) \geq \mathrm{Sp}_k(q)$ contains a nontrivial real r -element. Thus $k > n/2$ and k is odd. Now it is easy to see that $H := \mathrm{SL}_k(q)$ contains an r -element g with $\mathbf{C}_H(g) \cong C_{(q^k-1)/(q-1)}$. Embedding H naturally in $\mathrm{SL}_n(q)$, we get that

$$|g^G| = \frac{|\mathrm{GL}_n(q)|}{(q^k - 1) \cdot |\mathrm{GL}_{n-k}(q)|} = q^{\binom{n}{2} - \binom{n-k}{2}} \cdot \frac{\prod_{j=n-k+1}^n (q^j - 1)}{q^k - 1}$$

is even.

The same argument as above applies to the case $G = \mathrm{SU}_n(q)$ if we replace q by $-q$.

Suppose now that $S = P\Omega_{4n+2}^\epsilon(q)$ with $n \geq 2$. Then we replace S by $L = \Omega_{4n+2}^\epsilon(q)$. If $r \mid \prod_{j=1}^{2n} (q^{2j} - 1)$, then $\Omega_{4n+2}^\epsilon(q) > \Omega_{4n+1}(q)$ contains a nontrivial real r -element by [TZ, Prop. 3.1], a contradiction. So $r \nmid \prod_{j=1}^{2n} (q^{2j} - 1)$ but $r \mid (q^{2n+1} - \epsilon)$. Now it is easy to see that $H := \mathrm{SO}_{4n+2}^\epsilon(q)$ contains an r -element $g \in \Omega_{4n+2}^\epsilon(q)$ with $\mathbf{C}_H(g) \cong C_{q^{2n+1}-\epsilon}$. It follows that $|g^L|$ is even.

Finally, let $G := E_6^\epsilon(q)_{sc}$ with $\epsilon = +$ for $E_6(q)$ and $\epsilon = -$ for ${}^2E_6(q)$. If r divides $|F_4(q)|$, then $G > F_4(q)$ contains a nontrivial real r -element by [TZ, Prop. 3.1], a contradiction. So $r \nmid |F_4(q)|$ but $r \mid (q^5 - \epsilon)(q^9 - \epsilon)$. In particular, r is a Zsigmondy prime divisor for $q^5 - \epsilon$ or $q^9 - \epsilon$. Inspecting the centralizers of semisimple elements in G of order divisible by r , as described in [D], one sees that there exist r -elements h with $|C_G(h)|$ even. \square

We will now prove the following result, which, together with Lemmas 2.4, 2.5 and Theorem 2.2, implies Theorem 2.1.

Theorem 2.6. *Let S be a finite non-abelian simple group of Lie type defined over \mathbb{F}_q , q a power of a prime p , and let r and s be distinct odd prime divisors of $|S|$ different from p . Suppose that all the r -elements of S have s' -conjugacy class size and all the s -elements of S have r' -conjugacy class size. Then S has an abelian Hall $\{r, s\}$ -subgroup.*

Proposition 2.7. *Theorem 2.6 holds for $S = \mathrm{PSL}_n(q)$.*

Proof. By Lemma 2.3(i) we may replace S by $G = \mathrm{SL}_n(q)$. Let $k := \mathrm{ord}_r(q)$ and $l := \mathrm{ord}_s(q)$; also let $(q^k - 1)_r =: r^t$. Since r and s divide $|S|$, $n \geq k, l$. By symmetry we may assume $k \geq l$.

(a) Suppose first that $k \geq 2$, and so $r \nmid (q-1)$. Let $m \in \mathbb{Z}$ be such that $r^m \leq n/k < r^{m+1}$ and denote $\kappa := kr^m$. Since $r > 2$, $(q^\kappa - 1)_r = r^{m+t}$. Furthermore, since $r \nmid (q-1)$, the Singer cycle $\mathrm{GL}_1(q^\kappa) \leq \mathrm{GL}_\kappa(q)$ contains an r -element $h \in \mathrm{SL}_\kappa(q)$ of order r^{m+t} . Then for the r -element $g := \mathrm{diag}(h, I_{n-\kappa}) \in G$ we have that

$$|g^G| = \frac{|\mathrm{GL}_n(q)|}{(q^\kappa - 1) \cdot |\mathrm{GL}_{n-\kappa}(q)|} = \frac{|\mathrm{GL}_n(q)|}{|\mathrm{GL}_\kappa(q)| \cdot |\mathrm{GL}_{n-\kappa}(q)|} \cdot \frac{|\mathrm{GL}_\kappa(q)|}{q^\kappa - 1}$$

which is divisible by

$$\frac{|\mathrm{GL}_\kappa(q)|_{p'}}{q^\kappa - 1} = \prod_{j=1}^{\kappa-1} (q^j - 1).$$

Since $s \nmid |g^G|$, we must have that $l \geq \kappa = kr^m \geq k \geq l$. It follows that $k = l$, $m = 0$, i.e. $n/k < r$.

(b) Next we consider the case $k = l = 1$, i.e. $r, s \mid (q-1)$.

Suppose first that $n \geq r+1$. Then we can find $\alpha \in \overline{\mathbb{F}}_q^\times$ of order $(q^r - 1)_r = r^{t+1}$ and consider the r -element $g \in G$ conjugate (in $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}}_q)$) to

$$\mathrm{diag}(\alpha, \alpha^q, \dots, \alpha^{q^{r-1}}, \alpha^{-\frac{q^r-1}{q-1}}, \underbrace{1, \dots, 1}_{n-r-1}).$$

Note that $1 \neq \alpha^{-\frac{q^r-1}{q-1}} \in \mathbb{F}_q^\times \not\cong \alpha$. It follows that

$$|g^G| = \frac{|\mathrm{SL}_n(q)|}{(q^r - 1) \cdot |\mathrm{GL}_{n-r-1}(q)|} = \frac{|\mathrm{GL}_n(q)|}{|\mathrm{GL}_{r+1}(q)| \cdot |\mathrm{GL}_{n-r-1}(q)|} \cdot \frac{|\mathrm{GL}_{r+1}(q)|}{(q^r - 1)(q-1)}$$

which is divisible by

$$\prod_{j=2}^{r-1} (q^j - 1) \cdot (q^{r+1} - 1),$$

a multiple of s , a contradiction.

We have shown that $n \leq r$, and so $n \leq s$ as well by symmetry. Assume now that $n = r < s$. Then we can find $\beta \in \mathbb{F}_q^\times$ of order s and consider the s -element

$$h := \mathrm{diag}(\beta, \beta^{-1}, \underbrace{1, \dots, 1}_{r-2}) \in G.$$

Then

$$|h^G| = \frac{|\mathrm{SL}_r(q)|}{(q-1) \cdot |\mathrm{GL}_{r-2}(q)|} = q^{2r-3} \cdot \frac{q^r - 1}{q-1} \cdot \frac{q^{r-1} - 1}{q-1}$$

which is divisible by r , a contradiction.

(c) We have shown that $k = l$ and $n/k < \min(r, s)$. In this case, for $c := \lfloor n/k \rfloor$ we have that

$$\mathrm{GL}_n(q) \geq \mathrm{GL}_k(q)^c \geq \mathrm{GL}_1(q^k)^c$$

contains an abelian Hall $\{r, s\}$ -subgroup, and so does G . \square

Proposition 2.8. *Theorem 2.6 holds for $S = \mathrm{PSU}_n(q)$.*

Proof. By Lemma 2.3 we may replace S by $G = \mathrm{SU}_n(q)$. Let $k := \mathrm{ord}_r(-q)$ and $l := \mathrm{ord}_s(-q)$; also let $(q^k - (-1)^k)_r =: r^t$. Since r, s divide $|S|$, we have that $n \geq k, l$. By symmetry we may assume $k \geq l$. For any monic irreducible polynomial $f(x) \in \mathbb{F}_{q^2}[x]$ with a nonzero root γ , let \check{f} denote the (unique) monic irreducible polynomial over \mathbb{F}_{q^2} with γ^{-q} as a root.

(a) First we consider the case k is odd.

(a1) Suppose that $k \geq 3$ is odd, and so $r \nmid (q+1)$. Choose $m \in \mathbb{Z}$ such that $r^m \leq n/k < r^{m+1}$ and denote $\kappa := kr^m$. Since $r > 2$, $(q^\kappa + 1)_r = r^{m+t}$. Furthermore, since $r \nmid (q+1)$, the cyclic subgroup $\mathrm{GU}_1(q^\kappa) \leq \mathrm{GU}_\kappa(q)$ contains an r -element $h \in \mathrm{SU}_\kappa(q)$ of order r^{m+t} , with an eigenvalue $\theta \in \overline{\mathbb{F}}_q^\times$ of order r^{m+t} . One can check that the minimal polynomial $f(x) \in \mathbb{F}_{q^2}[x]$ for θ over \mathbb{F}_{q^2} has degree κ , and $f = \check{f}$. Now for the r -element $g := \mathrm{diag}(h, I_{n-\kappa}) \in G$ we have that

$$|g^G| = \frac{|\mathrm{GU}_n(q)|}{(q^\kappa + 1) \cdot |\mathrm{GU}_{n-\kappa}(q)|} = \frac{|\mathrm{GU}_n(q)|}{|\mathrm{GU}_\kappa(q)| \cdot |\mathrm{GU}_{n-\kappa}(q)|} \cdot \frac{|\mathrm{GU}_\kappa(q)|}{q^\kappa + 1}$$

which is divisible by

$$\frac{|\mathrm{GU}_\kappa(q)|_{p'}}{q^\kappa + 1} = \prod_{j=1}^{\kappa-1} (q^j - (-1)^j).$$

Since $s \nmid |g^G|$, we must have that $l \geq \kappa = kr^m \geq k \geq l$. It follows that $k = l$, $m = 0$, i.e. $n/k < r$.

(a2) Next we consider the case $k = l = 1$, i.e. $r, s \mid (q+1)$.

Suppose first that $n \geq r+1$. Then we can find $\alpha \in \overline{\mathbb{F}}_q^\times$ of order $(q^r + 1)_r = r^{t+1}$ and consider the r -element $g \in G$ conjugate (in $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}}_q)$) to

$$\mathrm{diag}(\alpha, \alpha^{-q}, \dots, \alpha^{(-q)^{r-1}}, \alpha^{-\frac{q^r+1}{q+1}}, \underbrace{1, \dots, 1}_{n-r-1}).$$

Note that $\alpha^{-\frac{q^r-1}{q-1}}$ has order $r^t \mid (q+1)$, and $f = \check{f}$ for the minimal polynomial $f(x)$ of α over \mathbb{F}_{q^2} (which has odd degree r). Now one can check that

$$|g^G| = \frac{|\mathrm{SU}_n(q)|}{(q^r + 1) \cdot |\mathrm{GU}_{n-r-1}(q)|} = \frac{|\mathrm{GU}_n(q)|}{|\mathrm{GU}_{r+1}(q)| \cdot |\mathrm{GU}_{n-r-1}(q)|} \cdot \frac{|\mathrm{GU}_{r+1}(q)|}{(q^r + 1)(q + 1)}$$

which is divisible by

$$\prod_{j=2}^{r-1} (q^j - (-1)^j) \cdot (q^{r+1} - 1),$$

a multiple of s , a contradiction.

We have shown that $n \leq r$, and so $n \leq s$ as well by symmetry. Assume now that $n = r < s$. Then we can find $\beta \in \mathbb{F}_{q^2}^\times$ of order $s|(q+1)$ and consider the s -element

$$h := \text{diag}(\beta, \beta^{-1}, \underbrace{1, \dots, 1}_{r-2}) \in G.$$

Then

$$|h^G| = \frac{|\text{SU}_r(q)|}{(q+1) \cdot |\text{GU}_{r-2}(q)|} = q^{2r-3} \cdot \frac{q^r + 1}{q+1} \cdot \frac{q^{r-1} - 1}{q+1}$$

which is divisible by r , a contradiction.

(a3) We have shown that $k = l$ and $n/k < \min(r, s)$. In this case, for $c := \lfloor n/k \rfloor$ we have that

$$\text{GU}_n(q) \geq \text{GU}_k(q)^c \geq \text{GU}_1(q^k)^c$$

contains an abelian Hall $\{r, s\}$ -subgroup, and so does G .

(b) Now we consider the case k is even: $k = 2k_1$; in particular, $r \nmid (q+1)$. Choose $m \in \mathbb{Z}$ such that $r^m \leq n/k_1 < r^{m+1}$ and denote $\kappa := k_1 r^m$. Since $r > 2$, $(q^{2\kappa} - 1)_r = r^{m+t}$. Furthermore, since $r \nmid (q+1)$, the cyclic subgroup

$$\text{GL}_1(q^{2\kappa}) \leq \text{GL}_\kappa(q^2) \leq \text{GU}_{2\kappa}(q)$$

contains an r -element $h \in \text{SU}_{2\kappa}(q)$ of order r^{m+t} , with an eigenvalue $\theta \in \overline{\mathbb{F}}_q^\times$ of order r^{m+t} . One can check that the minimal polynomial $f(x) \in \mathbb{F}_{q^2}[x]$ for θ over \mathbb{F}_{q^2} has degree κ , and $f \neq \check{f}$.

Now for the r -element $g := \text{diag}(h, I_{n-2\kappa}) \in G$ we have that

$$|g^G| = \frac{|\text{GU}_n(q)|}{(q^{2\kappa} - 1) \cdot |\text{GU}_{n-2\kappa}(q)|} = \frac{|\text{GU}_n(q)|}{|\text{GU}_{2\kappa}(q)| \cdot |\text{GU}_{n-2\kappa}(q)|} \cdot \frac{|\text{GU}_{2\kappa}(q)|}{q^{2\kappa} - 1}$$

which is divisible by

$$\frac{|\text{GU}_{2\kappa}(q)|_{p'}}{q^{2\kappa} - 1} = \prod_{j=1}^{2\kappa-1} (q^j - (-1)^j).$$

Since $s \nmid |g^G|$, we must have that $l \geq 2\kappa = kr^m \geq k \geq l$. It follows that $k = l$, $m = 0$, i.e. $n/k < r$. Hence, by symmetry, $n/k < \min(r, s)$. Now for $c := \lfloor n/k \rfloor$ we have that

$$\text{GU}_n(q) \geq \text{GU}_k(q)^c > \text{GL}_{k_1}(q^2)^c \geq \text{GL}_1(q^k)^c$$

contains an abelian Hall $\{r, s\}$ -subgroup, and so does G . \square

Proposition 2.9. *Theorem 2.6 holds for S a simple symplectic or orthogonal group.*

Proof. By Lemma 2.3(i) we may replace $S = \mathrm{PSp}_{2n}(q)$ by $G := \mathrm{Sp}_{2n}(q)$; set $d := 2n$, and $\mathrm{Cl} = \mathrm{Cl}^+ = \mathrm{Cl}^- := \mathrm{Sp}$ in this case. If $S = P\Omega_d^{(\epsilon)}(q)$, then, since both r and s are odd, we may replace S by $G := \mathrm{SO}_d^{(\epsilon)}(q)$; here $\epsilon = \pm$, and we set $d = 2n$ or $2n + 1$, and $\mathrm{Cl} := \mathrm{SO}$. Let $k := \mathrm{ord}_r(q)$ and $l := \mathrm{ord}_s(q)$; also let $(q^k - 1)_r =: r^t$. Since r divides $|S|$, we have that $2n \geq k$; moreover, $n \geq k$ if k is odd. The same conditions hold for l . By symmetry we may assume $k \geq l$. For any monic irreducible polynomial $f(x) \in \mathbb{F}_q[x]$ with a nonzero root γ , let \check{f} denote the (unique) monic irreducible polynomial over \mathbb{F}_q with γ^{-1} as a root.

(a) First we consider the case k is odd; in particular, $k \leq n$. Let $m \in \mathbb{Z}$ be such that $r^m \leq n/k < r^{m+1}$ and denote $\kappa := kr^m$. Since $r > 2$, $(q^\kappa - 1)_r = r^{m+t}$.

(a1) Assume that $k = l$ and $n/k < \min(r, s)$. Setting $c := \lfloor n/k \rfloor$, we get chains of subgroups of index coprime to rs

$$\mathrm{Sp}_{2n}(q) \geq \mathrm{Sp}_{2kc}(q) > \mathrm{GL}_{kc}(q) \geq \mathrm{GL}_k(q)^c \geq \mathrm{GL}_1(q^k)^c,$$

$$\mathrm{SO}_{2n+1}(q) \geq \mathrm{SO}_{2n}^+(q) \geq \mathrm{SO}_{2kc}^+(q) > \mathrm{GL}_{kc}(q) \geq \mathrm{GL}_k(q)^c \geq \mathrm{GL}_1(q^k)^c,$$

$$\mathrm{SO}_{2n}^-(q) \geq \mathrm{SO}_{2n-2}^+(q) \geq \mathrm{SO}_{2k(c-1)}^+(q) > \mathrm{GL}_{k(c-1)}(q) \geq \mathrm{GL}_k(q)^{c-1} \geq \mathrm{GL}_1(q^k)^{c-1},$$

and so G contains an abelian Hall $\{r, s\}$ -subgroup. So now we will assume that we are not in this case.

(a2) Suppose that $G \neq \mathrm{SO}_{2\kappa}^-(q)$, and so G contains a natural subgroup $\mathrm{Cl}_{2\kappa}^+(q)$. Then the cyclic subgroup

$$\mathrm{GL}_1(q^\kappa) \leq \mathrm{GL}_{\kappa}(q) < \mathrm{Cl}_{2\kappa}^+(q)$$

contains an r -element h of order r^{m+t} , with an eigenvalue $\theta \in \overline{\mathbb{F}}_q^\times$ of order r^{m+t} . One can check that the minimal polynomial $f(x) \in \mathbb{F}_q[x]$ for θ over \mathbb{F}_q has degree κ , and $f \neq \check{f}$. Now for the r -element $g := \mathrm{diag}(h, I_{d-2\kappa}) \in G$ we have that

$$|g^G| = \frac{|\mathrm{Cl}_d^\epsilon(q)|}{(q^\kappa - 1) \cdot |\mathrm{Cl}_{d-2\kappa}^\epsilon(q)|} = \frac{|\mathrm{Cl}_d^\epsilon(q)|}{|\mathrm{Cl}_{2\kappa}^+(q)| \cdot |\mathrm{Cl}_{d-2\kappa}^\epsilon(q)|} \cdot \frac{|\mathrm{Cl}_{2\kappa}^+(q)|}{q^\kappa - 1}$$

which is divisible by

$$\frac{|\mathrm{Cl}_{2\kappa}^+(q)|}{q^\kappa - 1},$$

a multiple of $\prod_{j=1}^{\kappa-1} (q^{2j} - 1)$. Since $s \nmid |g^G|$, we must have that $l \geq \kappa = kr^m \geq l$. It follows that $k = l$, $m = 0$, i.e. $n/k < r$.

As the roles of r and s are symmetric, repeating the above argument but with r replaced by s in the case $G \neq \mathrm{SO}_{2ks^i}^-(q)$ for $i \geq 0$, we see that $n/k < s$ and arrive at (a1).

(a3) It remains to consider the case $G = \mathrm{SO}_{2\kappa}^-(q)$ (so that $n = \kappa = kr^m$). Since $r \nmid (q^k + 1)$, we have that $m \geq 1$. Set $\kappa_1 := kr^{m-1}$. Then the cyclic subgroup

$$\mathrm{GL}_1(q^{\kappa_1}) \leq \mathrm{GL}_{\kappa_1}(q) < \mathrm{SO}_{2\kappa_1}^+(q)$$

contains an r -element h_1 of order r^{m+t-1} , with an eigenvalue $\theta_1 \in \overline{\mathbb{F}}_q^\times$ of order r^{m+t-1} . One can check that the minimal polynomial $f_1(x) \in \mathbb{F}_q[x]$ for θ_1 over \mathbb{F}_q has degree κ_1 , and $f_1 \neq \check{f}_1$. Now for the r -element $g_1 := \text{diag}(h_1, I_{2n-2\kappa_1}) \in G$ we have that

$$|(g_1)^G| = \frac{|\text{SO}_{2n}^-(q)|}{(q^{\kappa_1} - 1) \cdot |\text{SO}_{2n-2\kappa_1}^-(q)|} = \frac{|\text{SO}_{2n}^-(q)|}{|\text{SO}_{2\kappa_1}^+(q)| \cdot |\text{SO}_{2n-2\kappa_1}^-(q)|} \cdot \frac{|\text{SO}_{2\kappa_1}^+(q)|}{q^{\kappa_1} - 1}$$

which is divisible by

$$\frac{|\text{SO}_{2\kappa_1}^+(q)|_{p'}}{q^{\kappa_1} - 1} = \prod_{j=1}^{\kappa_1-1} (q^{2j} - 1).$$

Since $s \nmid |(g_1)^G|$, we must have that $l \geq \kappa_1 = kr^{m-1} \geq l$. It follows that $k = l$, $m = 1$, $n = kr$, $G = \text{SO}_{2kr}^-(q)$.

Now the roles of r and s are symmetric, and $G \neq \text{SO}_{2ks^i}^-(q)$ for any $i \geq 0$. So, repeating the argument in (a2) but with r replaced by s , we see that $r = n/k < s$. In this case, as shown in (a1), the subgroup $\text{SO}_{2kr-1}(q)$ of index coprime to rs in G contains an abelian Hall $\{r, s\}$ -subgroup H , and so H is also an abelian Hall $\{r, s\}$ -subgroup of G .

(b) Now we consider the case k is even; in particular, $n \geq k_1 := k/2$. Let $m \in \mathbb{Z}$ be such that $r^m \leq n/k_1 < r^{m+1}$ and denote $\kappa := k_1 r^m$. By the choice of k , $r \nmid (q^{k_1} - 1)$, whence $r \mid (q^{k_1} + 1)$. Since $r > 2$, $(q^\kappa + 1)_r = r^{m+t}$.

(b1) Assume that $k = l$ and $n/k_1 < \min(r, s)$. Setting $c := \lfloor n/k_1 \rfloor$, we get chains of subgroups of index coprime to rs

$$\begin{aligned} \text{Sp}_{2n}(q) &\geq \text{Sp}_{2k_1 c}(q) \geq \text{Sp}_{2k_1}(q)^c \geq \text{Sp}_2(q^{k_1})^c \geq \text{GU}_1(q^{k_1})^c, \\ \text{SO}_{2n+1}(q) &\geq \text{SO}_{2n}^{(-1)^c}(q) \geq \text{SO}_{2k_1 c}^{(-1)^c}(q) \geq \text{SO}_{2k_1}^-(q)^c > \text{GU}_1(q^{k_1})^c, \\ \text{SO}_{2n}^{(-1)^{c-1}}(q) &> \text{SO}_{2n-1}(q) \geq \text{SO}_{2k_1(c-1)}^{(-1)^{c-1}}(q) \geq \text{SO}_{2k_1}^-(q)^{c-1} \geq \text{GU}_1(q^{k_1})^{c-1}, \end{aligned}$$

and so G contains an abelian Hall $\{r, s\}$ -subgroup. So we will assume that we are not in this case.

(b2) Suppose that $G \neq \text{SO}_{2\kappa}^+(q)$, and so G contains a natural subgroup $\text{Cl}_{2\kappa}^-(q)$. Then the cyclic subgroup

$$\text{GU}_1(q^\kappa) < \text{Cl}_{2\kappa}^-(q)$$

contains an r -element h of order r^{m+t} , with an eigenvalue $\theta \in \overline{\mathbb{F}}_q^\times$ of order r^{m+t} . One can check that the minimal polynomial $f(x) \in \mathbb{F}_q[x]$ for θ over \mathbb{F}_q has degree 2κ , and $f = \check{f}$. Now for the r -element $g := \text{diag}(h, I_{d-2\kappa}) \in G$ we have that

$$|g^G| = \frac{|\text{Cl}_d^\epsilon(q)|}{(q^\kappa + 1) \cdot |\text{Cl}_{d-2\kappa}^{-\epsilon}(q)|} = \frac{|\text{Cl}_d^\epsilon(q)|}{|\text{Cl}_{2\kappa}^-(q)| \cdot |\text{Cl}_{d-2\kappa}^{-\epsilon}(q)|} \cdot \frac{|\text{Cl}_{2\kappa}^-(q)|}{q^\kappa + 1}$$

which is divisible by

$$\frac{|\text{Cl}_{2\kappa}^-(q)|}{q^\kappa + 1},$$

a multiple of $\prod_{j=1}^{\kappa-1} (q^{2j} - 1)$. Since $s \nmid |g^G|$, we must have that $l \geq \kappa = k_1 r^m$. If $m \geq 1$ in addition, then $l \geq k_1 r \geq 3k_1 > k$, a contradiction. So $m = 0$, $n/k_1 < r$, and $k/2 = k_1 \leq l \leq k$.

Now if $2|l < k$, then s divides $q^{2j} - 1$ with $1 \leq j := l/2 < \kappa$, again a contradiction. So when $2|l$, we must have that $k = l$. In this case, the roles of r and s are symmetric. Therefore, in the case $G \neq \mathrm{SO}_{2ks^i}^+(q)$ for $i \geq 0$, repeating the above argument but with r replaced by s , we see that $n/k < s$ and arrive at (b1).

(b3) Here we consider the case where $G \neq \mathrm{SO}_{2\kappa}^+(q)$ but l is odd. As shown in (b2), we now have $n/k_1 < r$, $k/2 = k_1 \leq l < k$. Write $n = ak_1 + b$ with $1 \leq a < r$ and $0 \leq b < k_1$. We again consider the element $h \in \mathrm{GU}_1(q^\kappa)$ as in (b2).

Assume in addition that $G \neq \mathrm{SO}_{2a\kappa}^{-\alpha}(q)$ with $\alpha := (-1)^a$, and so G contains a natural subgroup $\mathrm{Cl}_{2a\kappa}^{(-1)^a}(q) > \mathrm{GU}_1(q^\kappa)^a$. Now for the r -element

$$g_1 := \mathrm{diag}(\underbrace{h, \dots, h}_a, I_{d-2a\kappa}) \in G$$

we have that

$$|(g_1)^G| = \frac{|\mathrm{Cl}_a^\epsilon(q)|}{|\mathrm{GU}_a(q^\kappa) \cdot |\mathrm{Cl}_{d-2a\kappa}^{\alpha\epsilon}(q)|} = \frac{|\mathrm{Cl}_d^\epsilon(q)|}{|\mathrm{Cl}_{2a\kappa}^\alpha(q) \cdot |\mathrm{Cl}_{d-2a\kappa}^{\alpha\epsilon}(q)|} \cdot \frac{|\mathrm{Cl}_{2a\kappa}^\alpha(q)|}{|\mathrm{GU}_a(q^\kappa)|}$$

which is divisible by

$$\frac{|\mathrm{Cl}_{2a\kappa}^\alpha(q)|}{|\mathrm{GU}_a(q^\kappa)|}.$$

Since $s \nmid |(g_1)^G|$, we must have that

$$(1) \quad s \nmid \prod_{j=1, k_1 \nmid j}^{ak_1-1} (q^{2j} - 1) \cdot \prod_{i=1}^{a-1} (q^{ik_1} + (-1)^i).$$

In particular, if $a \geq 2$, then $s \nmid (q^{k_1} - 1)$ and so $l > k_1$. As $l \leq k$ and l is odd, it also follows that $k_1 \nmid l$. In this case, $s \mid (q^{2j} - 1)$ with $1 \leq j := l \leq k - 1 \leq ak_1 - 1$ and $k_1 \nmid j$, contrary to (1). So $a = 1$ and $k_1 \leq n < 2k_1 = k$. Also, recall that l is odd and $k_1 \leq l \leq n$. As s divides $|S|$, we must have that $G \neq \mathrm{SO}_{2l}^-(q)$, and so G contains a natural subgroup $\mathrm{Cl}_{2l}^+(q)$. Then the cyclic subgroup

$$\mathrm{GL}_1(q^l) < \mathrm{Cl}_{2l}^+(q)$$

contains an s -element h_2 of order $(q^l - 1)_s$, with an eigenvalue $\theta_2 \in \overline{\mathbb{F}}_q^\times$ of order $(q^l - 1)_s$. One can check that the minimal polynomial $f_2(x) \in \mathbb{F}_q[x]$ for θ_2 over \mathbb{F}_q has degree l , and $f_2 \neq \check{f}_2$. Now for the s -element $g_2 := \mathrm{diag}(h_2, I_{d-2l}) \in G$ we have that

$$(2) \quad |g_2^G| = \frac{|\mathrm{Cl}_d^\epsilon(q)|}{(q^l - 1) \cdot |\mathrm{Cl}_{d-2l}^\epsilon(q)|} = \frac{|\mathrm{Cl}_d^\epsilon(q)|}{|\mathrm{Cl}_{2l}^+(q) \cdot |\mathrm{Cl}_{d-2l}^\epsilon(q)|} \cdot \frac{|\mathrm{Cl}_{2l}^+(q)|}{q^l - 1}$$

which is divisible by

$$\frac{|\mathrm{Cl}_{2l}^+(q)|}{q^l - 1},$$

a multiple of $\prod_{j=1}^{l-1} (q^{2j} - 1)$. Since $r \nmid |g_2^G|$, we must have that $k_1 \geq l$, whence in fact $k_1 = l$. But in this case, $r \nmid (q^l - 1) \cdot |\mathrm{Cl}_{d-2l}^e(q)|$, and so (2) implies that r divides $|g_2^G|$, a contradiction.

Now suppose that $G = \mathrm{SO}_{2a\kappa}^{-\alpha}(q)$. Since $r \parallel |G|$, we must have that $a \geq 2$. Now for the r -element

$$g_3 := \mathrm{diag}(\underbrace{h, \dots, h}_{a-1}, I_{2\kappa}) \in G$$

we have that

$$|(g_3)^G| = \frac{|\mathrm{SO}_{2a\kappa}^{-\alpha}(q)|}{|\mathrm{GU}_{a-1}(q^\kappa) \cdot |\mathrm{SO}_{2\kappa}^+(q)||},$$

and so

$$(3) \quad s \nmid \frac{\prod_{j=1, k_1 \nmid j}^{ak_1-1} (q^{2j} - 1) \cdot \prod_{i=2}^a (q^{ik_1} + (-1)^i)}{\prod_{i=1}^{k_1-1} (q^{2i} - 1)}.$$

Since $l \geq k_1$ is odd, s does not divide the denominator in (3). On the other hand, if $k_1 \nmid l$, then s divides $q^{2l} - 1$ and so divides the numerator of (3), a contradiction. So $k_1 | l < k = 2k_1$, i.e. $l = k_1$. If $a \geq 3$, then s divides $q^{3k_1} - 1$ and so divides the numerator, again a contradiction. We conclude that $a = 2$, $G = \mathrm{SO}_{4l}^-(q)$. In this case,

$$\mathrm{GL}_1(q^l) \times \mathrm{GU}_1(q^l) < \mathrm{SO}_{2l}^+(q) \times \mathrm{SO}_{2l}^-(q) < \mathrm{SO}_{4l}^-(q)$$

contains an abelian Hall $\{r, s\}$ -subgroup of G .

(b4) It remains to consider the case $G = \mathrm{SO}_{2\kappa}^+(q)$ (so that $n = \kappa = k_1 r^m$). Since $r \nmid (q^{k_1} - 1)$, we have that $m \geq 1$. Set $\kappa_1 := k_1 r^{m-1}$. Then the cyclic subgroup

$$\mathrm{GU}_1(q^{\kappa_1}) < \mathrm{SO}_{2\kappa_1}^-(q)$$

contains an r -element h_1 of order r^{m+t-1} , with an eigenvalue $\theta_1 \in \overline{\mathbb{F}}_q^\times$ of order r^{m+t-1} . One can check that the minimal polynomial $f_1(x) \in \mathbb{F}_q[x]$ for θ_1 over \mathbb{F}_q has degree $2\kappa_1$, and $f_1 = \check{f}_1$. Now for the r -element

$$g_1 := \mathrm{diag}(\underbrace{h_1, \dots, h_1}_{r-1}, I_{2\kappa_1}) \in G$$

we have that

$$|g^G| = \frac{|\mathrm{SO}_{2\kappa_1 r}^+(q)|}{|\mathrm{GU}_{r-1}(q^{\kappa_1})| \cdot |\mathrm{SO}_{2\kappa_1}^+(q)|} = \frac{|\mathrm{SO}_{2\kappa_1 r}^+(q)|}{|\mathrm{SO}_{2\kappa_1(r-1)}^+(q)| \cdot |\mathrm{SO}_{2\kappa_1}^+(q)|} \cdot \frac{|\mathrm{SO}_{2\kappa_1(r-1)}^+(q)|}{|\mathrm{GU}_{r-1}(q^{\kappa_1})|}.$$

Since $s \nmid |g^G|$, we must have that

$$(4) \quad s \nmid \frac{|\mathrm{SO}_{2\kappa_1(r-1)}^+(q)|_{p'}}{|\mathrm{GU}_{r-1}(q^{\kappa_1})|_{p'}} = \prod_{j=1, \kappa_1 \nmid j}^{\kappa_1(r-1)-1} (q^{2j} - 1) \cdot \prod_{i=1}^{r-2} (q^{i\kappa_1} + (-1)^i).$$

Now if $\kappa_1 \nmid l$, then we have $s|(q^{2l} - 1)$ with $1 \leq l < k = 2k_1 \leq \kappa_1(r-1)$, contrary to (4). So $k_1 r^{m-1} = \kappa_1 |l| \leq k = 2k_1$, yielding $m = 1$ and $k_1 |l$. If furthermore $l = k_1$ then again $s|(q^{\kappa_1} - 1)$, contradicting (4). Thus $k = l$ and $G = \mathrm{SO}_{2k_1 r}^+(q)$.

Now the roles of r and s are symmetric, and $G \neq \mathrm{SO}_{2k_1 s^i}^+(q)$ for any $i \geq 0$. So, repeating the argument in (b2) but with r replaced by s , we see that $r = n/k_1 < s$. In this case, as shown in (b1), the subgroup $\mathrm{SO}_{2k_1 r-1}(q)$ of index coprime to rs in G contains an abelian Hall $\{r, s\}$ -subgroup H , and so H is also an abelian Hall $\{r, s\}$ -subgroup of G . \square

We now turn to the case of exceptional simple groups of Lie type.

Proposition 2.10. *Let \mathbf{G} be simple such that $G = \mathbf{G}^F$ is of exceptional type. Let $r \neq 2$ be a bad prime for \mathbf{G} , or $r = 3$ when $G = {}^3D_4(q)$. Let $S \leq G$ be a Sylow r -subgroup of G . Then $\mathbf{C}_G(S) = \mathbf{Z}(G)\mathbf{Z}(S)$, unless $G = E_7(q)$ and $r = 3$, when $\mathbf{C}_G(S) = \mathbf{Z}(S)T$ with T a torus of order $q - \epsilon$, where $q \equiv \epsilon \pmod{3}$.*

Proof. By assumption we have $r = 3$, or $r = 5$ when \mathbf{G} is of type E_8 . For almost all cases, the normalizer of a Sylow r -subgroup of G is given in [MN, Tab. 1], and the claim follows. The only remaining case is when $G = E_7(q)$ with $r = 3$. Here, S is contained in a Levi subgroup of type E_6 or 2E_6 , and the claim follows from the case of E_6 discussed before. \square

In what follows, we consider the cyclotomic polynomials, over $\mathbb{Q}(\sqrt{p})$ in the case of Suzuki-Ree groups (see [BM, §F]) and over \mathbb{Q} otherwise, that occur in the generic order of G .

Proposition 2.11. *Let \mathbf{G} be simple such that $G = \mathbf{G}^F$ is of exceptional type. Let $r \neq 2$ be a good prime for \mathbf{G} which divides two distinct cyclotomic polynomials $\Phi_d(q)$ occurring in the generic order of G . Then the following holds for a Sylow r -subgroup of G :*

- (a) $G = E_6(q)$, $r = 5$, $|\mathbf{C}_G(S)|$ divides $|\mathbf{Z}(S)|q(q-1)^2$;
- (b) $G = {}^2E_6(q)$, $r = 5$, $|\mathbf{C}_G(S)|$ divides $|\mathbf{Z}(S)|q(q+1)^2$;
- (c) $G = E_7(q)$, $r = 5$, $|\mathbf{C}_G(S)|$ divides $|\mathbf{Z}(S)|q(q \pm 1)^3$; or
- (d) $G = E_8(q)$, $r = 7$, $|\mathbf{C}_G(S)|$ divides $|\mathbf{Z}(S)|q(q \pm 1)^2$.

Proof. Note that the assumptions on r already exclude the Suzuki-Ree groups. Now, if r divides two distinct cyclotomic polynomials occurring in the generic order of G , then r divides the order of the Weyl group of G , see [BM, Cor. 3.13]. Since r was also assumed to be good, the only possibilities are that $r = 5$ and \mathbf{G} is of type E_6 or E_7 , or $r = 7$ and \mathbf{G} is of type E_8 .

First assume that $G = E_6(q)$. If $q \not\equiv 1 \pmod{5}$, then the Sylow 5-subgroups of G are cyclic. Else, a Sylow 5-subgroup of G is contained in a Levi subgroup of type A_4 , with

centralizer of type A_1 , whence the claim in this case. Entirely similar arguments apply for ${}^2E_6(q)$. If $G = E_7(q)$ with $r = 5$, then our assumption gives $q \equiv \pm 1 \pmod{5}$ and again a Sylow 5-subgroup of G lies inside a Levi subgroup of type A_4 , centralized by a Levi subgroup of type A_2 . Finally, in $E_8(q)$, a Sylow 7-subgroup S does not lie in maximal tori if and only if $q \equiv \pm 1 \pmod{7}$. In that case, S is contained in a Levi subgroup of type A_6 , with centralizer of type A_1 . \square

Next, for Suzuki-Ree groups let Φ_{d_1} be the cyclotomic polynomial occurring in the generic order of G such that $r|\Phi_{d_1}(q)$ as in [BM, App. 2], and similarly Φ_{d_2} for s . For other groups, let d_1 and d_2 be the order of q modulo r , respectively s .

Proposition 2.12. *Assume that the Sylow d_1 -tori of \mathbf{G} are maximal tori and that s divides a unique cyclotomic factor in the generic order of G . If every s -element of G centralizes a Sylow r -subgroup of G , then $d_1 = d_2$.*

Proof. Let g be an s -element such that $\mathbf{C}_G(g)$ contains a Sylow r -subgroup. Then by [BM, Cor. 3.13], g lies in a maximal torus \mathbf{T} containing a Sylow d_1 -torus \mathbf{T}_1 of \mathbf{G} . Since \mathbf{T}_1 is a maximal torus by assumption, $\mathbf{T} = \mathbf{T}_1$. So s divides $\Phi_{d_1}(q)$ which by our assumption implies that $d_2 = d_1$. \square

Proposition 2.13. *Assume that $d_1 = d_2$ and neither of r, s divides the order of the Weyl group of \mathbf{G} . Then there exists an abelian $\{r, s\}$ -Hall subgroup of G .*

Proof. Since neither r nor s divides the order of the Weyl group of \mathbf{G} , any Sylow d -torus \mathbf{T}_d of \mathbf{G} , where $d = d_1 = d_2$, has the property that \mathbf{T}_d^F contains a Sylow r -subgroup S_1 and a Sylow s -subgroup S_2 of G by [BM, Cor. 3.13]. In particular, $S_1 \times S_2$ is an abelian Hall $\{r, s\}$ -subgroup of G . \square

Now we can complete the proof of Theorem 2.6:

Theorem 2.14. *The assertion of Theorem 2.6 holds for S a simple exceptional group of Lie type.*

Proof. As before, there exists a simple, simply connected algebraic group \mathbf{G} with a Steinberg endomorphism F such that $S = G/\mathbf{Z}(G)$, where $G := \mathbf{G}^F$. By Lemma 2.3(i), we may replace S by G .

First assume that r is a bad prime for G . Then Proposition 2.10 yields the contradiction that $s \nmid |S|$, unless $G = E_7(q)$, $r = 3$ and $s|(q - \epsilon)$, where $q \equiv \epsilon \pmod{3}$. But $E_7(q)$ has a Levi subgroup of type D_6 , with center a torus of order $q - \epsilon$, and s -elements (which have order at least 5) in that center do not centralize a Sylow 3-subgroup of G .

Thus, both primes r, s are good for G . Next assume that r, s satisfies the assumptions of Proposition 2.11. Then necessarily s divides $q \pm 1$. In all of the cases (a)–(d) we will exhibit a Levi subgroup occurring as the centralizer of an s -element of order dividing $q \pm 1$ but not containing a Sylow r -subgroup. For $G = E_6(q)$, take a Levi of type $A_2^2 A_1$ (note that here $s \geq 7$), and similarly for ${}^2E_6(q)$. For $E_7(q)$, we may take a Levi of type $A_3 A_2 A_1$, and for $E_8(q)$ with $r = 7$ a Levi of type $E_6 A_1$ will do.

We may now assume that each of r and s divides a unique cyclotomic polynomial $\Phi_{d_i}(q)$ occurring in the generic order of G , but does not divide the order of the Weyl group of \mathbf{G} . If $d_1 = d_2$, we are done by Proposition 2.13. Otherwise, by Proposition 2.12 we have that neither of the Sylow d_i -tori are maximal tori of \mathbf{G} . Then d_1 and d_2 are as in Table 1. In these cases, the last column of the table gives certain centralizers of r -elements which do not contain a Sylow s -subgroup, a contradiction. \square

TABLE 1. Non-maximal non-cyclic Sylow tori

G	d_i	
${}^3D_4(q)$	1, 2	$(q-1).A_1(q^3)$
$E_6(q)$	2, 4, 6	$(q^2-1).{}^2D_4, (q^2+1)(q-1).{}^2A_3(q)$
${}^2E_6(q)$	1, 3, 4	$(q^2-1).{}^2D_4, (q^2+1)(q+1).A_3(q)$
$E_7(q)$	3, 4, 6	$(q^3 \pm 1).{}^3D_4(q)$

3. PROOF OF THEOREMS A, B AND C

We start with a trivial observation.

Lemma 3.1. *Suppose that $N \triangleleft G$, and let p and q be primes. If all the p -elements of G have conjugacy class size not divisible by q , then the same happens in G/N and N .*

Proof. If $x \in N$ is a p -element, then $|N : \mathbf{C}_N(x)|$ divides $|G : \mathbf{C}_G(x)|$, which is not divisible by q , by hypothesis. If $Nx \in G/N$ is a p -element, then $Nx = Ny$ where y is the p -part of x . If $D/N = \mathbf{C}_{G/N}(Nx)$, then $\mathbf{C}_G(y)N/N \leq D/N$ and $|G : D|$ divides $|G : \mathbf{C}_G(y)|$, which is not divisible by q , by hypothesis. \square

Lemma 3.2. *Let q be a prime. Suppose that a q -group Q acts coprimely on a finite group N . Let p be a prime and let P be a Q -invariant Sylow p -subgroup of N . Assume that for every $x \in Q$, there exists $n \in N$ such that $[x, P^n] = 1$. Then $[P, Q] = 1$.*

Proof. We argue by induction on $|Q|$. Suppose that R that is a maximal subgroup of Q . By induction, we have that $[R, P] = 1$. Suppose that S is another maximal subgroup of Q . Then $[S, P] = 1$, and therefore $[Q, P] = 1$ since $RS = Q$. Hence, we conclude that Q has a unique maximal subgroup. Then $Q/\Phi(Q)$ is cyclic, and therefore $Q = \langle x \rangle$. Now, by hypothesis, there is $n \in N$ such that $[Q, P^n] = 1$. In particular, P^n is Q -invariant. Since P is Q -invariant, by Glauberman's Lemma, there is $c \in \mathbf{C}_N(Q)$ such that $P = (P^n)^c \leq \mathbf{C}_N(Q)$, as desired. \square

We will use the following consequence in several places below.

Corollary 3.3. *Suppose that G is a finite group, and let p, q be different primes. Assume that every q -element of G has conjugacy class of size not divisible by p . Suppose that $N \triangleleft G$ is a q' -group. If $Q \in \text{Syl}_q(G)$ and $P \in \text{Syl}_p(N)$ is Q -invariant, then $[Q, P] = 1$.*

Proof. We have that $G = N\mathbf{N}_G(P)$ by the Frattini argument. Now, let $x \in Q$. By hypothesis, there is $P_1 \in \text{Syl}_p(G)$ such that $[x, P_1] = 1$. Now, $P^n \leq P_1$ for some $n \in N$, and thus $[x, P^n] = 1$. Now Lemma 3.2 applies. \square

Now, we are ready to prove Theorem A of the introduction. Recall that if a finite group G has a nilpotent Hall π -subgroup H , then every π -subgroup of G is contained in some G -conjugate of H by a well-known theorem of Wielandt.

Theorem 3.4. *Suppose that G is a finite group. Then G has nilpotent $\{p, q\}$ -Hall subgroups if and only if for every p -element $x \in G$, then $|G : \mathbf{C}_G(x)|$ is not divisible by q , and for every q -element $y \in G$, then $|G : \mathbf{C}_G(y)|$ is not divisible by p .*

Proof. We assume the condition *: for every p -element $x \in G$, $|G : \mathbf{C}_G(x)|$ is not divisible by q , and for every q -element $y \in G$, $|G : \mathbf{C}_G(y)|$ is not divisible by p . We prove by induction on $|G|$ that G has a nilpotent Hall $\{p, q\}$ -subgroup. Write $\pi = \{p, q\}$.

The condition * is inherited by quotients and normal subgroups, by Lemma 3.1.

Let $1 < N$ be a normal subgroup of G . By induction, we know that G/N has a nilpotent Hall π -subgroup H/N . Suppose that $|N|$ is not divisible by p or q . Then we use the Schur–Zassenhaus theorem in H to get a nilpotent Hall π -subgroup of G .

Suppose now that $|N|$ is not divisible by p . Let $P \in \text{Syl}_p(G)$ and let $Q \in \text{Syl}_q(N)$ be P -invariant (which we know to exist by coprime action). By Corollary 3.3, we have that $[P, Q] = 1$. Now, recall that G/N has a Hall nilpotent π -subgroup H/N . Thus, using the Frattini argument and the Schur–Zassenhaus theorem in the group $\mathbf{N}_H(Q)/Q$ with respect to the normal subgroup $\mathbf{N}_N(Q)/Q$, we have that $\mathbf{N}_H(Q)/Q$ has a nilpotent Hall π -subgroup U/Q , which we may assume contains P . Now, notice that U is a Hall π -subgroup of G . Write $U/Q = (S/Q) \times (PQ/Q)$, where $S \in \text{Syl}_q(U)$. (In particular, $S \in \text{Syl}_q(G)$.) Then $[P, S] \leq Q$ and $[S, P, P] = 1$. Thus $[S, P] = 1$ by coprime action.

Hence we may assume that the order of every proper normal subgroup is divisible by p and q .

Let N be a minimal normal subgroup of G . Hence $N = S_1 \times \cdots \times S_k$, where S_i is a non-abelian simple group of order divisible by pq , and G transitively permutes the set $\Omega = \{S_1, \dots, S_k\}$. Let B be the kernel of the action. We claim that G/B is a q' -group. Otherwise, let Bx be an element of order q , where x has q -power order. Now, by hypothesis, we have that $[x, P] = 1$ for some Sylow p -subgroup P of G . Then, $P \cap N \in \text{Syl}_p(N)$, and in fact $P \cap N = (P \cap S_1) \times \cdots \times (P \cap S_k)$. Now, let $y \in P \cap S_i$ be of order p . Then $[y, x] = 1$, and therefore we deduce that x has to normalize S_i . Then $x \in \bigcap_i \mathbf{N}_G(S_i) = B$ and this is a contradiction. This shows that G/B is a q' -group and by symmetry a p' -group. But then B contains both Sylow p -subgroups and q -subgroups of G and therefore by induction, we may assume that $B = G$. Thus N is a simple group of order divisible by pq .

We show now that G can be assumed to have no proper solvable quotients. Suppose that G/K has prime order, where $K \triangleleft G$. By induction, we know that K has nilpotent Hall π -subgroups. If G/K is π' -group, then the nilpotent Hall π -subgroups of G are Hall

subgroups of G and we are done. Therefore, we assume (by symmetry) that G/K has order p , so we may write $G = K\langle x \rangle$ for some p -element $x \in G$. Now, let $P \in \text{Syl}_p(G)$ be such that $x \in P$. By hypothesis, let $Q \in \text{Syl}_q(G)$ such that $[Q, x] = 1$. Notice that $Q \leq K$ and that $P \cap K \in \text{Syl}_p(K)$. Since K has nilpotent Hall π -subgroups, then $P \cap K$ is contained in some nilpotent Hall π -subgroup of K . Hence there is $k \in K$ such that $[Q^k, P \cap K] = 1$. In particular, $|K : \mathbf{C}_K(Q^k)|$ is not divisible by p . Since $x^k \in \mathbf{C}_G(Q^k)$, we have that $G = K\mathbf{C}_G(Q^k)$. Hence $|G : \mathbf{C}_G(Q^k)| = |K : \mathbf{C}_K(Q^k)|$ is not divisible by p , and there is some Sylow p -subgroup P_1 of G such that $[Q^k, P_1] = 1$. We deduce that G has nilpotent Hall π -subgroups, and in this case the theorem is proved.

Now, since $G/N\mathbf{C}_G(N)$ is isomorphic to a subgroup of $\text{Out}(N)$, then $G/N\mathbf{C}_G(N)$ is solvable and we conclude that $G = N\mathbf{C}_G(N)$. Since $\mathbf{C}_G(N)$ has nilpotent Hall π -subgroups by induction, then we conclude that N cannot be proper in G , because otherwise N and therefore G would have nilpotent Hall π -subgroups. Now Theorem 2.1 applies. \square

Theorem B immediately follows from Theorem A, by using the following.

Lemma 3.5. *Let G be a finite group and let π be a set of primes. Assume that π contains at least two prime divisors of $|G|$. If G has nilpotent Hall τ -subgroups for every $\tau \subseteq \pi$ with $|\tau| = 2$, then G has nilpotent Hall π -subgroups.*

Proof. This is [M, Lemma 3.4]. (See the comment that follows the proof.) \square

Proposition 2.3 of [KS] provides a different reduction to simple groups of Theorem B. However, it is easier to check Theorem 2.1 than Theorem B for simple groups.

Finally, Theorem C is now an obvious consequence of Theorem B and the Main Theorem in [NST].

4. SOME SOLVABILITY

Next, we show now that a version of Theorem A is possible under weaker hypotheses if we allow some solvability conditions.

Theorem 4.1. *Let p, q be primes, and let G be a finite group. Assume that all the q -elements have conjugacy class sizes not divisible by p . If G is p -solvable or q -solvable, then a Sylow p -subgroup of G normalizes some Sylow q -subgroup of G .*

Proof. We argue by induction on $|G|$. Assume first that G is p -solvable. Let $K = \mathbf{O}_{p'}(G)$ and let $L/K = \mathbf{O}_p(G/K)$. Let Kx be a q -element of G/K , where x is a q -element of G . Now, $[x, P] = 1$ for some Sylow p -subgroup P of G , by hypothesis. Thus $[Kx, PK/K] = 1$ and therefore Kx centralizes L/K . By Hall-Higman 1.2.3. Lemma, it follows that $Kx \in L/K$, and thus $Kx = K$. We conclude that G/K is a q' -group. In particular, K contains a Sylow q -subgroup of G . Now, P acts coprimely on K , and by coprime action, it follows that P normalizes some Sylow q -subgroup of K , which is a Sylow q -subgroup of G .

Assume now that G is q -solvable. If $1 < N \triangleleft G$, then by induction we know that there exists $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ such that P normalizes NQ . If $\mathbf{O}_q(G) > 1$, then

we set $N = \mathbf{O}_q(G)$, P normalizes $QN = Q$, and we are done. So we may assume that $\mathbf{O}_{q'}(G) = N > 1$. Now Q acts coprimely on N . By coprime action Q normalizes some $P_1 \in \text{Syl}_p(N)$. By Corollary 3.3, we have that $[Q, P_1] = 1$, in particular $|N : \mathbf{N}_N(Q)|$ is not divisible by p . Now $NQ \triangleleft NQP$ and by the Frattini argument, we have that $NQP = N\mathbf{N}_{NQP}(Q)$. Then $|NQP : \mathbf{N}_{NQP}(Q)| = |N : \mathbf{N}_N(Q)|$ is not divisible by p . Hence some Sylow p -subgroup of NQP (and hence of G) normalizes Q . \square

It is an interesting problem to study if the property that a Sylow p -subgroup normalizes some Sylow q -subgroup is detectable by the character table. (More generally, what does the character table of G know about $\nu_q(G)$, the number of Sylow q -subgroups of G ?) We can easily solve this question in p -solvable groups.

Theorem 4.2. *Suppose that G is p -solvable. Let $q \neq p$ be another prime. Then some Sylow p -subgroup of G normalizes some Sylow q -subgroup of G if and only if $G/\mathbf{O}_{p'}(G)$ is a q' -group.*

Proof. Suppose that G/N is a q' -group, where $N = \mathbf{O}_{p'}(G)$. Let $P \in \text{Syl}_p(G)$. Then P acts coprimely on N , and by coprime action it stabilizes some Sylow q -subgroup of N , which is a Sylow q -subgroup of G .

Conversely, suppose that $P \in \text{Syl}_p(G)$ normalizes $Q \in \text{Syl}_q(G)$. We show by induction on $|G|$ that $G/\mathbf{O}_{p'}(G)$ is a q' -group. If $N \triangleleft G$, then we have that PN/N normalizes QN/N , so by applying induction in $G/\mathbf{O}_{p'}(G)$, we may assume that $\mathbf{O}_{p'}(G) = 1$. Now, let $K = \mathbf{O}_p(G)$. By hypothesis, we have that K normalizes Q . Also Q normalizes K . Since $Q \cap K = 1$, then we conclude that $[Q, K] = 1$. Then $Q \leq \mathbf{C}_G(K) \leq K$ by Hall-Higman's Lemma 1.2.3, and $Q = 1$. This concludes the proof. \square

Finally, we prove that if the prime 2 is involved in the following form, then we can obtain a certain solvability.

Theorem 4.3. *Let q be an odd prime, and let G be a finite group. If all the q -elements of G have conjugacy class size not divisible by 2, then G is q -solvable. In particular, a Sylow 2-subgroup of G normalizes some Sylow q -subgroup of G .*

Proof. We argue by induction. If $1 < N$ is a proper normal subgroup of G , then G/N and N are q -solvable. So we may assume that G is a non-abelian simple group of order divisible by $2q$ and appeal to Theorem 2.2. The last part follows from Theorem 4.1. \square

We cannot reverse the primes in the previous theorem, not even to obtain normalizing conditions between Sylow subgroups. All the 2-elements of $G = J_1$ have conjugacy class not divisible by 5, and no Sylow 5-subgroup of G normalizes any Sylow 2-subgroup of G .

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