UNIVERSIDAD POLITÉCNICA DE VALENCIA

DEPARTAMENTO DE INGENIERÍA DE SISTEMAS Y AUTOMÁTICA

Tesis doctoral

Condiciones LMI relajadas para control de modelos no-lineales Takagi-Sugeno

Relaxed LMI conditions for control of nonlinear Takagi-Sugeno models

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RESUMEN

Los problemas de optimización de desigualdades matriciales lineales en control borroso se han convertido en la herramienta más utilizada en dicha área desde los años 90. Muchos sistemas no lineales pueden ser modelados como sistemas borrosos de modo que el control borroso puede considerarse como una técnica de control no lineal. Aunque se han obtenido muchos y buenos resultados, quedan algunas fuentes de conservadurismo cuando se comparan con otros enfoques de control no lineal. Esta tesis discute dichas cuestiones de conservadurismo y plantea nuevos enfoques para resolverlas.

La principal ventaja de la formulación mediante desigualdades matriciales lineales es la posibilidad de asegurar estabilidad y prestaciones de un sistema no lineal modelado como un sistema borroso Takagi-Sugeno. Estos modelos están formados por un conjunto de modelos lineales eligiendo el sistema a aplicar mediante el uso de unas reglas borrosas. Estas reglas se traducen en funciones de interpolación o de pertenecía que nos indican el grado de validez de un modelo lineal respecto del resto. El mayor problema que presentan estas técnicas basadas en desigualdades matriciales lineales es que las funciones de pertenencia no están incluidas en las condiciones de estabilidad del sistema, lo que significa que se prueba la estabilidad y prestaciones para cualquier forma de interpolación entre los diferentes modelos lineales. Esto genera una fuente de conservadurismo que sería conveniente limitar.

En la tesis doctoral se presentan varias metodologías capaces de trasladar la información de las funciones de pertenencia del sistema al problema basado en desigualdades matriciales lineales de estabilidad y prestaciones. Las dos principales aportaciones propuestas se basan, respectivamente, en introducir una serie de matrices de relajación que per-

mitan incorporar esta información y en aprovechar la descripción de una amplia clase de sistemas borrosos en productos tensoriales de modelos lineales, en los que cada función de pertenencia se obtiene del producto cartesiano de varias funciones.

Por otro lado, el problema de estabilidad y prestaciones para sistemas borrosos Takagi-Sugeno está basado en las condiciones de estabilidad de Lyapunov y no resulta equivalente al problema en desigualdades matriciales lineales obtenido, siendo este último conservativo respecto al primero. Así una de las mayores aportaciones de la tesis es la introducción de nuevas condiciones de estabilidad y prestaciones que mediante un parámetro de diseño permiten acercar esta equivalencia entre las expresiones, resultando equivalentes asintóticamente con el aumento del parámetro. Como inconveniente, el aumento del parámetro comporta un aumento en la complejidad del problema a resolver.

RESUM

Els problemes d'optimització de desigualtats matricials lineals en control borrós s'han convertit en la ferramenta més utilitzada a l'àrea des dels anys 90. Molts sistemes no lineals poden ser modelats com sistemes borrosos, de tal forma que el control borrós es pot considerar com una tècnica de control no lineal. Malgrat que s'han obtingut molts i bons resultats, queden algunes fonts de conservadorisme quan es comparen amb altres tècniques de control no lineal. Aquesta tesi discuteix les qüestions de conservadorisme i planteja noves tècniques per a resoldre-les.

El principal avantatge de la formulació mitjançant desigualtats matricials lineals és la possibilitat d'assegurar estabilitat i prestacions d'un sistema no lineal modelat com a un sistema borrós Takagi-Sugeno. Aquests models estan formats per un conjunt de models lineals triant el sistema a aplicar mitjançant l'ús d'unes regles borroses. Aquestes regles es tradueixen en funcions d'interpolació o de pertenència que ens indiquen el grau de validesa d'un model lineal respecte de la resta. El major problema que presenten aquestes tècniques basades en desigualtats matricials lineals és, que les funcions de pertenència no estan incloses en les condicions d'estabilitat del sistema, això significa que, es prova l'estabilitat i prestacions per a qualsevol forma d'interpolació entre els diferents models lineals. Açò genera una font de conservadorisme que seria convenient limitar.

En la tesis doctoral es presenten diverses metodologies que poden traslladar la informació de les funcions de pertenència del sistema al problema basat en desigualtats matricials lineals d'estabilitat i prestacions. Les dues principals aportacions proposades es basen, respectivament, en introduir una sèrie de matrius de relaxació que permeten incorporar aquesta informació, i en aprofitar la descripció d'una àmplia

classe de sistemes borrosos en productes tensorials de models lineals, en els que cada funció de pertenència s'obté del producte cartesià de diverses funcions.

D'altra banda, el problema d'estabilitat i prestacions per a sistemes borrosos Takagi-Sugeno està basat en les condicions d'estabilitat de Lyapunov i no és equivalent al problema en desigualtats matricials lineals obtingut, on aquest últim resulta conservatiu respecte al primer. Així una de les majors aportacions de la tesi és la introducció de noves condicions d'estabilitat i prestacions que mitjançant un paràmetre de disseny permeten aproximar aquesta equivalència entre les expressions, resultant equivalents assimptòticament amb l'augment del paràmetre. Com a inconvenient l'augment del paràmetre comporta un augment de la complexitat del problema a resoldre.

ABSTRACT

Linear Matrix Inequalities (LMI) optimization problems became the tool of choice for fuzzy control in the 1990's. Many nonlinear systems can be modelled as fuzzy systems, so fuzzy control may be considered as a nonlinear control technique. Useful results have been obtained using LMIs, however some sources of conservativeness remain when compared to other nonlinear approaches. This thesis deals with such issues of conservativeness and discusses some ideas on overcoming them.

The main advantage of Linear Matrix Inequalities formulations is that they can ensure stability and performance of a nonlinear system modelled by a Takagi-Sugeno fuzzy system. The system is described by fuzzy IF-THEN rules which present "local" linear systems of the nonlinear plant. These rules are numerically represented by a set of membership functions. As a drawback, current linear matrix inequalities methodologies do not include the shape of the membership functions. Therefore the stability is proved for any set of rules with any membership function that can be described by these linear models. This is a source of conservatism that can be reduced.

In this thesis some methodologies are presented which include the membership function information into the linear matrix inequalities stability and performance problem. We propose two main contributions in this area. The first method introduces a set of relaxation matrices that incorporates the information on the membership functions. The other uses the description of a wide class of Takagi-Sugeno fuzzy systems, labelled as Tensor-Product Takagi-Sugeno fuzzy systems. In these systems, each membership function is the product of several membership functions.

On the other hand, the problem of stability and performance for Takagi-Sugeno fuzzy systems is based on Lyapunov stability conditions

which are not equivalent to the linear matrix inequalities optimization problem, the second one is conservative compared to the first. That is why one of the main contributions of the thesis is a set of progressively less conservative sufficient conditions in order to hold stability or performance conditions for Takagi-Sugeno fuzzy systems. These conditions are asymptotically equivalent. But the problem complexity increases as the conditions are closer.

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Contents

Chapter 1
Summary of the thesis

1.1 Introduction

In contrast to conventional control, fuzzy control was initially introduced as a model-free control design method based on a representation of the knowledge and the reasoning process of a human operator (Zadeh, 1973; Assilian & Mamdani, 1975). Fuzzy logic can capture the continuous nature of human decision making processes. Practical applications of fuzzy control started to appear very quickly after the method had been introduced in publications. Moreover fuzzy modelling methodologies were developed and most nonlinear process could be modelled as fuzzy systems. A drawback of knowledge-based, model-free fuzzy control is that it does not allow for any kind of stability or robustness analysis, unless a model of the process is available.

In a parallel research line, new control methodologies appeared in the field of robust control in the 1990's. These technics were based on Lyapunov stability and convex optimization where the optimization problem is subject to a set of Linear Matrix Inequalities (LMI). These methodologies reached maturity after (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994) and were applied to fuzzy control theory by (Tanaka & Wang, 2001), in particular to Takagi-Sugeno fuzzy models, becoming widespread in the last 10 years. Results are available for control design of fuzzy systems with uncertainty, delay, descriptor forms.

LMIs have become a powerful tool in fuzzy control. They solved a wide variety of stability and performance problems for Takagi-Sugeno fuzzy models. The ability of fuzzy systems to approximate nonlinear models allowed control of large generic control problems.

As a drawback there are several sources of conservatism on mainstream fuzzy control. The choice of the structure of the Lyapunov function, conservativeness of the LMI conditions as they are not equivalent to the Lyapunov conditions, because they are independent of the membership shape.

LMI fuzzy control theory is not able to find any solution in some complex scenarios, where other nonlinear control theory is successful, due to the membership functions are not included in the LMI conditions. The aim of the thesis is to introduce some new conditions in order to reduce this conservatism.

1.2 Objectives

The main objective of the thesis is to reduce the above discussed gap between fuzzy and nonlinear control. Four ways have been explored in order to reduce this gap:

- Conservativeness of the positivity conditions for fuzzy summations.
- Local stability results for fuzzy systems.
- New stability conditions for Tensor product fuzzy systems.
- Membership-shape relaxation.

1.2.1 Conservativeness of the positivity conditions for fuzzy summations

Current literature (Tanaka & Wang, 2001; Liu & Zhang, 2003; Fang, Liu, Kau, Hong, & Lee, 2006) presents only sufficient conditions for ensuring stability or performance of Takagi-Sugeno Fuzzy models. There are many literature references in the area of stability and performance conditions. So we explore the different ways to reduce this conservativeness. These contributions have been published in (Sala & Arino, 2007b) and (Arino & Sala, 2007a).

1.2.2 Local stability approach

Local quadratic stability results can be obtained, even in the case where global quadratic-stability related LMIs are infeasible. Indeed, as commented in (Sugeno, 1992), it is an interesting question to determine for which initial conditions a fuzzy system is stable (or unstable) (Sugeno, 1992). Example 6 in (Tanaka & Wang, 2001)(Chapter 2) shows that the basin of attraction for fuzzy systems may be membership dependent. In this respect, the methodology presented here allows to determine the largest sphere around the origin for which quadratic stability may be provable in a given fuzzy system with *known* membership functions. These contributions have been published in (Ariño & Sala, 2006a) and (Ariño & Sala, 2006b).

1.2.3 Tensor-product fuzzy systems

Another example of conservativeness occurs when the membership functions can be expressed as the "tensor product" of simpler partitions, so that the fuzzy system can be written as a multi-dimensional fuzzy summation.

Removing part of the conservatism in current solutions for the tensor-product case above is indeed of interest; this product structure is often the case in many engineering applications of fuzzy control:

- in the systematic "sector nonlinearity" fuzzy modelling techniques reported in (Tanaka & Wang, 2001);
- in many man-made rulebases for multi-input fuzzy systems, where the rules are built via the *conjunction* of simpler concepts arising from fuzzy partitions on each of the input domains. A typical example are rulebases formed with rules in the form "if z_1 is *large* and z_2 is *small* and ... then ...", "if z_1 is *medium* and z_2 is *small* and ... then ...", *etc.*, with the antecedents covering all combinations of fuzzy sets on z_1, z_2, *etc.*.
- in approximate interpolation and model reduction techniques based in gridding and tensor-SVD algebra in (Baranyi, 2004).

These settings are a particular class of fuzzy models which will be denoted as *tensor-product* fuzzy systems.

In summary the objective is to define and analyzing the tensor-product fuzzy systems, by presenting fuzzy control design tools for them which explicitly use the tensor-product structure. The study of the properties of this class of systems is very relevant as most of the fuzzy systems in nontrivial engineering applications of fuzzy control belong to this class, as discussed above.

These contributions have been published in (Arino & Sala, 2007c)and (Ariño & Sala, 2007b).

1.2.4 Memberships shape

When the expressions of the memberships as a function of some premise variables are actually known, some zones of the possible membership space can be excluded. Reducing the size of the (multi-dimensional) set where the membership functions take values should obtain less conservative conditions than those expressed for any membership. However, LMI conditions in current literature do not take into account that fact. These contributions have been published in (Sala & Ariño, 2007a).

1.2.5 Uncertain memberships

In most applications, we need to approximate the membership functions and in these cases, it is difficult to take into account the knowledge of the membership functions shape. Therefore the ability to incorporate a wider class of constraints on the membership shape will improve the current stability and performance results.

1.3 Structure of the thesis

The document is organized in three parts. **Part I** outlines with the state of the art, and it is composed of two chapters. **Chapter 2** introduces the reader to Lyapunov stability and linear matrix inequalities problems and **Chapter 3** presents the stability and performance problem in Takagi-Sugeno Fuzzy systems.

Part II presents the contributions and is organized in seven chapters:

- **Chapter 4** exposes the problems with the LMI theory and deals with the different points of view to tackle the problem.
- **Chapter 5** presents asymptotical necessary and sufficient conditions for quadratic stability in Takagi-Sugeno fuzzy models.
- **Chapter 6** presents local stability conditions to apply when global stability can not be proved
- **Chapter 7** deals with the introduction of the membership functions shape into the LMI stability conditions in order to relax this expressions and approach to nonlinear control specific results.
- **Chapter 8** develops new stability and performance LMI conditions for TS fuzzy models with uncertain memberships functions.
- **Chapter 9** deals with the Tensor Product Takagi-Sugeno models and presents a new stability results that take into account the special structure of these models.

Part III closes the thesis with a summary of contributions and open research lines.

Part I

State of the art

Chapter 2

Linear Matrix Inequalities stability analysis

2.1 Introduction

The most useful and general approach for studying the stability of nonlinear control systems is the theory introduced in the late 19th century by the Russian mathematician Alexander Mikhailovich Lyapunov. Lyapunov's work, The General Problem of Motion Stability, includes two methods for stability analysis (the so-called linearization method and the direct method) and was first published in 1892. The linearization method draws conclusions about a nonlinear system's local stability around an equilibrium point from the stability properties of its linear approximation. The direct method is not restricted to local motion, and determines the stability properties of a nonlinear system by constructing a scalar "energy-like" function for the system and examining the function's time variation. However Lyapunov's pioneering work on stability received little attention outside Russia, although it was translated into French in 1908 (at the instigation of Poincare), and reprinted by Princeton University Press in 1947. The publication of the work by Lur'e and a book by La Salle and Lefschetz brought Lyapunov's work to the attention of the larger control engineering community in the early 1960's. Many refinements of Lyapunov's methods have since been developed. Today, Lyapunov's linearization method has come to represent the theoretical justification of linear control, while Lyapunov's direct method has become the most important tool for nonlinear system analysis and design. Together, the linearization method and the direct method constitute the so-called Lyapunov stability theory.

The objective of this chapter is to present Lyapunov stability theory and illustrate its use in the analysis and the design of control systems.

2.2 Stability concept

The concept of stability and instability (Slotine & Li, 1991) are useful in a wide field of knowledge: Finance, Medicine, Construction, Control, Chemistry, etc... So a clear definition of the concept of stability is needed in the control theory.

Qualitatively, a system is described as stable if starting the system somewhere near its desired operating point implies that it will stay around the point ever after. And this point is called the equilibrium point. From this qualitative definition, we can classify three kinds of equilibrium points (Slotine & Li, 1991; Boyd et al., 1994):

- Asymptotically stable: the system returns to the equilibrium point after a small perturbation.
- Non-asymptotically stable: the system remains in the proximity of the equilibrium point after a small perturbation.
- Unstable equilibrium: the system keeps away from the equilibrium point after a small perturbation.

2.2.1 Nonlinear systems

Let us consider a class of nonlinear dynamic systems (Slotine & Li, 1991), which can be represented by a set of differential equations in the form

$$\dot{x} = f(x,t) \tag{2.1}$$

Where x is the $(n \times 1)$ state space vector, and f is a $(n \times 1)$ vector function. The number of the states is also called the order of the system. And a solution $x(t)$ of the equation is a trajectory.

If f is only a function of x, the system is said to be autonomous, otherwise the system is called non-autonomous.

A special class of nonlinear systems are linear systems (Antsaklis & Mitchel, 1997). The dynamics of linear systems have the form

$$\dot{x} = A(t)x \tag{2.2}$$

where $A(t)$ is an $(n \times n)$ matrix. If $A(t)$ is constant the system is called Linear time-invariant (LTI) system, otherwise it is denoted as linear time variant (LTV) system.

Equilibrium points
It is possible for a system trajectory to correspond to only a single point. Such a point is called an equilibrium point. As we shall see later, many stability problems are naturally formulated with respect to equilibrium points.

In autonomous systems state x' is an equilibrium state (or equilibrium point) of the system if once $x(t)$ is equal to x, it remains equal to x for all future time. Mathematically, this means that the constant vector x' satisfies

$$0 = f(x')$$

In a linear time-invariant system $0 = Ax'$. So $x' = 0$ is the single equilibrium point if A is not singular. On the other hand, nonlinear systems can have several isolated equilibrium points.

2.2.2 Lyapunov stability

In the beginning of this chapter, an intuitive notion of stability was introduced, as a kind of behavior around an equilibrium point. However, since nonlinear systems may have much more complex and exotic behavior than linear systems, the mere notion of stability is not enough to describe the essential features of their motion. A number of more refined stability concepts, such as asymptotic stability, exponential stability and global asymptotic stability, are needed. In this section, we define these stability concepts formally, for autonomous systems, and explain their practical meanings.

Essentially, stability in the sense of Lyapunov means that the system trajectory can be kept arbitrarily close to the equilibrium point, starting sufficiently close to it. So let us define local stability as

Definition 2.1 *The equilibrium state $x = 0$ is said to be stable if, for any $\varepsilon > 0$, there exists $\eta > 0$, such that if $||x_0|| < \eta$ then $||y(t,x_0)|| < \varepsilon$ for all t>0 . Otherwise, the equilibrium point is unstable.*

Also, Asymptotic stability in the sense of Lyapunov (Slotine & Li, 1991) means that the equilibrium is stable and that, in addition, if the states started close to the equilibrium point, they actually converge to the equilibrium point.

Definition 2.2 *An equilibrium point $x = 0$ is asymptotically stable if it is stable, and if in addition there exists some $\varepsilon > 0$ such that $||x(0)|| < \varepsilon$ implies that $x(t) \to 0$ as $t \to \infty$.*

2.2.3 Lyapunov stability theorem

Inside the ball $\mathbf{B}(\varepsilon)$, the equilibrium point $x = 0$ of the system (2.1) is locally stable, if there is a scalar function $V(x)$ continuous and differentiable such that

$$\begin{aligned} V(x) &\geq 0 \ \forall x \in \mathbf{B}(\varepsilon) \\ \frac{dV}{dt} &\leq 0 \ t \geq 0 \\ V(0) &= 0 \end{aligned}$$

where x is the state space vector of the system (2.1). And the function $V(x)$ is called Lyapunov function.

The result can be derived using the geometric interpretation of a Lyapunov function, as illustrated in figure 2.1 in which is drawn the equipotentials of V on the state space of the system. In order to show stability, it must be shown that given any strictly positive number $\varepsilon > 0$, there exists a smaller positive number η such that any trajectory starting inside the ball $\mathbf{B}(\eta)$ remains inside the ball $\mathbf{B}(\varepsilon)$ for all future time. Let m be the minimum of V on the ball $\mathbf{B}(\varepsilon)$. Since V is continuous and positive definite, m exists and is strictly positive. Furthermore, since $V(0) = 0$, there exists a ball $\mathbf{B}(\eta)$ around the origin such that $V(x) < m$ for any x inside the ball. Consider now a trajectory whose initial point is in the ball $\mathbf{B}(\eta)$. Since V is non-increasing along system trajectories, V remains strictly smaller than m, and therefore the trajectory cannot possibly cross the outside sphere $\mathbf{B}(\varepsilon)$. Thus, any trajectory starting inside the ball $\mathbf{B}(\eta)$ remains inside the ball $\mathbf{B}(\varepsilon)$, and therefore Lyapunov stability is guaranteed.

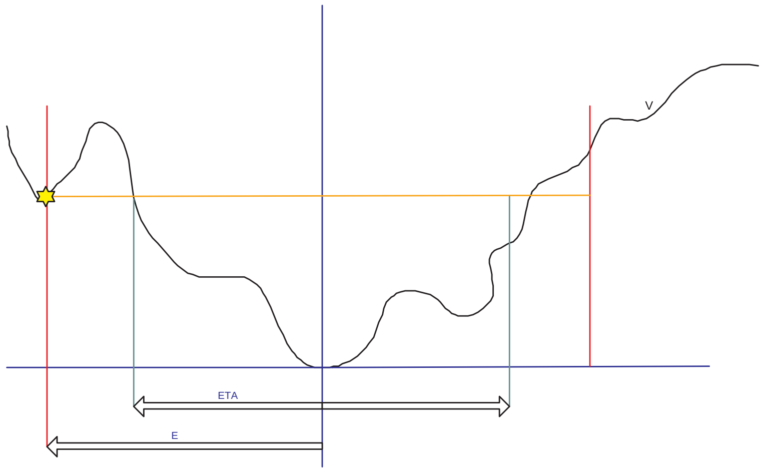


Figure 2.1: Lyapunov stability in the neighborhood of V(0)

In applying the above theorem for analysis of a nonlinear system, one goes through the two steps of choosing a positive definite function $V(x)$, and then determining its derivative along the path of the nonlinear systems. The following example illustrates this procedure for the linear system (2.2).

Example 2.1 *We propose a quadratic function* $V(x) = x^T P x$ *as Lyapunov function candidate.*

Then V is definite positive if P is definite positive. And evaluating the second condition w obtain:

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = \tag{2.3}$$

$$= x^T PAx + x^T A^T Px = x^T (PA + A^T P)x \tag{2.4}$$

$$\tag{2.5}$$

Then $\dot{V}(x) < 0$ if $PA + A^T P$ is definite negative. Therefore the linear system (2.2) is stable if we can find a definite positive matrix P such that $PA + A^T P$ is definite negative. This introduces an LMI condition as discussed below.

2.3 Linear matrix inequalities (LMI)

A linear matrix inequality (Gahinet & Apkarian, 1994; Boyd et al., 1994; Anderson, Kraus, Mansour, & Dasgupta, 1995; Scherer, Gahinet, & Chilali, 1997) is an expression of the form:

$$A(x) = A_0 + x_1 A_1 + ... + x_N A_N < 0 \tag{2.6}$$

where

- $x = (x_1, ..., x_N)$ is a vector of N real numbers called the decision variables.
- $A_0, ..., A_N$ are real symmetric matrices.
- the inequality < 0 in (2.6) means definite negative. That is equivalent to saying that all eigenvalues $\lambda(A(x))$ are negative, or equivalently, the maximum eigenvalue is negative.

2.3.1 Properties of linear matrix inequalities

Convex set. The linear matrix inequality (2.6) defines a convex constraint on x. That is, the set

$$\Phi = \{x | A(x) < 0\} \tag{2.7}$$

of solutions of the LMI $A(x) < 0$ is convex. In fact, if x_1 and x_2 are in Φ then $\alpha x_1 + (1-\alpha)x_2$ is in Φ, where $0 \leq \alpha \leq 1$.

$$A(\alpha x_1 + (1-\alpha)x_2) = \alpha A(x_1) + (1-\alpha)A(x_2) < 0$$

where we used that $A(x)$ is affine and where the inequality follows from the fact that $0 \leq \alpha \leq 1$.

A set of linear matrix inequalities (LMIs) is a finite set of linear matrix inequalities:

$$A_1(x) < 0, \ldots, A_k(x) < 0 \tag{2.8}$$

Any set of linear matrix inequalities can be expressed as a linear matrix inequality in the form

$$A(x) = \begin{pmatrix} A_1(x) & 0 & \dots & 0 \\ 0 & A_2(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_k(x) \end{pmatrix} < 0 \tag{2.9}$$

The last inequality indeed makes sense as $A(x)$ is symmetric for any x. Further, since the set of eigenvalues of $A(x)$ is simply the union of the eigenvalues of $A_1(x), \dots, A_k(x)$, any x that satisfies $A(x)$ also satisfies the set of LMIs (2.8).

Affine constrains. A third important property amounts to incorporating affine constraints in linear matrix inequalities. By this, we mean that combined constraints (in the unknown x) of the form $x = Au + b$ for some u. The LMI constrain $A(x) <$ subject to $x = Au + b$ can be rewritten as an LMI of u as $\hat{A}(u) < 0$.

Congruence transformation. If M is a square matrix and T is non-singular, then the product $T^T MT$ is called a congruence transformation of M. For symmetric matrices M this transformation does not change the number of positive and negative eigenvalues of M. Indeed, if an LMI $A(x) < 0$ holds for some x then $u^T A(x) u < 0$ holds for any nonzero u and x in Φ, (2.7). In particular if $u = Tv$, $v^T T^T A(x) T v < 0$, therefore $T^T A(x) T < 0$ for any x in Φ, (2.7).

2.3.2 Schur complement

Let $A(x)$ be an affine function which is partitioned according to

$$A(x) = \begin{pmatrix} A_{11}(x) & A_{12}(x) \\ A_{12}(x)^T & A_{22}(x) \end{pmatrix} \tag{2.10}$$

then $A(x) < 0$ is equivalent to:

$$\begin{cases} A_{11}(x) < 0 \\ A_{22}(x) - A_{12}(x)^T A_{22}(x)^{-1} A_{12}(x) < 0 \end{cases} \tag{2.11}$$

and

$$\begin{cases} A_{22}(x) < 0 \\ A_{11}(x) - A_{12}(x) A_{11}(x)^{-1} A_{12}(x)^T < 0 \end{cases} \tag{2.12}$$

The second inequalities in (2.11) and (2.12) are nonlinear in x. Using that result follows that nonlinear matrix inequalities in the form (2.11) can be converted into linear matrix inequalities. The proof can be obtained with the congruence transformation

$$T = \begin{pmatrix} I & -A_{22}^{-1}A_{12}^T \\ 0 & I \end{pmatrix}$$

2.3.3 S-procedure

Sometimes a quadratic function should be negative whenever some other quadratic functions are all negative. This constraint can be expressed as an LMI defining the quadratic functions. Sometimes this LMI is conservative, but often a useful approximation of the constraint.

The S-procedure for quadratic functions and non-strict inequalities: Let $F_0,\ldots,F_p$ the quadratic function of the variable $u \in \mathbf{R}^n$:

$$F_i(u) = u^T T_i u + 2b_i u + v_i \;\; i = 0,\ldots,p \tag{2.13}$$

where T_i is a symmetric matrix. We consider the following condition on $F_0,\ldots,F_p$:

$$F_0(u) > 0 \;\forall u \mid F_i(u) \geq 0,\; i = 0,\ldots,p \tag{2.14}$$

Obviously if there exist $\tau_1 \geq 0,\ldots,\tau_p \geq 0$ such that for all u,

$$F_0(u) - \sum_{i=1}^{p} \tau_i F_i(u) \geq 0 \tag{2.15}$$

then (2.14) holds.

If p=1, the converse holds, provided that there is some u_0 such that $F_1(u_0) > 0$, See (Wolkowicz & Stern, 1995; Nocedal & Wright, 1999; Boyd & Vandenberghe, 2004)

The inequality (2.15) can be rewritten as an LMI as

$$\begin{pmatrix} T_0 & b_0 \\ b_0^T & v_0 \end{pmatrix} - \sum_{i=1}^{p} \tau_i \begin{pmatrix} T_i & b_i \\ b_i^T & v_i \end{pmatrix} \geq 0 \tag{2.16}$$

Farkas Lemma is a particular case of the s-procedure, where the functions F_i are affine. In that case (2.14) and (2.15) are equivalent. See (Boyd & Vandenberghe, 2004)

The S-procedure for quadratic forms and strict inequalities:

Another variation of the s-procedure involves quadratic forms and strict inequalities. Let $T_0, \dots, T_p \in \mathbf{R}^{n\times n}$ be symmetric matrices. We consider the following conditions on $T_0, \dots, T_p$

$$u^T T_0 u > 0 \ \forall u \neq 0 \mid u^T T_i u \geq 0, \ i = 0, \dots, p \tag{2.17}$$

it is obvious that if there exist $\tau_1 \geq 0, \dots, \tau_p \geq 0$ such that

$$T_0 - \sum_{i=1}^{p} \tau_i T_i > 0 \tag{2.18}$$

then (2.17) holds.

2.3.4 Finsler Lemma

Let $x \in \mathbf{R}^n$, $Q = Q^T \in \mathbf{R}^{n\times n}$ and $R \in \mathbf{R}^{m\times n}$. The following statements are equivalent (Boyd et al., 1994):

1. $x^T Q x < 0$ for all $x \neq 0$ such that $Rx = 0$
2. $R_\perp^T Q R_\perp < 0$ where $RR_\perp = 0$
3. $Q - \sigma R^T R < 0$ for some scalar $\sigma \in \mathbf{R}$
4. $Q + XR + R^T X^T < 0$ for some matrix $X \in \mathbf{R}^{n\times m}$

2.3.5 Linear matrix inequalities problems

There are two generic problems related to the study of linear matrix inequalities:

1. **The LMI optimization problem.** Let an objetctive function $f : \Phi \to \mathbf{R}$ where $\Phi = \{x | A(x) < 0\}$. The problem to determine $\min_x f(x)$ is called an optimization problem with an LMI constraint. If the function f is linear $f = c_1 x_n + \cdots + c_n x_n$, the optimization problem is a generalization of linear programming to a cone of positive semidefinite matrices. It is also called semidefinite programming.

2. **The LMI feasibility problem.** Test whether exist $x \in \mathbf{R}^n$ such that $A(x) < 0$. That is, $A(x) < 0$ is feasible if and only if $\min_x \lambda_{max}(A(x)) < 0$. Therefore involves minimizing the function $f : x \to \lambda_{max}(A(x))$. That is possible because this function is convex as it has been shown above.

2.3.6 Solving LMIs

A solution method for an LMI optimization problems is an algorithm that computes a solution of the problem (to some given accuracy), given a particular problem from the class, i.e., an instance of the problem. Since the late 1940s, a large effort has gone into developing algorithms for solving various classes of optimization problems, analyzing their properties, and developing good software implementations. The effectiveness of these algorithms varies considerably, and depends on factors such as the particular forms of the objective and constraint functions, how many variables and constraints there are, and special structure, such as sparsity. (A problem is sparse if each constraint function depends on only a small number of the variables).

High quality implementations of LMI optimization problems are available in (Gahinet, Nemirovski, Laub, & Chilali, 1995; Sturm, 1999) software.

Chapter 3

Takagi-Sugeno Models

3.1 Introduction

This Chapter outlines model-based fuzzy control systems. The central subject is a systematic framework for the stability and design of nonlinear fuzzy control systems. Building on the so-called Takagi-Sugeno fuzzy model, a number of the most important issues in fuzzy control systems are addressed. These include stability analysis, systematic design procedures, incorporation of performance specifications, robustness and observer design.

3.2 Definitions

This section shows the definition of the Takagi-Sugeno fuzzy model (TS fuzzy model). followed by construction procedures of such models. Then a model-based fuzzy controller design utilizing the concept of "parallel distributed compensation" is described. The main idea of the controller design is to derive each control rule so as to compensate each rule of a fuzzy system. The design procedure is conceptually simple and natural. Moreover, it is shown in this chapter that the stability analysis and control design problems can be reduced to linear matrix inequality (LMI) problems.

First we present the Takagi-Sugeno Fuzzy model as an approximation of a given nonlinear plant, described by fuzzy IF-THEN rules which present "local" linear systems of the nonlinear plant (Babuska, 1998; Babuska, Fantuzzi, & Verbruggen, 1996). So each rule expresses a significant feature of the plant, expressed as a linear system. Then the whole plant is described by the interpolation among these linear systems. This feature is very useful in order to reduce the control problem into linear matrix inequalities conditions.

3.2.1 Takagi-Sugeno Fuzzy Models

Takagi-Sugeno (Takagi & Sugeno, 1985) (TS) fuzzy model is defined by IF-THEN rules that represent local linear dynamics of a nonlinear system. The rules of the TS fuzzy models are usually expressed as:

IF z is M_i **THEN**

$$\begin{cases} \delta x_i = A_i x_i + B_i u \\ y_i = C_i x_i \end{cases}$$

The local models represented by $A_i x_i + B_i u$ are used to compute the final output as follows:

$$\delta x = \sum_{i=1}^{r} \mu_i(z)(A_i \cdot x + B_i \cdot u) \tag{3.1}$$

$$y = \sum_{i=1}^{r} \mu_i(z) C_i x \tag{3.2}$$

$$\sum_{i=1}^{r} \mu_i(z) = 1, \quad \mu_i(z) > 0 \ \forall z \ i = 1 \dots r \tag{3.3}$$

In continuous models, δ represents the derivative operator and in discrete case, it represents the advance operator. $x(t) \in \mathbb{R}^n$ is the state space vector, $u(t) \in \mathbb{R}^m$ is the input vector of the model and $y(t) \in \mathbb{R}^p$ is the output vector. Matrices $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$ and $C_i \in \mathbb{R}^{p \times n}$ for $i = 1 \dots r$ represents a set of linear models. $z(t)$ is the premise variables and $\mu_i(z)$ represents the membership functions of fuzzy set M_i that hold (3.3)

3.2.2 PDC-controller

The parallel distributed compensation (PDC) controller began with a model-based design prodedure proposed by (Sugeno & Kang, 1986) without stability analysis. The LMI stability analysis was done in (Tanaka & Sugeno, 1992) and it was named PDC in (Wang, Tanaka, & Griffin, 1995).

In the PDC controller design, each control rule is designed for the corresponding rule of a TS fuzzy model. Therefore the designed fuzzy controller shares the fuzzy sets with the fuzzy model, that are the membership functions in the inference procedure. For the TS fuzzy models the PDC-controller is defined as:

IF z is M_i **THEN**

$$u_i(t) = -F_i x_i(t)$$

So Each rule represents a linear controller (state feedback law). Other controllers such as output feedback controllers and dynamic output feedback controllers can also be used. As is shown in Chapter 5 they can be used in our approaches. The fuzzy controller is expressed by the following control input vector:

$$u(t) = -\sum_{i=1}^{r} \mu_i F_i x(t) \tag{3.4}$$

The above expression is usually denoted as a *parallel distributed compensation* controller, PDC-controller.

3.3 Sector nonlinearity

In order to design a fuzzy controller, we need a Takagi-Sugeno fuzzy model for a nonlinear system. Therefore the construction of a fuzzy model represents an important and basic procedure in this approach. In this section we discuss the issue of how to construct such a fuzzy model. In general there are two approaches for constructing fuzzy models:

- Experimental identification using input and output data.

- Mathematical approximation from nonlinear equations.

There has been an extensive literature on fuzzy modelling using input-output data following Takagi's, Sugeno's, and Kang's excellent work (Takagi & Sugeno, 1985; Sugeno & Nishida, 1985; Sugeno & Kang, 1986, 1988). The procedure mainly consists of two parts: structure identification and parameter identification. The identification approach to fuzzy modelling is suitable for plants that are unable or too difficult to be represented by analytical physical models. On the other hand, nonlinear dynamic models for mechanical systems can be readily obtained by, for example, the Lagrange method and the Newton-Euler method. In such cases, the second approach, which derives a fuzzy model from given nonlinear dynamical models, is more appropriate. This section focuses on this second approach. This approach utilizes the idea of **sector nonlinearity**.

The idea of using sector nonlinearity in fuzzy model construction first appeared in (Kawamoto, Tada, Ishigame, & Taniguchi, 1992) and expanded in (Tanaka & Wang, 2001). Sector nonlinearity is based on the following idea. Consider a nonlinear system $\dot{x} = f(x),\ x \in \mathscr{R}$. The aim is to find a global sector such that $a_1 x \leq f(x) \leq a_2 x$. Figure 3.1 illustrates the sector nonlinearity approach. This approach guarantees an exact fuzzy model construction. However, it is sometimes difficult to find global sectors for general nonlinear systems. In this case, we can consider local sector nonlinearity. This is reasonable as variables of physical systems are always bounded.

Example 3.1 *Let us considerer the nonlinear model*

$$\dot{x} = \cos(x)x \tag{3.5}$$

then as $\cos(x) \in [-1,1]$

$$\cos(x) = \frac{\cos(x)+1}{2} * 1 + \frac{1-\cos(x)}{2}(-1)$$

defining the membership functions $\mu_1 = \frac{1+\cos(x)}{2}$ and $\mu_2 = \frac{1-\cos(x)}{2}$ clearly μ_1 and μ_2 fulfill (3.3). Then a TS fuzzy representation of the system is:

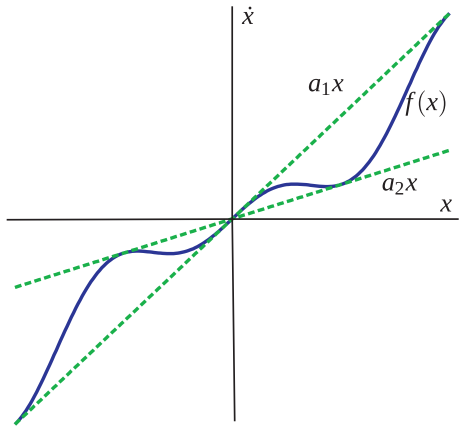


Figure 3.1: Sector nonlinearity representation.

IF x is $2n\pi$ **THEN**

$$\dot{x} = x$$

IF x is $(2n+1)\pi$ **THEN**

$$\dot{x} = -x$$

$$\dot{x} = \sum_{x=1}^{2} \mu_i a_i x \tag{3.6}$$

where $n \in \mathbb{N}$, $a_1 = 1$ and $a_2 = -1$.

We can generalize to any bounded nonlinear function such that: $f(x) \in [\underline{f}, \overline{f}]$, then

$$f(x) = \frac{f(x) - \underline{f}}{\overline{f} - \underline{f}} * \overline{f} + \frac{\overline{f} - f(x)}{\overline{f} - \underline{f}} \underline{f} \tag{3.7}$$

The drawbacks of the methodology are:

- If $f(x)$ is unbounded this is only true on a compact set.
- If there are N nonlinear functions on the model then there are 2^N rules and they describe a multiaffine or tensor product Fuzzy system which is widely discussed in Chapter 9.

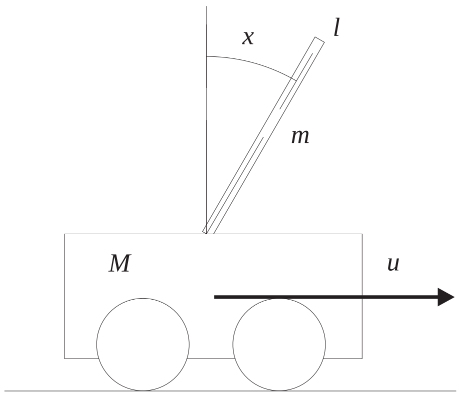


Figure 3.2: Inverted pendulum system.

Example 3.2 *Inverted pendulum TS model*

The equations of motion for the inverted pendulum (Cannon, 2003) are

$$\dot{x}_1 = x_2 \tag{3.8}$$

$$\dot{x}_2 = \frac{g\sin(x_1) - amlx_2^2\sin(2x_1)/2 - a\cos(x_1)u}{4l/3 - aml\cos^2(x_1)} \tag{3.9}$$

where x_1 denotes the angle in radians of the pendulum from the vertical and x_2 is the angular velocity; g is the acceleration due to gravity, m is the mass of the pendulum, M is the mass of the cart, l is the length of the pendulum, and u is the force applied to the cart; $a = 1/(m+M)$. following the reasoning in (Tanaka & Wang, 2001) we choose the nonlinear functions

$$\begin{aligned} f_1 &= \frac{1}{4l/3 - aml\cos^2(x_1)} \\ f_2 &= \sin(x_1)/x_1 \\ f_3 &= x_2\sin(2x_1) \end{aligned}$$

$$f_4 = \cos(x_1)$$

where $x_1 \in (-\pi/2, \pi/2)$ and $x_2 \in [-\alpha, \alpha]$. Defining the membership functions μ_{im} as in (3.7), where the subindex i corresponds to the nonlinear linear function f_i and m is the membership function number for f_i, that is 1 or 2. We obtain the TS model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} \mu_{1i}\mu_{2j}\mu_{3k}\mu_{4l} \left[A_{ijkl} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + B_{ijkl} u \right] \tag{3.10}$$

where A_{ijkl} and B_{ijkl} are defined from the bounds of f_1, f_2, f_3, f_4.

There are several contributions on nonlinear systems are modelled as TS models (Sugeno & Nishida, 1985; Sugeno & Kang, 1986; Babuska, 1998; Hori, Tanaka, & Wang, 2002; Baranyi, 2004; Feng, 2006) etc... There are also interesting results (Fantuzzi & Rovatti, 1996; Ying, 1998; Wang, Li, Niemann, & Tanaka, 2000; Tanaka & Wang, 2001) that show TS models are universal approximators.

3.4 Stability of fuzzy models

Lyapunov stability theory proves that such a system is stable if there exist positive α, β and a function $V(x)$ such that:

$$\beta\|x\| \geq V(x) \geq \alpha\|x\|, \ \frac{dV}{dx} < 0, \ V(0) = 0, \ \forall \ x$$

The most popular Lyapunov Functions proposed in literature are quadratic forms:

$$V(x) = x^T P x \tag{3.11}$$

with matrix P being symmetric and definite positive. So we review the basic results on quadratic stability of the Takagi-Sugeno fuzzy model (3.1).

3.4.1 Stability of open-loop system

Let us first consider the stability of continuous system without inputs (u), that is:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(x) A_i \cdot x \tag{3.12}$$

In this case, we take the quadratic form shown above $V(x) = x^T P x$ as our Lyapunov function. Due to P being definite positive the first condition holds. So we only need to prove that $\dot{V} < 0$. First we obtain the value of $\dot{V}$, using (3.12), that is:

$$\dot{V} = x^T [\sum_{i=1}^{r} \mu_i (A_i^T P + PA_i)] x$$

as the membership functions are greater than 0 a sufficient condition for stability is:

$$A_i^T P + PA_i < 0, \ i : 1..r \tag{3.13}$$

where $<$ means definite negative. As it can be seen these conditions are independent for any membership function. That makes these conditions conservative. This thesis contributes new conditions in order to reduce the conservatism of the conditions.

The above equations are LMIs, hence widely available LMI optimization software either finds a P or determines that the LMI is infeasible.

Remark: Note that the membership functions μ do not appear in the LMI conditions. Hence, the same P defines a quadratic Lyapunov function for multiple nonlinear systems with the same "vertex models" as the original one. Such generality is good in case a feasible P is found but, on the contrary,it is too restrictive a condition as in some cases a solution can not be reached despite the underlying system being stable.

3.4.2 Stability of PDC closed-loop systems

The objective here is to illustrate the basic ideas of stability analysis and stable fuzzy control via LMIs.

First we define the nonlinear control law by a PDC controller (3.4). The substituting (3.4) into (3.1), The closed-loop system can be rewritten as:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (A_i - B_i F_j) x \tag{3.14}$$

Note that we have used the equality $\sum_{i=1}^{r} \mu_i = 1$ to obtain the above expression.

Denote:

$$G_{ij} = A_i - B_i F_j \tag{3.15}$$

As $\sum_i \sum_j \mu_i \mu_j = 1$, it may be considered as an open-loop one and stability addressed by (3.13), i.e., looking for a common P such that:

$$G_{ij}^T P + PG_{ij} < 0 \quad \forall\ i, j$$

However, there are *much less* conservative conditions. Stability analysis in such a case has been discussed in (Tanaka & Wang, 2001) and (Kim & Lee, 2000), among others. The main results in those works to be applied in this contribution are the following.

Theorem 3.1 *(Tanaka & Wang, 2001) The equilibrium of the system described by* (3.1) *is globally asymptotically stable if there exists a common positive definite matrix* P *such that:*

$$G_{ii}^T P + PG_{ii} \quad < \quad 0 \tag{3.16}$$

$$\left(\frac{G_{ij}+G_{ji}}{2}\right)^T P + P\left(\frac{G_{ij}+G_{ji}}{2}\right) \quad \leq \quad 0 \tag{3.17}$$

where (3.17) *must hold only if* $\mu_i \mu_j \neq 0$.

Proof: note that system (3.14) can be rewritten as:

$$\dot{x} = \sum_{i=1}^{r} \mu_i^2 (A_i - B_i F_i)x + \sum_{i=1}^{r} \sum_{j>i} \mu_i \mu_j (A_i - B_i F_j + A_j - B_j F_i)x \tag{3.18}$$

then we take the quadratic Lyapunov function defined in (3.11).

$$\dot{V} = \sum_{i=1}^{r} \mu_i^2 (G_{ii}^T P + PG_{ii})x + \sum_{i=1}^{r} \sum_{j>i} \mu_i \mu_j [(G_{ij}+G_{ji})^T P + P(G_{ij}+G_{ji})]x \tag{3.19}$$

so $\dot{V}$ is definite positive if (3.16) and (3.17) hold.

■

Theorem 3.2 *(Kim & Lee, 2000) The equilibrium of the fuzzy control system* (3.1) *is globally quadratically stable if there exist symmetric positive matrix* P *and symmetric matrices* X_{ij} *such that*

$$\Lambda_{ii}^T P + P\Lambda_{ii} + X_{ii} \quad < \quad 0, \quad i: 1, \ldots, n \tag{3.20}$$

$$\Lambda_{ij}^T P + P\Lambda_{ij} + X_{ij} \quad < \quad 0, \quad i < j \leq r \tag{3.21}$$

$$\begin{pmatrix} X_{11} & \ldots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \ldots & X_{nn} \end{pmatrix} \quad > \quad 0 \tag{3.22}$$

where

$$\Lambda_{ii} = G_{ii}$$
$$\Lambda_{ij} = \frac{G_{ij} + G_{ji}}{2}$$

■

3.5 Control Design for Takagi-Sugeno Fuzzy Systems

In many situations, Lyapunov-based conditions for stability or performance of a fuzzy control system may be expressed in the form

$$\Xi(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z(t))\mu_j(z(t))x(t)^T Q_{ij} x(t) > 0 \quad \forall\, x \neq 0 \tag{3.23}$$

where $z(t)$ are denoted as premise variables (usually measurable) and r denotes the number of fuzzy "rules" or "local models". Symmetry ($Q_{ij} = Q_{ji}^T$), and a fuzzy partition condition (3.3) are assumed to hold. Notation μ_i will be used as shorthand for $\mu_i(z(t))$. Also, in most cases, "positive" in the text below should be understood as shorthand for "positive for $x \neq 0$". Note that, the fuzzy partition condition (3.3) also implies:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j = 1 \tag{3.24}$$

A typical example of the use of condition (3.23) is proving quadratic stability of the fuzzy system (3.1) with a fuzzy PDC state-feedback controller (3.4)

$$\dot{V} = \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i(z(t))\mu_j(z(t))x(t)^T((A_i - B_iF_j)^T P + P(A_i - B_iF_j))x(t) \tag{3.25}$$

Introducing the notation $G_{ij} = A_i - B_iF_j$, $\Lambda_{ij} = \frac{1}{2}(G_{ij} + G_{ji})$, matrices Q_{ij} in (3.23) are (Kim & Lee, 2000):

$$Q_{ii} = -G_{ii}^T P - PG_{ii} \tag{3.26}$$
$$Q_{ij} = -\Lambda_{ij}^T P - P\Lambda_{ij} \tag{3.27}$$

Where $P > 0$ is a symmetric matrix, which defines a Lyapunov function $V(x) = x^T Px$, to be obtained via LMI algorithms (Boyd et al., 1994; Tanaka & Wang, 2001; Boyd & Vandenberghe, 2004).

Stabilization design

If F_j in (3.4) is also to be designed, applying the change of variable $\xi = P^{-1}x$, (3.25) can be rewritten as

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z(t))\mu_j(z(t))\xi(t)^T(X(A_i - B_iF_j)^T + (A_i - B_iF_j)X)\xi(t) \quad (3.28)$$

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z(t))\mu_j(z(t))\xi(t)^T(XA_i^T - XF_j^T B_i^T + A_iX - B_iF_jX)\xi(t) \quad (3.29)$$

where $X = P^{-1}$ and defining M_i as $M_i = F_iX$, Q_{ij} may be se to

$$Q_{ij} = -(A_iX + XA_i{}^T - B_iM_j - M_j^T B_i^T) \quad (3.30)$$

Decay-rate performance requirement

The speed of response is related to the decay rate, that is, the largest Lyapunov exponent α (Boyd et al., 1994) such that

$$\lim_{t\to\infty} e^{\alpha t}\|x(t)\| = 0 \quad (3.31)$$

A sufficient condition for (3.31) with Lyapunov candidate $V = x^T Px$ is

$$\dot{V} < -2\alpha V \quad (3.32)$$

for any initial point. Then if $V(t) < V(x_0)e^{-2\alpha t}$ and $P > 0$ then (3.31) holds.

In Fuzzy control it can be applied introducing by Q_{ij} such that

$$Q_{ij} = -(A_iX + XA_i{}^T - B_iM_j - M_j^T B_i^T + 2\alpha X) \quad (3.33)$$

where $X > 0$ and $M_i = F_iX$ are LMI decision variables (for details, see (Tanaka & Wang, 2001; Kim & Lee, 2000)).

Controller and observer design

In many applications, the state is not readily available. Then a state space observer is needed to apply the fuzzy controller. A fuzzy observer is defined in (Tanaka & Wang, 1997; Tanaka, Ikeda, & Wang, 1998) as

$$\dot{\bar{x}} = \sum_{i=1}^{r} \mu_i(z)(A_i\bar{x} + B_i u + K_i(y - \bar{y})) \tag{3.34}$$

$$\bar{y} = \sum_{i=1}^{r} C_i\bar{x} \tag{3.35}$$

Therefore, the PDC controller is computed using the state space estimation $\bar{x}$ as

$$u = \sum_{i=1}^{r} \mu_i F_i \bar{x}$$

In (Tanaka et al., 1998), the augmented close-loop plant (3.36) is proposed

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A_i - B_iF_j & B_iF_j \\ 0 & A_i - K_jC_i \end{pmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} \tag{3.36}$$

where $\tilde{x} = x - \bar{x}$. Then with the Lyapunov function candidate

$$V = (x^T \ \tilde{x}^T) \begin{pmatrix} P_1 & 0 \\ 0 & \gamma P_2 \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \tag{3.37}$$

where γ is a positive constant. The systems (3.36) can be stabilized if (3.23) holds, with Q_{ij} defined as

$$Q_{ij} = A_iX + XA_i^T - B_iM_j - M_j^T B_i^T \tag{3.38}$$

where $X = P_1^{-1}$, $M_j = F_jX$ and if (3.23) holds, with Q_{ij} defined as

$$Q_{ij} = A_i^T P_2 + P_2A_i - N_jC_i - C_i^T N_j^T \tag{3.39}$$

where $N_j = P_2K_j$. This result is similar to the separation principle for linear systems. It has also been proved for TS fuzzy systems in (Ma, Sun, & He, 1998). Note that, in the presented results the system model do not include stability conditions of output stabilization for uncertain fuzzy models. This has been reported in (Guerra, Kruszewski, Vermeiren, & Tirmant, 2006).

Disturbance rejection

Considerer the following TS fuzzy system with a disturbance w, in (Tuan, Apkarian, Narikiyo, & Yamamoto, 2001)

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)(A_i x + B_{1i} w + B_{2i} u) \tag{3.40}$$

$$y = \sum_{i=1}^{r} \mu_i(z)(C_i x + D_{11i} w + D_{12i} u) \tag{3.41}$$

The disturbance rejection can be realized by minimizing γ such that

$$\sup_{\|w\|_2 \neq 0} \frac{\|y\|_2}{\|w\|_2} \leq \gamma \tag{3.42}$$

The related performance condition in (Tuan et al., 2001) uses as Q_{ij} the matrix:

$$\begin{pmatrix} PA_i^T + R_j^T B_{2i}^T + A_i P + B_{2i} R_j & B_{1i} & PC_i^T + R_j^T D_{12i}^T \\ B_{1i}^T & -\gamma I & D_{11i}^T \\ C_i P + D_{12i} R_j & D_{11i} & -\gamma I \end{pmatrix} \tag{3.43}$$

Then the H_∞ disturbance rejection is lower than γ

Other performance settings

The reader is referred to (Tanaka & Wang, 2001; Fang et al., 2006; Sala, Guerra, & Babuska, 2005) and references therein for details on the possibilities of different Q_{ij}. Also the reader can consult (Chen, Chang, Su, Chung, & Lee, 2005; Hsiao, Hwang, Chen, & Tsai, 2005) for fuzzy delay systems , (Guerra & Vermeiren, 2004; Choi & Park, 2003) for uncertain ones, non-quadratic Lyapunov functions, (Chen, Tseng, & Uang, 2000) for output feedback control, *etc.*

3.5.1 Positiveness conditions

Of course, requiring $Q_{ij} > 0$ is a trivial sufficient condition for positiveness of (3.23), but much less conservative conditions appear in literature. One of the first was proposed in

Theorem 3.3 ***(Tanaka & Sano, 1994).*** *Expression* (3.23) *under fuzzy partition condition holds if*

$$Q_{ii} > 0 \quad i = 1 \dots r \tag{3.44}$$

$$Q_{ij} + Q_{ji} > 0 \quad i = 1 \dots r \;\; i < j \tag{3.45}$$

In (Tuan et al., 2001) more relaxed conditions for (3.23) are presented

Theorem 3.4 ***(Tuan et al., 2001).*** *Expression* (3.23) *under fuzzy partition condition holds if*

$$Q_{ii} > 0 \quad i = 1 \dots r \tag{3.46}$$

$$\frac{2}{r-1} Q_{ii} + Q_{ij} + Q_{ji} > 0 \quad i = 1 \dots r \;\; i \neq j \tag{3.47}$$

Another popular one is Theorem 2 in (Liu & Zhang, 2003), which is mainly based on the scheme in (Kim & Lee, 2000). The conditions in (Liu & Zhang, 2003) may be proved by means of reordering (3.23) as

$$\Xi = \sum_{i=1}^{r} \mu_i^2 x^T Q_{ii} x + \sum_{i=1}^{r} \sum_{i<j\leq r} \mu_i \mu_j x^T (Q_{ij} + Q_{ji}) x > 0 \tag{3.48}$$

Hence, if $X_{ii} \leq Q_{ii}$ and $X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji}$ for $i \neq j$, as $\mu_i \mu_j$ is always greater or equal than 0,

$$\Xi \geq \sum_{i=1}^{r} \mu_i^2 x^T X_{ii} x + \sum_{i=1}^{r} \sum_{i<j\leq r} \mu_i \mu_j x^T (X_{ij} + X_{ji}) x > 0 \tag{3.49}$$

which may be expressed in matrix form as in Theorem 7 in (Kim & Lee, 2000), yielding:

Theorem 3.5 ***(Liu & Zhang, 2003, Theorem 2).*** *Expression* (3.23) *under fuzzy partition condition holds if there exist matrices* $X_{ij} = X_{ji}^T$ *such that:*

$$X_{ii} \leq Q_{ii} \tag{3.50}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} \;\; i \neq j \tag{3.51}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} > 0 \tag{3.52}$$

Note: In (Kim & Lee, 2000), X_{ij} are forced to be symmetric (more conservative than (Liu & Zhang, 2003)). Also, it is well-known that conditions (3.51) need to be enforced only when $\mu_i \mu_j \neq 0$, *i.e.*, referring to overlapping fuzzy sets. The reader is also referred to (Teixeira, Assuncao, & Avellar, 2003) for related conditions.

Part II

Researched issues and contributions

Chapter 4

Roadmap: Reducing the gap between the fuzzy and nonlinear control

As have been shown in the introduction, LMI formulations for fuzzy control became the tool of choice in the 1990s. Many nonlinear systems can be modelled as fuzzy systems (sector-nonlinearity), so fuzzy control may be considered as a technique for nonlinear control. In spite of the obtained results some sources of conservativeness remain when compared to other nonlinear approaches. This chapter, inspired by the work exposed in (Sala, 2007), can be understood as a brief compilation of the main contributions of this thesis.

4.1 Introduction

Fuzzy control started as a heuristic methodology in the 1970's, coding control rules by hand, trying to embed heuristic and reasoning into the control block. However, most of the widespread heuristic rules have no fundamental differences with standard PID regulators (the fuzzy-PD, fuzzy-PI and alike) and many other heuristic designs, which fuzzify operation rules for a complex plant, are one-of-a-kind tailored developments which therefore have little interest for a broad audience. Due to these reasons, the emphasis on heuristics and logic reasoning in fuzzy control has almost disappeared, in favor of rigorous mathematical tools, in order to guarantee control specifications expressed in terms of stability, performance, robustness to modelling errors, *etc.* for a class of *nonlinear systems*, for which a systematic modelling methodology (sector nonlinearity (Tanaka & Wang, 2001)) is available to transform them into Takagi-Sugeno form (3.1):

$$\dot{x} = \sum_{i=1}^{r} \mu_i f_i, \quad x_{k+1} = \sum_{i=1}^{r} \mu_i f_i$$

with f_i linear. In this way, nonlinear systems may be "embedded" into a linear time-varying (LTV) dynamics and LTV design tools applied readily.

The advantages of the fuzzy approach are that efficient semidefinite-programing tools (LMI (Boyd et al., 1994; Boyd & Vandenberghe, 2004), sum-of-squares (Prajna, Papachristodoulou, & Parrilo, 2002)), widely used in linear systems, can be almost directly applied to nonlinear control problems. Unfortunately, there is a price to pay: the objective of this Chapter is discussing the specific disadvantages of the fuzzy approach versus "direct nonlinear" alternatives. The reader is referred to (Sala et al., 2005) for other considerations, trends and open issues in fuzzy modelling, identification and control.

Nowadays, Linear Matrix Inequality (LMI) techniques have become the tool of choice in order to design fuzzy controllers when a fuzzy model of the process is available in the Takagi-Sugeno form: LMIs were introduced by (Tanaka & Sugeno, 1992) in the fuzzy control community, becoming widespread in the last 10 years. Results are available for systems with uncertainty, delay, descriptor forms, etc.

As it has been shown in Chapter 3, the most frequently considered setting is the so called parallel-distributed-compensation (PDC) in which a fuzzy TS controller shares the membership functions with the plant to be controlled (3.4). Conditions for decrescence of a Lyapunov functions in such settings end up requiring to prove positiveness of a so-called double fuzzy summation[1] (Tanaka & Wang, 2001; Liu & Zhang, 2003), in expressions such as (3.23).

4.2 Some Shadows

Basically, there are several sources of conservatism on mainstream fuzzy control results. The following will be chosen for discussion below:

[1]Other settings (fuzzy observers, descriptor systems, fuzzy Lyapunov functions, *etc.*) may require a higher summation dimension.

1. choice of the Lyapunov function family,
2. conservativeness on the most used theorems on "positivity of fuzzy summations",
3. results are independent of the membership shape,
4. computational power

and some recent ideas to overcome them will be discussed in Section 4.3.

Other issues, such as uncertainty structures, adaptive approaches, observers, etc. are also important, deserving further attention, but they have not been elaborated here. The reader is referred to (Labiod & Guerra, 2007; ?, ?) for recent relevant contributions in some of those topics.

Let us discuss the above-selected issues in some detail:

4.2.1 Conservatism of the Lyapunov approach

In fuzzy control, the search of Lyapunov functions is only made on a particular family of candidate functions, hence a stable system may not have a Lyapunov function in the particular class being sought. This problem is common to nonlinear control theory: the Lyapunov approach is not constructive.

The approach using the quadratic Lyapunov function $x^T Px$ has been deeply explored. Improvements are available: a piecewise quadratic function (Feng, 2003; Johansson, 1999), and fuzzy Lyapunov functions for continuous (Tanaka, Hori, & Wang, 2003) or discrete (Guerra & Vermeiren, 2004) systems.

4.2.2 Conservativeness of the positivity conditions for fuzzy summations

Current literature (Tanaka & Wang, 2001; Liu & Zhang, 2003; Fang et al., 2006) presents only sufficient conditions for (3.23) to hold. Hence a Lyapunov function fulfilling (3.23) may exist but the conditions above may fail to find it. There is a long-term quest in the fuzzy community to find necessary and sufficient conditions for positivity of (3.23).

4.2.3 Intrinsic conservativeness of the fuzzy approach versus a nonlinear one

The last open issue regards the knowledge of the membership function shape. Indeed, in order to implement a fuzzy PDC controller, the values of μ_i should be a known function of some measurable variables. However, the LMI conditions in the above works do *not* depend on the shape of the membership functions. This is good, because the results apply to a set of nonlinear time-varying systems (as long as they share the same "vertices"), but it is, evidently, conservative *for a particular system.* On the contrary, Lyapunov-based nonlinear control usually takes into account the *exact shape* of the nonlinearity in order to derive results, being less conservative than the fuzzy approach when used *for a particular system.*

As a trivial example, consider $\dot{x} = \mu_1(z) \cdot x + (1 - \mu_1(z)) \cdot (-x)$. It cannot be proved stable for an arbitrary μ_1, $0 \leq \mu_1(z) \leq 1$ (it is unstable for $\mu_1(z) = 1$). However, it *is* stable for, say, $\mu_1 = 0.2 + 0.2sin(x)$ as $\dot{x} = (-1+2\mu_1)x$ is, trivially, an exponentially stable first-order nonlinear system when $\mu_1 \leq b < 0.5$, $b \in \mathbb{R}$. The example is a clear case indicating that this situation may happen even in first-order systems: by using nonlinear control ideas, once the explicit formula of the membership functions has been included in the TS model, Lyapunov functions may be obtained where fuzzy methodologies fail.

4.2.4 Computational power

The number of decision variables in some of the latest LMI results is huge. Even if LMI solvers use polynomial-time algorithms, the exponent of the system order is large and many results can only be implemented for simple systems.

4.3 Some Lights

There are some ideas reducing conservativeness on the first three cases above, but heavily increasing the fourth problem (computational power) so there is a tradeoff. The results, however, diminish the gap between fuzzy and nonlinear control, at least in theory.

4.3.1 Arbitrarily complex Lyapunov functions

It is interesting to mention three possibilities of getting arbitrarily complex Lyapunov functions:

(1) higher-degree homogeneous Lyapunov functions (Chesi, Garulli, Tesi, & Vicino, 2003)

(2) exacerbate the piecewise idea (the point-wise approach in (Johansen, 2000));

(3) use "resampling", checking for $V(t+k)-V(t) \leq 0$, $k>1$ (Kruszewski & Guerra, 2005) (indeed, if a system is stable, there will exist a k so that even $V(x) = x^T x$ will do; complexity lies in the predictors needed in the approach and in an underlying non-delayed Lyapunov function expression).

4.3.2 Asymptotically necessary and sufficient conditions for fuzzy summations. Chapter 5

Consider a multi-dimensional index variable $\mathbf{i} \in \{1,\ldots,r\}^n$ where r is the number of rules and n is an arbitrary complexity parameter. Denote the permutations of $\mathbf{i}$ by $\mathscr{P}(\mathbf{i})$. Then, results in (Liu & Zhang, 2003; Fang et al., 2006) are particular cases of finding a multi-dimensional arrangement of matrices (tensor) fulfilling, for all $\mathbf{i}$:

$$\sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_{j_1 j_2} > \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} \frac{1}{2}(X_{\mathbf{j}} + X_{\mathbf{j}}^T) \tag{4.1}$$

and the inequality (with complexity $n-2$):

$$\sum_{\mathbf{k}\in B_{n-2}} \mu_{\mathbf{k}} \xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)} & \cdots & X_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)} & \cdots & X_{(\mathbf{k},r,r)} \end{pmatrix} \xi > 0 \tag{4.2}$$

In a suitable recursive framework, it can be proved that the above conditions become *necessary and sufficient* with $n \to \infty$, and establish some "tolerance" parameter for finite n as it is shown in Chapter 5

The main issue with these conditions is, however, that they are necessary only asymptotically; hence, unfeasibility does not imply that the original fuzzy control problem is not solvable as they are only sufficient. In (Kruszewski, Sala, Guerra, & Arino, 2007) a complementary approach (necessary and asymptotically sufficient condition) is presented.

4.3.3 Membership shape techniques

Incorporating membership-shape information may relax conservativeness. There are two types of such information:

- restrictions on the memberships themselves (which would apply to LTV systems and, hence, to nonlinear systems embedded as LTV, *i.e.*, Takagi-Sugeno models), say $\mu_1\mu_2 < 0.1$
- Nonlinearity-related restrictions (restrictions on the membership shape in particular regions of the state space), such as $x_1 \geq 0 \Rightarrow \mu_2 < 0.4$ *i.e.*, in a certain zone of the state space, a particular model is *not* totally active so the Lyapunov function may take it into account.

Each of those two classes will be discussed in more detail in Chapters 7 and 9. Another promising possibility is *designing* the memberships of the fuzzy controller (not necessarily the same and the same number than those of the plant) to achieve some performance objectives. A first approach in that direction appears in (Lam & Leung, 2007).

Membership-only restrictions (LTV)

(a) Overlap bounds. Chapter 7. As the membership functions for fuzzy controllers are known, the following set of bounds can be easily computed:

$$0 \leq \mu_i(z)\mu_j(z) \leq \beta_{ij} \ \forall \ z \tag{4.3}$$

The bounds β_{ij} may be used to set up some relaxed LMIs. From (Sala & Ariño, 2007a), expression (3.23) holds if there exist matrices $X_{ij} = X_{ji}^T$

and symmetric R_{ij}, $i \leq j$, such that, defining $\Lambda = \sum_{k=1}^{r} \sum_{k \leq l \leq r} \beta_{kl} R_{kl}$:

$$X_{ii} \leq Q_{ii} + R_{ii} - \Lambda \tag{4.4}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + R_{ij} - 2\Lambda \tag{4.5}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} > 0, \qquad R_{ij} \geq 0 \tag{4.6}$$

In this way, adding information about the "overlap" between membership functions allows additional performance to be gained from a PDC controller for a particular system.

(b) Tensor-product fuzzy systems, Chapter 9. Very frequently, fuzzy systems are in the form:

$$\dot{x} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \dots \sum_{i_p=1}^{n_p} \mu_{1i_1} \mu_{2i_2} \dots \mu_{pi_p} (A_{i_1 i_2 \dots i_p} x + B_{i_1 i_2 \dots i_p} u) \tag{4.7}$$

where the fuzzy antecedents are conformed as a tensor product of simpler fuzzy partitions. A well-known case are those using the sector-nonlinearity modelling technique in (Tanaka & Wang, 2001), where $n_1 = \dots = n_p = 2$ from interpolations between maximum and minimum. The recursive methodologies in Chapter 9 exploit such tensor structure. However, it is more computationally demanding than using the previous approach, knowing that the tensor-product systems have overlap bounds which are powers of 0.25, say

$$\mu_{12}\mu_{23} = (\mu_{11}\mu_{22}) \times (\mu_{12}\mu_{23}) \leq 0.25^2$$

4.3.4 Local stability approach, Chapter 6

When global stability conditions are unfeasible, it is interesting to archive *local* stability results in a zone around the equilibrium as large as possible. this is motivated by the first Lyapunov theorem for local stability, Chapter 2. In that case we approximate the global TS fuzzy system to a *local* TS fuzzy model which is valid in a defined region. The idea is obtain a region as large as possible where the global system remains stable.

If the membership functions $\mu(x)$ of the fuzzy system (3.14) in a region Ω can be expressed as

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p, \quad \forall\, x \in \Omega \tag{4.8}$$

then the system can be equivalently expressed as:

$$\dot{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \beta_j (A_i^* - B_i^* F_j^*) x \tag{4.9}$$

where

$$A_p^* = \sum_{i=1}^{n} v_{pi} A_i \tag{4.10}$$

$$B_p^* = \sum_{i=1}^{n} v_{pi} B_i \tag{4.11}$$

$$F_p^* = \sum_{i=1}^{n} v_{pi} F_i \tag{4.12}$$

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \;\; \beta_p(x) > 0 \;\; \forall x \in \Omega \;\; p = 1 \dots n_v$$

This model is valid only in the region Ω. If the region is close to $x = 0$ then the TS model tends to the linearised model of the system at the equilibrium point. Therefore if the linearised system is stable with this local modelling we will find an ellipsoidal region $\Omega^* \subset \Omega$ where the system is stable.

4.3.5 Uncertain memberships, Chapter 8

The majority of works on fuzzy control for TS models assume that the membership functions are known. But, in most cases, the applied memberships functions are only an approximation. As the memberships are needed to compute the PDC controller that stabilize the plant, the conditions in Section 3.5 cannot be applied to prove stability. New shape-dependent LMI conditions have been developed and the allowed uncertainty description is more general than that in (Lam & Leung, 2005), which did consider only multiplicative uncertainty.

The system (3.1) will be controlled via a state-feedback fuzzy controller:

$$u = -\sum_{i=1}^{r} \eta_i K_i x \quad \sum_{i=1}^{r} \eta_i = 1,\ \eta_i \geq 0 \tag{4.13}$$

The controller yields a closed-loop (Lam & Leung, 2005):

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z) \eta_j(z')(A_i - B_i K_j)x \tag{4.14}$$

With a given set of p restrictions on the shape of the membership functions of the plant, μ_i, and controller, η_i

$$\mathbf{c}_k^T \eta + \mathbf{a}_k^T \mu + b_k \leq 0 \; k = 1 \dots p \tag{4.15}$$

Then we can apply the s-procedure introducing new positive definite matrices R_{ij} and R_{ij}^* to the conditions of Theorem 3.5

$$\sum_{k=1}^{p} \left(a_{jk} R_{ik} + a_{ik} R_{jk} \right) \geq X_{ij} + X_{ji} \tag{4.16}$$

$$Q_{ij} + \sum_{k=1}^{p} \left(c_{jk} R_{ik} + a_{ik} R_{jk}^* + b_k (R_{ik} + R_{jk}^*) \right) \geq X_{i(j+r)} + X_{(j+r)i} \tag{4.17}$$

$$\sum_{k=1}^{p} \left(c_{ik} R_{jk}^* + c_{jk} R_{ik}^* \right) \geq X_{(i+r)(j+r)} + X_{(j+r)(i+r)} \tag{4.18}$$

$$\begin{pmatrix} X_{11} & \dots & X_{1(2r)} \\ \vdots & \ddots & \vdots \\ X_{(2r)1} & \dots & X_{(2r)(2r)} \end{pmatrix} > 0 \tag{4.19}$$

4.4 Conclusions

LMI fuzzy control approaches nonlinear control if the actual shape of the membership functions is used in the LMIs (exploiting some nonlinearity knowledge) and some "fuzzy" sufficient conditions are made also (asymptotically) necessary or, conversely, necessary ones are made asymptotically sufficient. That fact, jointly with the use of more general Lyapunov functions, looks promising but, in practice, it requires a lot of computing power.

Chapter 5

Relaxed LMI conditions, an asymptotically approach

5.1 Introduction

The aim of this chapter is to provide a set of progressively less conservative sufficient conditions in order to prove stability or performance conditions for Takagi-Sugeno Fuzzy systems. These conditions are asymptotically necessary. This is not a new area in Fuzzy control, there are many literature references in the area of stability and performance conditions of Takagi-Sugeno (TS) (Takagi & Sugeno, 1985) fuzzy control systems. Currently, most of the significant results use a linear matrix inequality (LMI) approach (Boyd et al., 1994), which reached maturity in the fuzzy area with (Tanaka & Wang, 2001).

Basically, the most frequently considered setting is parallel-distributed-compensation (PDC), in which a fuzzy TS controller shares the membership functions with the plant to be controlled. In many cases, PDC fuzzy control design involves checking positivity of "double" fuzzy summations in the form (3.23).

The first nontrivial sufficient conditions for such positivity were reported in the 1990's (Tanaka & Wang, 2001). Later research has focused in conceiving less conservative LMI conditions, such as the ones in (Kim & Lee, 2000; Teixeira et al., 2003; Liu & Zhang, 2003). Recently, (Fang et al., 2006) provided sufficient conditions which are less conservative than those in (Kim & Lee, 2000; Teixeira et al., 2003; Liu & Zhang, 2003). However, the problem of finding necessary and sufficient positivity conditions (*i.e.*, the "least conservative sufficient conditions") for fuzzy summations remains open. Note that, even if necessary and sufficient conditions for fuzzy summations were found, conservatism would remain in many cases due to the limited choices of Lyapunov functions (for instance, quadratic stability is more conservative than other possibilities, such as the ones in (Guerra & Vermeiren, 2004)).

This chapter provides an infinite family of sufficient conditions for positivity of fuzzy summations which improve over the ones in (Kim & Lee, 2000; Teixeira et al., 2003; Liu & Zhang, 2003; Fang et al., 2006) (actually, the cited conditions are particular cases). A theorem is presented which introduces a parameter, n, related to the complexity of the resulting LMI conditions. As n increases, the conditions are asymptotically exact, *i.e.*, they become necessary for a large enough n. The results in this work are based on Polya's theorems on positivity of homogeneous forms in the standard simplex which date back to the 1920's (Pólya & Szegö, 1928), refined in (Powers & Reznick, 2001; Loera & Santos, 1996; Scherer, 2005).

The presented results close an open theoretical problem, in an asymptotic sense. In practice, however, there are two shortcomings: first, bounds on n to achieve necessity and sufficiency can be stated for fixed Q_{ij} in (3.23), but that is not the case if Q_{ij} contain LMI decision variables (then, a sort of "tolerance" parameter needs to be introduced); second, the reachable value of n is limited by numerical precision of solver software and the available computing power.

5.2 Problem statement

Widely-used conditions (Tanaka & Wang, 2001; Sala et al., 2005; Guerra & Vermeiren, 2004; Fang et al., 2006) for stability or performance of a closed-loop fuzzy control system may be expressed, for some matrices Q_{ij} (assumed symmetric, without loss of generality), in the form (3.23)

$$\Xi(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z)\mu_j(z)x^T Q_{ij} x > 0 \quad \forall x \neq 0$$

where z and x are system variables which change with time and $\mu_i(z)$ denotes the membership functions of a fuzzy partition

$$\{\mu_1(z), \mu_2(z), \ldots, \mu_r(z)\}$$

being z a set of known variables, and r the number of "rules".

A well-known example (Tanaka & Wang, 2001) of conditions for quadratic stabilizability of a continuous-time Takagi-Sugeno (TS) fuzzy system

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)(A_i x + B_i u)$$

with a parallel-distributed compensator (3.4) $u = -\sum_{i=1}^{r} \mu_i(z) K_i x$ may be written, for some symmetric matrix $T > 0$, in the form (3.23) as it is shown in Chapter 3. Where the most important stability and performance conditions are expressed in this way.

Importantly, if Q_{ij} are linear in some matrix decision variables, linear matrix inequality (LMI) techniques (Boyd et al., 1994; Tanaka & Wang, 2001) may be used to find feasible values for the unknown matrices, if *sufficient* LMI conditions for (3.23) to hold are stated; the objective of this chapter is studying such sufficient conditions. Historically, conditions in (Tanaka & Wang, 2001):

$$Q_{ii} > 0, \quad Q_{ij} + Q_{ji} > 0 \qquad \forall\ i, j = 1, \ldots, r \quad i \leq j \tag{5.1}$$

were the first proposed ones, which have been generalised in literature (Kim & Lee, 2000; Liu & Zhang, 2003; Teixeira et al., 2003). Recently, (Fang et al., 2006) provided some sufficient conditions which improved over the previously cited ones. Section 5.3 will further improve such conditions.

As a generalization of (3.23), other fuzzy control results require positiveness of a p-dimensional fuzzy summation, *i.e.*, checking

$$\Xi(t) = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_p=1}^{r} \mu_{i_1}(z)\mu_{i_2}(z)\ldots\mu_{i_p}(z) x^T Q_{i_1 i_2 \ldots i_p} x > 0 \quad \forall x \neq 0 \tag{5.2}$$

The case $p = 2$ reduces to (3.23). Conditions requiring $p = 3$ are, for instance, the fuzzy dynamic controllers in (Li, Niemann, Wang, & Tanaka, 1999; Tanaka & Wang, 2001), using $Q_{ijk} = E_{ijk} + E_{ijk}^T$, with

$$E_{ijk} = \begin{pmatrix} A_i Q_{11} + B_i \mathscr{C}_{jk} & A_i + B_i \mathscr{D}_j C_k \\ \mathscr{A}_{ijk} & A_i P_{11} + \mathscr{B}_{ij} C_k \end{pmatrix} < 0 \tag{5.3}$$

or the output-feedback ones in (Fang et al., 2006; Chen et al., 2005). For convenience, shorthand μ_i denoting $\mu_i(z)$ will be used in the from now on.

5.2.1 Multi-index notation.

In order to streamline notation in multi-dimensional summations (5.2), the following notation will be used:

$$\mathbb{I}_p = \{\mathbf{i} = (i_1, i_2, \dots, i_p) \in \mathbb{N}^p \mid 1 \leq i_j \leq r \ \forall \ j = 1, 2, \dots, p\} \tag{5.4}$$

$$\sum_{\mathbf{i} \in \mathbb{I}_p} \gamma_{\mathbf{i}} = \sum_{i_1=1}^{r} \sum_{i_2=1}^{r} \cdots \sum_{i_p=1}^{r} \gamma_{i_1 i_2 \dots i_p} \tag{5.5}$$

For some suitably defined multi-dimensional $\gamma_{\mathbf{i}} = \gamma_{i_1 i_2 \dots i_p}$ ($\gamma_{i_1 i_2 \dots i_p}$ will be either a number or a matrix), *i.e.*, boldface symbol $\mathbf{i}$ will denote a multi-index in a p-dimensional index set $\mathbb{I}_p$ ($\mathbb{I}_p$ has r^p elements). For instance, triple-summations in a four-rule fuzzy system will be spanned by a multi-index $\mathbf{i} \in \mathbb{I}_3$, where $\mathbb{I}_3$ has 4^3 elements:

$$\{1,1,1\}, \{1,1,2\}, \dots, \{1,1,4\}, \{1,2,1\}, \dots, \{4,4,4\}$$

By convention, the juxtaposition of several multi-indices, resulting in a higher-dimensional one, will be symbolized by parentheses:

$$\mathbf{i} \in \mathbb{I}_p, \ \mathbf{j} \in \mathbb{I}_q, \dots, \mathbf{m} \in \mathbb{I}_t \ \Rightarrow \mathbf{k} = (\mathbf{i}, \mathbf{j}, \dots, \mathbf{m}) \in \mathbb{I}_{p+q+\dots+t} \tag{5.6}$$

When the context is clear, by mere concatenation $\mathbf{k} = \mathbf{ij}\dots\mathbf{m}$. One-dimensional indices, say $\mathbf{j} \in \mathbb{I}_1$ are ordinary integer index variables: they will be typed in italic typeface (say, j, $1 \leq j \leq r$) when its one-dimensionality should be emphasized.

Multi-dimensional fuzzy summations. The purpose of multi-index notation is to compactly represent multi-dimensional fuzzy summations, as follows.

First, let us define the following notation, specific for membership functions as a shorthand for a product:

$$\mu_{\mathbf{i}} = \prod_{l=1}^{p} \mu_{i_l} = \mu_{i_1} \mu_{i_2} \dots \mu_{i_p} \quad \mathbf{i} \in \mathbb{I}_p \tag{5.7}$$

For instance $\mu_{(3,5,1,1)} = \mu_3 \mu_5 \mu_1^2$ will be the membership associated to the term $Q_{(3,5,1,1)}$ in a 4-dimensional fuzzy summation with 5 or more rules. Note that if $\mathbf{t} = (\mathbf{i}, \mathbf{k})$

$$\mu_{\mathbf{t}} = \mu_{\mathbf{i}} \mu_{\mathbf{k}} \tag{5.8}$$

With the above notation, p-dimensional fuzzy summations (5.2) may be written as follows:

$$\Xi(t) = \sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x \tag{5.9}$$

where the basic memberships $\mu = \{\mu_1, \ldots, \mu_r\}$ from which $\mu_{\mathbf{i}}$ stem fulfill the partition condition $\mu \in \Delta^{r-1}$. The expression $\sum_{\mathbf{i}\in\mathbb{I}_0} \mu_{\mathbf{i}} Q$ will be defined to be equal to Q.

Permutations. Given a multi-index $\mathbf{i} \in \mathbb{I}_p$, let us denote by $\mathscr{P}(\mathbf{i}) \subset \mathbb{I}_p$ the set of permutations (with, possibly, repeated elements) of the multi-index $\mathbf{i}$. For instance, for $\mathbf{i} = (3,3,1,1)$, the permutations are:

$$\mathscr{P}(\mathbf{i}) = \{(3,3,1,1),(3,1,3,1),(3,1,1,3),(1,3,1,3),(1,1,3,3),(1,3,3,1)\}$$

The number of permutations of $\mathbf{i}$, $\mathbf{i} \in \mathbb{I}_p^+$ will be denoted by $c(\mathbf{i})$. Such numbers can be computed by using well-known combinatoric expressions (in the above case, $4!/(2!2!) = 6$). Of course, if $\mathbf{i} \in \mathscr{P}(\mathbf{j})$, then $\mathbf{j} \in \mathscr{P}(\mathbf{i})$.

The permutations will be used to group elements in multiple fuzzy summations which share the same “antecedent”: it is an evident fact that

$$\mathbf{j} \in \mathscr{P}(\mathbf{i}) \;\Rightarrow\; \mu_{\mathbf{j}} = \mu_{\mathbf{i}}$$

For instance,

$$\mu_{(1,1,3,4)} = \mu_1^2 \mu_3 \mu_4 = \mu_{(3,1,4,1)} = \mu_{(4,1,1,3)} = \ldots$$

Then, the idea can be stated in formal terms via equivalence classes.

Equivalence class: Consider a set-theoretic relation between multi-indices which is defined to be true if and only if $\mu_{\mathbf{i}} = \mu_{\mathbf{j}}$ for all $\mu \in \Delta^{r-1}$, *i.e.*, true if $\mathbf{i} \in \mathscr{P}(\mathbf{j})$. It can be easily shown that such a relation is an equivalence one, which partitions the set of multi-indices in disjoint classes. The equivalence class of a multi-index $\mathbf{i}$ is, evidently, the set of its permutations $\mathscr{P}(\mathbf{i})$.

The following subset of $\mathbb{I}_p$:

$$\mathbb{I}_p^+ = \{\mathbf{i} \in \mathbb{I}_p \mid i_k \le i_{k+1}, \quad k = 1 \ldots p-1\}$$

may be interpreted as the generalization of “upper triangular” indices in two-dimensional ordinary matrices. Its importance lies in the usefulness of the proposition below.

Proposition 5.1 *Given any index* $\mathbf{j} \in \mathbb{I}_p$*, there exists a unique permutation of it, say* $\mathbf{i}$*, which belongs to* $\mathbb{I}_p^+$ *(*i.e, $\mathscr{P}(\mathbf{j}) \cap \mathbb{I}_p^+$ *has only one element). Hence, given any* $Q_\mathbf{j}$*,* $\mathbf{j} \in \mathbb{I}_p$*:*

$$\sum_{\mathbf{j}\in\mathbb{I}_p} \mu_\mathbf{j} Q_\mathbf{j} = \sum_{\mathbf{i}\in\mathbb{I}_p^+} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} \mu_\mathbf{j} Q_\mathbf{j} = \sum_{\mathbf{i}\in\mathbb{I}_p^+} \mu_\mathbf{i} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_\mathbf{j} =$$

$$= \sum_{\mathbf{i}\in\mathbb{I}_p^+} \mu_\mathbf{i} \tilde{Q}_\mathbf{i} = \sum_{i_1=1}^{r} \sum_{i_2=i_1}^{r} \dots \sum_{i_p=i_{p-1}}^{r} \mu_{i_1}\mu_{i_2}\dots\mu_{i_p} \tilde{Q}_\mathbf{i} \quad (5.10)$$

where $\tilde{Q}_\mathbf{i} = \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_\mathbf{j}$.

Proof is evident by arranging the elements of $\mathbf{j}$ in increasing order, giving a unique permutation $\mathbf{i}$ in $\mathbb{I}_p^+$ which belongs to the equivalence class of $\mathbf{j}$. The number of elements in $\mathbb{I}_p^+$ is the number of equivalence classes. Then, $\tilde{Q}_\mathbf{i}$ is just the sum of all $Q_\mathbf{j}$ for all $\mathbf{j}$ in the equivalence class of $\mathbf{i}$.

Remark: Expression (5.10) is a compact way of writing a multiple fuzzy summation as an homogeneous polynomial in μ, with coefficients $x^T \tilde{Q}_\mathbf{i} x$. For instance, for $p = 3$ and $r = 2$ and some $Q_\mathbf{j}$:

$$\sum_{\mathbf{j}\in\mathbb{I}_3} \mu_\mathbf{j} x^T Q_\mathbf{j} x = x^T \left(\mu_1^3 \tilde{Q}_{111} + \mu_1^2\mu_2 \tilde{Q}_{112} + \mu_1\mu_2^2 \tilde{Q}_{122} + \mu_2^3 \tilde{Q}_{222}\right) x$$

is an homogeneous polynomial[1] of degree 3 in μ, where

$$\tilde{Q}_{111} = Q_{111}, \quad \tilde{Q}_{112} = Q_{112} + Q_{121} + Q_{211}$$
$$\tilde{Q}_{122} = Q_{122} + Q_{212} + Q_{221}, \quad \tilde{Q}_{222} = Q_{222}$$

5.3 Relaxed positivity conditions via dimensionality expansion

As previously discussed, the most trivial sufficient check of (3.23) is checking positive-definiteness of Q_{ij}, which often fails. This section discusses two possible paths in order to obtain less conservative conditions;

[1]evidently, it is also an homogeneous polynomial of degree 2 in x, *i.e.*, a quadratic form.

the first option will resort to splitting the number of positivity conditions, the second option will discuss the addition of artificial decision variables.

The results below will be based on the evident fact that

$$1 = \sum_{i=1}^{r} \mu_i = \left(\sum_{i=1}^{r} \mu_i\right)^p = \sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} \tag{5.11}$$

for any positive integer p.

5.3.1 Relaxation by increasing the number of positivity conditions

Now, consider the 2-dimensional fuzzy summation (3.23). Based on (5.11), proving (3.23) is equivalent to proving, for $n \geq 2$:

$$[\Xi]_n(t) = \left(\sum_{i=1}^{r} \mu_i\right)^{n-2} \cdot \ \Xi(t) = \sum_{\mathbf{i}\in\mathbb{I}_n} \mu_{\mathbf{i}} x^T Q_{i_1 i_2} x > 0 \tag{5.12}$$

i.e., a 2-dimensional fuzzy summation may be expanded to any desired level of nested sums. Of course, $[\Xi]_n = \Xi$, *i.e.*, notation $[\Xi]_n$ will denote the equivalent expression of Ξ as an homogeneous polynomial of degree n appearing in (5.12) above. $[\Xi]_n$ will be referred to as the *dimensionality expansion* of degree n.

The parameter n may be considered as a sort of *complexity parameter*. The results in this section will show that the larger n is, the less conservative the proposed conditions are, but the more computationally demanding the procedure is.

Note that, by using Proposition 5.1:

$$[\Xi]_n = \sum_{\mathbf{i}\in\mathbb{I}_n} \mu_{\mathbf{i}} x^T Q_{i_1 i_2} x = \sum_{\mathbf{i}\in\mathbb{I}_n^+} \mu_{\mathbf{i}} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} x^T Q_{j_1 j_2} x = \sum_{\mathbf{i}\in\mathbb{I}_n^+} \mu_{\mathbf{i}} x^T \tilde{Q}_{\mathbf{i}} x \tag{5.13}$$

where

$$\tilde{Q}_{\mathbf{i}} = \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_{j_1 j_2} \tag{5.14}$$

For instance, given $\mathbf{i} = \{1,1,1,3\}$, which has four permutations

$$\{1,1,1,3\}, \quad \{1,1,3,1\}, \quad \{1,3,1,1\}, \quad \{3,1,1,1\}$$

results in $\tilde{Q}_{\mathbf{i}} = Q_{11} + Q_{11} + Q_{13} + Q_{31} = 2Q_{11} + Q_{13} + Q_{31}$.

Let us now consider conditions for proving positivity of Ξ in (3.23).

Proposition 5.2 *For a fixed n, the positive-definiteness conditions*

$$\tilde{Q}_{\mathbf{i}} = \sum_{\mathbf{j} \in \mathscr{P}(\mathbf{i})} Q_{j_1 j_2} > 0 \quad \forall\, \mathbf{i} \in \mathbb{I}_n^+ \tag{5.15}$$

are sufficient for positivity of Ξ in (3.23)*.*

Proof: it is almost evident as, $\Xi = [\Xi]_n = \sum_{\mathbf{i} \in \mathbb{I}_p^+} \mu_{\mathbf{i}} x^T \tilde{Q}_{\mathbf{i}} x$. Hence, Ξ is expressed as a sum of positive terms if (5.15) holds (as $\mu_{\mathbf{i}} \geq 0$). ■

Theorem 5.1 (Matrix Polya Theorem) *For each collection of matrices Q_{ij} fulfilling* (3.23)*,* i.e., $\Xi > 0$ *for any fuzzy partition μ and any $x \neq 0$, there exists a finite n such that condition* (5.15) *holds,* i.e., (5.15) *become necessary and sufficient for some finite value of n.*

Proof: The proof follows the lines of a similar result in (Scherer, 2005) in a different context. Consider $\|x\|$ to be the Euclidean norm of vector x in (3.23). Denote as

$$\kappa = \min_{\|x\|=1, \mu \in \Delta^{r-1}} \Xi \tag{5.16}$$

which, by assumption exists and is positive (existence is proved by continuity of polynomials and compactness of the finite-dimensional unit balls and the simplex Δ^{r-1}). Denote as

$$L = \max_{i \leq j} \frac{1}{c(ij)} \max(\lambda_{max}(\tilde{Q}_{ij}), -\lambda_{min}(\tilde{Q}_{ij})) \tag{5.17}$$

where $\tilde{Q}_{ii} = Q_{ii}$, $\tilde{Q}_{ij} = Q_{ij} + Q_{ji}$, for $i \neq j$, arise from the permutation equivalence classes of 2-dimensional indices, and the cardinals are $c(ii) = 1$, and for $j \neq i$, $c(ij) = 2$. For any $i \leq j$, as $\lambda_{max}(\tilde{Q}_{ij}) < c(ij)L$ and $\lambda_{min}(\tilde{Q}_{ij})\|x\| \leq x^T \tilde{Q}_{ij} x \leq \lambda_{max}(\tilde{Q}_{ij})\|x\|$, the constant L is positive (otherwise Ξ would be non-positive for all x, contradicting the assumptions) and $|x^T \tilde{Q}_{ij} x| \leq c(ij)L\|x\|$ holds for any vector x.

For a fixed x, expression (3.23) is an homogeneous polynomial in μ of degree 2, whose coefficients are $x^T \tilde{Q}_{ii} x = x^T Q_{ii} x$ (those multiplying μ_i^2)

and $x^T \tilde{Q}_{ij} x = x^T (Q_{ij} + Q_{ji}) x$ (those multiplying $\mu_i \mu_j$ for $i \leq j$). Hence, for any x with unit norm, (3.23) is an homogeneous polynomial in μ whose lower value is greater than κ (by definition) in the standard simplex, and whose coefficients (divided by the cardinal of the corresponding permutation) are upper bounded by L in absolute value. Then, from (Powers & Reznick, 2001) (use notation $\lambda = \kappa$, $d = 2$, $a_\alpha = x^T \tilde{Q}_{ij} x$, $N = n - 2$ in (Powers & Reznick, 2001)), for any value of n such that:

$$n > \frac{L}{\kappa}, \quad n \geq 2$$

all coefficients of the corresponding dimensionality expansion of degree n (*i.e.*, $x^T \tilde{Q}_{\mathbf{i}} x$, $\mathbf{i} \in \mathbb{I}_n^+$) will be positive. As x is an arbitrary unit norm vector, $x^T \tilde{Q}_{\mathbf{i}} x > 0$ entails $\tilde{Q}_{\mathbf{i}}$ being a positive definite matrix, denoted as $\tilde{Q}_{\mathbf{i}} > 0$ in (5.15). ■

The reader is referred to Example 5.1 in the following section, which illustrates conditions (5.15) for different settings.

Remark: it is easily shown that if (5.15) holds for some n_0, it does for any $n \geq n_0$ (indeed, the coefficients of $[\Xi]_n$ are a positive linear combination of those of $[\Xi]_m$ if $n \geq m$). Hence, the conditions in Proposition 5.2 are less and less conservative as n increases, becoming asymptotically exact: if (3.23) holds, a large enough dimensionality expansion n will hold. Hence, when $n \to \infty$, (5.15) and (3.23) become equivalent conditions.

Using the bound[2] for n provided in the proof of Theorem 5.1, a version of it can be rewritten as stated below, where $floor(a)$ denotes the largest integer s such that $s \leq a$.

Theorem 5.2 *The first of the following statements implies the second one:*

1. *For any fuzzy partition $\mu \in \Delta^{r-1}$, for any vector $x \neq 0$, given any tolerance level $\kappa > 0$*

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T Q_{ij} x \geq \kappa x^T x$$

[2] The upper bound for finding the required n is not tight. Providing tighter bounds than L/κ in the second condition is a matter of current research. In (Loera & Santos, 2001) other bounds and conjectures are put forward.

2. *There exists* $n \in \mathbb{N}$, $2 \leq n \leq floor(\frac{L}{\kappa}+1)$ *(L as defined in* (5.17)*), such that*
$$\sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_{j_1 j_2} > 0 \quad \forall\ \mathbf{i} \in \mathbb{I}_n^+$$

The "gap" between Theorem 5.2 and the necessary and sufficient conditions sought after in the fuzzy control community is now expressed via the tolerance level κ, which may be chosen to be as small as desired.

Note that, in the case Q_{ij} includes some decision variables (for subsequent LMI optimizations), the constraint $-L\cdot I \leq \tilde{Q}_{ij}/c(ij) \leq L\cdot I$ for a predefined L should be enforced[3] (in addition to the tolerance κ) to ensure that (5.17) holds, in order to compute the bound for n.

5.3.2 Relaxation via artificial decision variables

This subsection discusses a closely related path to obtaining sufficient conditions for positivity of fuzzy summations, based on the introduction of artificial decision variables. These will allow using a much lower value for n, compared to the one required in the previous theorems.

Theorem 5.3 *A sufficient condition for positivity of* Ξ *in* (3.23) *is the positivity condition below, for* $n \geq 2$*:*

$$\sum_{\mathbf{k}\in\mathbb{I}_{n-2}} \mu_{\mathbf{k}}\xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)} & \cdots & X_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)} & \cdots & X_{(\mathbf{k},r,r)} \end{pmatrix} \xi > 0 \tag{5.18}$$

if there exist matrices $X_{\mathbf{j}}$, $\mathbf{j} \in \mathbb{I}_n$ *so that*

$$\sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_{j_1 j_2} \geq \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} \frac{1}{2}(X_{\mathbf{j}} + X_{\mathbf{j}}^T) \quad \forall\ \mathbf{i} \in \mathbb{I}_n^+ \tag{5.19}$$

Proof: Starting from (5.12):

$$\Xi = [\Xi]_n = \sum_{\mathbf{i}\in\mathbb{I}_n} \mu_{\mathbf{i}} x^T Q_{i_1 i_2} x = \sum_{\mathbf{i}\in\mathbb{I}_n^+} \mu_{\mathbf{i}} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} x^T Q_{j_1 j_2} x \tag{5.20}$$

[3]Usually, in Lyapunov-function-based fuzzy control setups, such constraint can be enforced with no loss of generality as Lyapunov functions can be arbitrarily scaled.

Hence, if (5.19) holds, as $\mu_{\mathbf{i}} \geq 0$,

$$\begin{aligned}\sum_{\mathbf{i}\in\mathbb{I}_n^+} \mu_{\mathbf{i}} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} x^T Q_{j_1 j_2} x \geq \sum_{\mathbf{i}\in\mathbb{I}_n^+} \mu_{\mathbf{i}} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} x^T \frac{1}{2}(X_{\mathbf{j}} + X_{\mathbf{j}}^T) x = \\ = \sum_{\mathbf{i}\in\mathbb{I}_n^+} \mu_{\mathbf{i}} \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} x^T X_{\mathbf{j}} x = \sum_{\mathbf{i}\in\mathbb{I}_n} \mu_{\mathbf{i}} x^T X_{\mathbf{i}} x = \\ = \sum_{\mathbf{k}\in\mathbb{I}_{n-2}} \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_{\mathbf{k}} \mu_i \mu_j x^T X_{(\mathbf{k},i,j)} x = \\ = \sum_{\mathbf{k}\in\mathbb{I}_{n-2}} \mu_{\mathbf{k}} \xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)} & \dots & X_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)} & \dots & X_{(\mathbf{k},r,r)} \end{pmatrix} \xi \qquad (5.21)\end{aligned}$$

where $\xi = (\mu_1 x^T \ \mu_2 x^T \ \dots \ \mu_r x^T)^T$. Hence, if (5.18) holds, (3.23) also does. For $n = 2$, the leftmost summation over $\mathbb{I}_{n-2}$ in the last two expressions must be deleted (this case is a well-known result in (Liu & Zhang, 2003)).

■

Note that, without loss of generality, the equality restrictions $X_{(\mathbf{k},i,i)} = X_{(\mathbf{k},i,i)}^T$ and $X_{(\mathbf{k},i,j)} = X_{(\mathbf{k},j,i)}^T$ can be enforced, as the sign of (5.18) only depends on the symmetric component. In this way, the number of decision variables is reduced.

Corollary 5.1 *Given some Q_{ij} fulfilling* (3.23)*, if conditions* (5.15) *hold for some n, then there exist feasible $X_{\mathbf{i}}$ in Theorem 5.3 for a lower or equal value of n.*

Proof: As (5.15) holds by assumption, there exists n_0 such that the left-hand side of (5.19) is positive-definite for all $\mathbf{i} \in \mathbb{I}_n$ for any $n \geq n_0$. Then, consider the same value of n_0 for Theorem 5.3, and denote by I the identity matrix. Take $X_{(\mathbf{k},i,i)} = \varepsilon I > 0$, and $X_{(\mathbf{k},i,j)} = 0$ when $i \neq j$. It's easy to check that condition (5.19) holds for a sufficiently small ε. Furthermore, all matrices being added in (5.18) are positive definite (they are εI). In this way, a feasible set of decision variables for Theorem (5.3) has been found. ■

The examples in the following section will show that the actually needed value of n is much lower for Theorem 5.3 than Theorem 5.1, due to the new decision variables introduced.

Theorem 5.3 is still not useful, because it replaces a positivity condition (3.23) with another similar one (5.18) with a possibly higher dimension. Hence, some extra developments (useful for their own sake, anyway) are needed in order to turn Theorem 5.3 into a practically applicable solution.

5.3.3 Generalization to higher-dimensional fuzzy summations

Polya's theorem applies to homogeneous forms (Loera & Santos, 1996, 2001; Powers & Reznick, 2001) of any dimension (5.2), and not only to quadratic forms (3.23). Hence, the results can be generalised to other fuzzy control setups involving p-dimensional fuzzy summations for any p, as proposed below.

Theorem 5.4 *Consider proving positivity of a p-dimensional fuzzy summation ($p \geq 2$) for some fuzzy partition $\sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} = 1$, given by* (5.2). *Consider now its dimensionality expansion $[\Xi]_n = (\sum_{i=1}^r \mu_i)^{n-p}\Xi$, with $n \geq p$. Then, a sufficient condition for $\Xi > 0$ when $x \neq 0$ is*

$$\sum_{\mathbf{k}\in\mathbb{I}_{n-2}} \mu_{\mathbf{k}} \xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)} & \dots & X_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)} & \dots & X_{(\mathbf{k},r,r)} \end{pmatrix} \xi > 0 \quad \text{for } \xi \neq 0 \tag{5.22}$$

if there exist matrices $X_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_n$, so that:

$$\sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_{j_1 j_2 \dots j_p} > \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} \frac{1}{2}(X_{\mathbf{j}} + X_{\mathbf{j}}^T) \quad \forall\, \mathbf{i} \in \mathbb{I}_n^+ \tag{5.23}$$

Furthermore, the condition is asymptotically exact with a large enough n so a statement analogous to Theorem 5.2 can be made.

Proof: The proof of the above theorem follows identical lines to that of Theorem 5.3, just by changing $j_1 j_2$ in (5.20) and (5.21) to $j_1 j_2 \dots j_p$.

In order to prove the asymptotic exactness, matrix Polya theorem can be also set up for p greater than 2, resulting in a bound $n > \frac{p(p-1)L}{2\kappa}$ (Powers & Reznick, 2001) for which positivity of all $\tilde{Q}_{\mathbf{i}}$ holds. Hence, it is easy to show that, given some $Q_{\mathbf{i}}$, $\mathbf{i} \in \mathbb{I}_p$, if (5.2) holds, then Theorem

5.4 will hold for a large enough n, *i.e.*, it is asymptotically exact, and n will be lower or equal than $\frac{p(p-1)L}{2\kappa}+1$. Details are omitted for brevity, as they are a replica of those discussed for $p=2$. ■

Theorem 5.4 reduces to Theorem 5.3 for $p=2$. The case $p=3$ would directly apply to the cubic parametrization dynamic controllers in (Li et al., 1999; Tanaka & Wang, 2001) or the output feedback fuzzy controllers in (Chen et al., 2005; Fang et al., 2006).

Remark: note that n can be equal to p in Theorem 5.4. The sufficient conditions arising from this choice will be denoted as *non-expanded* conditions: such conditions state positivity of a p-dimensional summation if a particular $(p-2)$-dimensional positivity condition holds.

5.3.4 Recursive procedure

The previous remark provides the missing link in order to express a version of Theorems 5.3 and 5.4 computable in practice. This section discusses a methodology to set up a collection of LMI constraints via a recursive procedure. Note that the LMIs cannot be solved until all the constraints, for all the recursion steps, have been set up.

Given a value of the complexity parameter $n \geq 2$, a super-index in square brackets will be used to indicate recursion steps, denoted by $h=0,1,\dots,h_{max}=floor((n-1)/2)$.

Theorem 5.5 *Computable sufficient conditions for proving positiveness of a p-dimensional fuzzy summation may be obtained as follows, for any desired value of the complexity parameter n:*

1. *(*initialisation*) Obtain the dimensionality expansion of degree n of the fuzzy summation, $n \geq p$, setting $Q_{\mathbf{i}}^{[0]}=Q_{i_1i_2\dots i_p}$, $\mathbf{i}\in\mathbb{I}_n$. The dimension of the multi-indices in iteration step h will be denoted by d_h, starting with $d_0=n$.*

2. *(*recursive procedure*) Consider now iteration step $h\geq 0$, trying to set up sufficient conditions for $\sum_{\mathbf{i}\in\mathbb{I}_{d_h}} x^T Q_{\mathbf{i}}^{[h]} x>0$, with $d_h>2$. Apply Theorem 5.4 by setting a constraint* (5.23) *and then considering a*

new ($d_h - 2$)*-dimensional summation* (5.22), i.e.*:*

$$\sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} Q_{\mathbf{j}}^{[h]} > \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} \frac{1}{2}(X_{\mathbf{j}}^{[h]} + (X_{\mathbf{j}}^{[h]})^T) \quad \forall\, \mathbf{i} \in \mathbb{I}_{d_h}^{+} \tag{5.24}$$

$$\sum_{\mathbf{k}\in\mathbb{I}_{d_h-2}} \mu_{\mathbf{k}} \xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)}^{[h]} & \cdots & X_{(\mathbf{k},1,r)}^{[h]} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)}^{[h]} & \cdots & X_{(\mathbf{k},r,r)}^{[h]} \end{pmatrix} \xi > 0 \quad \text{for } \xi \neq 0 \tag{5.25}$$

Note that any sufficient condition for (5.25) *implies positivity of the original summation. Then, considering* $d_{h+1} = d_h - 2$*, and*

$$Q_{\mathbf{k}}^{[h+1]} = \begin{pmatrix} X_{(\mathbf{k},1,1)}^{[h]} & \cdots & X_{(\mathbf{k},1,r)}^{[h]} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)}^{[h]} & \cdots & X_{(\mathbf{k},r,r)}^{[h]} \end{pmatrix} \quad \mathbf{k} \in \mathbb{I}_{d_{h+1}} \tag{5.26}$$

expression (5.25) *may be written as*

$$\sum_{\mathbf{i}\in\mathbb{I}_{d_{h+1}}} \mu_{\mathbf{i}} \xi^T Q_{\mathbf{i}}^{[h+1]} \xi > 0 \tag{5.27}$$

i.e.*, sufficient conditions for* (5.25) *may be stated applying non-expanded conditions* (5.24) *and* (5.25) *recursively, until* $h = h_{max}$ *is reached, and* $d_{h_{max}} = 1$ *or* $d_{h_{max}} = 2$*, depending on whether the starting complexity parameter* n *is odd or even, respectively.*

3. *(*termination*) Use either the well-known sufficient conditions in (Tanaka & Wang, 2001) for 1-dimensional sums when* $d_{h_{max}} = 1$*, or those in (Liu & Zhang, 2003) for 2-dimensional ones when* $d_{h_{max}} = 2$.

The result of the above procedure is a set of conditions which are sufficient for the positivity of Ξ *in* (3.23)*. LMI conditions are obtained if the original* $Q_{\mathbf{i}}$ *are affine in some matrix decision variables.*

Proof: Proof is evident and almost outlined in the theorem statement, which describes an algorithm. Indeed, given a p-dimensional summation Ξ, for a chosen n, the recursion above sets up a "ladder" of sufficient conditions with dimension $n-2$, $n-4$, $\ldots$, so that each one is a sufficient condition for the preceding one, *i.e.*, the following chain of implications can be built:

$$
\begin{array}{rcl}
[p\text{-dimensional sum}]: \Xi > 0 & \Leftrightarrow & [n\text{-dimensional sum}]: [\Xi]_n > 0 \\
& & \Uparrow \\
& & (5.25)[n-2\text{-dimensional sum}] \text{ and } (5.24),\ d_0 = n \\
& & \Uparrow \\
& & (5.25)[n-4\text{-dimensional sum}] \text{ and} (5.24),\ d_1 = n-2 \\
& & \Uparrow \\
& & \dots
\end{array}
$$

Hence, proving the final conditions , i.e., $\sum_{\mathbf{i}\in\mathbb{I}_{d_{h_{max}}}} \mu_{\mathbf{i}} \xi^T Q_{\mathbf{i}}^{[h_{max}]} \xi > 0$ entails proving the initial p-dimensional condition and, as those conditions have dimensionality one or two, standard literature results can be used (most of which, in fact, are particular cases of Proposition 5.2 and Theorem 5.3, see Section 5.3.6). Once all conditions have been set up, an LMI solver can be used. ■

5.3.5 Number of conditions and decision variables

The number of conditions needed in Proposition 5.2 is the number of elements in $\mathbb{I}_n^+$, *i.e.*, the combinatorial number:

$$NC_1 = \begin{pmatrix} r+n-1 \\ n \end{pmatrix}$$

Proving positivity of fuzzy summations via the sufficient conditions in Proposition 5.2 does not need any new decision variable apart from those in Q_{ij}. NC_1 grows approximately as $O(n^{r-1})$ when n grows.

The above expression also gives the number of conditions (5.24) in a particular iteration (replacing n by d_h). Hence, the total number of conditions (5.24) is:

$$NC = \sum_{h=0}^{h_{max}} \begin{pmatrix} r+n-2h-1 \\ n-2h \end{pmatrix}$$

and, after applying the termination conditions:

$$NC_2 = NC + r(r+1)/2 + 1, \quad NC_2 = NC + r$$

if n was even or odd, respectively. NC_2 grows at the rate $O(n^r)$ when n grows.

With r rules, if $Q_{\mathbf{i}}$ were $t \times t$, the matrices $Q_{\mathbf{i}}^{[h]}$ are of size $(t \cdot r^h) \times (t \cdot r^h)$. Hence, after symmetry considerations, a number of $tr^h(tr^h - 1)/2$ decision variables is introduced for each h, *i.e.*, the recursive procedure needs NV decision variables with NV given by

$$NV = \sum_{h=1}^{h_{max}} tr^h(tr^h - 1)/2 \tag{5.28}$$

If $d_{h_{max}} = 2$, then (Liu & Zhang, 2003) needs additional decision variables: the total number is obtained by replacing h_{max} by $h_{max} + 1$ in (5.28). The rate of growth with n is $O(r^n)$.

In summary, the number of conditions is polynomial-complexity in n, but the number of decision variables in the recursive procedure is exponential in n. To avoid such explosion of decision variables, some simplifications can be made, such as choosing a lower-dimensional arrangement for X variables, *i.e.*, $X_{\mathbf{i}} = X_{i_1 \ldots i_q}$ with $\mathbf{i} \in \mathbb{I}_p$, $q \leq p$ (see Example 5.1 in Section 5.4).

5.3.6 Comparison to previous literature

The presented approach includes previous literature results on relaxed LMI conditions for fuzzy control as a particular case.

Corollary 5.2 *Proposition 5.2 with* $n = 2$ *are the conditions stated in Theorem 7 in (Tanaka & Wang, 2001);*

Proof: Indeed, restating Proposition 5.2 for the particular case in consideration, it says that $\sum_{i=1}^{r} \sum_{i=1}^{r} \mu_i \mu_j x^T Q_{ij} x > 0$ if $Q_{ii} > 0$ (for the only permutation of the multi index $\{i,i\}$) and $Q_{ij} + Q_{ji} > 0$ (when $i < j$, corresponding to the two permutations of $\{i,j\} \in \mathbb{I}_2^+$) . ■

Corollary 5.3 *if Theorem 5.3 were adapted to* $n = 2$, *conditions in (Liu & Zhang, 2003) would have been obtained.*

Proof: It's evident just considering, as done above, the unique permutation if $\{i,i\}$, yielding $Q_{ii} > \frac{1}{2}(X_{ii} + X_{ii}^T)$ in (5.19) and the permutations $\{i,j\}$ and $\{j,i\}$ when $i < j$, yielding

$$Q_{ij} + Q_{ji} < \frac{1}{2}\left((X_{ij} + X_{ij}^T) + (X_{ji} + X_{ji}^T)\right)$$

Now consider a symmetric matrices X_{ijj} and $X_{ij} = X_{ji}^T$, the sufficient condition is expressed by $Q_{ii} \leq X_{ii}$ and $Q_{ij} + Q_{ji} \leq X_{ij} + X_{ji}$, jointly with the positive-definiteness of the matrix whose blocks are X_{ij}. ■

Corollary 5.4 *Theorem 5.5 with $p = 2$, $n = 3$ sets up the conditions in (Fang et al., 2006).*

Proof: Denote $Sym(X) = \frac{1}{2}(X + X^T)$. Consider a double summation ($p = 2$), after a dimensionality increase ($n = 3$). The results in this work assert that positivity of the double summation holds when conditions (5.23) are applied, *i.e.*,

- $Q_{ii} \leq Sym(X_{iii})$ (for permutations $\{i,i,i\}$),
- $Q_{ii} + Q_{ij} + Q_{ji} \leq Sym(X_{iij} + X_{iji} + X_{jii})$ (for permutations of i,i,j, $i \neq j$)
- $Q_{ij} + Q_{ji} + Q_{ik} + Q_{ki} + Q_{jk} + Q_{kj} \leq Sym(X_{ijk} + X_{jik} + X_{ikj} + X_{kij} + X_{jki} + X_{kji})$ (for permutations of i,j,k,l $i \neq j$, $i \neq k$, $j \neq k$)

and, subsequently, positivity of a one-dimensional summation $\sum_{i=1}^{r} \mu_i Q_i$ is proved, where the new Q_i is formed from the decision variables X_{ijk} as in (5.26).

Considering now X_{ijj} symmetric and $X_{ijk} = X_{ikj}^T$ the conditions result in:

- $Q_{ii} \leq X_{iii}$,
- $Q_{ii} + Q_{ij} + Q_{ji} \leq X_{iij} + X_{iij}^T + X_{jii}$,
- $Q_{ij} + Q_{ji} + Q_{ik} + Q_{ki} + Q_{jk} + Q_{kj} \leq X_{ijk} + X_{jik} + X_{kij} + X_{ijk}^T + X_{jik}^T + X_{kij}^T$

and the positivity of the remaining one-dimensional fuzzy summation is proved if each Q_i is positive definite. The resulting conditions are analogous (we use a more general notation) to the ones in Theorem 5 in (Fang et al., 2006). ■

5.4 Examples

In this section, three examples are presented. The first one illustrates how the various $\tilde{Q}_\mathbf{i}$ previously discussed are computed via combinatoric permutation formulae. The second one applies the conditions in section 5.3 to a numerical example. And the third applies these conditions to a specific $Q_\mathbf{i}$ formed of a Quadratic Parallel Distributed Compensation (QPDC) controller. This example illustrates that the additional degrees of freedom in QPDC controllers allow to achieve a more stringent performance requirements than of a PDC controller.

Example 5.1 *Consider proving* $\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_i\mu_j x^T Q_{ij}x > 0$, i.e., $n=2$, $r=3$ *with Proposition 5.2. For* $\mathbf{i}=(1,1)$, $\mathbf{i}=(1,2)$, $\mathbf{i}=(1,3)$, $\mathbf{i}=(2,2)$, $\mathbf{i}=(2,3)$, *and* $\mathbf{i}=(3,3)$ *(which are the six elements of* $\mathbb{I}_2^+$*), the conditions* (5.15) *are, respectively,*

$$Q_{11} > 0,\ Q_{12}+Q_{21} > 0,\ Q_{13}+Q_{31} > 0, \tag{5.29}$$

$$Q_{22} > 0,\ Q_{23}+Q_{32} > 0,\ Q_{33} > 0 \tag{5.30}$$

which are well-known conditions stated in (Tanaka et al., 1998; Tanaka & Wang, 2001).

Consider now a dimensionality expansion to $n=3$. In order to prove

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_i\mu_j x^T Q_{ij}x = \sum_{k=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{3}\mu_k\mu_i\mu_j x^T Q_{ij}x > 0$$

the permutations of

$$\mathbb{I}_3^+ = \{(1,1,1),(1,1,2),(1,1,3),(1,2,2),(1,2,3),\\ (1,3,3),(2,2,2),(2,2,3),(2,3,3),(3,3,3)\} \tag{5.31}$$

must be considered. The result are the conditions

$$\begin{aligned}
\tilde{Q}_{111} = Q_{11} > 0,\ \tilde{Q}_{112} = Q_{11} + Q_{12} + Q_{21} > 0,\\
\tilde{Q}_{113} = Q_{11} + Q_{13} + Q_{31} > 0,\ \tilde{Q}_{122} = Q_{12} + Q_{21} + Q_{22} > 0,\\
\tilde{Q}_{123} = Q_{12} + Q_{21} + Q_{13} + Q_{31} + Q_{23} + Q_{32} > 0,\\
\tilde{Q}_{133} = Q_{13} + Q_{31} + Q_{33} > 0,\ \tilde{Q}_{222} = Q_{22} > 0,\\
\tilde{Q}_{223} = Q_{22} + Q_{23} + Q_{32} > 0,\\
\tilde{Q}_{233} = Q_{23} + Q_{32} + Q_{33} > 0,\ \tilde{Q}_{333} = Q_{33} > 0
\end{aligned} \tag{5.32}$$

Observe that the obtained conditions are sums with positive coefficients of those for $n = 2$ (*cf.* remark in page 55).

Considering now $n = 8$, the condition associated to the equivalence class of, for instance, $\mathbf{i} = (1,1,1,2,2,2,3,3) \in \mathbb{I}_8^+$ is:

$$\begin{aligned}
\tilde{Q}_{11122233} = \frac{6!}{1!3!2!}Q_{11} + \frac{6!}{2!2!2!}(Q_{12} + Q_{21}) + \frac{6!}{3!1!2!}Q_{22}+\\
+\frac{6!}{2!3!1!}(Q_{13} + Q_{31}) + \frac{6!}{3!2!1!}(Q_{23} + Q_{32}) + \frac{6!}{3!3!0!}Q_{33} =\\
60Q_{11} + 90(Q_{12} + Q_{21}) + 60(Q_{13} + Q_{31}) + 60(Q_{23} + Q_{32}) + 60Q_{22} + 20Q_{33} > 0
\end{aligned}$$

That of $\mathbf{i} = (1,1,1,1,1,2,3,3)$ is:

$$\begin{aligned}
\tilde{Q}_{11111233} = \frac{6!}{3!1!2!}Q_{11} + \frac{6!}{4!0!2!}(Q_{12} + Q_{21}) + \frac{6!}{4!1!1!}(Q_{13} + Q_{31})+\\
+\frac{6!}{5!0!1!}(Q_{23} + Q_{32}) + \frac{6!}{5!1!0!}Q_{33} =\\
60Q_{11} + 15(Q_{12} + Q_{21}) + 30(Q_{13} + Q_{31}) + 6(Q_{23} + Q_{32}) + 6Q_{33} > 0
\end{aligned}$$

and so on. Of course, for large values of n, generation of the vast number of resulting conditions must be carried out automatically by means of a suitable computer program. For instance, the number of positivity conditions required in Proposition 5.2 for $n = 50$ is 1326.

As discussed in Section 5.3.5, Theorem 5.5 involves a high number of decision variables in $X_{\mathbf{i}}$ which, however, may be reduced by sticking to a lower-dimensional X. For instance, conditions for $n = 4$ for a 2-rule fuzzy system may be stated, taking $X_{12} = X_{21}^T$, $X_{11} = X_{11}^T$, $X_{22} = X_{22}^T$, as:

$$\begin{array}{lc}
\{1111\}: & Q_{11} \geq X_{11} \\
\{1112\}: & 2Q_{11} + Q_{12} + Q_{21} \geq 2X_{11} + X_{12} + X_{21} \\
\{1122\}: & Q_{11} + Q_{22} + 2Q_{12} + 2Q_{21} \geq X_{11} + X_{22} + 2X_{12} + 2X_{21} \\
\{1222\}: & 2Q_{22} + Q_{12} + Q_{21} \geq 2X_{22} + X_{12} + X_{21} \\
\{2222\}: & Q_{22} \geq X_{22} \\
 & \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} > 0
\end{array}$$

where a 2-dimensional decision variable X has been chosen instead of the 4-dimensional one which arises from direct application of Theorem 5.5 (see next example).

Example 5.2 *Consider a continuos fuzzy system, taken from Example 1 in (Fang et al., 2006), defined by:*

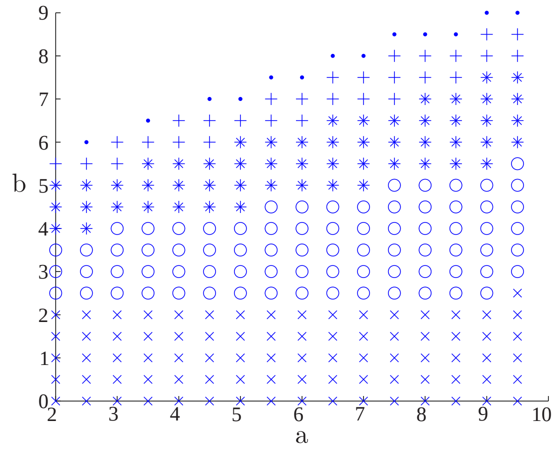
$$\dot{x} = \sum_{i=1}^{3} \mu_i (A_i x + B_i u)$$

where $A_1 = \begin{pmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,
$A_2 = \begin{pmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{pmatrix}$, $B_2 = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$,
$A_3 = \begin{pmatrix} -a & -4.33 \\ 0 & 0.05 \end{pmatrix}$, $B_3 = \begin{pmatrix} -b+6 \\ -1 \end{pmatrix}$.

Analogously to (Fang et al., 2006), existence of a stabilising controller is cast as a quadratic Lyapunov condition $\sum_{i=1}^{3} \sum_{j=1}^{3} \mu_i \mu_j x^T Q_{ij} x > 0$ where Q_{ij} are defined by (3.1).

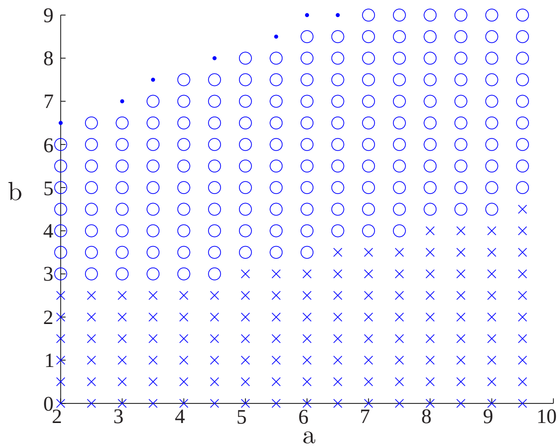
Feasibility of the sufficient conditions arising from Proposition 5.2 are evaluated for different values of a, b. The obtained feasible points for $n = \{5, 10, 20, 35, 50\}$ are depicted in Figure 5.1. The LMI solver was Matlab LMI Toolbox (Gahinet et al., 1995).

Similarly, results with Theorem 5.5 (n ranging from 2 to 4) for the same grid of values for a and b are depicted in Figure 5.2. Indeed, results for $n = 2$ and $n = 3$ coincide with those reported in (Fang et al., 2006).



Legend: $\times \rightarrow n = 5$, $\circ \rightarrow n = 10$, $\star \rightarrow n = 20$, $+ \rightarrow n = 35$, $\bullet \rightarrow n = 50$

Figure 5.1: Stabilization region based on Proposition 5.2 with different values of n



Legend: $\times \rightarrow n = 2$, $\circ \rightarrow n = 3$, $\bullet \rightarrow n = 4$

Figure 5.2: Stabilization region based on Theorem 5.5 with different values of n

As expected, $n=4$ improves over those results.

Let us detail the procedure used to set up the conditions for $n=4$ (the notation $Sym(N)=\frac{1}{2}(N+N^T)$ will be used).

First, Theorem 5.3 with $n=4$ is applied. As a result, if the following conditions hold:

$$\tilde{Q}_{1111} = Q_{11} > Sym(X_{1111}) \tag{5.33}$$

$$\begin{aligned}\tilde{Q}_{1112} = 2Q_{11} + 1(Q_{12}+Q_{21}) > \\ Sym(X_{1112}+X_{1121}+X_{1211}+X_{2111})\end{aligned} \tag{5.34}$$

$$\vdots$$

$$\begin{aligned}\tilde{Q}_{1122} = Q_{11} + 2(Q_{12}+Q_{21}) + Q_{22} > \\ Sym(X_{1122}+X_{1212}+X_{1221}+X_{2112}+X_{2121}+X_{2211})\end{aligned} \tag{5.35}$$

$$\vdots$$

$$\begin{aligned}\tilde{Q}_{1223} = 2(Q_{12}+Q_{21}) + (Q_{13}+Q_{31}) + \\ 2(Q_{23}+Q_{32}) + 2Q_{22} > \\ Sym(X_{1223}+X_{1232}+X_{1322}+X_{2123}+X_{2132}+X_{2231}+ \\ +X_{2213}+X_{2312}+X_{2321}+X_{3122}+X_{3212}+X_{3221})\end{aligned} \tag{5.36}$$

$$\vdots$$

$$\tilde{Q}_{3333} = Q_{33} > Sym(X_{3333}) \tag{5.37}$$

(the total number of elements of $\mathbb{I}_4^+$ for $r=3$ is 15, but only 5 inequalities have been shown, for brevity) then positivity of

$$\sum_{k=1}^{3}\sum_{l=1}^{3} \mu_{kl}\xi^T H_{kl}\xi > 0 \tag{5.38}$$

where

$$H_{kl} = \begin{pmatrix} X_{kl11} & X_{kl12} & X_{kl13} \\ X_{kl21} & X_{kl22} & X_{kl23} \\ X_{kl31} & X_{kl32} & X_{kl33} \end{pmatrix}$$

ensures that the original quadratic stabilisation conditions do also hold.

Finally, in order to test the positivity of (5.38) the conditions of (Liu & Zhang, 2003) are added, introducing decision variables Y_{ij} fulfilling

$Y_{ij} = Y_{ji}^T$ and

$$Sym(H_{11}) > Y_{11} \tag{5.39}$$

$$Sym(H_{22}) > Y_{22}, \quad Sym(H_{33}) > Y_{33}, \tag{5.40}$$

$$Sym(H_{12} + H_{21}) > Y_{12} + Y_{21} \tag{5.41}$$

$$Sym(H_{13} + H_{31}) > Y_{13} + Y_{31} \tag{5.42}$$

$$Sym(H_{23} + H_{32}) > Y_{23} + Y_{32} \tag{5.43}$$

and

$$\begin{pmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{23} & Y_{33} \end{pmatrix} > 0 \tag{5.44}$$

As a result, the problem is cast as a set of LMIs with decision variables T and M_j in Q_{ij}, from (3.1). Also X_{ikjl} (81 matrices of size 3×3) and the Y_{mn} (9 matrices of size 9×9), *i.e.*, a total of 23 LMI's (adding $T > 0$) with 504 scalar decision variables.

Note that the recursive procedure in Theorem 5.5 with $n = 4$ achieved better results than the original Polya conditions (Proposition 5.2) with $n = 50$. Larger values of n in the above computations resulted in numerical problems (slow progress) with the LMI solver, due to the large number of LMI constraints (in Proposition 5.2) or decision variables (in Theorem 5.5). This did not allow additional feasiblility points to be found. The average computation times in seconds per test point (304 tests were needed to plot the whole figure) were, for Figure 1 0.11, 0.45, 2.49, 11.5 and 33.7 s, for $n = 5$, 10, 20, 35 and 50, respectively (increasing the LMI solver iteration limit to 650). Regarding Figure 2 (with default options), with $n = 3$ the average time per test point was 0.13 s, whereas that for $n = 4$ was 11.7 s (*i.e.*, it took about 1 hour to compute the full figure). The computer was a dual Pentium 4, 3.0 GHz, Matlab 7.1, 1 GB RAM.

Example 5.3 ***(Quadratic-Parametrisation Controllers.)***

Once a criterion for positivity of (5.2) *is described, it will be applied to the design of quadratic-parametrisation controllers, to be denoted as QPDC controllers:*

$$u = -\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j F_{ij} x \tag{5.45}$$

which are a generalisation of the PDC ones, inspired on some proposals in (Tanaka & Wang, 2001). Clearly, the PDC control law is a particular case of the QPDC law (setting $F_{ij} = F_i$).

A Takagi-Sugeno system (3.1) under QPDC control yields a closed loop given by:

$$\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r} \mu_i \mu_j \mu_k (A_i - B_i F_{jk}) x \tag{5.46}$$

Therefore, following the same reasoning than in Section 3.5, The equilibrium of the continuous TS fuzzy system (3.1) is quadratically stabilizable via a QPDC control law (5.45) if there exist matrices M_{jk} such that:

$$\sum_{ijk} \mu_{ijk} \xi^T Q_{ijk} \xi > 0$$

where

$$Q_{ijk} = -(A_i X - B_i M_{jk} + X A_i^T - M_{jk}^T B_i^T)$$

Clearly, Theorem 5.5 can be applied to solve previous positivity conditions.

The system to be controlled will be taken from (Kim & Lee, 2000), and it will be given by:

$$\begin{aligned}
\dot{x}_1 &= x_2 + \sin x_3 - 0.1 x_4 + (x_1^2 + 1) u \\
\dot{x}_2 &= x_1 + 2x_2 && (5.47) \\
\dot{x}_3 &= x_1^2 x_2 + x_1 && (5.48) \\
\dot{x}_4 &= \sin x_3 && (5.49)
\end{aligned}$$

Under the assumption that $x_1 \in [-a, a]$ and $x_3 \in [-b, b]$, the above nonlinear system is exactly represented by the TS model:

$$
\begin{aligned}
A_1 &= \begin{pmatrix} 0 & 1 & 1 & -0.1 \\ 1 & 2 & 0 & 0 \\ 1 & a^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
A_2 &= \begin{pmatrix} 0 & 1 & \sin b/b & -0.1 \\ 1 & 2 & 0 & 0 \\ 1 & a^2 & 0 & 0 \\ 0 & 0 & \sin b/b & 0 \end{pmatrix} \\
A_3 &= \begin{pmatrix} 0 & 1 & 1 & -0.1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
A_4 &= \begin{pmatrix} 0 & 1 & \sin b/b & -0.1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \sin b/b & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
B_1 = B_2 &= \begin{pmatrix} (1+a^2) & 0 & 0 & 0 \end{pmatrix}^T \\
B_3 = B_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T
\end{aligned}
$$

with membership functions:

$$
\begin{aligned}
M_1(x) &= \frac{x_1^2}{a^2} \\
M_3(x) &= \begin{cases} \frac{b \sin x_3 - x_3 \sin b}{x_3(b - \sin b)} & x_3 \neq 0 \\ 1 & x_3 = 0 \end{cases} \\
\mu_1(x) &= M_1(x) M_3(x) \\
\mu_2(x) &= M_1(x)(1 - M_3(x)) \\
\mu_3(x) &= (1 - M_1(x)) M_3(x) \\
\mu_4(x) &= (1 - M_1(x))(1 - M_3(x))
\end{aligned}
$$

In order to compare the achieved QPDC controllers with those in (Kim & Lee, 2000), the same constraints on the control input as in

(Kim & Lee, 2000) will be enforced. The results applying Theorem 2 in (Liu & Zhang, 2003), which relaxes some symmetry conditions in (Kim & Lee, 2000), will also be compared. The constrain $\|u\|_2 \leq \delta$ can be expressed in terms of QPDC controllers as

$$\begin{pmatrix} 1 & x(0)^T \\ x(0) & X \end{pmatrix} \geq 0 \tag{5.50}$$

$$\begin{pmatrix} X & M_{ii}^T \\ M_{ii} & \delta^2 I \end{pmatrix} \geq Z_{ii}, \; i = 1...r \tag{5.51}$$

$$\begin{pmatrix} X & M_{ij}^T \\ M_{ij} & \delta^2 I \end{pmatrix} + \begin{pmatrix} X & M_{ji}^T \\ M_{ji} & \delta^2 I \end{pmatrix} \geq Z_{ij} + Z_{ij}^T, \quad i = 1...r, \; i < j \leq r \tag{5.52}$$

$$\begin{pmatrix} Z_{11} & \dots & Z_{1r} \\ \vdots & \ddots & \vdots \\ Z_{r1} & \dots & Z_{rr} \end{pmatrix} > 0 \tag{5.53}$$

where $X = P^{-1}$, $M_{ij} = F_i X$, $Z_{ij} = Z_{ji}^T$ and Z_{ii} symmetric.
Where the initial states in this example are $x(0) = \begin{pmatrix} -1.2 & 0.5 & 0.7 & -0.6 \end{pmatrix}^T$, $a = 1.4$ and $b = 0.7$.

Results: Using PDC control and (Kim & Lee, 2000), the lower bound of δ for which the resulting LMIs were feasible was 5.116. With (Liu & Zhang, 2003), the limit of δ was 4.65. For a QPDC controller, the feasibility bound obtained with Theorem 5.2 with $n = 3$ is $\delta \geq 4.484$. Theorem 5.5 obtains a new QPDC controller with $\delta \geq 4.272$, significantly improving over previous results.

In order to show the additional flexibility of the QPDC controller, figures 5.3 (PDC) and 5.4 (QPDC) compare the first element of the 1×4 state-feedback matrix $\hat{F}$ below:

$$u = -\hat{F}(x)x = -\left(\hat{F}_1(x)\ \hat{F}_2(x)\ \hat{F}_3(x)\ \hat{F}_4(x)\right) x$$

as a function of M_1 and M_3 above, used to define the membership functions. Indeed, the peak “gain” is reduced in the QPDC case.

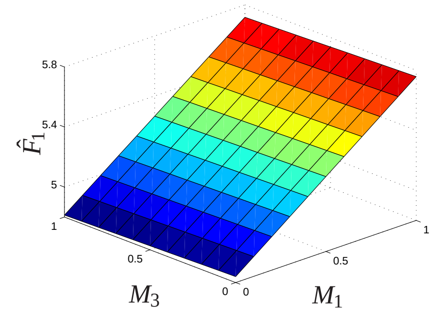


Figure 5.3: The first component of the state-feedback matrix $\hat{F}$ with a PDC controller computed with Theorem 3.5, $\delta = 4.65$.

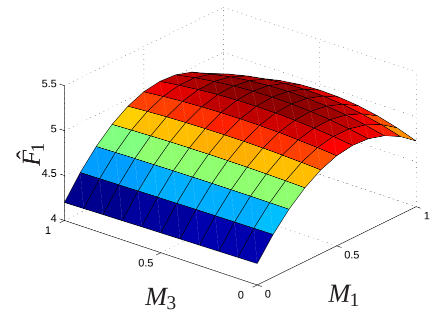


Figure 5.4: The first component of the state-feedback matrix $\hat{F}$ with QPDC controller, $\delta = 4.272$.

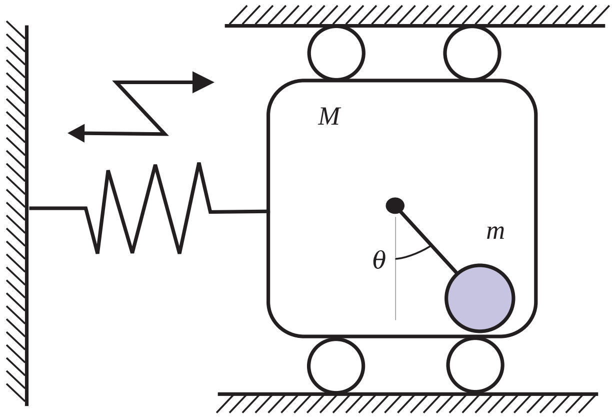


Figure 5.5: TORA model.

Example 5.4 *Consider the system shown in Figure 5.5, which represents a translational oscillator with an eccentric rotational proof mass actuator (TORA) (Jankovic, Fontaine, & Kokotovic, 1996; Tanaka & Wang, 2001).*

It consists of a platform that can oscillate without damping in the horizontal plane (no gravity effect). On the platform a rotating eccentric mass is actuated by a dc motor. Its motion applies a force to the platform which can be used to damp the translational oscillations. Assuming that the motor torque is the control variable, the task is to find a control law to asymptotically stabilize the system at a desired equilibrium, with restrictions on the control action. Let x_1 and x_2 denote the translational position and velocity of the cart with $x_2 = \dot{x}_1$. Let $x_3 = \theta$ and $x_4 = \dot{x}_3$ denote the angular position and velocity of the rotational proof mass. Then the system dynamics can be described by the equation

$$\dot{x} = f(x) + g(x)u \tag{5.54}$$

where u is the torque applied to the eccentric mass and

$$f(x) = \begin{pmatrix} x_2 \\ \frac{-x+\varepsilon x_4^2 \sin x_3}{1-\varepsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1-\varepsilon^2 \cos^2 x_3} \end{pmatrix} \quad (5.55)$$

$$g(x) = \begin{pmatrix} 0 \\ \frac{-\varepsilon \cos x_3}{1-\varepsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1-\varepsilon^2 \cos^2 x_3} \end{pmatrix} \quad (5.56)$$

with $\varepsilon = 0.1$.

As in (Jankovic et al., 1996; Tanaka & Wang, 2001), introduce new state variables $z_1 = x_1 + \sin x_3$, $z_2 = x2 + \varepsilon x_4 \cos x_3$, $y_1 = x_3$, $y_2 = x_4$, and employ the feedback transformation

$$v = \frac{1}{1-\varepsilon^2 \cos^2 y_1}[\varepsilon \cos y_1 (z_1 - (1+y_2^2)\varepsilon \sin y_1) + u] \quad (5.57)$$

the system can be described in the following form

$$\dot{z}_1 = z_2 \quad (5.58)$$
$$\dot{z}_2 = -z_1 + \varepsilon \sin y_1 \quad (5.59)$$
$$\dot{y}_1 = y_2 \quad (5.60)$$
$$\dot{y}_2 = v \quad (5.61)$$

The T-S model of the TORA system can be constructed as in (Tanaka & Wang, 2001) from (5.58)-(5.61) by using the sector nonlinearity fuzzy model construction described in Chapter 3.

$$\dot{x} = \sum_{i=1}^{4} \mu_i (A_i x + B_i u) \quad (5.62)$$

where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \varepsilon\frac{\alpha\pi}{\alpha\pi} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\varepsilon}{1-\varepsilon^2} & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{1-\varepsilon^2} \end{pmatrix} \tag{5.63}$$

$$A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \varepsilon\frac{2}{\pi} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.64}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\varepsilon}{1-\varepsilon^2} & 0 & \frac{-\varepsilon^2}{1-\varepsilon^2} & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{1-\varepsilon^2} \end{pmatrix} \tag{5.65}$$

$$A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\varepsilon}{1-\varepsilon^2} & 0 & \frac{-\varepsilon^2(1+a^2)}{1-\varepsilon^2} & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{1-\varepsilon^2} \end{pmatrix} \tag{5.66}$$

$$\tag{5.67}$$

In this simulation, as in (Tanaka & Wang, 2001), $x_4 \in [-a, a]$, with $a = 4$ and $0 < \alpha < 1$ we take for instance $alpha = 0.99$.

As in the previous example, in order to compare the achieved PDC controller computed with the multidimensional summation conditions of Theorem 5.3, with those in (Liu & Zhang, 2003), we impose a constrain on the control input. $\delta = 500$ and $x_0 = [0101]^T$. The archived results with Theorem 3.5 are the maximum decay rate $\alpha = 0.008$ and for the presented theorem 5.3 and the parameter $n = 3$ the maximum obtained decay rate is $\alpha = 0.0147$. As is shown, in mechanical systems the presented procedure shows a great improvement with previous technics.

5.5 Conclusions

This chapter has shown how to improve some results in previous literature, achieving less conservative sufficient conditions on positive definiteness of fuzzy summations (related to stability and performance criteria in fuzzy control). Based on Polya's theorem, it can be shown that the conditions are progressively less conservative as a complexity parameter n increases, becoming asymptotically exact. Bounds for n can be computed if a tolerance parameter is introduced. The number of conditions is polynomial in n; if decision variables are introduced, the number of them may be exponential in n.

The achievable value of n in a particular fuzzy control problem depends on solver accuracy and available computing resources. Previously reported conditions in literature correspond to $n = 2$ and $n = 3$ in the presented approach.

Chapter 6

Local stability results

In most literature contributions, LMI stability conditions are devised in order to prove stability and performance of Takagi-Sugeno (Takagi & Sugeno, 1985) fuzzy systems as has been shown in The Chapter 1. However, such laws are usually independent of the values of membership functions, and valid for any arbitrary shapes of them. Knowledge of the shape of the membership functions may allow to lift some conservativeness.

The LMI conditions in the above works do not dependent on the shape of the membership functions. The values of the membership are needed when implementing a fuzzy controller, but LMI conditions in controller design are usually stated as valid for *any* underlying fuzzy partition. The conditions may be conservative: a particular nonlinear system (modelled as a fuzzy TS one (Tanaka & Wang, 2001)) may be stable, but the LMI conditions may fail to pinpoint the fact.

In summary, there is still some conservativeness to be lifted if knowledge of the shape of the membership functions for a articular TS model is introduced in the LMI framework.

As a conclusion from the above, "pure nonlinear" control strategies (feedback linearisation, backstepping, Lyapunov synthesis (Khalil, 1996)) on an original nonlinear model may find better solutions than "fuzzy control" ones on an "equivalent" fuzzy TS model. This is due to fuzzy control conditions usually been set in order to hold for a "convex family" of nonlinear systems, making them conservative if a "particular one" in such family is the only target for control design.

6.1 Introduction

When LMI stability conditions (3.13) are unfeasible, other alternative conditions must be sought. Fuzzy or piecewise Lyapunov functions are discussed in (Johansson, 1999), Fuzzy Lyapunov functions are discussed in (Oliveira, Bernussou, & Geromel, 1999), non-PDC regulator are discussed in (Guerra & Vermeiren, 2004) and some ellipsoidal bounds are discussed in (Calaÿore & El Ghaoui, 2004).

A different alternative, not commonly explored in current literature, is trying to achieve *local* stability results in a zone around the equilibrium as large as possible. Such a result is motivated on the first Lyapunov theorem for local stability: if the linearised system in $x = 0$ is exponentially stable, then so it is the nonlinear one, for initial conditions in a sufficiently small neighborhood of $x = 0$. Ellipsoidal characterisations of subsets of those basins of attraction may be expressed via LMI conditions, as discussed in the following section.

6.2 Local Fuzzy Models

In order to analyze the local stability of a TS fuzzy model (3.1) within a region, the original model is modified using the information of the membership functions.

Lemma 6.1 *if the membership functions $\mu(x)$ of a fuzzy system described in* (3.12) *in a region of Ω can be themselves expressed as a convex sum of some vectors v_p:*

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p, \quad \forall\ x \in \Omega \tag{6.1}$$

where:

$$\mu(x) = [\mu_1(x), \mu_2(x), \ldots, \mu_n(x)]$$

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \ \beta_p(x) > 0 \ \forall x \in \Omega \ \ p = 1 \ldots n_v$$

Then the system can be transformed to:

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) A_p^* \cdot x \tag{6.2}$$

where

$$A_p^* = \sum_{i}^{n} v_{pi} A_i \tag{6.3}$$

Proof: *The expression* (6.1) *can be substituted in the system equation* (3.12)*:*

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p \tag{6.4}$$

$$v_p = [v_{p1}, v_{p2}, \ldots, v_{pn}] \tag{6.5}$$

$$\mu_i(x) = \sum_{p=1}^{n_v} \beta_p(x) v_{pi} \tag{6.6}$$

$$\dot{x} = \sum_{i=1}^{n} \sum_{p=1}^{n_v} \beta_p(x) v_{pi} A_i \cdot x \tag{6.7}$$

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) \sum_{i}^{n} v_{pi} A_i \cdot x \tag{6.8}$$

so the local representation of the system in Ω

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) A_p^* \cdot x \quad \forall x \in \Omega$$

where:

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \;\; \beta_p(x) > 0 \;\; \forall x \in \Omega \;\; p : 1 \ldots n_v$$

■

Lemma 6.2 *If the membership functions* $\mu(x)$ *of the fuzzy system* (3.14) *in a region* Ω *can be expressed as* (6.1)*, then the system can be equivalently expressed as:*

$$\dot{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \beta_j (A_i^* - B_i^* F_j^*) x \tag{6.9}$$

where

$$A_p^* = \sum_{i=1}^{n} v_{pi} A_i \qquad (6.10)$$

$$B_p^* = \sum_{i=1}^{n} v_{pi} B_i \qquad (6.11)$$

$$F_p^* = \sum_{i=1}^{n} v_{pi} F_i \qquad (6.12)$$

■

The proof uses the same procedure as Lemma 6.1 above.

The convex-combination conditions for the membership functions required in the above lemmas are easy to meet. Indeed μ_i are assumed to be known in fuzzy systems. Then, the result below may be applied to obtain a (possibly conservative) vertex set.

Note 6.1 *Let us consider a region* Ω*. If bounds* μ_i^M *and* μ_i^m *on the extremum values of the membership functions in* Ω *can be computed, in such a way that:*

$$\mu_i^M \geq \max_{x \in \Omega} \mu_i(x) \quad \mu_i^m \leq \min_{x \in \Omega} \mu_i(x) \qquad (6.13)$$

then there exist a set of $\beta_p(x)$*,* $p = 1, \ldots, n_v$ *so that the vector of membership functions*

$$\mu(x) = [\mu_1(x), \mu_2(x), \ldots, \mu_n(x)]$$

may be expressed in Ω *as:*

$$\mu(x) = \sum_{p=1}^{n_v} \beta_p(x) v_p, \quad x \in \Omega \qquad (6.14)$$

where:

$$\sum_{p=1}^{n_v} \beta_p(x) = 1 \;\; \beta_p(x) > 0 \;\; \forall x \in \Omega \;\; p : 1 \ldots n_v$$

Indeed, the linear restrictions $\mu_i^M \geq \mu_i \geq \mu_i^m$*,* $\sum_i \mu_i = 1$ *describe a bounded polytope with a finite number of vertices (Luenberger, 2003).*

Well-known linear-programming-related methods to obtain the membership vector vertices may be used (related to the obtention of the basic feasible solutions in an LP problem (Luenberger, 2003)). A related alternative is described below.

Lemma 6.3 *Consider the set* Σ_i *of at most* 2^{n-1} *vectors defined by:*

$$\Sigma_i = \{[\tilde{\mu}_1, ..., \tilde{\mu}_{i-1}, X, \tilde{\mu}_{i+1}, ..., \tilde{\mu}_n], \quad X = 1 - \sum_{\substack{1 \le j \le n \\ j \ne i}} \tilde{\mu}_j \quad \text{such that } \tilde{\mu}_j \in \{\mu_j^M, \mu_j^m\} \; j \ne i, \; \mu_i^m \le X \le \mu_i^M\} \tag{6.15}$$

Then, the vectors belonging to the set

$$\Sigma = \bigcup_{i=1}^{n} \Sigma_i \tag{6.16}$$

satisfy (6.14) *for some* β_p.

Indeed, as there is only one equality restriction in memberships, all except one of them are "free" to attain an extremum value; the remaining one must fulfill the add-1 restriction *and* be inside its required bounds. The above lema produces the union of all the "all minus one" combinations, and the sought vertices will belong to such set.

Example. For instance, if three memberships have minimum and maximum values given by {0.15,0.3,0.35} and {0.6,0.5,0.4}, the set Σ_1 is originated by the four combinations:

$$\{(X_1, 0.3, 0.35), (X_2, 0.5, 0.35), (X_3, 0.3, 0.4), (X_4, 0.5, 0.4)\}$$

with $X_1 = 1 - 0.65 = 0.35$, $X_2 = 0.15$, $X_3 = 0.3$, $X_4 = 0.1$. As X_4 is out of the required range, the candidate vertices kept are:

$$\Sigma_1 = \{(0.35, 0.3, 0.35), (0.15, 0.5, 0.35), (0.3, 0.3, 0.4)\}$$

The set Σ_2 is generated by:

$$\{(0.15, X_1, 0.35), (0.6, X_2, 0.35), \\ (0.15, X_3, 0.4), (0.6, X_4, 0.4)\}$$

with $X_1 = 0.5$, $X_2 = 0.05$, $X_3 = 0.45$ and $X_4 = 0$. Hence,

$$\Sigma_2 = \{(0.15, 0.5, 0.35), (0.15, 0.45, 0.4)\}$$

Regarding the third membership,

$$\{(0.15, 0.3, X_1), (0.6, 0.3, X_2), \\ (0.15, 0.5, X_3), (0.6, 0.5, X_4)\}$$

results in

$$\Sigma_3 = \{(0.15, 0.5, 0.35)\}$$

hence the resulting set of vertices to compute the local models is:

$$\Sigma = \{(0.35, 0.3, 0.35), (0.15, 0.5, 0.35), \\ (0.3, 0.3, 0.4), (0.15, 0.45, 0.4)\}$$

6.2.1 Stability analysis

By using the transformed models discussed in the above section, local stability results may be obtained.

Lemma 6.4 *The ellipsoidal region* $\Omega^* \subset \Omega$

$$\Omega^* = \{x \mid x^T P x \leq V_M,\ P > 0\} \tag{6.17}$$

is a basin of attraction of the equilibrium point $x = 0$ *of the system* (3.1)*, i.e., all trajectories with initial state in* Ω^* *converge assymptotically to* $x = 0$*, if*

$$\tag{6.18}$$

$$V_M \quad \leq \quad min\{x^T P x \mid x \in\ \partial\Omega\} \tag{6.19}$$

where $\partial\Omega$ *denotes the boundary of* Ω *and* P *verifies:*

$$A_p^{*T} P + P A_p^* < 0 \quad p : 1, \ldots, n_v \tag{6.20}$$

Proof: *As, by Lemma 6.1, the system can be expressed in* Ω *as:*

$$\dot{x} = \sum_{p=1}^{n_v} \beta_p(x) A_p^* \cdot x$$

if the LMI (6.20) *is feasible for a positive definite matrix* P*,* $V(x) = x^T P x$ *is a decreasing function with time, so a Lyapunov function has been obtained ensuring that* Ω^* *is an invariant set. La Salle's theorem (Khalil, 1996) ensures that every solution starting in* Ω^* *will approach* $x = 0$*.*

As the expression of the local system (6.2) *is not valid outside* Ω*, then the local stability can only be proved in the largest ellipsoid* Ω^* *contained in* Ω*, which will be defined by a value of* V_m *equal to the minimum value of* $V(x)$ *in the boundary of* Ω *(*$\partial\Omega$*).* ■

The following lemma is useful in order to set up an LMI characterisation of the largest ellipsoid in Ω which is a Lyapunov equipotential[1].

Suppose Ω defined as a symmetric polytope that contains $x = 0$:

$$\Omega = \{x \mid |a_i^T x| \leq 1 i : 1, ..., n_p\} \tag{6.21}$$

Lemma 6.5 $\Theta = \{x \mid x^T Q^{-1} x \leq 1\}$, $Q = Q^T > 0$ *is an ellipsoid contained in* Ω *which itself contains the maximum volume sphere centered at* $x = 0$ *if the LMI problem*

minimize λ
subject to $\lambda I > Q^{-1} > 0$
$Q > 0,\ a_1^T Q a_i \leq 1, \quad i = 1, ... n_p$

is feasible. Then, no other ellipsoid in Ω *contains a larger centered sphere.* ■

The proof appears in (Boyd et al., 1994) chapter 3.

[1] Largest is here understood as containing the largest spherical ball around $x = 0$, i..e, guaranteeing stability for the largest initial distance from the origin.

Theorem 6.1 *Consider the system* (3.12)*. The largest spherical basin of attraction of $x = 0$ provable by a quadratic Lyapunov function in a symmetric polytopic region Ω has a radius $\lambda^{-\frac{1}{2}}$ given by the solution of the following LMI problem:*

minimize λ subject to

$$\lambda I > P \quad > \quad 0 \tag{6.22}$$

$$P \quad > \quad 0 \tag{6.23}$$

$$\begin{pmatrix} P & a_j \\ a_j^T & 1 \end{pmatrix} \quad > \quad 0, \; j : 1 \ldots n_p \tag{6.24}$$

$$A_p^{*T} P + P A_P^* \quad < \quad 0, \; p : 1 \ldots n_v \tag{6.25}$$

and Ω is defined as (6.21)*. The ellipsoid $\Theta = \{x \mid x^T P x \leq 1\}$ is, of course, also contained in the basin of attraction of $x = 0$.*
Proof: *Conditions* (6.25) *imply that trajectories inside any equipotential region defined by P converge to the point $x = 0$, as shown in Lemma 6.4. Applying the Schur complement, the conditions* (6.24) *are equivalent to*

$$a_p^T P^{-1} a_p < 1, i : 1..n_p$$

Then, conditions (6.24) *keep Θ inside Ω and the condition* (6.22) *along with the LMI objective, maximize the radius of the quadratically invariant sphere contained in Θ, from Lemma 6.5.* ■

6.3 Stability analysis of Feedback Systems

Some interesting results for local stability of feedback systems may be found by using the theorems in Chapter 3 and adding the conditions of Lemma 6.5.

The theorem below adapts Theorem 3.1 to the local model framework previously presented.

Theorem 6.2 *Consider the feedback fuzzy system* (3.14)*.The largest spherical basin of attraction of $x = 0$ provable by a quadratic Lyapunov function in a symmetric polytopic region Ω has a radius $\lambda^{-\frac{1}{2}}$ given by the solution of the following LMI problem:*

minimize λ *subject to*

$$
\begin{aligned}
G_{ii}^{*T}P + PG_{ii}^{*} &< 0 \quad (6.26)\\
\left(\frac{G_{ij}^{*}+G_{ji}^{*}}{2}\right)^{T} P + P\left(\frac{G_{ij}^{*}+G_{ji}^{*}}{2}\right) &\leq 0 \quad (6.27)\\
P &> 0 \quad (6.28)\\
\begin{pmatrix} P & a_j \\ a_j^T & 1 \end{pmatrix} &> 0 \quad (6.29)\\
\lambda I &> P \quad (6.30)
\end{aligned}
$$

where:

$$G_{ij}^{*} = A_i^{*} - B_i^{*}F_j^{*}$$

■

The theorem below is based on Theorem 3.5 for local models.

Theorem 6.3 *The feedback fuzzy system* (3.14)*. The largest spherical basin of attraction of* $x = 0$ *provable by a quadratic Lyapunov function in a symmetric polytopic region* Ω *has a radius* $\lambda^{-\frac{1}{2}}$ *given by the solution of the following LMI problem:*

minimize λ *subject to*

$$
\begin{aligned}
G_{ii}^{*T}P + PG_{ii}^{*} + X_{ii} &< 0 \quad (6.31)\\
\left(\frac{G_{ij}^{*}+G_{ji}^{*}}{2}\right)^{T} P + P\left(\frac{G_{ij}^{*}+G_{ji}^{*}}{2}\right) + X_{ij} + X_{ji} &\leq 0 \quad (6.32)\\
\begin{pmatrix} X_{11} & \dots & X_{1n} \\ \vdots & \ddots & \vdots \\ X_{n1} & \dots & X_{nn} \end{pmatrix} &> 0 \quad (6.33)\\
P &> 0 \quad (6.34)\\
\begin{pmatrix} P & a_j \\ a_j^T & 1 \end{pmatrix} &> 0 \quad (6.35)\\
\lambda I &> P \quad (6.36)
\end{aligned}
$$

where: P is a definite positive matrix and matrices $X_{ij} = X_{ji}^T$.

■

6.4 Algorithm

These results may be combined in order to obtain an algorithm to compute the largest ball around $x = 0$ for which attraction is ensured.

Basically, the procedure will first check the extreme cases: (1) checking for feasibility of LMI problems as stated in Section 3.5 (2) checking for stability of the linearised model around $x = 0$.

If the first one is unfeasible but the second one is feasible, selecting a polytopic region on the state space and a scaling factor ρ allows to set up a bisection procedure in order to determine the largest feasible ρ.

6.5 Examples

Example 6.1 *Let us have a fuzzy system given by:*

$$\dot{x} = \sum_{i=1}^{2} \mu_i(x) A_i x \tag{6.37}$$

$$A_1 = \begin{pmatrix} -0.5 & -1 \\ -1 & -0.5 \end{pmatrix} \tag{6.38}$$

$$A_2 = \begin{pmatrix} -0.5 & 1 \\ 1 & -0.5 \end{pmatrix} \tag{6.39}$$

Figure 6.1 shows the membership functions μ_1 and μ_2 which, for simplicity, depend only on x_2. The value of $a = 1$ will be assumed.

Define Ω_k as a rectangle bounded in x_2, unbounded in x_1:

$$\Omega_k = \{x \mid \ |(0 \ 1/\rho_k)x| \leq 1\}$$

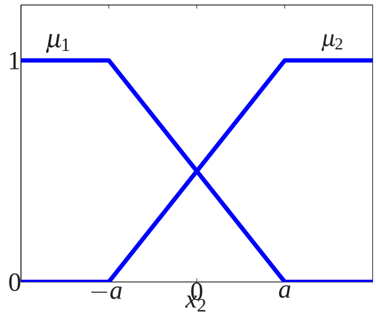


Figure 6.1: membership functions $\mu_1(x_2)$, $\mu_2(x_2)$

where k is the iteration number.
Note that the maximum and minimum values of μ_i in Ω are easily obtained, and the Lemma 6.5 can be applied.

In the proposed procedure, the LMIs for $\rho = 1$ are unfeasible. However, the linearised model is:

$$\dot{x} = (0.5A_1 + 0.5A_2)x = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix} \tag{6.40}$$

which is stable. Hence, there exists a zone around $x = 0$ (possibly small) where local stability holds. The procedures in this chapter allow to determine the largest sphere around $x = 0$ for which local quadratic stability holds.

Let us consider for the fist iteration $\rho_1 = 0.1$. The maximum and minimum values of μ are, in that case: $\mu_1^M = 0.55$, $\mu_1^m = 0.45$, $\mu_2^M = 0.55$, $\mu_2^m = 0.45$
Then the vertices obtained in the region Ω_1 are:

$$\begin{aligned} v_1 &= \begin{pmatrix} 0.45 & 0.55 \end{pmatrix} \\ v_2 &= \begin{pmatrix} 0.55 & 0.45 \end{pmatrix} \end{aligned}$$

The local fuzzy model from Lemma 6.1 is described by:

$$\begin{aligned} A_1^* &= \begin{pmatrix} -0.5 & 0.1 \\ 0.1 & -0.5 \end{pmatrix} \\ A_2^* &= \begin{pmatrix} -0.5 & -0.1 \\ -0.1 & -0.5 \end{pmatrix} \end{aligned}$$

And, solving the LMIs:

$$\begin{aligned} A_1^{*T}P + PA_1^* &< 0 \\ A_2^{*T}P + PA_2^* &< 0 \\ X &> 0 \end{aligned}$$

local stability in a certain ellipsoidal region inside $\mathbf{\Omega}_1$ is proved.

When the same procedure is applied to $\rho = 0.5$ the LMIs are unfeasible. The LMIs are, however, feasible for any $\rho < 0.5$. for instance, $\rho_n = 0.499$ results in the following LMI conditions:

$$\begin{aligned} A_1^{*T}P + PA_1^* &< 0 \\ A_2^{*T}P + PA_2^* &< 0 \\ \begin{pmatrix} \begin{matrix} & P & \end{matrix} & \begin{matrix} 0 \\ 1/\rho_n \end{matrix} \\ \begin{matrix} 0 & 1/\rho_n \end{matrix} & 1 \end{pmatrix} &> 0 \\ P &> 0 \\ \lambda I &> P \end{aligned}$$

which are feasible for the above value of ρ_n and, the matrix P obtained for the minimum λ defines an ellipsoid:

$$\Theta = \{x \mid x^T P x \leq 1\}$$

$$P = \begin{pmatrix} 3.8274 & 0 \\ 0 & 4.016 \end{pmatrix}$$

which conforms the guaranteed basin of attraction.

Trajectories with starting points inside the ellipsoid Θ are guaranteed to converge to the origin. Points outside the ellipsoid may lead to either convergent or non-convergent trajectories.

Example 6.2 *Let us have a fuzzy system given by:*

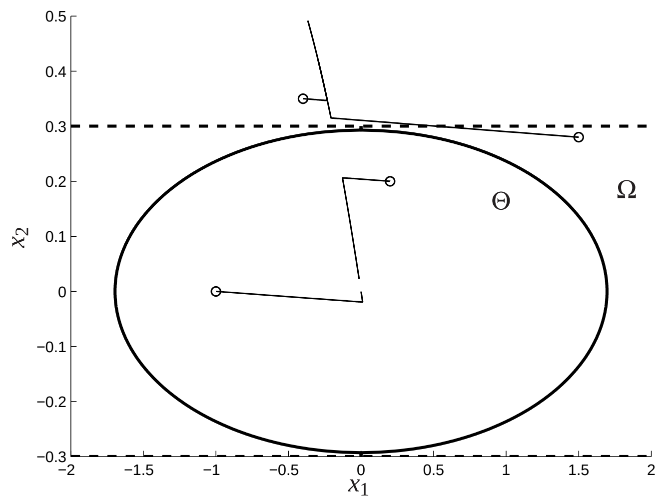


Figure 6.2: The Basin of attraction of $x = 0$ provable by quadratic stability (Example 2).

$$\dot{x} = \sum_{i=1}^{2} \mu_i(x)(A_i x + B_i u) \tag{6.41}$$

$$A_1 = \begin{pmatrix} 2 & -10 \\ 1 & 0 \end{pmatrix} \tag{6.42}$$

$$A_2 = \begin{pmatrix} 6 & -10 \\ 1 & 1 \end{pmatrix} \tag{6.43}$$

$$B_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}^T \tag{6.44}$$

$$B_2 = \begin{pmatrix} 10 & 0 \end{pmatrix}^T \tag{6.45}$$

Figure 6.1 shows the membership functions μ_1 and μ_2 which, for simplicity, depend only on x_2. The value of $a = 1$ will be assumed.
The control action u is given by (3.4) where F_1 and F_2 are:

$$F_1 = \begin{pmatrix} 18 & 5 \end{pmatrix} \tag{6.46}$$

$$F_2 = \begin{pmatrix} 2.3 & 2.2 \end{pmatrix} \tag{6.47}$$

These values of F_i are designed to place the closed-loop poles of $A_i - B_i F_i$ at -1 and -15.

Conveniently, we take the same region shape Ω as in Example 1. Then the maximum ρ obtained is 0.3 and the maximum ellipsoid θ that ensures stability for its points is illustrated in Figure 6.2, with some examples of initial points.

6.6 Conclusions

This chapter shows how *local* stability results (the largest sphere around $x = 0$ for which a quadratic Lyapunov function can be proven via LMI) may be obtained in fuzzy systems via the knowledge of the membership functions, even when no feasible quadratic Lyapunov function can be found to prove *global* stability. The discovered sphere is part of a larger ellipsoidal guaranteed basin of attraction.

The methodology used is based on transformation of the membership functions by expressing them as a convex combination of some points in the membership space. These points are obtained from the knowledge of the maximum and minimum values of the memberships in the zone under study.

Chapter 7

Relaxed LMI conditions: Overlap in membership functions

7.1 Introduction

The objective of this chapter is to reduce, the gap between fuzzy and nonlinear control discussed in Chapter 4 by looking at the membership functions shape. Indeed, for a particular application, when the expressions of the memberships as a function of some premise variables are actually known, some zones of the possible membership space can be excluded (only *for that particular case*, of course). Reducing the size of the (multi-dimensional) set where the membership functions take values "should" obtain less conservative conditions than those expressed for "any" membership. However, LMI conditions in current literature do not take into account that fact.

This contribution presents results generalizing ones presented in Chapter 5 in order to incorporate any multivariate polynomial constraint on the membership shapes, such as $\mu_1^2 - \mu_2^3\mu_3 - 4\mu_2\mu_4 - 0.1 \geq 0$. The most interesting particular cases are: (a) asserting that the membership vector is always inside (or outside) a particular ellipse (or set of them); (b) asserting that the overlap (measured as a product) of some memberships is below a particular value, say $\mu_1\mu_4\mu_2 \leq 0.03$.

In summary, this work provides membership-shape dependent conditions for fuzzy closed-loop analysis and controller design problems, which are less conservative than other approaches in literature. The results allow specialization of the designs to a particular nonlinear plant, and give room to greater flexibility of the designs (such as multiple performance objectives in different operation regions).

The structure of the chapter is as follows: next section relaxes the conditions in theorem 3.5 using bounds on the products of the membership functions (measuring the degree of overlap) and introducing additional artificial LMI variables. The approach is directly applicable with minor modifications to other settings where stability or performance (decay rate, $\mathscr{H}_\infty$, *etc.*) requires positivity of a double fuzzy summation $\sum_i \sum_j \mu_i \mu_j x^T Q_{ij} x > 0$, for instance (Tuan et al., 2001). The required bounds on the membership functions are easily obtained in practice, as examples will illustrate. As an additional result, relaxed LMI conditions for single fuzzy summations $\sum_i \mu_i x^T Q_i x > 0$ are also proposed from bounds on the maximum value of the memberships. An alternative approach, based on changes of variable, as is reported in Chapter 6, but that approach does not apply to the product bounds. The next section will generalize these results to any shape that can be expressed as a quadratic form of the membership functions. And finally an approach for relaxing the conditions for any polynomial constrain of the membership functions' constraints using the stability and performance results from Chapter 5 will be shown. Numerical examples are provided in section 7.5.

7.2 Relaxed conditions with overlap information

This section presents relaxations of the above results, which apply when nontrivial bounds on the maximum firing strength of each membership function (or their products) are known.

7.2.1 Single sum relaxation

If a bound for μ_j is known, so that

$$\mu_j(z) \leq \beta_j \ \forall \ z$$

the following theorem allows to set up stability and performance conditions which are less conservative that those in Chapter 3.

Theorem 7.1 *Expression* $\Xi(t) = \sum_{i=1}^{r} \mu_i(z(t)) x(t)^T Q_i x(t)$ *is positive if there exist symmetric positive definite matrices* $N_1, \ldots, N_r$ *such that,* *for all* i*:*

$$Q_i + N_i - \sum_{j=1}^{r} \beta_j N_j > 0 \tag{7.1}$$

If $Q_i > 0$ *is a linear matrix inequality (LMI (Boyd et al., 1994; Tanaka & Wang, 2001)), the above expressions are LMIs, too.*

Proof: Consider an arbitrary positive definite N_j. Then, by using the partition condition (3.3) and the fact that $x^T N_j x \geq 0$ for any x:

$$\mu_j \leq \beta_j \sum_{i=1}^{r} \mu_i \tag{7.2}$$

$$\mu_j x^T N_j x \leq \beta_j x^T N_j x \sum_{i=1}^{r} \mu_i = \sum_{i=1}^{r} \mu_i \beta_j x^T N_j x \tag{7.3}$$

Hence, the term

$$H_j = \mu_j x^T N_j x - \sum_{i=1}^{r} \mu_i \beta_j x^T N_j x \tag{7.4}$$

is negative semi-definite, *i.e.*, $H_j \leq 0$. As a result, it may be added to Ξ and positiveness of Ξ will be proved if positiveness of $\Xi + H_j$ is shown, *i.e.*,

$$\begin{aligned}\Xi \geq \Xi + H_j = \sum_{i=1}^{r} \mu_i x^T Q_i x + \mu_j x^T N_j x - \sum_{i=1}^{r} \mu_i \beta_j x^T N_j x = \\ = x^T \left(\mu_j (Q_j + (1-\beta_j) N_j) + \sum_{i \neq j} \mu_i (Q_i - \beta_j N_j) \right) x\end{aligned} \tag{7.5}$$

Such positivity occurs if

$$Q_i - \beta_j N_j > 0 \quad \forall i \neq j \tag{7.6}$$

and, for the case $i = j$

$$Q_j + (1-\beta_j) N_j > 0 \tag{7.7}$$

In order to complete the proof, if a bound β_j is known for all j, the conditions may be stated similarly, *i.e.*:

$$\Xi \geq \Xi + \sum_{j=1}^{r} H_j = \sum_{i=1}^{r} \mu_i x^T Q_i x + \sum_{j=1}^{r} \left(\mu_j x^T N_j x - \sum_{i=1}^{r} \mu_i \beta_j x^T N_j x \right) =$$

$$= x^T \left(\sum_{i=1}^{r} \mu_i (Q_i + N_i - \sum_{j=1}^{r} \beta_j N_j) \right) x \quad (7.8)$$

from which conditions (7.1) are readily obtained. ■

The theorem indicates that a negative definite Q_j may be "compensated" by addition of a positive matrix to it, in expression (7.7), if it fires only up to a limited strength β_j and the rest of fuzzy models (which will fire with strengths adding at least $1-\beta_j$) are positive enough, fulfilling (7.6).

Note that the trivial bound $\beta_j = 1$ is useless. Bounds lower than 1 arise with the use of non-normal fuzzy sets. Note also that, by fixing $N_j = 0$, Theorem 7.1 reduces to the standard conditions in literature.

The ideas arising from Theorem7.1 can be extended to a more interesting and useful result involving double sums, as discussed below.

7.2.2 Double-sum relaxation

Assume that knowledge of the specific shape of the membership functions allows to set up a bound:

$$0 \leq \mu_i(z)\mu_j(z) \leq \beta_{ij} \ \forall \ z \quad (7.9)$$

The bounds β_{ij} may be used to set up some relaxations of Theorem 1, similarly to the previously considered single-sum case. The trivial bounds in this case are obtained with: $\mu_i\mu_j \leq \mu_i(1-\mu_i) \leq 0.25$ and, for $i = j$, $\beta_{ii} = 1$ if the involved fuzzy sets are normal.

Theorem 7.2 *Consider an antecedent fuzzy partition fulfilling the overlap bounds* (7.9)*. Expression* (3.23) *holds if there exist matrices* $X_{ij} = X_{ji}^T$

and symmetric R_{ij}*,* $i \le j$*, such that:*

$$X_{ii} \le Q_{ii} + R_{ii} - \Lambda \tag{7.10}$$

$$X_{ij} + X_{ji} \le Q_{ij} + Q_{ji} + R_{ij} - 2\Lambda \tag{7.11}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} > 0, \qquad R_{ij} \ge 0 \tag{7.12}$$

where

$$\Lambda = \sum_{k=1}^{r} \sum_{k \le l \le r} \beta_{kl} R_{kl}$$

Proof. Consider an arbitrary symmetric positive definite R_{kl}. Using condition (3.24),

$$\mu_k \mu_l \le \beta_{kl} \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \tag{7.13}$$

$$\mu_k \mu_l x^T R_{kl} x \le \beta_{kl} x^T R_{kl} x \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T \beta_{kl} R_{kl} x \tag{7.14}$$

Then, the term:

$$H_{kl} = \mu_k \mu_l x^T R_{kl} x - \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T \beta_{kl} R_{kl} x$$

is negative-semidefinite, *i.e.*, $H_{kl} \le 0$, so it may be added to Ξ in (3.23), *i.e.*:

$$\Xi \ge \Xi + H_{kl} = \mu_k \mu_l x^T R_{kl} x + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T (Q_{ij} - \beta_{kl} R_{kl}) x \tag{7.15}$$

and, if the resulting $\Xi + H_{kl}$ were proved to be positive, Ξ would evidently be positive. Reordering the above expression as in (3.48) results in

$$\Xi \ge \mu_k \mu_l x^T R_{kl} x + \sum_{i=1}^{r} \mu_i^2 x^T (Q_{ii} - \beta_{kl} R_{kl}) x + \\ + \sum_{i=1}^{r} \sum_{i<j \le r} \mu_i \mu_j x^T (Q_{ij} + Q_{ji} - 2\beta_{kl} R_{kl}) x \tag{7.16}$$

Considering all the different pairs of membership functions, yielding different β_{kl} (for $k : 1 \dots r$, $k \leq l \leq r$), an expression of $\Xi + \sum_{k=1}^{r} \sum_{k \leq l \leq r} H_{kl}$ can be easily obtained:

$$\Xi \geq \Xi + \sum_{k=1}^{r} \sum_{k \leq l \leq r} H_{kl} = \sum_{i=1}^{r} \sum_{i \leq j \leq r} \mu_i \mu_j x^T R_{ij} x + \sum_{i=1}^{r} \mu_i^2 x^T (Q_{ii} - \sum_{k=1}^{r} \sum_{k \leq l \leq r} \beta_{kl} R_{kl}) x + \sum_{i=1}^{r} \sum_{i < j \leq r} \mu_i \mu_j x^T (Q_{ij} + Q_{ji} - 2 \sum_{k=1}^{r} \sum_{k \leq l \leq r} \beta_{kl} R_{kl}) x \tag{7.17}$$

By straightforward manipulations of the expression (7.17), if conditions (7.10) and (7.11) hold, then

$$\Xi \geq \sum_{i=1}^{r} \mu_i^2 x^T (Q_{ii} + R_{ii} - \sum_{k=1}^{r} \sum_{k \leq l \leq r} \beta_{kl} R_{kl}) x + \sum_{i=1}^{r} \sum_{i < j \leq r} \mu_i \mu_j x^T (Q_{ij} + Q_{ji} + R_{ij} - 2 \sum_{k=1}^{r} \sum_{k \leq l \leq r} \beta_{kl} R_{kl}) x \geq \sum_{i=1}^{r} \mu_i^2 x^T X_{ii} x + \sum_{i=1}^{r} \sum_{i < j \leq r} \mu_i \mu_j x^T (X_{ij} + X_{ji}) x \tag{7.18}$$

Finally, using an argumentation analogous to that in (3.49), a sufficient condition for $\Xi > 0$ is stated in the theorem. ∎

Proposition 7.1 *For the trivial off-diagonal bound $\beta_{kl} = 0.25$, Theorem 7.2 does not provide better results than Theorem 3.5,* i.e., *Theorem 7.2 with normal partitions is only useful in fuzzy models with 3 or more rules.*

Proof. Indeed, assume that feasible X_{ij}, R_{kl}, $k < l$ have been obtained with Theorem 7.2, so that

$$X_{ii} \leq Q_{ii} - 0.25 R_{kl} \tag{7.19}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} - 0.5 R_{kl} \quad (i, j) \neq (k, l) \tag{7.20}$$

$$X_{kl} + X_{lk} \leq Q_{kl} + Q_{lk} + 0.5 R_{kl} \tag{7.21}$$

Take $X'_{ij} = X_{ij}$ for all pairs (i, j) except (k,k), (l,l), (k,l), (l,k), and take $X'_{kk} = X_{kk} + 0.25 R_{kl}$, $X'_{ll} = X_{ll} + 0.25 R_{kl}$, $X'_{kl} = X_{kl} - 0.25 R_{kl}$, $X'_{lk} = X_{lk} -$

$0.25R_{kl}$. Then, conditions (3.50)–(3.51) are fulfilled for X'_{ij}. Consider now an arbitrary vector $\xi = (\xi_1^T \ldots \xi_k^T \ldots \xi_l^T \ldots \xi_r^T)^T$ where ξ_i is itself a vector compatible with the dimensions of X_{ii}. Then,

$$\xi^T \mathbf{X}'\xi = \xi^T \mathbf{X}\xi + \left(\xi_k^T\, \xi_l^T\right) \begin{pmatrix} 0.25R_{kl} & -0.25R_{kl} \\ -0.25R_{kl} & 0.25R_{kl} \end{pmatrix} \begin{pmatrix} \xi_k \\ \xi_l \end{pmatrix} = \\ = \xi^T \mathbf{X}\xi + 0.25(\xi_k - \xi_l)^T R_{kl}(\xi_k - \xi_l) \quad (7.22)$$

proves that $\mathbf{X}' \geq \mathbf{X}$, as $(\xi_k - \xi_l)^T R_{kl}(\xi_k - \xi_l) \geq 0$. Hence, the matrix $\mathbf{X}'$ in (3.52) formed with the X'_{ij} is positive definite, *i.e.*, the procedure has obtained a feasible solution for Theorem 3.5. ■

The numerical examples in Section 7.5 will show that Theorem 3 improves over Theorem 3.5 when $\beta_{ij} < 0.25$.

Remark: if $\beta_{kl} = 0$ (*i.e.*, fuzzy sets μ_k and μ_l are disjoint and non-overlapping), R_{kl} dissapears from Λ. Hence, the LMI (7.10) or (7.11) in which R_{kl} appears may be eliminated as R_{kl} can be as large as needed. In this way, the usual conditions regarding completely non-overlapping memberships in literature (Kim & Lee, 2000) are recovered as a particular case.

7.3 Shape-dependent positivity conditions for double fuzzy summations

Denote by $\mu(z)$ the column vector of membership functions $\mu(z) = (\mu_1(z), \mu_2(z), \ldots, \mu_r(z))^T$ in (3.23). On the sequel, the shorthand notation μ will be used instead of $\mu(z)$, as previously introduced for the individual membership components.

Assume that knowledge of:

- the specific shape of the membership functions,
- the set of values Ω taken by premise variables z,

allows to set up a bound, for some known S, t and v, in the form:

$$\mu^T S\mu + t\mu + v \leq 0 \quad \forall\, z \in \Omega \tag{7.23}$$

where parameters S, t and v are, respectively, a matrix (of dimensions $r \times r$), a row vector $(1 \times r)$ and a scalar. The elements of S will be denoted by s_{ij}, and β_{ij} will denote:

$$\beta_{ij} = (s_{ij} + t_i + v)$$

The left-hand side term in (7.23) is a second-order polynomial in the membership functions. Particular examples are, for instance, knowledge on degrees of membership function overlap (say, $\mu_1\mu_2 < 0.15$, $\mu_1\mu_3 = 0$, $(\mu_1 + \mu_2) * \mu_4 \leq 0.4$), ellipsoidal sets (such as $(\mu_1 - 0.9)^2 + 2(\mu_2 - 0.1)^2 \leq 0.05^2$) or drilling ellipsoidal "holes" (such as $(\mu_1 - 0.9)^2 + 2(\mu_2 - 0.1)^2 \geq 0.05^2$); see section 7.3.1 for further discussion.

Proposition 7.2 *If the membership functions conform a fuzzy partition, then the matrix $\mathscr{B}$ whose elements are β_{ij} fulfills:*

$$\mu^T \mathscr{B} \mu \leq 0 \tag{7.24}$$

Proof: Indeed, using condition (3.3),

$$\begin{aligned} 0 \geq \mu^T S \mu + t\mu + v &= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j s_{ij} + \sum_{i=1}^{r} t_i \mu_i + v = \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j s_{ij} + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j t_i + \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j v = \\ &= \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (s_{ij} + t_i + v) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \beta_{ij} \end{aligned} \tag{7.25}$$

Hence, any second-order polynomial restriction on the membership shape can be expressed as an homogeneous quadratic form.

Theorem 7.3 *Assume knowledge about the particular membership function shape is available via a constraint matrix $\mathscr{B}$, with elements β_{ij}, fulfilling* (7.24)*. Then, expression* (3.23) *is proved if there exists a symmetric matrix $R \geq 0$ so that the condition:*

$$\Xi'(t) = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T Q'_{ij} x > 0 \quad \forall\, x \neq 0 \tag{7.26}$$

holds, where

$$Q'_{ij} = Q_{ij} + \beta_{ij} R$$

i.e. Q_{ij} *is replaced in* (3.23) *by* Q'_{ij} *involving an additional matrix variable.*

Proof: consider an arbitrary symmetric positive semi-definite R. Then, the term

$$H = x^T R x \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j \beta_{ij} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T \beta_{ij} R x \tag{7.27}$$

verifies $H \leq 0$, so it may be added to Ξ in (3.23) and, if the resulting sum is positive, Ξ will evidently be positive, *i.e.*,

$$\Xi \geq \Xi + H = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j x^T (Q_{ij} + \beta_{ij} R) x = \Xi' \tag{7.28}$$

■

Multiple restrictions can be incorporated by repeated application of Theorem 7.3, *i.e.*:

Corollary 7.1 *Assume that knowledge about the particular membership function shape is available via a set of matrices* $\mathscr{B}^{[k]}$, $k = 1, \ldots, t$, *with elements* $\beta_{ij}^{[k]}$. *Expression* (3.23) *is proved if there exist positive semi-definite matrices* R_k, $k = 1, \ldots, t$, *so that the condition* (7.26) *holds, where*

$$Q'_{ij} = Q_{ij} + \sum_{k=1}^{t} \beta_{ij}^{[k]} R_k$$

In order to check condition (7.26), well-known expressions can be used, such as the ones in (Tuan et al., 2001), or (Liu & Zhang, 2003). For instance, straightforward application of (Liu & Zhang, 2003) yields the following theorem:

Theorem 7.4 *Consider an antecedent fuzzy partition fulfilling the bounds* (7.23). *Expression* (3.23), *under shape constraints* $\mathscr{B}[k]$, $k =$

$1,\ldots,t$, *holds if there exist matrices* $X_{ij} = X_{ji}^T$ *and symmetric* $R_k \geq 0$ *such that:*

$$X_{ii} \leq Q_{ii} + \sum_{k=1}^{t} \beta_{ii}^{[k]} R_k \tag{7.29}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + \sum_{k=1}^{t} (\beta_{ij}^{[k]} + \beta_{ji}^{[k]}) R_k \tag{7.30}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \ldots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \ldots & X_{rr} \end{pmatrix} > 0 \tag{7.31}$$

7.3.1 Particular cases

Membership function overlap. Assume that a bound on the overlap between fuzzy sets defined by μ_k and μ_l, for $k \neq l$, is expressed as:

$$\mu_k \mu_l \leq \gamma_{kl} \tag{7.32}$$

It is a particular case of (7.23) with $t_i = 0$ and

$$s_{kl} = s_{lk} = \frac{1}{2}, \quad \nu = -\gamma_{kl} \tag{7.33}$$

$$s_{ij} = 0 \ \forall \ (i,j) \neq (k,l) \tag{7.34}$$

resulting in $\beta_{kl} = \beta_{lk} = \frac{1}{2} - \gamma_{kl}$, the rest of $\beta_{ij} = -\gamma_{kl}$.

The conditions of Theorem 7.4 result in:

$$X_{ii} \leq Q_{ii} - \gamma_{kl} R \tag{7.35}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} - 2\gamma_{kl} R \ \ (i,j) \neq (k,l) \tag{7.36}$$

$$X_{kl} + X_{lk} \leq Q_{ij} + Q_{ji} + (1 - 2\gamma_{kl}) R \tag{7.37}$$

$$\mathbf{X} = \begin{pmatrix} X_{11} & \ldots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \ldots & X_{rr} \end{pmatrix} > 0, \qquad R \geq 0 \tag{7.38}$$

As it can be seen, the results are equivalent to ones obtained in the previous section.

Ellipsoidal hole. The interior (or exterior) of any ellipse, parabola or hiperbola can be considered via a suitable $\mathscr{B}$. As a simple example, Let us assume that the condition

$$\sum_{i=1}^{r}(\mu_i - c_i)^2 \geq \delta^2 \tag{7.39}$$

is known to hold, *i.e.*, the membership functions are known to lie outside of a particular hyper-sphere.

Then,

$$\sum_{i=1}^{r}(\mu_i^2 + c_i^2 - 2c_i\mu_i) \geq \delta^2 \tag{7.40}$$

$$\sum_{i=1}^{r}\mu_i^2 + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j(\sum_{k=1}^{r}c_k^2) - 2\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j c_i \geq \delta\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \tag{7.41}$$

so we have, denoting by $\phi = -\delta^2 + \sum_{k=1}^{r} c_k^2$,

$$\sum_{i=1}^{r}\mu_i^2(1-2c_i) + \phi + \sum_{i=1}^{r}\sum_{j=i+1}^{r}\mu_i\mu_j(2\phi - 2c_i - 2c_j) \geq 0 \tag{7.42}$$

so the previous results apply with

$$\beta_{ii} = -(1-2c_i) - \phi \tag{7.43}$$

$$\beta_{ij} = -\phi + 2c_i \quad i \neq j \tag{7.44}$$

In general, one may specify that the membership functions are outside of an arbitrary ellipsoidal quadratic form, in the form:

$$\mu^T S\mu - v \leq 0 \tag{7.45}$$

where S is a positive definite matrix. A set of such ellipsoids may be used to exclude any zones which are known to be out of the range of the membership function vector (Figure 7.1). Using a negative definite S is equivalent to stating that the membership vector lies inside a particular ellipsoid.

7.3.2 Obtention of bounds in practice.

As the shape of the membership functions is known in fuzzy control[1], obtaining the bounds for any second-order polynomial in the form (7.23)

[1] See (Lam & Leung, 2005) for conditions with uncertain memberships.

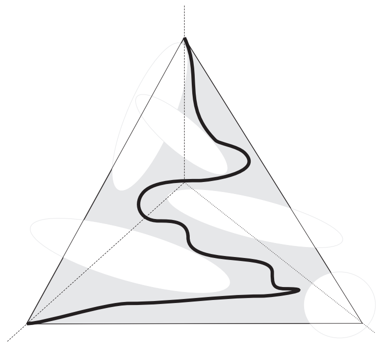


Figure 7.1: Ellipsoidal holes on the add-1 simplex in $\mathbb{R}^3$.

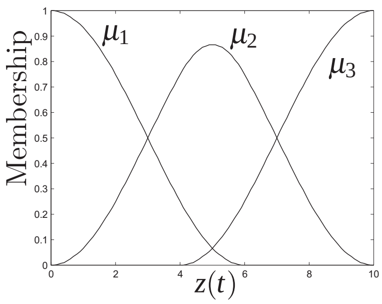


Figure 7.2: Fuzzy partition with limited overlap.

is, in principle, easy, as the problem can be approached as an optimization one. Indeed, choosing arbitrary S and t, maximising $J(z) = \mu(z)^T S\mu(z) + t\mu(z)$ over the expected range of values of the premise variables z, previously denoted as Ω, yields a value $J_{max} = \max_{z\in\Omega} J(z)$. Once $J_{\max}$ is available[2], an expression in the form (7.23) can be obtained from $J(z) - J_{max} \leq 0$, *i.e.*, the polynomial $\mu^T S\mu + t\mu - J_{\max} \leq 0$.

As an example, consider the fuzzy partition in Figure 7.2. In this case, the bounds $\mu_2 - 0.86 \leq 0$ and $\mu_1\mu_3 - 0.0045 \leq 0$ may be easily computed by line-search on the one-dimensional set where the premise variable takes values.

In other common cases, membership functions are the cartesian product of simpler ones, describing either fuzzy partitions on individual vari-

[2]The optimization on Ω can be carried out by using any optimization technique; even a brute-force approach evaluating the memberships on a dense-enough grid on z may suffice.

ables or basic nonlinearities in the system equations, following the modelling methodology in (Tanaka & Wang, 2001). In that case, if the membership functions are the cartesian product of w fuzzy partitions, it can be shown that certain products of memberships can be bounded by a power of 0.25 (because $\mu(1-\mu) \leq 0.25$). For instance, consider "high" and "low" to be contrary concepts defined on pressure and temperature universes. If a membership μ_1 is "temperature is *low* and pressure is *low*", and other membership μ_2 is "temperature is *high* and pressure is *high*", then $\mu_1\mu_4 \leq 0.25^2$. Details and examples of these situations appear in (Sala & Ariño, 2007a).

The reader may browse at this moment, if so wished, Example 7.1 in Section 7.5, which shows some numeric results.

7.4 Generalization to higher dimensions

As a generalization of (3.23), other fuzzy control results require positiveness of a p-dimensional fuzzy summation, *i.e.*, checking

$$\Xi(t) = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\dots\sum_{i_p=1}^{r}\mu_{i_1}(z)\mu_{i_2}(z)\dots\mu_{i_p}(z)x^T Q_{i_1 i_2 \dots i_p} x \geq 0 \tag{7.46}$$

The case $p=2$ reduces to (3.23). Conditions requiring $p=3$ are, for instance, the fuzzy dynamic controllers in (Li et al., 1999; Tanaka & Wang, 2001), using $Q_{ijk} = E_{ijk} + E_{ijk}^T$, with

$$E_{ijk} = \begin{pmatrix} A_i Q_{11} + B_i \mathscr{C}_{jk} & A_i + B_i \mathscr{D}_j C_k \\ \mathscr{A}_{ijk} & A_i P_{11} + \mathscr{B}_{ij} C_k \end{pmatrix} < 0 \tag{7.47}$$

for suitably defined $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$. Triple fuzzy summations are needed also in the output-feedback settings in (Fang et al., 2006; Chen et al., 2005).

Consider also the possibility that a restriction on the shape of the membership functions is given by a multivariate polynomial of degree p, for instance

$$\mu_1^4 - 2\mu_3 + \mu_1\mu_2 - 0.3 < 0 \tag{7.48}$$

with $p=4$, where monomials of degree four, one, two and zero appear.

Now, choose any arbitrary integer n, $n \geq p$. By multiplying each of the atomic monomials of degree q ($0 \leq q \leq p$) by $(\sum_{i=1}^{r}\mu_i)^{n-q}$ (which is identically equal to one), any polynomial of degree p can be converted to an homogeneous polynomial of degree n. For instance, the above (7.48) gets converted in

$$\mu_1^4 - 2\mu_3(\sum_i \mu_i)^3 + \mu_1\mu_2(\sum_i \mu_i)^2 - 0.3(\sum_i \mu_i)^4 < 0 \tag{7.49}$$

So, let us restate a generalised version of the theorem, introducing some notation from (Sala & Arino, 2007b).

7.4.1 Multi-dimensional index notation

In order to streamline notation in multi-dimensional summations (7.46), we will follow the multi-index notation in Section 5.2.1 to handle p-dimensional vectors of natural numbers (denoted by boldfaced variables), and its associated p nested summations:

$$\begin{aligned} \mathbb{I}_p &= \{(i_1, i_2, \ldots, i_p) \mid 1 \le i_j \le r, j = 1, 2, \ldots, p\} \\ \sum_{\mathbf{i}\in\mathbb{I}_p} \gamma_{\mathbf{i}} &= \sum_{i_1=1}^{r}\sum_{i_2=1}^{r}\cdots\sum_{i_p=1}^{r} \gamma_{i_1 i_2 \ldots i_p} \end{aligned} \tag{7.50}$$

By convention, the cartesian product of several multi-indices, resulting in a higher-dimensional one, will be symbolized by parentheses:

$$\mathbf{i} \in \mathbb{I}_p,\ \mathbf{j} \in \mathbb{I}_q, \ldots, \mathbf{m} \in \mathbb{I}_t \ \Rightarrow \mathbf{k} = (\mathbf{i}, \mathbf{j}, \ldots, \mathbf{m}) \in \mathbb{I}_{p+q+\cdots+t} \tag{7.51}$$

and sometimes by mere juxtaposition, such as in $\gamma_{i_1\ldots i_p}$ in (7.50) One-dimensional indices, say $\mathbf{j} \in \mathbb{I}_1$ are ordinary integer index variables: they will be typed in italic typeface as j, $1 \le j \le r$ when its one-dimensionality should be emphasised.

Multi-dimensional fuzzy summations. The purpose of multi-index notation is to compactly represent multi-dimensional fuzzy summations, as follows.

First, let us define the following notation, specific for membership functions as a shorthand for a product:

$$\mu_{\mathbf{i}} = \prod_{l=1}^{p} \mu_{i_l} = \mu_{i_1}\mu_{i_2}\ldots\mu_{i_p} \quad \mathbf{i} \in \mathbb{I}_p \tag{7.52}$$

With the above notation, p-dimensional fuzzy summations (7.46) may be written as follows:

$$\Xi(t) = \sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x \tag{7.53}$$

where the basic memberships $\mu = \{\mu_1, \ldots, \mu_r\}$ from which $\mu_{\mathbf{i}}$ stem fulfill the add-1 partition condition.

Permutations. Given a multi-index $\mathbf{i} \in \mathbb{I}_p$, let us denote by $\mathscr{P}(\mathbf{i}) \subset \mathbb{I}_p$ the set of permutations (with, possibly, repeated elements) of the multi-index $\mathbf{i}$. The permutations will be used to group elements in multiple fuzzy summations which share the same "antecedent": it's an evident fact that $\mathbf{j} \in \mathscr{P}(\mathbf{i}) \;\Rightarrow\; \mu_{\mathbf{j}} = \mu_{\mathbf{i}}$.

The following subset of $\mathbb{I}_p$ will be used in later developments:

$$\mathbb{I}_p^+ = \{\mathbf{i} \in \mathbb{I}_p \mid i_k \le i_{k+1}, \quad k = 1 \ldots p-1\} \tag{7.54}$$

7.4.2 Relaxed conditions

This section discusses how to obtain relaxed conditions for positivity of fuzzy summations under information on membership shape given by a degree-q constraint:

$$\sum_{\mathbf{i}\in\mathbb{I}_q} \mu_{\mathbf{i}} \beta_{\mathbf{i}} \le 0 \tag{7.55}$$

The coefficients $\beta_{\mathbf{i}}$ may be considered elements of a multi-dimensional array (tensor (Temple, 2004)) $\mathscr{B}$, generalising the matrix appearing in (7.24).

Note that any polynomial of degree lower or equal to q may be expressed as an homogeneous q-dimensional summation (7.55), by following the methodology used to obtain (7.49).

Theorem 7.5 *Consider a p-dimensional fuzzy summation condition* (7.46)*, jointly with shape-dependent knowledge expressed as constraints of degree q* (7.55)*. Choose any arbitrary integer n so that $n \ge \max(p,q)$. The positivity condition* (7.46) *(in the region determined by the constraints* (7.55)*) is fulfilled if there exists a positive definite matrix R so that the condition $\sum_{\mathbf{i}\in\mathbb{I}_n} \mu_{\mathbf{i}} x^T Q'_{\mathbf{i}} x \ge 0$, $\mathbf{i} = (i_1, i_2, \ldots, i_n)$, holds with:*

$$Q'_{\mathbf{i}} = Q_{i_1 i_2 \ldots i_p} + \beta_{i_1 i_2 \ldots i_q} R \tag{7.56}$$

Proof: As $\sum_{k=1}^{r} \mu_k = 1$, it's straightforward that:

$$\sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x \geq \sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x + x^T R x \sum_{\mathbf{i}\in\mathbb{I}_q} \mu_{\mathbf{i}} \beta_{\mathbf{i}} =$$

$$= (\sum_{k=1}^{r} \mu_k)^{n-p} \sum_{\mathbf{i}\in\mathbb{I}_p} \mu_{\mathbf{i}} x^T Q_{\mathbf{i}} x + (\sum_{k=1}^{r} \mu_k)^{n-q} \sum_{\mathbf{i}\in\mathbb{I}_q} \mu_{\mathbf{i}} x^T \beta_{\mathbf{i}} R x =$$

$$= \sum_{\mathbf{i}\in\mathbb{I}_n} \mu_{\mathbf{i}} x^T Q'_{\mathbf{i}} x \quad (7.57)$$

so positivity of the last term implies positivity of the first one. ■

Note that an extension to multiple restrictions (say, t restrictions) is straightforward, by using:

$$Q'_{\mathbf{i}} = Q_{i_1 i_2 \ldots i_p} + \sum_{k=1}^{t} \beta^{[k]}_{i_1 i_2 \ldots i_q} R_k \quad (7.58)$$

for some positive-definite R_k associated to each constraint tensor $\mathscr{B}^{[k]}$.

Once the new $Q'_{\mathbf{i}}$ are defined as above, any sufficient condition to prove positivity of an n-dimensional fuzzy summation may be used. For instance, $n = 3$ may be handled by adapting the conditions in (Fang et al., 2006, Theorem 5).

Theorem 7.6 *Given a set of t degree-3 polynomial restrictions, expressed as t homogeneous forms* (7.55)*, and any Q_{ijk} expressing some fuzzy control requirements, the 3-dimensional summation*

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \sum_{k=1}^{r} \mu_i \mu_j \mu_k x^T Q_{ijk} x$$

is positive (in the region determined by the constraints) if there exist X_{ijk}, R_l so that, being $Q'_{ijk} = Q_{ijk} + \sum_{l=1}^{t} \beta^{[l]}_{ijk} R_l$:

$$R \geq 0 \quad X_{ijk} = X_{ikj}^T \quad (7.59)$$

$$Q'_{iii} \leq X_{iii} \quad (7.60)$$

$$Q'_{iij} + Q'_{iji} + Q'_{jii} \leq X_{iij} + X_{iji} + X_{jii} \quad i < j \quad (7.61)$$

$$Q'_{ijk} + Q'_{ikj} + Q'_{jik} + Q'_{jki} + Q'_{kij} + Q'_{kji} \leq$$
$$X_{ijk} + X_{ikj} + X_{jik} + X_{jki} + X_{kij} + X_{kji} \quad i < j < k \quad (7.62)$$

$$\begin{pmatrix} X_{i11} \ldots X_{i1r} \\ \vdots \ddots \vdots \\ X_{ir1} \ldots X_{irr} \end{pmatrix} > 0 \quad \forall i \quad (7.63)$$

In fact, any $n \geq 2$ may be handled by using Theorems 4 and 5 in (Sala & Arino, 2007b). Substituting Q' from (7.58) in the theorems of (Sala & Arino, 2007b), the result is:

Theorem 7.7 *A sufficient condition for positivity of* Ξ *in* (7.46)*, under as set of* t *constraints in the form* (7.55)*, is the positivity condition below, for* $n \geq \max(p,q)$*:*

$$\sum_{\mathbf{k}\in\mathbb{I}_{n-2}} \mu_{\mathbf{k}} \xi^T \begin{pmatrix} X_{(\mathbf{k},1,1)} & \dots & X_{(\mathbf{k},1,r)} \\ \vdots & \ddots & \vdots \\ X_{(\mathbf{k},r,1)} & \dots & X_{(\mathbf{k},r,r)} \end{pmatrix} \xi > 0 \tag{7.64}$$

if there exist matrices $X_{\mathbf{j}}$*,* $\mathbf{j} \in \mathbb{I}_n$*, and positive-definite* R_l*,* $l = 1, \dots, t$*, so that, for all* $\mathbf{i} \in \mathbb{I}_n^+$

$$\sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} (Q_{j_1 j_2 \dots j_p} + \sum_{l=1}^{t} \beta^{[l]}_{j_1 j_2 \dots j_q} R_l) \geq \sum_{\mathbf{j}\in\mathscr{P}(\mathbf{i})} \frac{1}{2}(X_{\mathbf{j}} + X_{\mathbf{j}}^T) \tag{7.65}$$

The proof is a trivial adaptation of those in (Sala & Arino, 2007b) (changing Q by Q') and, hence, omitted. The number of decision variables can be reduced by assuming $X_{(\mathbf{k},i,j)} = X^T_{(\mathbf{k},j,i)}$ with no loss of generality.

The above theorem must be applied recursively as, given a starting value of n, it provides sufficient conditions for the positivity of the n-dimensional sum expressed as an $(n-2)$-dimensional one. Hence, repeated application of the theorem is needed until: (a) 2-dimensional fuzzy summations are obtained (using Theorem 2 in (Liu & Zhang, 2003) as a last step) when starting from an even n; (b) in the odd-n case, one-dimensional fuzzy summations are obtained, stating then the condition that each of the elements in the sum must be positive. The reader is referred to (Sala & Arino, 2007b) for details.

Remark: Theorem 5 in (Fang et al., 2006) is the particular case of Theorem 7.7 for $n = 3$ and $\beta^{[l]}_{\mathbf{i}} = 0$; Theorem 2 in (Liu & Zhang, 2003) is Theorem 7.7 for $n = 2$. Note, for the first case, that (7.60), (7.61) and (7.62) correspond to the 1, 3 and 6 permutations of indexes iii, ijj and ijk, respectively, which appear in (7.65); then, (7.63) is the condition to prove positivity of the remaining 1-dimensional sum (7.64). The particularisation to (Liu & Zhang, 2003) is also straightforward (there are only 2 permutation of the index ij).

7.5 Examples

Example 7.1 *Let us consider the fuzzy system* (3.1) *where*

$$A_1 = \begin{pmatrix} 2 & -10 \\ 1 & 0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7.66}$$

$$A_2 = \begin{pmatrix} a & -10 \\ 1 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} b \\ 0 \end{pmatrix} \tag{7.67}$$

$$A_3 = \begin{pmatrix} -2 & -10 \\ 1 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 1 \\ 0.334 \end{pmatrix} \tag{7.68}$$

with the knowledge that $\mu_1\mu_3 \leq \beta_{13}$ (the actual value of β_{13} depends on the particular application; several of such values will be tested below).

A controller K_i is designed to place the closed-loop poles of $A_i - B_iK_i$ at -1 and -15, and the overall control law is obtained from (3.4). Note that solution for K_i is unique.

Setting Q_{ij} as dictated by (3.26) and (3.27), stability analysis is carried out: theorems 3.5 and 7.2 are compared by evaluating the feasibility of the associated LMI problems (with decision variables P, X_{ij}, R_{ij}) for varying values of a and b. If a and b yield a feasible LMI, stability of the resulting closed-loop system is proved.

The results appear in Figure 7.3, indicating that Theorem 7.2 clearly outperforms Theorem 3.5: the lower β_{ij} are, the more the difference between Theorems 3.5 and 7.2 is (*i.e.*, the larger the set of feasible values of a and b is with Theorem 7.2). Feasible values for the different conditions are:

- Theorem 3.5 and Theorem 7.2 ($\beta_{13} = 0.25$): [*]
- Theorem 7.2 ($\beta_{13} = 0.2$): [*,∘]
- Theorem 7.2 ($\beta_{13} = 0.05$): [*,∘,×]

Example 7.2 *Consider a nonlinear model* $\dot{x} = A(x)x + B(x)u$ *where*

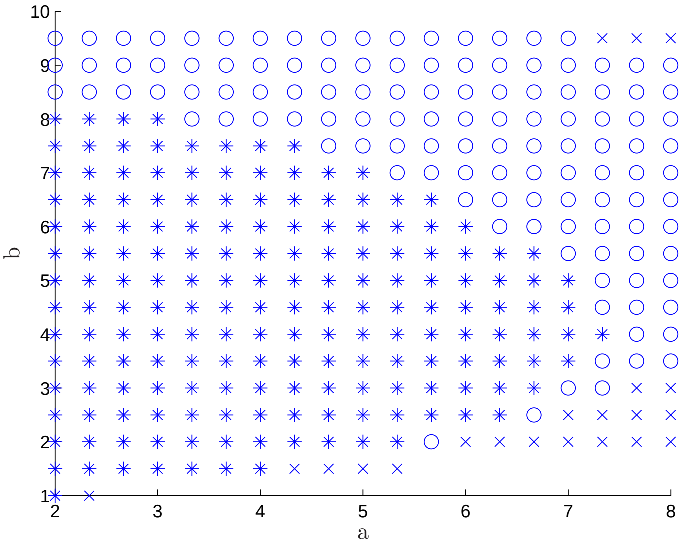


Figure 7.3: LMI feasibility as a function of a and b: $-\ast$: Feasible with Theorem 3.5 and Theorem 7.2 with $\beta_{13} = 0.25$, $-\circ$: additional feasible points obtained with Theorem 7.2 and $\beta_{13} = 0.20$, $-\times$: further feasible points obtained with Theorem 7.2 and $\beta_{13} = 0.05$, – unfeasible grid points have been left blank.

$$A(x) = 0.75x - 2.25sin(x) + sin(x)x - 2.5 \tag{7.69}$$
$$B(x) = 0.42x + 1.25sin(x) - 0.42sin(x)x - 0.25 \tag{7.70}$$

for which PDC fuzzy controllers are to be designed when $x \in [-\pi, \pi]$. In this case, x may be written as $x = \sum_{i=1}^{2} \nu_i p_i$, and $sin(x)$ as $sin(x) = \sum_{i=1}^{2} \eta_i q_i$, with:

$$x = \nu_1(x) \cdot \pi + \nu_2(x) \cdot (-\pi), \quad sin(x) = \eta_1(x) \cdot 1 + \eta_2(x) \cdot (-1)$$

where membership functions are $\nu_1 = \frac{1}{2\pi}(x + \pi)$, $\nu_2 = 1 - \mu_1$, $\eta_1 = \frac{1}{2}(sin(x) + 1)$, $\eta_2 = 1 - \eta_1$, resulting in

$$A(x) = 0.75 \sum_{i=1}^{2} \nu_i p_i - 2.25 \sum_{i=1}^{2} \eta_i q_i + (\sum_{i=1}^{2} \eta_i q_i)(\sum_{i=1}^{2} \nu_i p_i) - 2.5$$

$$A(x) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j (0.75p_i - 2.25q_j + p_i q_j - 2.5) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j a_{ij}$$

where

$$a_{11} = 0.748, a_{12} = -1.035, a_{21} = -10.247, a_{22} = 0.536$$

and similarly

$$B(x) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j (0.42p_i + 1.25q_j - 0.42p_i q_j - 0.25) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j b_{ij}$$

where:

$$b_{11} = 1, b_{12} = 1.139, b_{21} = 1, b_{22} = -4.139$$

Then, if $\mu_1 = \nu_1\eta_1$, $\mu_2 = \nu_1\eta_2$, $\mu_3 = \nu_2\eta_1$ and $\mu_4 = \nu_2\eta_2$ were defined, a fuzzy TS with four models:

$$\dot{x} = \sum_{i=1}^{4} \mu_i (a_i x + b_i u) \tag{7.71}$$
$$a_1 = 0.748, a_2 = -1.035, a_3 = -10.247, a_4 = 0.536 \tag{7.72}$$
$$b_1 = 1, b_2 = 1,139, b_3 = 1, b_4 = -4.139 \tag{7.73}$$

will exactly describe the nonlinear system under analysis for $x \in [-\pi, \pi]$. The reader is referred to Chapter 9 and (Tanaka & Wang, 2001) for more examples of this cartesian-product nonlinear modelling methodology.

If Theorem 3.5 is applied with the obtained set of four models and Q_{ij} given by (3.33), no feasible stabilizing PDC state-feedback controller $u = -\sum_{i=1}^{4} \mu_i K_i x$ can be found by the LMIs.

However, note that:

$$\mu_1\mu_4 = \mu_3\mu_2 = \nu_1\nu_2\eta_1\eta_2 \leq 0.25^2$$

as $\nu_1\nu_2 \leq 0.25$ and $\eta_1\eta_2 \leq 0.25$. Hence, Theorem 3 may be used with $\beta_{14} = \beta_{23} = 0.0625$. In this case, Theorem 3 shows that there exists a feasible stabilizing PDC controller for the nonlinear system expressed as a cartesian TS one which achieves a decay rate performance of 0.1369.

Regarding $\mathscr{H}_\infty$ performance, the model $\dot{x} = A(x)x + B(x)u + w$, $y = x$ has been tested with conditions (3.43), searching for the minimum γ. Without the relaxations considered in Theorem 3 the problem was, evidently, unfeasible from the considerations above. With the relaxations arising from β_{14} and β_{23}, a value of $\gamma_{min} = 7.3002$ was obtained as the LMI-guaranteed $\mathscr{H}_\infty$ bound.

Example 7.3 *Consider the system* (3.1), *with* $r = 3$ *and*

$$A_1 = \begin{pmatrix} -0.74 & 0.61 & 0.87 \\ 0.39 & -0.26 & 0.56 \\ 0.99 & 0.05 & -0.16 \end{pmatrix} \quad B_1 = \begin{pmatrix} 0.99 & 0.65 \\ 0.2 & 0.87 \\ 0.76 & 0.12 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.69 & 0.07 & 0.35 \\ 0.48 & 0.86 & 0.37 \\ 0.1 & 0.31 & 0.3 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0.96 & 0.36 \\ 0.76 & 0.17 \\ 0.04 & 0.20 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0.28 & 0.39 & 0.58 \\ 0.32 & 0.13 & 0.20 \\ 0.68 & 0.34 & 0.19 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0.32 & 0.02 \\ 0.72 & 0.15 \\ 0.29 & 0.06 \end{pmatrix}$$

The fastest decay rate[3] with a state feedback PDC law provable by Theorem 2 in (Liu & Zhang, 2003), using the Q_{ij} in (3.33), is $\alpha = 0.51$.

Now, the procedures in this chapter will be applied to achieve improved decay rates when some knowledge about the membership function shape is available (Figure 7.4).

[3]Note that the chosen performance measure has been decay rate, for simplicity, but other features such as robustness margins or $\mathscr{H}_\infty$ bounds may be tested by selecting a different Q_{ij}, as discussed in Section 3.5.

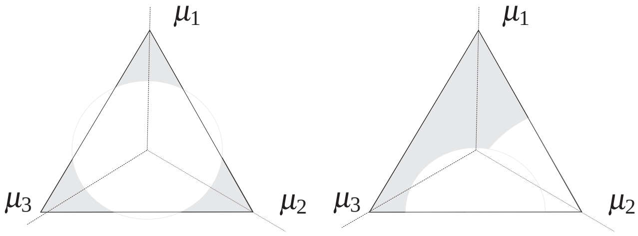


Figure 7.4: Example 1: (left) case 1, (right) case 2. Shaded area denotes possible values of membership functions inside the triangle $\mu_1 + \mu_2 + \mu_3 = 1$, $\mu_i \geq 0$. Isometric projection.

Case 1. Assume that, for a particular system, the membership vector *does not* lie inside a sphere centered at the origin ($c_i = 0$ in (7.39)) with radius δ (Figure 7.4-left), and that fact is known to the designer to take advantage of it.

Then, conditions (7.43) and (7.44) in Theorem 7.4 result in:

$$X_{ii} \leq Q_{ii} + (\delta^2 - 1)R \tag{7.74}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + 2\delta^2 R \quad i < j \tag{7.75}$$

Note that, for $0 \leq \delta \leq 1$, the conditions are less conservative as δ increases, because there is a larger addition to the off-diagonal terms (proportional to δ) and a smaller subtraction to the diagonal ones (proportional to $1 - \delta$). In fact, the extreme case $\delta = 1$ indicates that only the canonical vertices $(\mu_1, \mu_2, \mu_3) \in \{(1,0,0), (0,1,0), (0,0,1)\}$ are possible values of memberships (corresponding, for instance, to a switching linear system). In that extreme case, conditions (7.74) and (7.75) result in:

$$X_{ii} \leq Q_{ii} \tag{7.76}$$

$$X_{ij} + X_{ji} \leq Q_{ij} + Q_{ji} + 2R \quad i < j \tag{7.77}$$

Hence, with a large enough R, $X_{ij} = 0$ for $i \neq j$ can be made to be a solution of (7.77), so (3.23) holds if $Q_{ii} > 0$, which is a well known result in "crisp" switching linear systems.

Let us check different cases for δ with the above system. In particular, values of $\delta > 1/\sqrt{2}$ produce faster decay rates than the case with arbitrary memberships: for $\delta = 0.72$ the decay rate is $\alpha = 0.96$; for $\delta = 1$ it is $\alpha = 3.4$.

Case 2. Using expressions (7.43)-(7.44) as before, the knowledge that the membership vector lies outside of a sphere with center (0, 0.5, 0.5) and radius 0.4 is cast into the LMIs. The result is a decay $\alpha = 0.62$. If a sphere with center (0,0,1) and radius 0.49 is used, the achievable decay is $\alpha = 0.56$. With both circles and two relaxation variables R_1, R_2, the achievable decay rate is $\alpha = 0.66$.

Case 3: Multi-objective design (dual specifications). Consider the problem of achieving different levels of specifications in different regions of the working space[4]. For instance, only stability may suffice for certain infrequent cases whereas a faster decay may be desired near a particular operation point. The operation point will be assumed to be given by a particular condition on the membership functions (stating that the membership vector μ is inside or outside a region with a polynomial boundary).

Under these assumptions, the LMI conditions for mere stability, stated for any unrestricted membership shape, may be adjoined with the decay rate ones incorporating membership shape information.

For instance, $\alpha = 1.11$ is the fastest decay when the distance of the membership vector to the point $(1,0,0)$ is smaller than 0.75 (*i.e.*, conditions on the membership vector being *inside* a sphere are setup), simultaneously ensuring overall stability via a shared Lyapunov function: conditions in Theorem 7.4 with Q_{ij} in (3.33) are stated with $\alpha = 1.11$ and suitable β_{ij} (obtained from (7.43) and (7.44)); additionally, conditions in (Liu & Zhang, 2003, Theorem 2) using $\alpha = 0$ are also stated, with different decision variables X_{ij}. The LMI solver is able to find a feasible solution for both sets of constraints simultaneously; when $\alpha > 1.11$ on the first set of (shape-dependent) restrictions, the problem renders

[4]The above multi-objective design is different, and complementary, to the usual approach in literature of achieving different sorts of performance bounds on *all* the state space (mixed $\mathscr{H}_2/\mathscr{H}_\infty$, $\mathscr{H}_\infty$ plus decay rate, etc. (Gahinet et al., 1995; Li et al., 1999)): the same performance type but with different level in different regions is suggested here. Note also that some definitions are needed in order to rigorously define the meaning of "local" decay rate in terms of basins of attraction and Lyapunov level sets. The definition of, say, a "local" $\mathscr{H}_\infty$ norm would also be cumbersome. These issues are, however, omitted for brevity as they are not the main objective of the chapter. In the example in consideration, in a set of premise variables Ω, a "local" decay rate α will be said to have been proved when a Lyapunov function is found fulfilling $\dot{V}(x) \leq -2\alpha V(x)$ for all $z \in \Omega$ (indeed, z may include some or all of the components of x, as usual in TS modelling (Tanaka & Wang, 2001)).

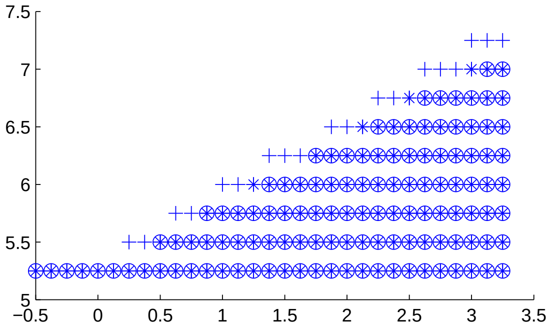


Figure 7.5: Feasible points (example 2) under different restrictions with Theorem 7.7, $n=3$. '$\otimes$': $\beta_{ijk}=0$; '*': $\mu_1\mu_2\mu_3 \leq 0.0004$; '+' $\mu_1\mu_2\mu_3=0$. Horizontal axis denotes parameter a, vertical axis denotes parameter b.

unfeasible.

Example 7.4 *Consider the system:*

$$A_1 = \begin{pmatrix} 1.59 & -7.29 \\ 0.01 & 0 \end{pmatrix} \quad B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.02 & -4.64 \\ 0.35 & 0.21 \end{pmatrix} \quad B_2 = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -a & -4.33 \\ 0 & 0.05 \end{pmatrix} \quad B_3 = \begin{pmatrix} -b+6 \\ -1 \end{pmatrix}$$

A stabilising PDC controller is to be designed. The ranges of values of parameters a and b yielding a feasible LMI solution are compared following several methods, for $a \in [-0.5, 3.25]$, $b \in [5.25, 7.25]$.

The procedure in (Liu & Zhang, 2003, Theorem 2) does not yield any feasible value in this parameter range.

Even with no restriction ($\beta_{\mathbf{i}}=0$), Theorem 7.6 (*i.e.*, (Fang et al., 2006)), produces feasible stabilising regulators for the values of a and b indicated by the "cart wheel" symbol in Figure 7.5.

When the restriction $\mu_1\mu_2\mu_3 \leq 0.0004$ is enforced, a few more feasible points appear (indicated by a star) using Theorem 7.6. The restriction

$\mu_1\mu_2\mu_3 = 0$ produces a larger set of feasible points; the + sign on the figure pinpoints those combinations of parameter values yielding feasible controllers only under the last restriction.

In all cases, the LMI solver was Matlab LMI Toolbox with default options.

7.6 Conclusions

This chapter shows how to relax stability and performance conditions for fuzzy control models with knowledge of membership function overlap; two theorems relaxing some LMI stability and performance conditions in (Tanaka & Wang, 2001; Kim & Lee, 2000; Liu & Zhang, 2003) have been presented. The conditions consider a set of known bounds in μ_i and $\mu_i\mu_j$, which generalize the relaxations when $\mu_i\mu_j = 0$ previously reported in literature for non-overlapping fuzzy sets. As a result, more freedom in guaranteeing control requirements is available. A numerical example shows how the feasibility regions for typical fuzzy control problems become larger as the overlap bounds become smaller, showing the improvements over previous work.

The proposed technique may prove useful in fuzzy control applications. In fuzzy PDC control techniques, membership functions are assumed to be known so the required bounds may be easily obtained. Importantly this technique is particularly well suited to widespread cartesian-product Takagi-Sugeno modelling of nonlinear systems.

Chapter 8

Stability Analysis with uncertain membership functions

8.1 Introduction

Fuzzy control has reached maturity and acceptance nowadays via a formalisation of the performance requirements and controller design techniques. In particular, there is a vast amount of literature on control design for Takagi-Sugeno (TS) (Takagi & Sugeno, 1985) fuzzy systems via linear matrix inequalities (LMI) (Boyd et al., 1994; Tanaka & Wang, 2001; Liu & Zhang, 2003). There may be other (possibilistic) interpretations of fuzziness in a control context (Bondia, Sala, Pico, & Sainz, 2006). The reader is referred to (Sala et al., 2005; Feng, 2006) for a review of the current trends and open issues in fuzzy modelling, identification and control.

The majority of works on fuzzy control for TS models assume the parallel distributed compensation (PDC) paradigm (Tanaka & Wang, 2001), *i.e.*, the membership functions of the controller, say η_i, are the same as the ones from the process, say μ_i. Furthermore, the proposed stability and performance conditions are shape-independent, *i.e.*, valid for any non-negative membership function setup.

Recent contributions in the non-PDC case are (Lam & Leung, 2005, 2007; Guerra & Vermeiren, 2004). In particular, in (Lam & Leung, 2005), LMI stability conditions were given for non-PDC fuzzy systems with uncertain degrees of membership expressed as a multiplicative uncertainty inequality $\rho_i^m \mu_i \leq \eta_i \leq \rho_i^M \mu_i$. Lam and Leung's conditions are shape-dependent, in the sense that they achieve a reduction of conservativeness by setting up conditions which are only valid for membership functions having a constrained shape. In the same spirit, (Sala & Ariño, 2007a) presents some shape-dependent conditions for the PDC case.

The main objective of this chapter is presenting new shape-dependent LMI conditions to design controllers for Takagi-Sugeno fuzzy systems with uncertain memberships. The allowed uncertainty description is more general than that in (Lam & Leung, 2005), which did consider only multiplicative uncertainty.

The structure of the chapter is as follows. The next section will describe the fuzzy systems and closed-loop equations to be discussed. Section 8.3 presents the main result which extends the uncertainty descriptions in literature. Section 8.4 applies it to particular cases of additive and multiplicative uncertainty. Section 8.5 will show numerical examples illustrating the possibilities of the approach. A conclusion section closes the chapter.

8.2 Problem statement

Consider a Takagi-Sugeno fuzzy system (3.1) of order n with r rules:

$$\dot{x} = \sum_{i=1}^{r} \mu_i(z)(A_i x + B_i u) \qquad \sum_{i=1}^{r} \mu_i(z) = 1,\ \mu_i(z) \geq 0 \tag{8.1}$$

The above system will be controlled via a state-feedback fuzzy controller:

$$u = -\sum_{i=1}^{r} \eta_i(z') K_i x \qquad \sum_{i=1}^{r} \eta_i(z') = 1,\ \eta_i(z') \geq 0 \tag{8.2}$$

where z' denotes measurable scheduling variables. The controller yields a closed-loop (Lam & Leung, 2005):

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(z)\eta_j(z')(A_i - B_i K_j)x \tag{8.3}$$

On the following μ_i, η_i will be used as shorthand for $\mu_i(z)$, $\eta_i(z')$, respectively.

Conditions for stability and performance of the above closed-loop system may be cast as positivity of fuzzy summations, similar that ones in (3.23), in the form:

$$\psi^T \Theta \psi = \psi^T \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \eta_j Q_{ij} \right) \psi \geq 0 \quad \forall \psi \neq 0 \tag{8.4}$$

where Q_{ij} are symmetric $\mathbb{R}^{n\times n}$ matrices, possibly including unknown decision variables to be found via optimization algorithms (usually LMI (Boyd et al., 1994; Tanaka & Wang, 2001)). For instance, stability is proved if (8.4) holds with $\psi = x$ and (Lam & Leung, 2005):

$$Q_{ij} = -\left((A_i - B_i K_j)P + P(A_i - B_i K_j)^T\right) \tag{8.5}$$

Apart from mere stability, if performance is sought, there are also LMI conditions (Tanaka & Wang, 2001; Tuan et al., 2001) resulting in different expressions[1] for Q_{ij} above. In the same way, different expressions for Q_{ij} may be proposed to deal with discrete systems (Tanaka & Wang, 2001; Guerra & Vermeiren, 2004). The reader is referred to the cited references and the review articles (Feng, 2006; Sala et al., 2005) for details on the different options for Q_{ij}.

In this context, there are two extreme situations:

- *Parallel Distributed Compensation (PDC).* If the membership functions μ_i are perfectly known and z is measurable, a usual approach is setting $\eta_i = \mu_i$; this approach is the widely-known parallel distributed compensation (3.4), for which the LMI framework, introduced by (Tanaka & Wang, 2001), is nowadays well developed. Widely-used positivity conditions for (8.4) in the PDC case are the adaptation of (Liu & Zhang, 2003, Theorem 2).

- *Shape-independent conditions.* If μ_i and η_j may be arbitrary, the only possibility for (8.4) to hold is enforcing the naive conditions

$$Q_{ij} > 0 \quad \forall\ i, j \tag{8.7}$$

[1] Note that most of the cited literature considers Q_{ij} for cases where $\eta_i = \mu_i$, *i.e.*, applied to

$$\psi^T \Theta \psi = \psi^T \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j Q_{ij} \right) \psi \geq 0 \quad \forall \psi \neq 0 \tag{8.6}$$

However, such Q_{ij} also apply to (8.4) without modification.

The difference between the PDC and non-PDC cases are in the LMIs needed to prove (8.4), which are different to those needed for (8.6). Of course, all LMI's proving (8.4) –such as the ones in this chapter– prove as well (8.6), but they may be *very* conservative for the PDC case (8.6).

as, for instance, say $\mu_3 = \eta_5 = 1$ (the rest being zero) involves $\Theta = Q_{35}$ in (8.4) and the numbers 3, 5 may be arbitrarily replaced by any i or j.

However, there are intermediate cases where the membership functions μ_i are not perfectly known, hence $\eta_i \neq \mu_i$, yielding non-PDC setups but, on the other hand, some knowledge on them is available which might be used for stating conditions which are less conservative than (8.7). In particular, (Lam & Leung, 2005) states conditions which guarantee closed-loop stability[2] of (8.3) when:

$$\rho_i^m \leq \frac{\mu_i}{\eta_i} \leq \rho_i^M \tag{8.8}$$

given known values of the bounds ρ_i^m and ρ_i^M.

Lemma 8.1 ***(Lam & Leung, 2005)*** *Expression* (8.4) *holds if there exists* $P > 0$ *and matrices* $X_{ij} = X_{ji} = X_{ij}^T$ *such that:*

$$\rho_i^M Q_{ii} - X_{ii} > 0, \quad \rho_i^m Q_{ii} - X_{ii} > 0 \tag{8.9}$$

$$\rho_j^M Q_{ij} + \rho_i^M Q_{ji} - 2X_{ij} > 0, \quad \rho_j^m Q_{ij} + \rho_i^m Q_{ji} - 2X_{ij} > 0 \tag{8.10}$$

$$\rho_j^m Q_{ij} + \rho_i^M Q_{ji} - 2X_{ij} > 0, \quad \rho_j^M Q_{ij} + \rho_i^m Q_{ji} - 2X_{ij} > 0 \tag{8.11}$$

$$\begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} > 0 \tag{8.12}$$

The developments in next section will present less conservative conditions for the above case, as numerical examples in Section 8.5 will later show. The case will be denoted as multiplicative uncertainty, being a particularization of a more general uncertainty description to be discussed in next section.

8.3 Main Result

Consider a set of p restrictions on the shape of the membership functions of the plant, μ_i, and controller, η_i, given by:

$$\mathbf{c}_k^T \eta + \mathbf{a}_k^T \mu + b_k \leq 0 \;\; k = 1 \dots p \tag{8.13}$$

[2]The cited work discussed only the particular case of Q_{ij} in (8.5) but it is, trivially, generalisable to any other performance-related expression of Q_{ij}.

where η and μ denote the membership functions arranged as a column vector, $\mathbf{c}_k$, $\mathbf{a}_k$ are also column vectors and b_k are scalars. Notation a_{ik} and c_{ik} denote the i-th component of vectors $\mathbf{a}_k$ and $\mathbf{c}_k$, respectively. For instance, the restriction $\mu_2+\mu_1 \le 2\eta_1+0.05$, in a 3-rule fuzzy system, is trivially expressed in the form (8.13) with $\mathbf{c}=(-2\ 0\ 0)^T$, $\mathbf{a}=(1\ 1\ 0)^T$, $b=-0.05$.

Theorem 8.1 *If* (8.13) *is known to hold, Expression* (8.4) *holds if there exist matrices* $X_{ij}=X_{ji}^T$ $i,j=1...2r$, *and symmetric definite positive matrices* R_{jk} *and* R^*_{jk} *such that for all* $j=1...r$, $k=1\dots p$

$$\sum_{k=1}^{p}\left(a_{jk}R_{ik}+a_{ik}R_{jk}\right)\ge X_{ij}+X_{ji} \tag{8.14}$$

$$Q_{ij}+\sum_{k=1}^{p}\left(c_{jk}R_{ik}+a_{ik}R^*_{jk}+b_k(R_{ik}+R^*_{jk})\right)\ge X_{i(j+r)}+X_{(j+r)i} \tag{8.15}$$

$$\sum_{k=1}^{p}\left(c_{ik}R^*_{jk}+c_{jk}R^*_{ik}\right)\ge X_{(i+r)(j+r)}+X_{(j+r)(i+r)} \tag{8.16}$$

$$\begin{pmatrix} X_{11} & \dots & X_{1(2r)} \\ \vdots & \ddots & \vdots \\ X_{(2r)1} & \dots & X_{(2r)(2r)} \end{pmatrix} > 0 \tag{8.17}$$

If Q_{ij} *are linear in some matrix decision variables, then Theorem 8.1 provides LMI conditions for checking* (8.4) *in a shape-dependent framework.*

Proof: Expression (8.13) may be written as

$$\sum_{i=1}^{r} c_{ik}\eta_i+a_{ik}\mu_i+b_k\le 0 \quad k=1\dots p \tag{8.18}$$

Consider now, for a particular fixed k, the matrix $\Gamma_k=\sum_{j=1}^{r}(\mu_j R_{jk}+\eta_j R^*_{jk})$. Evidently, $\Gamma_k\ge 0$ because it is a sum with positive coefficients of positive definite matrices. For a particular k, multiplying (8.18) above by Γ_k we get:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}(c_{ik}R_{jk}\eta_i\mu_j+c_{ik}R^*_{jk}\eta_i\eta_j+a_{ik}R_{jk}\mu_i\mu_j+a_{ik}R^*_{jk}\mu_i\eta_j)+ \\ +b_k\sum_{j=1}^{r}(\mu_j R_{jk}+\eta_j R^*_{jk})\le 0 \tag{8.19}$$

Subsequently, by using the equalities $\sum_{i=1}^{r}\mu_i = 1$ and $\sum_{i=1}^{r}\eta_i = 1$ in

$$b_k\Gamma_k = b_k\sum_{j=1}^{r}(\sum_{i=1}^{r}\eta_i\mu_j R_{jk} + \sum_{i=1}^{r}\mu_i\eta_j R^*_{jk})$$

we get a negative-semi-definite matrix to be denoted as H_k given by:

$$H_k = \sum_{i=1}^{r}\sum_{j=1}^{r}(c_{ik}R_{jk}\eta_i\mu_j + c_{ik}R^*_{jk}\eta_i\eta_j + a_{ik}R_{jk}\mu_i\mu_j + \\ + a_{ik}R^*_{jk}\mu_i\eta_j + b_k(\eta_i\mu_j R_{jk} + \mu_i\eta_j R^*_{jk})) \leq 0 \quad (8.20)$$

As (8.20) holds for each k, denoting by $H = \sum_{k=1}^{p} H_k$, evidently $H \leq 0$, *i.e.*:

$$H = \sum_{k=1}^{p}\sum_{i=1}^{r}\sum_{j=1}^{r}(c_{ik}R_{jk}\eta_i\mu_j + c_{ik}R^*_{jk}\eta_i\eta_j + a_{ik}R_{jk}\mu_i\mu_j + \\ + a_{ik}R^*_{jk}\mu_i\eta_j + b_k(\eta_i\mu_j R_{jk} + \mu_i\eta_j R^*_{jk})) \leq 0 \quad (8.21)$$

Taking H above and Θ from (8.4), it is evident that if $\Theta + H > 0$ can be proved, then $\Theta > 0$. Then, conveniently grouping terms:

$$\Theta + H = \sum_{i=1}^{r}\sum_{j=1}^{r}\left(\mu_i\mu_j\sum_{k=1}^{p}a_{ik}R_{jk} + \eta_i\eta_j\sum_{k=1}^{p}c_{ik}R^*_{jk} + \right. \\ \left. + \mu_i\eta_j\left(Q_{ij} + \sum_{k=1}^{p}c_{jk}R_{ik} + a_{ik}R^*_{jk} + b_k(R_{ik} + R^*_{jk})\right)\right) = \\ = \sum_{i=1}^{r}\left(\mu_i^2\sum_{k=1}^{p}a_{ik}R_{ik} + \eta_i^2\sum_{k=1}^{p}c_{ik}R^*_{ik}\right) + \sum_{i<j\leq r}\left(\mu_i\mu_j\sum_{k=1}^{p}a_{ik}R_{jk} + a_{jk}R_{ik} + \right. \\ \left. + \eta_i\eta_j\sum_{k=1}^{p}c_{ik}R^*_{jk} + c_{jk}R^*_{ik}\right) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\eta_j(Q_{ij} + \\ + \sum_{k=1}^{r}c_{jk}R_{ik} + a_{ik}R^*_{jk} + b_k(R_{ik} + R^*_{jk})) \quad (8.22)$$

Consider now the variables $X_{ij} = X_{ji}^T$ $i, j = 1\ldots r$, which fulfill (8.14)-(8.16). By suitably grouping terms, and associating (8.14) to the terms where $\mu_i\mu_j$ appears, (8.15) to those with $\mu_i\eta_j$ and (8.16) to those with

$\eta_i\eta_j$, we have[3]:

$$\Theta + H \geq \sum_{i=1}^{r}\left(\mu_i^2 X_{ii} + \eta_i^2 X_{ii}\right) + \sum_{i=1}^{r}\sum_{i<j\leq r}(\mu_i\mu_j(X_{ij}+X_{ji})+ \\ + \eta_i\eta_j(X_{(i+r)(j+r)}+X_{(j+r)(i+r)})) + \sum_{i=1}^{r}\sum_{j=1}^{r}\left(\mu_i\eta_j(X_{i(j+r)}+X_{(j+r)i})\right) \quad (8.23)$$

By taking the same ψ as in (8.4), and defining $\xi = [\mu_1\psi^T \ \dots \ \mu_r\psi^T \ \eta_1\psi^T \ \dots \ \eta_r\psi^T]^T$, the terms in the right-hand side of (8.23) may be expressed as follows:

$$\psi^T(\Theta+H)\psi \geq \xi^T \begin{pmatrix} X_{11} & \dots & X_{1(2r)} \\ \vdots & \ddots & \vdots \\ X_{(2r)1} & \dots & X_{(2r)(2r)} \end{pmatrix} \xi \quad (8.24)$$

Hence, given (8.24), if (8.17) holds, $\psi^T(\Theta+H)\psi > 0$ for $\psi \neq 0$ and, hence, (8.4) holds. ■

8.4 Particular cases

Let us now consider some particular cases of Theorem 8.1.

Multiplicative uncertainty Consider now the multiplicative uncertainty case, also discussed in (Lam & Leung, 2005):

$$\rho_i^m \leq \frac{\eta_i}{\mu_i} \leq \rho_i^M \quad (8.25)$$

Corollary 8.1 *If* (8.25) *is known to hold, Expression* (8.4) *holds if there exist matrices* $X_{ij} = X_{ji}^T$ $i,j = 1...2r$, *and symmetric definite positive*

[3]Note that (8.14) implies $\sum_{k=1}^{p} a_{ik}R_{ik} \geq X_{ii}$, and an analogous consideration may be made with (8.16): the diagonal terms μ_i^2, η_i^2 need not be explicitly written in the theorem statement.

matrices R_{ij}*,* N_{ji}*,* R^*_{ij} *and* N^*_{ji} *such that for all* $i, j = 1...r$*:*

$$R_{ij}\rho_j^m - N_{ij}\rho_j^M + R_{ji}\rho_i^m - N_{ji}\rho_i^M \geq X_{ij} + X_{ji} \tag{8.26}$$

$$Q_{ij} - (R_{ij} - N_{ij}) - (R^*_{ij} - N^*_{ij}) \geq X_{i(j+r)} + X_{(j+r)i} \tag{8.27}$$

$$\frac{R^*_{ij}}{\rho_i^M} - \frac{N^*_{ij}}{\rho_i^m} + \frac{R^*_{ji}}{\rho_j^M} - \frac{N^*_{ji}}{\rho_j^m} \geq X_{(i+r)(j+r)} + X_{(j+r)(i+r)} \tag{8.28}$$

$$\begin{pmatrix} X_{11} & \dots & X_{1(2r)} \\ \vdots & \ddots & \vdots \\ X_{(2r)1} & \dots & X_{(2r)(2r)} \end{pmatrix} > 0 \tag{8.29}$$

Proof: The uncertainty description can be expressed as:

$$\eta_k - \rho_k^M \mu_k \leq 0 \quad -\eta_k + \rho_k^m \mu_k \leq 0 \tag{8.30}$$

Hence, the theorem will be proved by using Theorem 8.1 with $2r$ constraints, divided in two groups (both with $b_k = 0$):

- One with $a_{kk} = -\rho_k^M$, $a_{ik} = 0$ for $i \neq k$ and $c_{kk} = +1$, $c_{ik} = 0$ for $i \neq k$; Theorem 8.1 will be applied using M_{ik} and M^*_{ik} as relaxation variables,
- Another group (consider $a'_k = a_{k+r}$, $c'_k = c_{k+r}$) with $a'_{kk} = \rho_k^m$, $a'_{ik} = 0$ for $i \neq k$ and $c'_{kk} = -1$, $c'_{ik} = 0$ for $i \neq k$; Theorem 8.1 will be applied using T_{ik} and T^*_{ik} as relaxation variables.

Note that (8.14)–(8.16) in this case reduce to

$$a_{jj}M_{ij} + a_{ii}M_{ji} + a'_{jj}T_{ij} + a'_{ii}T_{ji} \geq X_{ij} + X_{ji} \tag{8.31}$$

$$Q_{ij} + \left(c_{jj}M_{ij} + a_{ii}M^*_{ji} + c'_{jj}T_{ij} + a'_{ii}T^*_{ji}\right) \geq X_{i(j+r)} + X_{(j+r)i} \tag{8.32}$$

$$c_{ii}M^*_{ji} + c_{jj}M^*_{ij} + c'_{ii}T^*_{ji} + c'_{jj}T^*_{ij} \geq X_{(i+r)(j+r)} + X_{(j+r)(i+r)} \tag{8.33}$$

because most of the a_{ik}, c_{ik}, a'_{ik}, c'_{ik} are zero.

The conditions for the theorem being proved arise immediately once the values of a_{ik}, c_{ik}, a'_{ik}, c'_{ik} are substituted above and the following changes of variable are made:

$$M_{ij} = N_{ij} \quad T_{ij} = R_{ij} \quad \rho_i^M M^*_{ij} = R^*_{ij} \quad \rho_i^m T^*_{ij} = N^*_{ij}$$

■

Additive uncertainty Consider a set of known additive bounds on the membership functions, δ_k, so that, given (8.4), it is known that:

$$|\mu_k - \eta_k| < \delta_k \quad k = 1 \dots r \tag{8.34}$$

Corollary 8.2 *If the membership functions satisfy* (8.34)*, expression* (8.4) *holds if there exist matrices* R_{ij}*,* N_{ij}*,* $X_{ij} = X_{ji}^T$ *and* $X_{i(j+r)} = X_{(j+r)i}^T$ $i, j = 1...r$*, such that:*

$$M_{ij} = R_{ij} - N_{ij}, \quad M_{ij}^{+} = R_{ij} + N_{ij} \tag{8.35}$$

$$M_{ij} + M_{ji} \geq X_{ij} + X_{ji} \tag{8.36}$$

$$Q_{ij} - 2M_{ij} - \sum_{k=1}^{r} \delta_k (M_{ik}^{+} + M_{kj}^{+}) \geq X_{i(j+r)} + X_{(j+r)i} \tag{8.37}$$

for all $i = 1 \dots, r,\ j = 1 \dots, r$ *and*

$$Y_{11} = \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} \quad Y_{12} = \begin{pmatrix} X_{1(r+1)} & \dots & X_{1(2r)} \\ \vdots & \ddots & \vdots \\ X_{r(r+1)} & \dots & X_{r(2r)} \end{pmatrix} \tag{8.38}$$

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{11} \end{pmatrix} > 0 \tag{8.39}$$

Proof: The uncertainty description can be expressed as:

$$\eta_k - \mu_k - \delta_k \leq 0 \quad -\eta_k + \mu_k - \delta_k \leq 0 \tag{8.40}$$

Hence, the theorem will be proved by using Theorem 8.1 with $2r$ constraints, divided in two groups:

- One with $a_{kk} = 1$, $a_{ik} = 0$ for $i \neq k$ and $c_{kk} = -1$, $c_{ik} = 0$ for $i \neq k$; Theorem 8.1 will be applied using R_{ik} and R_{ik}^* as relaxation variables,
- Another one with $a_{kk} = -1$, $a_{ik} = 0$ for $i \neq k$ and $c_{kk} = +1$, $c_{ik} = 0$ for $i \neq k$; Theorem 8.1 will be applied using N_{ik} and N_{ik}^* as relaxation variables.

The conditions for the theorem being proved arise almost immediately once those for Theorem 8.1 are written as (8.31)–(8.33) and the following equalities are enforced: $R^*_{ij} = N_{ji}$, $N^*_{ij} = R_{ji}$. Details are omitted for brevity. ■

The following two lemmas show that: (a) the PDC case is recovered from the above corollaries under no uncertainty and, (b) as expected, the conditions proposed with the additive or multiplicative uncertainty bounds are less conservative than the trivial ones $Q_{ij} > 0$.

Lemma 8.2 *When the membership error bounds δ_i is equal to zero, or the multiplicative bounds are equal to one (*i.e.*, $\mu_i = \eta_i$), a feasible set of variables for corollaries 8.1 and 8.2 may be obtained if (Liu & Zhang, 2003, Theorem 2) (which applies to the PDC case) is feasible.*

Proof: Note that, when $\delta = 0$ and $\rho_i^m = \rho_i^M = 1$, enforcing $R^*_{ij} = R_{ij}$, $N^*_{ij} = N_{ij}$, $X_{(i+r)(j+r)} = X_{ij}$ in corollary 8.1 leaves conditions (8.26)–(8.29) identical to (8.36)–(8.39) in corollary 8.2 so a unified analysis is possible, considering only corollary 8.2 in the sequel.

If $\delta_i = 0$ for all i, then (8.37) can be rewritten as

$$Q_{ij} - 2M_{ij} \geq X_{i(j+r)} + X_{(j+r)i} \tag{8.41}$$

Taking $M_{ij} = Q_{ij}/2$ and $X_{i(j+r)} = 0$, which fulfill (8.41), the inequality (8.36) gets converted into:

$$Q_{ij}/2 + Q_{ji}/2 \geq X_{ij} + X_{ji} \tag{8.42}$$

and the matrix Y_{12} is equal to 0. Finally (8.39) is

$$\begin{pmatrix} Y_{11} & 0 \\ 0 & Y_{11} \end{pmatrix} > 0 \tag{8.43}$$

equivalent to $Y_{11} > 0$.This condition and (8.42) are the ones in (Liu & Zhang, 2003, Theorem 2): if the latter is feasible, corollaries 8.1 and 8.2 without uncertainty will be feasible as well. ■

Lemma 8.3 *If $Q_{ij} > 0$ for all $i, j = 1 \ldots r$, then corollaries 8.1 and 8.2 are satisfied.*

Proof: Regarding corollary 8.2, take all $N_{ij} = 0$, and all $X_{(i+r)j} = X_{i(j+r)} = 0$. Then, take $R_{ij} = 0$ for $i \neq j$ and $R_{ii} = \varepsilon_i I$ for $i = 1 \dots r$, choosing a small enough $\varepsilon_i > 0$ so that $Q_{ii} \gg 2\varepsilon_i I$ (I denotes the identity matrix). Then $X_{ij} = 0$, $i \neq j$, fulfills (8.36) and (8.37), and for $i = j$ $X_{ii} = \varepsilon_i I$ fulfills (8.36). Finally all X_{ij} form a diagonal positive matrix that satisfies (8.39) if ε is small enough.

Corollary 8.1 is also satisfied with the same choice of N_{ij} and R_{ij} plus $N^*_{ij} = N_{ij}$, $R^*_{ij} = R_{ij}$. Details are analogous to those above for Corollary 8.2. ∎

8.5 Examples

Example 8.1 *Let us consider the system:*

$$\dot{x} = \sum_{i=1}^{3} \mu_i(x)(A_i x + B_i u) \tag{8.44}$$

$$A_1 = \begin{pmatrix} 0.39 & 0.85 & 0.48 \\ 0.81 & 0.010 & 0.34 \\ 0.51 & 0.28 & 0.078 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.1 & 0.58 \\ 0.016 & 0.32 \\ 0.80 & 0.58 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0.0089 & 0.35 & 0.96 \\ 0.76 & 0.54 & 0.38 \\ 0.14 & 0.85 & 0.25 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.031 & 0.036 \\ 0.87 & 0.53 \\ 0.75 & 0.78 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0.84 & 0.094 & 0.8 \\ 0.19 & 0.2 & 0.13 \\ 0.82 & 0.58 & 0.33 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0.054 & 0.16 \\ 0.21 & 0.84 \\ 0.47 & 0.64 \end{pmatrix}$$

A state-feedback fuzzy controller with the structure

$$u = -\sum_{j=1}^{2} \eta_j(x) F_j x$$

is proposed, where functions $\eta_j(x)$ are an approximation of $\mu_j(x)$ fulfilling (8.34), for a shared $\delta_k = \delta$. Several values of the uncertainty δ will be tested, ranging from $\delta = 0$ (which is the well-known PDC case $\eta_i = \mu_i$) to $\delta = 1$ (indicating absolute ignorance on the shape of μ_i).

Table 8.1: Decay rate α achievable as a function of the uncertainty δ

δ	0	0.05	0.1	0.15	0.2	0.25	0.3	...	1
α	0.36	0.35	0.33	0.31	0.30	0.29	0.28	...	0.28

The control objective will be finding the F_j maximizing the achievable quadratic decay rate α (roughly, analogous to the dominant pole of a linear system), by checking (8.4) with conditions from (Tanaka & Wang, 2001)

$$Q_{ij} = -A_iY - YA_i^T + B_iM_j + M_j^T B_i - 2\alpha Y \tag{8.45}$$

where Y is a positive-definite matrix related to a Lyapunov function and $M_j = F_jY$. The sufficient conditions provided in corollary 8.2 will be used, searching for the maximum value of α for which a feasible LMI solution exists (which is a generalized eigenvalue problem). The maximum α achieved for different values of δ appears in Table 8.1.

The results in the referred table show that the more precise the knowledge of μ is (lower δ), the faster decay rates can be achieved. The results for $\delta = 0$ are coincident with those (Liu & Zhang, 2003, Theorem 2) for the PDC case, as discussed in Lemma 8.2. Furthermore, the results for $\delta = 1$ are coincident with the ones obtained by using a non-fuzzy linear regulator $u = -Kx$ robustly stabilising a polytopic system (Boyd et al., 1994) via the LMI conditions:

$$Q_i = -A_iY - YA_i^T + B_iM + M^T B_i - 2\alpha Y > 0$$

Such conditions are, in fact, equivalent to the shape-independent ones $Q_{ij} > 0$ (indeed, conditions for M_1 in (8.45) are the same as those for M_2, etc. so there is no loss of generality by assuming $M_1 = M_2 = \cdots = M$).

In summary, with the methodology in this chapter, a smooth transition between a full-PDC fuzzy controller and a robust linear one is achieved: as uncertainty increases the feasible performance decreases. If the uncertainty is greater than 0.3, the performance of fuzzy and non-fuzzy controllers is the same.

Example 8.2 *Let us consider an example of DC motor controlling an inverted pendulum on a gear train (Kuschewski, Hui, & Zak, 1993):*

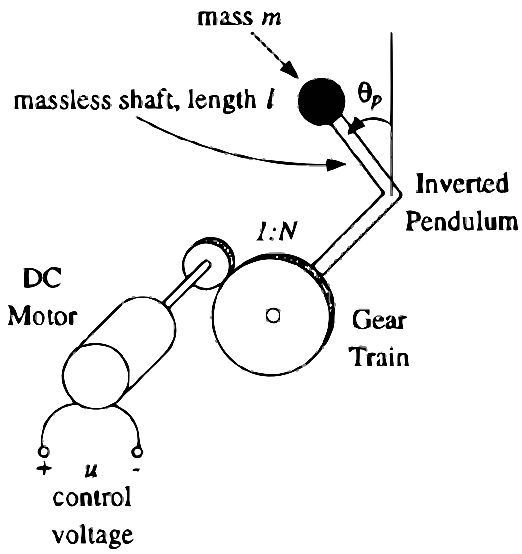


Figure 8.1: Inverted pendulum controlled by a DC motor

The state equations describing the plant are

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{g}{l}\sin x_1 + \frac{NK_m}{ml^2}x_3 \\ -\frac{K_bN}{L_a}x_2 - \frac{R_a}{L_a}x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{L_a} \end{pmatrix} u \tag{8.46}$$

where the state variables are: $x_1 = \theta_p$, $x_2 = \dot{\theta}_p$, x_3 the dc motor current and the input u is the Voltage of the dc motor supposed less than 100V. K_m is the motor torque constant, K_b is the back emf constant, and N is the gear ratio. Reasonable parameters for the plant are $g = 9.8\ m./s.^2$, $l = 1\ m.$, $m = 1\ Kg.$, $N = 10$, $K_m = 0.1\ Nm/A$, $K_b = 0.1\ Vs/rad$, $R_a = 1\ \Omega$ and $L_a = 100\ mH$. this parameters lead to

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ 9.8\sin x_1 + x_3 \\ -10x_2 - 10x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} u \tag{8.47}$$

Fuzzy modelling for the nonlinear system was done in (Tanaka & Sano, 1994; Kawamoto, 1996), following sector nonlinearity procedure described in Section 3.3. The fuzzy model is as follows

$$\dot{x} = \mu_1(A_1x + B_1u) + \mu_2(A_2x + B_2u) \tag{8.48}$$

Here,

$$x = (x_1\ x_2\ x_3)^T \tag{8.49}$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ 0 & -10 & -10 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \tag{8.50}$$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & -10 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \tag{8.51}$$

$$\tag{8.52}$$

the membership functions are defined as

$$\mu_1(x) = \begin{cases} \frac{\sin x_1}{x_1}, & x_1 \neq 0 \\ 1 & x_1 = 0 \end{cases} \tag{8.53}$$

$$\mu_2 = 1 - \mu_1 \tag{8.54}$$

This fuzzy model exactly represents the dynamics of the nonlinear mechanical system under $-\pi \leq x_1 \leq \pi$.
We design a disturbance rejection fuzzy controller. We can rewrite the system as:

$$\dot{x} = \sum_{i=1}^{2} \mu_i (A_i x + B_i u) + B_w w \tag{8.55}$$

$$y = x_1 \tag{8.56}$$

where $B_w = (1\ 1\ 0)^T$ is the direction of the disturbance w. Then positivity conditions of the closed loop system controlled by a PDC controller (8.2) can be formulated as

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \mu_1 \eta_j x^T Q_{ij} x \tag{8.57}$$

where Q_{ij} are the conditions for disturbance rejection obtained in Section 3.5. For a proper comparison, considerer the system without knowledge of the membership functions. Then the best controller is a Linear controller. In this case the best disturbance rejection coefficient is $\gamma_M = 0.31102$. On the other hand let us considerer $\mu_i = \eta_i$ then the performance conditions yield to Theorem 3.5 and the obtained coefficient is $\gamma_m = 0.28434$. So our system with some information on the η_i boundaries must be between γ_M and γ_m.

Considering the relative uncertainty (8.25)

$$\rho^m \leq \frac{\eta_i}{\mu_i} \leq \rho^M \tag{8.58}$$

with $\rho^m = 1/1.5$ and $\rho^M = 1.5$ Then the best disturbance rejection coefficient obtained for Lemma 8.1 is $\gamma = 0.291$ and for Corollary 8.1 is $\gamma = 0.289$. So the result of Corollary 8.1 improves the result of Corollary Lemma 8.1.

Example 8.3 *Let us now discuss the same example as in (Lam & Leung, 2005) regarding multiplicative uncertainty. Consider a TS fuzzy system with 2 rules, with matrices*

$$A_1 = \begin{pmatrix} 2 & -10 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a & -10 \\ 1 & 1 \end{pmatrix}$$
$$B_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Analogously to (Lam & Leung, 2005), a two-rule state-feedback fuzzy controller is built by designing the F_i, $i = 1, 2$, by pole-placement so that the closed-loop poles of $A_i - B_i F_i$ are at -1 and -15 (unique solution). Then, stability of the overall closed-loop system (8.3) is tested for different values of a and b, and different values of ρ_i^M and ρ_i^m in (8.25).

Figure 8.2 shows the values of a and b for which the closed loop can be proved stable, for different uncertainty levels: the left plot presents the results with Corollary 8.1, whereas the right one presents the results with Lemma 8.1; the tested uncertainty values were $\rho_i^M = \varepsilon$, $\rho_i^m = 1/\varepsilon$ for ε taking values in $\{2, 1.5, 1.2, 1.1, 1\}$. All points feasible for a particular value of ε were as well feasible for lower values of it.

Clearly, Corollary 8.1 in this work achieves better results than (Lam & Leung, 2005) in all tested uncertain cases, *i.e.*, it finds a larger set of values for a and b yielding a stable closed loop. As expected, for no uncertainty in memberships ($\rho_i^M = \rho_i^m = 1$) the PDC case is recovered in both cases (denoted with $\bullet$ in the figure), with results coincident to those from Theorem 3.5.

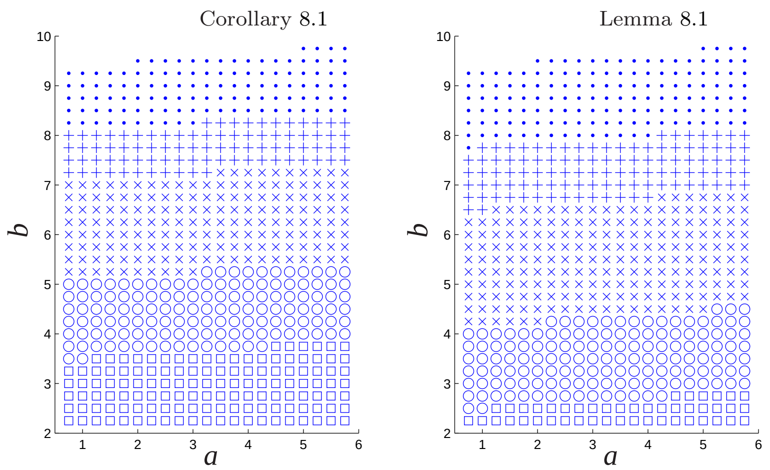


Figure 8.2: Feasible values of parameters a and b for Example 2: – (left): Corollary 8.1, – (right) Lemma 8.1. Legend: [□ symbol]: $\rho_i^M = 2$, $\rho_i^m = 1/2$; [◯ symbol]: $\rho_i^M = 1.5$, $\rho_i^m = 1/1.5$; [× symbol]: $\rho_i^M = 1.2$, $\rho_i^m = 1/1.2$; [+ symbol]: $\rho_i^M = 1.1$, $\rho_i^m = 1/1.1$; [• symbol]: $\rho_i^M = \rho_i^m = 1$, *i.e.*, PDC controller

8.6 Conclusions

This chapter presents an extension of the methodology in (Lam & Leung, 2005) to consider arbitrary linear constraints in the shape of uncertain membership functions in a non-PDC fuzzy control setup. The proposed extensions apply to various stability and performance requirements in continuous and discrete systems, by making different choices for Q_{ij}.

The main contribution of the chapter is, thus, the ability to incorporate a wider class of constraints on the membership shape than (Lam & Leung, 2005). Interestingly, with the same type of restrictions, numerical examples illustrate that improvements over (Lam & Leung, 2005) may also be achieved, at least in the particular cases of the examples.

The examples in the chapter also illustrate the gradual loss of performance from a “full-PDC” fuzzy controller to a “robust linear” one as uncertainty in the memberships increases.

Chapter 9

Tensor product TS systems

In many fuzzy models, membership functions with multiple arguments are defined as the product of simpler ones, where all possible combinations of such products conform a fuzzy partition. In particular, such situations arise with widely-used fuzzy modelling techniques for nonlinear systems (Tanaka & Wang, 2001; Babuska et al., 1996). These type of fuzzy models will be denoted as tensor-product fuzzy systems, because their expressions can be understood as operations on multi-dimensional arrays.

This chapter discusses the generalization to tensor-product fuzzy systems of the results on stability and performance of Takagi-Sugeno fuzzy systems with LMIs. They were shown in Chapter 3.

9.1 Introduction

In many situations in fuzzy systems the membership functions can be expressed as the "tensor product" of simpler partitions, so that the fuzzy system can be written as a multi-dimensional fuzzy summation, for instance $\dot{x} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\mu_i\mu_j\mu_k(A_{ijk}x + B_{ijk}u)$. The tensor notation to be used in this chapter is motivated by the use of multidimensional arrays to describe this class of fuzzy systems (see (Baranyi, Tikk, Yam, & Patton, 2003)).

Removing part of the conservatism in current solutions for the tensor-product case above is indeed of interest; this product structure is often the case in many engineering applications of fuzzy control:

- in the systematic "sector nonlinearity" fuzzy modelling techniques reported in (Tanaka & Wang, 2001);

- in many man-made rulebases for multi-input fuzzy systems, where the rules are built via the *conjunction* of simpler concepts arising from fuzzy partitions on each of the input domains. A typical example are rulebases formed with rules in the form "if z_1 is *large* and z_2 is *small* and ... then ...", "if z_1 is *medium* and z_2 is *small* and ... then ...", *etc.*, with the antecedents covering all combinations of fuzzy sets on z_1, z_2, *etc.*.

- in approximate interpolation and model reduction techniques based on gridding and tensor-SVD algebra in (Baranyi, 2004).

These settings will give rise to a particular class of fuzzy models which will be denoted, following the nomenclature in (Baranyi, 2004), as *Tensor-Product* (**TP**) fuzzy systems. The reader is referred to the above references and later sections in this chapter for a more precise definition of TP fuzzy systems. In particular, a tensor-product structure of Takagi-Sugeno fuzzy systems will be the object of study, denoted as Tensor-Product Takagi-Sugeno fuzzy systems (TPTS).

In summary, the objective of this contribution is defining and analyzing the tensor-product fuzzy systems, presenting fuzzy control design tools for them which explicitly use the tensor-product structure. The study of the properties of this class of systems is very relevant, in the authors' opinion, as most of the fuzzy systems in nontrivial engineering applications of fuzzy control belong to this class, as discussed above.

In particular, a generalization of Theorem 3.5, exploiting the particular structure of the TPTS systems will be presented. The result provides less conservative conditions than other approaches in literature on closed-loop analysis and controller design problems. A numerical example will illustrate the achieved improvement.

9.2 Problem Statement

Tensor calculus originated in 19th century physics as a way of working with multilinear transformations, even in non-Euclidean geometries (Spain, 2003). When the multilinear transformations have arguments in $\mathbb{R}^n$ with the usual Euclidean metric and Hilbert space structure, tensors may be considered as multi-dimensional arrays. This is the case in this work.

In the definitions below, the notation I_q will refer to array index sets in the form $I_q = \{1, 2, \ldots, n_q\}$ for some n_q. Several values of q will be used in defining multi-dimensional arrays.

Definition 9.1 *A tensor T is a multilinear application which can be represented as a multidimensional array $T \in \mathbb{R}^{I_1 \times \cdots \times I_p}$ relative to the basis vectors being chosen on each array dimension. The number p is denoted as* tensor rank. *When the tensor structure is to be made explicit, the notation $T_{I_1 \times \cdots \times I_p}$ will be used, or even $T_{n_1 \times n_2 \times \cdots \times n_p}$ to describe both the rank and the sizes on each dimension. The tensor elements are real numbers, denoted by a lowercase symbol, indexed by a multi-dimensional index variable (to be denoted as* multi-index*):*

$$t_{i_1 i_2 \ldots i_p} \qquad 1 \leq i_q \leq n_q, \quad q = 1, \ldots, p \tag{9.1}$$

Note that rank-1 tensors may be considered transpose-free *vectors* and rank-2 ones are *matrices*. In the same way that matrices can be considered as a collection of vectors, a tensor can be considered a collection of lower-rank ones. On the sequel, when a rank-p tensor $T \in \mathbb{R}^{I_1 \times \cdots \times I_p}$ is indexed by an index with less than p components, the result will be a tensor (thus, denoted by uppercase), symbolised by the notation, for $q < p$:

$$T_{i_1 i_2 \ldots i_q} \in \mathbb{R}^{I_{q+1} \times \cdots \times I_p} \tag{9.2}$$

For instance a rank-5 tensor may be considered as a 3-dimensional array of matrices or a 4-dimensional array of vectors.

Definition 9.2 (Outer tensor product) . *The outer tensor product of $U_{n_1 \times \cdots \times n_p}$ and $T_{n'_1 \times \cdots \times n'_s}$ is a tensor $V_{n_1 \times \cdots \times n_{p+s}} = U \otimes T$, where $n_{p+q} \equiv n'_q$, $q = 1, \ldots, s$. The elements of V are:*

$$v_{i_1 \ldots i_p i_{p+1} \ldots i_{p+s}} = u_{i_1 \ldots i_p} t_{i_{p+1} \ldots i_{p+s}} \tag{9.3}$$

Definition 9.3 (Multi-indices) *On the following, as in Chapters 5 and 7, boldface symbols will denote multi-indices when its structure is clear from the context:*

$$\mathbf{i} = i_1 i_2 \ldots i_p \qquad 1 \leq i_q \leq n_q, \quad q = 1, \ldots, p \tag{9.4}$$

and, similarly, the cartesian product of index sets will be referred to by the notation:

$$\mathbb{I}_p = I_1 \times \cdots \times I_p \tag{9.5}$$

For instance, if either the dimensions have been suitably defined beforehand or they are not relevant to a particular discussion, the elements referred to in (9.1) will be denoted as $t_{\mathbf{i}}$, $i \in \mathbb{I}_p$ for convenience. The multi-index will be said to have rank p, as the tensor it indexes. Note that, the multi-indices sets $\mathbb{I}_p$ are formed of the cartesian product of different index sets. On the other hand, the multi-indices sets defined in Chapters 5 and 7 were formed of the cartesian product of the same index set.

Multi-indices of higher rank will also be represented by the juxtaposition of indices of smaller rank. For instance, the elements of the tensor resulting from the outer product in (9.3) will be denoted, when convenient, by $v_{\mathbf{ij}} = u_{\mathbf{i}} t_{\mathbf{j}}$, for suitably defined $\mathbf{i} \in \mathbb{I}_p$, $\mathbf{j} \in \mathbb{I}'_s$.

The following definition extends the usual matrix product along p shared dimensions[1].

Definition 9.4 (product) *The ordinary product of two tensors* $U \in \mathbb{R}^{\mathbb{I}''_s \times \mathbb{I}_p}$ *and* $V \in \mathbb{R}^{\mathbb{I}_p \times \mathbb{I}'_q}$*, which share the dimensions* $\mathbb{I}_p$*, is a tensor* $T \in \mathbb{R}^{\mathbb{I}''_s \times \mathbb{I}'_q}$ *which will be denoted as* $T = U \cdot_p V$ *whose elements are:*

$$t_{\mathbf{i}''\mathbf{i}'} = t_{i''_1 \dots i''_s i'_1 \dots i'_q} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_p=1}^{n_p} u_{i'_1 \dots i''_s i_1 i_2 \dots i_p} v_{i_1 i_2 \dots i_p i'_1 \dots i'_q} = \sum_{\mathbf{i} \in \mathbb{I}_p} u_{\mathbf{i}''\mathbf{i}} v_{\mathbf{i}\mathbf{i}'}$$

The notation $U \cdot V$ *(or* UV*) will be used to represent the product with shared index of rank 1,* i.e.*,* $UV = U \cdot V = U \cdot_1 V$ *(if* U *and* V *are rank-2 tensors,* UV *is the usual matrix product).*

As an example of the use of the product notation, a quadratic form ($x^T Q x$ in matrix notation) may be expressed as $(x \otimes x) \cdot_2 Q$.

[1]There are other alternative definitions and notations for (inner) tensor products (De Lathauwer, De Moor, & Vandewalle, 2000; Baranyi, 2004), as the number and position of the shared dimensions may vary. The one presented here has been adopted for convenience.

Proposition 9.1 *Given rank-p tensors A_1, A_2, a rank-q tensor B and a rank-$(p+q)$ tensor C, the ordinary product and the outer product verify:*

$$A_1 \cdot_p A_2 = A_2 \cdot_p A_1 \tag{9.6}$$

$$(A \otimes B) \cdot_{p+q} C = A \cdot_p (C \cdot_q B) \tag{9.7}$$

Note that $A_1 \cdot_p A_2$ is a real number, which is the generalization of the vector scalar product. For a rank-2 tensor (matrix) $\sqrt{A \cdot_2 A}$ is the Frobenius norm.

Definition 9.5 (unfolding) *The unfolding operation ("flattening"), denoted as $fl_{r \leftarrow q} V$ reduces the rank of a tensor $V \in R^{I_1 \times \cdots \times I_r \times \cdots \times I_q \times \cdots \times I_p}$ by one, converting it to a new tensor $U \in R^{I_1 \times \cdots \times I_r' \times \cdots \times I_{q-1} \times I_{q+1} \times \cdots \times I_p}$ whose elements are given by:*

$$u_{i_1 \ldots i_r \ldots i_p} = v_{i_1 \ldots i_{r-1} j_r i_{r+1} \ldots i_{q-1} j_q i_{q+1} \ldots i_p} \tag{9.8}$$

where $(i_r - 1) = (j_r - 1) * n_q + (j_q - 1)$,i.e., $j_r - 1$ is the integer part of the quotient $(i_r - 1)/n_q$, and $(j_q - 1)$ is the remainder.

As unfolding can be nested, successive applications of the operator can rearrange the tensor as a matrix or even as a vector. The notation

$$fl_{p \leftarrow q \leftarrow r \leftarrow \cdots \leftarrow t \leftarrow s} = fl_{p \leftarrow q} fl_{q \leftarrow r} \ldots fl_{t \leftarrow s} \quad p < q < r < \ldots t < s$$

will be later used.

Example 9.1 *Consider the tensor of rank 3, with $n_1 = 2$, $n_2 = 3$, $n_3 = 2$ given by: $t_{i_1 i_2 i_3} = 2^{i_1 - 1} 3^{i_2 - 1} 5^{i_3 - 1}$. Then,*

$$fl_{2 \leftarrow 3} T = \begin{pmatrix} 1 & 5 & 3 & 15 & 9 & 45 \\ 2 & 10 & 6 & 30 & 18 & 90 \end{pmatrix}$$

and

$$fl_{1 \leftarrow 2 \leftarrow 3} T = fl_{1 \leftarrow 2} fl_{2 \leftarrow 3} T = (1\ 5\ 3\ 15\ 9\ 45\ 2\ 10\ 6\ 30\ 18\ 90)$$

Example 9.2 *Unfolding a rank-3 tensor T may produce 6 different matrices: $fl_{1 \leftarrow 2} T$, $fl_{1 \leftarrow 3} T$, $fl_{2 \leftarrow 1} T$, $fl_{2 \leftarrow 3} T$, $fl_{3 \leftarrow 1} T$ and $fl_{3 \leftarrow 2} T$. The n-mode matrix of a rank-p tensor T (Definition 4 in (Baranyi, 2004)) is, for $n > 2$, the transpose of the matrix resulting from the unfolding $fl_{2 \leftarrow 1 \leftarrow 3 \leftarrow 4 \leftarrow \cdots \leftarrow n-1 \leftarrow n+1 \leftarrow p} T$.*

As unfolding is just a reordering of the tensor elements, it's easy to prove the following proposition (details omitted for brevity).

Proposition 9.2 *The inner product of tensors remains invariant under unfolding on any of the shared dimensions,* i.e.,

$$(fl_{r\leftarrow q}U)\cdot_{p-1}(fl_{r\leftarrow q}V) = U\cdot_p V$$

In particular, the above proposition generalises the transformation from (9.14) to (9.15), which used the fact that

$$(\mu\otimes\mu\otimes x\otimes x)\cdot_4 X = fl_{1\leftarrow 3}fl_{2\leftarrow 4}(\mu\otimes\mu\otimes x\otimes x)\cdot_2 fl_{1\leftarrow 3}fl_{2\leftarrow 4}X$$

There are many other definitions in tensor algebra (n-mode tensor-matrix products (De Lathauwer et al., 2000; Baranyi, 2004), etc.) which are out of the scope of this work. The reader is referred to the just cited works and textbooks (Spain, 2003; Temple, 2004) for further information about tensor algebra.

9.2.1 Sufficient Positivity Conditions

Sufficient conditions for positivity of Ξ in (3.23) are discussed in Chapter 3. For convenience, some of them are reviewed below.

Lemma 9.1 *If there exist matrices* $X_{ij} = X_{ji}^T$ *such that:*

$$X_{ii} \leq Q_{ii} \tag{9.9}$$

$$X_{ij}+X_{ji} \leq Q_{ij}+Q_{ji} \quad i<j \tag{9.10}$$

defining

$$\Theta(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i(z(t))\mu_j(z(t))x(t)^T X_{ij}x(t) \tag{9.11}$$

then $\Xi(t)$ *in* (3.23) *fulfills*

$$\Xi(t) \geq \Theta(t) \tag{9.12}$$

Proof: The proof is evident after reordering (3.23) and (9.11) as

$$\Xi(t) = \sum_{i=1}^{r} \mu_i^2 x^T Q_{ii} x + \sum_{i=1}^{r} \sum_{j=i+1}^{r} \mu_i \mu_j x^T (Q_{ij} + Q_{ji}) x \tag{9.13}$$

$$\Theta(t) = \sum_{i=1}^{r} \mu_i^2 x^T X_{ii} x + \sum_{i=1}^{r} \sum_{j=i+1}^{r} \mu_i \mu_j x^T (X_{ij} + X_{ji}) x \tag{9.14}$$

respectively. In this way, (9.9) and (9.10) indicate that each term in the summations in Ξ in (9.13) is larger than the corresponding one in the reordered Θ in (9.14). ■

Note that, in addition to an expression in the form (9.14), another expression for Θ is:

$$\Theta(t) = (\mu_1 x^T \ \mu_2 x^T \dots \mu_n x^T) \begin{pmatrix} X_{11} & \dots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \dots & X_{rr} \end{pmatrix} \begin{pmatrix} \mu_1 x \\ \mu_2 x \\ \vdots \\ \mu_n x \end{pmatrix} \tag{9.15}$$

which yields the well-known theorem 3.5.

9.2.2 Tensor-product fuzzy systems

This section will first present the fuzzy systems and fuzzy summations in Chapter 3 with tensor notation, and then generalize the expressions via a new definition, which will encompass widely used classes of fuzzy systems. Basically, a so-called rank-p tensor will denote a p-dimensional array of real numbers. The reader is referred to section 9.2 for tensor definitions, notation and operations with them.

Tensor expression for fuzzy systems

Note that the Takagi-Sugeno system (3.1) may be considered, by juxtaposing A_i and B_i as a matrix with size $n \times (n+w)$, as:

$$\dot{x} = \sum_{i=1}^{r} \mu_i (A_i \ B_i) \begin{pmatrix} x \\ u \end{pmatrix} \tag{9.16}$$

Consider now the one-dimensional array of matrices $(A_i \ B_i)$ to be the components of a suitably defined rank-3 tensor S so that the element s_{ijk}

is the element (j,k) of the matrix $(A_i\ B_i)$, for $j=1,\ldots,n$, $k=1,\ldots,(n+w)$. Consider also the membership functions to be arranged as a vector (rank-1 tensor). Then, (9.16) may be written as a tensor product

$$\dot{x} = (\mu \cdot_1 S)\begin{pmatrix} x \\ u \end{pmatrix} \tag{9.17}$$

because the tensor product $\mu \cdot_1 S$ produces the so-called system matrix (rank-2 tensor):

$$\mu \cdot_1 S = \sum_{i=1}^{r} \mu_i (A_i\ B_i) \tag{9.18}$$

As the memberships are a rank-1 tensor, the above fuzzy systems will be also denoted as rank-1 fuzzy systems. The case of higher dimensionality (higher tensor rank) will be discussed later in this section.

It's also straightforward to check that the double fuzzy summations in (3.23) may also be expressed as:

$$\Xi = (\mu \otimes \mu \otimes x \otimes x) \cdot_4 Q \tag{9.19}$$

where Q is a rank-4 tensor (a "matrix" of matrices Q_{ij}), *i.e.*, element q_{ijkl} is equal to the element at position (k,l) of the matrix Q_{ij}. Note that Ξ is a scalar.

9.2.3 Multi-dimensional tensor-product fuzzy systems

In many applications, membership functions in a multi-input fuzzy model are chosen to be the product of simpler memberships with a linguistic interpretation, and all the possible products of such simpler memberships appear as rule antecedents[2]. Let us discuss a couple of simple motivating examples.

Example 9.3 *Consider a so-called fuzzy-PD regulator built by setting up a fuzzy partition on an "error (e)" variable (say, a partition with 5 sets given by {negative large, negative, zero, positive, positive large}), and another (different) partition in the "error derivative (de)" (say, a partition with 3 sets {negative, zero, positive }).*

[2]Such simpler functions usually refer to a reduced number of input variables (but the definitions later in this section allow for any set of variables in any membership).

For convenience, the membership functions on the error partition will be denoted by $(\mu_{11}(e),\ldots,\mu_{15}(e))$*, respectively, and those on the error derivative, by* $(\mu_{21}(de),\mu_{22}(de),\mu_{23}(de))$*. The partitions are assumed to verify* $\sum_{i=1}^{5}\mu_{1i}=1$*,* $\sum_{i=1}^{3}\mu_{2i}=1$*.*

Once such partitions have been defined, rules are stated in a form such as:

IF e is *negative large* **and** de is *positive* **THEN** $u=u_{13}$
IF e is *negative* **and** de is *zero* **THEN** $u=u_{22}$
$\vdots$

In this example, the total number of rules is $5\times 3=15$. If the conjunction is interpreted as the algebraic product, the output of the controller may be expressed as:

$$u=\sum_{i_1=1}^{5}\sum_{i_2=1}^{3}\mu_{1i_1}(e)\mu_{2i_2}(de)u_{i_1i_2} \tag{9.20}$$

Now, consider the tensor outer product of the vectors (i.e., rank-1 tensors) $\mu_1=(\mu_{11}(e),\ldots,\mu_{15}(e))$ and $\mu_2=(\mu_{21}(de),\mu_{22}(de),\mu_{23}(de))$. Then, considering the following "membership tensor",

$$\mu_1(e)\otimes\mu_2(de)=\begin{pmatrix}\mu_{11}(e)\mu_{21}(de) & \mu_{11}(e)\mu_{22}(de) & \mu_{11}(e)\mu_{23}(de)\\ \mu_{12}(e)\mu_{21}(de) & \mu_{12}(e)\mu_{22}(de) & \mu_{12}(e)\mu_{23}(de)\\ \mu_{13}(e)\mu_{21}(de) & \mu_{13}(e)\mu_{22}(de) & \mu_{13}(e)\mu_{23}(de)\\ \mu_{14}(e)\mu_{21}(de) & \mu_{14}(e)\mu_{22}(de) & \mu_{14}(e)\mu_{23}(de)\\ \mu_{15}(e)\mu_{21}(de) & \mu_{15}(e)\mu_{22}(de) & \mu_{15}(e)\mu_{23}(de)\end{pmatrix}$$

it's easy to see that (9.20) may be expressed as an inner product of two tensors:

$$u=(\mu_1\otimes\mu_2)\cdot_2 U \tag{9.21}$$

for a suitably crafted matrix (rank-2 tensor) U of size 5×3 whose elements are the corresponding rule consequents u_{ij} for $i=1,\ldots,5,\ j=1,2,3$.

Example 9.4 *Consider a nonlinear model* $\dot{x}=A(x)x+B(x)u$ *where*

$$A(x) = 0.75x - 2.25sin(x) + sin(x)x - 2.5 \tag{9.22}$$
$$B(x) = 0.42x + 1.25sin(x) - 0.42sin(x)x - 0.25 \tag{9.23}$$

for which a fuzzy model is to be set up for $x \in [-\pi, \pi]$. In this case, x may be written as $x = \sum_{i=1}^{2} \nu_i p_i$, and $sin(x)$ as $sin(x) = \sum_{i=1}^{2} \eta_i q_i$, with:

$$x = \nu_1(x) \cdot \pi + \nu_2(x) \cdot (-\pi), \quad sin(x) = \eta_1(x) \cdot 1 + \eta_2(x) \cdot (-1)$$

where membership functions are $\nu_1 = \frac{1}{2\pi}(x + \pi)$, $\nu_2 = 1 - \mu_1$, $\eta_1 = \frac{1}{2}(sin(x) + 1)$, $\eta_2 = 1 - \eta_1$, resulting in

$$A(x) = 0.75 \sum_{i=1}^{2} \nu_i p_i - 2.25 \sum_{i=1}^{2} \eta_i q_i + (\sum_{i=1}^{2} \eta_i q_i)(\sum_{i=1}^{2} \nu_i p_i) - 2.5$$

$$A(x) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j (0.75 p_i - 2.25 q_j + p_i q_j - 2.5) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j a_{ij}$$

where

$$a_{11} = 0.748, a_{12} = -1.035, a_{21} = -10.247, a_{22} = 0.536$$

and similarly

$$B(x) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j (0.42 p_i + 1.25 q_j - 0.42 p_i q_j - 0.25) = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j b_{ij}$$

where:

$$b_{11} = 1, b_{12} = 1.139, b_{21} = 1, b_{22} = -4.139$$

Hence, the fuzzy system can be expressed as:

$$\dot{x} = \sum_{i=1}^{2} \sum_{j=1}^{2} \nu_i \eta_j (a_{ij} x + b_{ij} u) = \sum_{\mathbf{i} \in \mathbb{I}_2} \tilde{\mu}_{\mathbf{i}} (a_{\mathbf{i}} x + b_{\mathbf{i}} u) \tag{9.24}$$

where $\mathbf{i}$ is a two-dimensional index variable (i_1, i_2) taking values in the set $\mathbb{I}_2 = \{1, 2\} \times \{1, 2\}$, and $\tilde{\mu}_{\mathbf{i}} = \nu_{i_1} \eta_{i_2}$, using the multiindex notation in the Appendix9.2. In an analogous way to (9.20) in the previous example, a tensor notation can be thought of (see below).

Motivated by the above examples, let us consider now a definition for a general tensor-product fuzzy model in the Takagi-Sugeno (TS) framework (TS fuzzy systems are the most frequently used process model for fuzzy control in current literature), in order give a compact notation to fuzzy systems whose expression is a multi-dimensional sum, as in the above examples.

Definition 9.6 ***Tensor-product Takagi-Sugeno fuzzy systems.*** *Consider a vector of measurable variables, z, in an universe of discourse Z. Consider also p fuzzy partitions defined on Z, each of them with n_1, $\dots n_p$ fuzzy sets, respectively.*

The fuzzy sets will be assumed to have linguistic labels denoted by M_{1i_1}, $i_1 = 1,\dots,n_1$ for the first partition, M_{2i_2}, $i_2 = 1,\dots,n_2$ for the second partition, etc. *and membership functions arranged in rank-1 tensors:*

$$\begin{aligned} \mu_1 &= (\mu_{11}(z)\ \mu_{12}(z)\ \dots\ \mu_{1n_1}(z)) \\ \mu_2 &= (\mu_{21}(z)\ \mu_{22}(z)\ \dots\ \mu_{1n_2}(z)) \\ &\vdots \\ \mu_p &= (\mu_{p1}(z)\ \mu_{p2}(z)\ \dots\ \mu_{pn_p}(z)) \end{aligned} \tag{9.25}$$

fulfilling

$$\sum_{k=1}^{n_l} \mu_{lk} = 1 \quad 0 \le \mu_{lk} \le 1 \quad l = 1,\dots,p$$

Then, a rank-p continuous-time tensor-product Takagi-Sugeno fuzzy system (TPTS) built on the above fuzzy sets will be defined as the one described by the rules[3]*:*

> ***IF*** *z* ***is*** *(M_{1i_1}* ***and*** *M_{2i_2}* ***and*** *M_{pi_p})* ***THEN*** *$\dot{x} = A_{i_1 i_2 \dots i_p} x + B_{i_1 i_2 \dots i_p} u$*

being its output evaluated with:

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}_p} \tilde{\mu}_{\mathbf{i}} (A_{\mathbf{i}} x + B_{\mathbf{i}} u) \tag{9.26}$$

---

[3]In many applications, such as the one in Example 9.3, the rules have the form:

> **IF** z_1 **is** M_{1i_1} **and** z_2 **is** M_{2i_2} **and** ... **and** z_p **is** M_{pi_p} **THEN** $\dot{x} = A_{i_1 i_2 \dots i_p} x + B_{i_1 i_2 \dots i_p} u$

i.e., fuzzy partitions are defined over universes of discurse of smaller dimension, so that $Z = Z_1 \times Z_2 \times \dots Z_p$. However, that's not necessary, in principle, for the results in this chapter to apply. For instance if Z is $\mathbb{R}^2$, we could have $p = 3$, with three fuzzy partitions defined on, say, $z_1 + z_2$, $z_1 - \sqrt{z_2}$ and $(\sin(z_1) + 1)/(\cos(z_2) + 1)$. Hence, the rules above in this footnote are a particular case of the ones in Definition 9.6.

where x *and* u *are the TPTS state and input variables, respectively,* $\mathbf{i} = i_1 i_2 \ldots i_p$*, and*

$$\tilde{\mu}_{\mathbf{i}} = \prod_{k=1}^{p} \mu_{k i_k} \tag{9.27}$$

Remark: Analogous definitions may be cast for discrete-time TPTS systems and also for systems incorporating output equations, but they are omitted for brevity.

Using tensor notation, the following definition for TPTS systems is equivalent to the previous one (proof is omitted as it is just an issue of notation).

Definition 9.7 *Consider a state vector* x *with dimension* d*, and an input vector* u *with dimension* w*, and form a vector of dimension* $n+w$ *by juxtaposing* x *and* u*. Consider a set of* p *fuzzy partitions defined on a universe* Z*, each of them arranged as a rank-1 tensor* μ_i*,* $i = 1, \ldots, p$*,* i.e.*, as in* (9.25) *above. Then, a TPTS fuzzy system is described by:*

$$\dot{x} = (\tilde{\mu} \cdot_p S) \begin{pmatrix} x \\ u \end{pmatrix} \tag{9.28}$$

where S *is a tensor with rank* $p+2$ *and dimensions* n_1*,* n_2*,* $\ldots$*,* n_p*,* $n_{p+1} = d$*,* $n_{p+2} = d+w$ *and*

$$\tilde{\mu} = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_p$$

is a tensor with dimensions n_1*,* n_2*,* $\ldots$*,* n_p *(whose elements are, evidently, given by* (9.27)*), denoted as* membership tensor*.* S *will be denoted as* consequent tensor[4]*.*

An example of a membership tensor element is, for instance $\mu_{3,4,1,1} = \mu_{13}\mu_{24}\mu_{31}\mu_{41}$, which will denote a particular rule in a rank-4 TPTS fuzzy system.

Note that $\tilde{\mu} \cdot_p S$ is a rank-2 tensor (*i.e.*, a matrix which multiplies the state-input vector with the ordinary matrix-vector multiplication).

[4]Notation in (9.28) is somehow different from that in (Baranyi, 2004), but equivalent. We wanted to emphasise the concept of "membership tensor" (generated via an outer product) whereas Baranyi used n-mode products (De Lathauwer et al., 2000) for subsequent singular-value-related computations.

Obviously, the notations (9.26) and (9.28) are equivalent to an expression such as:

$$\dot{x} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_p=1}^{n_p} \mu_{1i_1} \mu_{2i_2} \ldots \mu_{pi_p} (A_{i_1 i_2 \ldots i_p} x + B_{i_1 i_2 \ldots i_p} u) \qquad (9.29)$$

For instance, the fuzzy system (9.24) may be considered a rank-2 TPTS one.

Remarks on TPTS modelling: Many fuzzy systems in practice have the tensor-product structure:

- Example 9.3 shows how they naturally arise from man-made rules.
- Another paradigmatic example is the "sector nonlinearity" modelling methodology in (Tanaka & Wang, 2001); Example 9.4 in this work is one of the simplest cases of the referred modelling technique. The reader is also referred to Example 3, in section 2.2.1 of the referred book, which results in a 16-rule model TPTS described by a membership tensor of dimensions $2 \times 2 \times 2 \times 2$ (of course, the authors there do not use the notation introduced here), *i.e.*, a rank-4 TPTS system.
- Last, (Baranyi, 2004) proposes a tensor-product based methodology to approximate functions of multiple variables via Takagi-Sugeno fuzzy systems. The procedure, instead of being based on the previously-discussed sector-nonlinearity approach, is based on multi-dimensional gridding, lookup and interpolation. A subsequent step of complexity reduction based on higher-order singular value decomposition (De Lathauwer et al., 2000) is needed in order to get a reduced number of rules.

In fact, fuzzy system without a TSTP structure are seldom present in applications, except in the simplest cases (even some first-order single-input TS systems can be better modelled as TSTP, by using the sector-nonlinearity methodology above cited, as demonstrated in Example 9.4 in this work).

Proposition 9.3 *Standard TS fuzzy systems are rank-1 TPTS fuzzy systems. Conversely, TPTS systems are a subclass of standard TS fuzzy systems.*

Proof: The first affirmation is evident from the definitions, and it has already been discussed in Section 9.2.2. Regarding the second one, consider the well-known identity

$$\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_p=1}^{n_p}\mu_{1i_1}\ldots\mu_{pi_p}=1 \tag{9.30}$$

It shows that the tensor product conforms a fuzzy partition composed of $q=n_1\times n_2\times\cdots\times n_p$ fuzzy sets. Such partition is given by the rank-1 membership functions obtained by unfolding (flattening) the tensor $\tilde{\mu}$ onto a vector. The idea can be formalised by using proposition 9.2, as:

$$\dot{x}=(\tilde{\mu}\cdot_p S)\begin{pmatrix} x \\ u \end{pmatrix}=(fl_{1\leftarrow\cdots\leftarrow p}\tilde{\mu}\cdot_1 fl_{1\leftarrow\cdots\leftarrow p}S)\begin{pmatrix} x \\ u \end{pmatrix} \tag{9.31}$$

Hence, the original TPTS fuzzy system is expressed as a standard TS one because the membership tensor has been unfolded onto a vector, and the consequent tensor S has been suitably rearranged by the fl operator as a rank 3 tensor. Such rank-3 tensor $fl_{1\leftarrow\cdots\leftarrow p}S$ produces an ordinary matrix when subject to the product with the unfolded $fl_{1\leftarrow\cdots\leftarrow p}\tilde{\mu}$ (rank 1). ■

Example 9.5 *Consider a TP fuzzy system with $p=2$, $n_1=2$, $n_2=3$. It may be equivalently considered as an "unfolded" fuzzy system with 6 membership functions, denoted as $\beta_k(z)$ given by:*

$$\begin{array}{ll}
k=1 & \beta_1(z)=\mu_{11}(z)\mu_{21}(z)\\
k=2 & \beta_2(z)=\mu_{11}(z)\mu_{22}(z)\\
k=3 & \beta_3(z)=\mu_{11}(z)\mu_{23}(z)\\
k=4 & \beta_4(z)=\mu_{12}(z)\mu_{21}(z)\\
k=5 & \beta_5(z)=\mu_{12}(z)\mu_{22}(z)\\
k=6 & \beta_6(z)=\mu_{12}(z)\mu_{23}(z)
\end{array}$$

As another example, consider the fuzzy model of Example 9.4. In the same way as above, if $\mu_1=\nu_1\eta_1$, $\mu_2=\nu_1\eta_2$, $\mu_3=\nu_2\eta_1$ and $\mu_4=\nu_2\eta_2$ were defined, a fuzzy TS with four models:

$$\dot{x}=\sum_{i=1}^{4}\mu_i(a_ix+b_iu) \tag{9.32}$$

$$a_1=0.748, a_2=-1.035, a_3=-10.247, a_4=0.536 \tag{9.33}$$

$$b_1=1, b_2=1,139, b_3=1, b_4=-4.139 \tag{9.34}$$

will exactly describe the nonlinear system under analysis for $x \in [-\pi, \pi]$.

The reader is referred to (Tanaka & Wang, 2001) for more examples of this tensor-product nonlinear modelling methodology (although tensor notation is not used and the final model is always unfolded).

Remark: Proposition (9.3) seems to make ill-fated any attempt to approach fuzzy control design for TPTS systems because TPTS are TS systems and vice-versa. However, a crucial fact is overlooked in this argumentation: most results for stability and performance of TS fuzzy systems are *independent* of the membership shapes – particularly those in (Tanaka & Wang, 2001; Kim & Lee, 2000; Liu & Zhang, 2003). However, an unfolded TPTS system does *not* sweep over all possible membership values[5]. Hence, such membership-independent stability and performance conditions are conservative in the case of TPTS systems. This is the key issue motivating the work in Section 9.4.

9.3 Closed-loop tensor-product fuzzy systems

Definition 9.8 (tensor-product controller) *Given a rank-p TPTS system* (9.26)*, a controller in the form:*

$$u = -\sum_{\mathbf{j} \in \mathbb{I}_p} \tilde{\mu}_{\mathbf{j}}(z) F_{\mathbf{j}} x = -(\tilde{\mu} \cdot_p F) x \tag{9.35}$$

will be denoted as rank-p tensor product PDC controller (F is a rank-$(p+2)$ tensor formed by suitably arranging matrices $F_{\mathbf{j}}$).

By analogy with (3.14), it is straightforward to prove that, when a rank-p tensor-product PDC controller is used to control a rank-p system (9.26), the closed loop equations are given by:

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}_p} \sum_{\mathbf{j} \in \mathbb{I}_p} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} G_{\mathbf{ij}} = ((\tilde{\mu} \otimes \tilde{\mu}) \cdot_{2p} G) x \tag{9.36}$$

where $G_{\mathbf{ij}} = A_{\mathbf{i}} - B_{\mathbf{i}} F_{\mathbf{j}}$ defines a tensor G with rank $2p+2$ (note that, for fixed $\mathbf{i}$ and $\mathbf{j}$, $B_{\mathbf{i}}$ and $F_{\mathbf{j}}$ are rank-2 tensors, following notation (9.2), so the product is well defined, being the usual matrix product).

[5] for instance, it's impossible to have $\beta_1 = 0.1$ and β_2 or β_3 larger than 0.1 in Example 9.5.

In general, analogously to (3.23), many stability and performance criteria for tensor-product closed-loop fuzzy systems can be expressed as requiring, for any $x \neq 0$:

$$\Theta = \sum_{\mathbf{i} \in \mathbb{I}_p} \sum_{\mathbf{j} \in \mathbb{I}_p} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} x^T Q_{\mathbf{ij}} x > 0 \tag{9.37}$$

For instance, it's almost evident to check that a condition for quadratic stability of a TPTS fuzzy system is (9.37) with $Q_{\mathbf{ij}}$ given by (3.1) but replacing the i and j with its boldfaced counterparts.

In tensor notation, stability and performance conditions (9.37) look like

$$\Theta = (\tilde{\mu} \otimes \tilde{\mu} \otimes x \otimes x) \cdot_{2p+2} Q > 0 \tag{9.38}$$

for a suitably defined tensor Q with rank $2p+2$. Indeed,

$$\Theta = \sum_{\mathbf{i} \in \mathbb{I}_p} \sum_{\mathbf{j} \in \mathbb{I}_p} \sum_{k=1}^{n} \sum_{l=1}^{n} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} x_k x_l Q_{\mathbf{ij}kl}$$

Unfolding to a TS system. A possibility to work with TPTS systems is considering them as ordinary TS systems (Proposition 9.3) and design fuzzy controllers for them. Indeed, this is the commonly considered option in literature which this chapter seeks to improve.

The above argumentation may be equivalently stated by using Proposition 9.2 on (9.38), which results in stating:

$$\Theta = (fl_{1\leftarrow\cdots\leftarrow p}\tilde{\mu} \otimes fl_{1\leftarrow\cdots\leftarrow p}\tilde{\mu} \otimes x \otimes x) \cdot_4 fl_{1\leftarrow\cdots\leftarrow p} fl_{(p+1)\leftarrow\cdots\leftarrow 2p} Q \tag{9.39}$$

where $fl_{1\leftarrow\cdots\leftarrow p}\tilde{\mu}$ is a rank-1 tensor (*i.e.*, the memberships of an ordinary TS system arranged as a vector, suitably ordered) so (9.39) may be written as (9.19), *i.e.*, (3.23). Hence, LMIs for such conditions can be applied, such as Theorem 3.5 (details are omitted for brevity).

The next section discusses an explicit use of the tensor-product form of the memberships in order to produce conditions less conservative than the "unfolding + Theorem 3.5" procedure used in literature.

9.4 Relaxed stability and performance conditions for TPTS fuzzy systems

Theorem 9.1 *Expression* (9.37) *(equiv.* (9.38)*) holds if there exists a rank-*$(2p+2)$ *tensor* X *such that the conditions stated below hold. For ease of notation, note that* $X_{\mathbf{i}k\mathbf{j}s}$ $\mathbf{i},\mathbf{j} \in \mathbb{I}_{p-1}$, $k,s \in I_p$ *is a rank-2 tensor (matrix), and the same applies to* $Q_{\mathbf{i}k\mathbf{j}s}$*. The conditions are:*

$$X_{\mathbf{i}k\mathbf{j}s} = X_{\mathbf{i}s\mathbf{j}k}^T \tag{9.40}$$

$$X_{\mathbf{i}k\mathbf{j}k} \le Q_{\mathbf{i}k\mathbf{j}k} \tag{9.41}$$

$$X_{\mathbf{i}k\mathbf{j}s} + X_{\mathbf{i}s\mathbf{j}k} \le Q_{\mathbf{i}k\mathbf{j}s} + Q_{\mathbf{i}s\mathbf{j}k} \tag{9.42}$$

$$Y = fl_{p\leftarrow 2p+1} fl_{2p\leftarrow 2p+2} X, \text{ i.e.,}$$

$$Y_{\mathbf{ij}} = \begin{pmatrix} X_{\mathbf{i}1\mathbf{j}1} & \dots & X_{\mathbf{i}1\mathbf{j}n_p} \\ \vdots & \ddots & \vdots \\ X_{\mathbf{i}n_p\mathbf{j}1} & \dots & X_{\mathbf{i}n_p\mathbf{j}n_p} \end{pmatrix} \tag{9.43}$$

$$\sum_{\mathbf{i}\in\mathbb{I}_{p-1}} \sum_{\mathbf{j}\in\mathbb{I}_{p-1}} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} \xi(t)^T Y_{\mathbf{ij}} \xi(t) > 0 \tag{9.44}$$

where Y *is a rank-*$(2p)$ *tensor (hence* $Y_{\mathbf{ij}}$ *is a matrix).*

If (9.44) *can be proved by a set of LMI sufficient conditions, then such conditions jointly with* (9.41)*–*(9.43) *are still an LMI problem stating sufficient conditions for* (9.37)*.*

Proof. Note that, for $\mathbf{i} \in \mathbb{I}_{p-1}$, $k \in I_p$, for any tensor T of rank greater than p:

$$\sum_{\mathbf{h}\in\mathbb{I}_p} \tilde{\mu}_{\mathbf{h}} T_{\mathbf{h}} = \sum_{\mathbf{i}\in\mathbb{I}_{p-1}} \sum_{k=1}^{n_p} \tilde{\mu}_{\mathbf{i}k} T_{\mathbf{i}k} = \sum_{\mathbf{i}\in\mathbb{I}_{p-1}} \tilde{\mu}_{\mathbf{i}} \sum_{k=1}^{n_p} \mu_{pk} Q_{\mathbf{i}k}$$

Similarly, (9.37) may be written as:

$$\Theta = \sum_{\mathbf{i}\in\mathbb{I}_{p-1}} \sum_{\mathbf{j}\in\mathbb{I}_{p-1}} \tilde{\mu}_{\mathbf{i}} \tilde{\mu}_{\mathbf{j}} \sum_{k=1}^{n_p} \sum_{s=1}^{n_p} \mu_{pk} \mu_{ps} x^T Q_{\mathbf{i}k\mathbf{j}s} x \tag{9.45}$$

Then, Lemma 9.1 can be applied to

$$\delta_{\mathbf{ij}} = \sum_{k=1}^{n_p} \sum_{s=1}^{n_p} \mu_{pk} \mu_{ps} x^T Q_{\mathbf{i}k\mathbf{j}s} x \tag{9.46}$$

considering $\mathbf{i}$ and $\mathbf{j}$ as fixed, so that, if (9.41), (9.42) hold, then (considering the analogous formulas to (9.13) and (9.14)):

$$\delta_{\mathbf{ij}} \geq p_{\mathbf{ij}} = \sum_{k=1}^{n_p} \sum_{s=1}^{n_p} \mu_{pk}\mu_{ps} x^T X_{\mathbf{i}k\mathbf{j}s} x \tag{9.47}$$

and, hence, building the matrix $Y_{\mathbf{ij}}$ in (9.43),

$$p_{\mathbf{ij}} = \xi^T Y_{\mathbf{ij}} \xi \tag{9.48}$$

where $\xi = fl_{1\leftarrow 2}(\mu_p \otimes x) = (\mu_{p1}x_1 \ \dots \ \mu_{p1}x_n \ \mu_{p2}x_1 \ \dots \ \mu_{pn_p}x_n)$ expressed as a column vector. As the elements of the membership tensors are all positive, we have

$$\Theta \geq \sum_{\mathbf{i}\in\mathbb{I}_{p-1}} \sum_{\mathbf{j}\in\mathbb{I}_{p-1}} \tilde{\mu}_{\mathbf{i}}\tilde{\mu}_{\mathbf{j}} p_{\mathbf{ij}} \tag{9.49}$$

and the proof is complete. □

The above theorem is a generalization of Theorem 3.5. It provides a sufficient condition which transforms computation of positivity conditions for a "double p-dimensional sum" (9.37) into computations with a "double $(p-1)$-dimensional sum" and larger matrices (the size of $Y_{\mathbf{ij}}$ is $(n_p \cdot n) \times (n_p \cdot n)$, where n is the size of the square matrices $Q_{\mathbf{ij}}$).

From a computational point of view, recursive application of the above theorem allows to reach $p = 1$, and directly applying Theorem 3.5 as a last step. Then, Theorem 2 allows to assert that (9.37) holds if a certain $nq \times nq$ matrix is positive definite.

Note that the size of the final matrix is the same as the one obtained by unfolding (9.37) and applying Theorem 3.5: the number of elements of tensors X, Y and Q are the same, but arranged differently. However, the larger number of relaxation variables X in Theorem 9.1, with various sizes, allows to produce less conservative results as the example in next section shows.

Recursive application of Theorem 9.1 for a rank-p TP fuzzy system needs $p-1$ tensors of decision variables (of rank $2p+2$, $2p$, $\dots$, 4). All of these tensors have the same number of elements as the original Q.

9.5 Robust stability of LTI multiaffine systems

An application of this Tensor-Product Description and relaxation may also be done to prove robust stability on linear time-invariant (LTI) multiaffine or Tensor-Product systems (Ariño & Sala, 2007b; Anderson et al., 1995). Robust control, understood as the study of the properties of open-and closed-loop systems with uncertain parameters. Robust control techniques for a class of dynamics systems in state-space form, for which some "sector bounds" can be computed, are well known for at least 20 years. The sector bounds may arise either from uncertain parameters (even time-varying) (Gahinet, Apkarian, & Chilali, 1996; Amato, Mattei, & Pironti, 1998; Bliman, 2004) in linear systems, or from sector bounded nonlinearities from first-principle equations (such as those used in gain scheduling or fuzzy control developments) (Rugh & Shamma, 2000; Apkarian & Adams, 1998; Tanaka, Ikeda, & Wang, 1996; Ebihara & Hagiwara, 2005; Dettori & Scherer, 2000)

Definition 9.9 *Let us define a multiaffine system as the one described by*

$$,Boyd:1994\dot{x} = \sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\dots\sum_{i_p=1}^{n_p} \alpha_{1i_1}\alpha_{2i_2}\dots\alpha_{pi_p}A_{i_1i_2\dots i_p}x \tag{9.50}$$

where $\alpha_j \in \Omega_{n_j}$*, where* Ω_{n_j} *is the* n_j*-dimensional simplex given by:*

$$\Omega_{n_j} = \left\{ \alpha_j \in \mathbb{R}^{n_j} \mid \sum_{i=1}^{n_j} \alpha_{ji} = 1,\ \alpha_{ji} \geq 0,\ i = 1\dots n_j \right\}$$

and j ranges from 1 to p.

Note that, in fact, this system looks like the Tensor-Product Tanaka-Sugeno Fuzzy System proposed in section 9.2.2. Definition 9.9 may be recast as follows.

Definition 9.10 *A multiaffine linear system can be described using the multiindex notation by*

$$\dot{x} = \sum_{\mathbf{i}\in\mathbb{I}_p} \alpha_{\mathbf{i}}A_{\mathbf{i}}x \tag{9.51}$$

where $\mathbf{i} = \{i_1, \dots, i_p\} \in \mathbb{I}_p$*, and* $\alpha_{\mathbf{i}}$ *is obtained from* (9.27) *for some pre-defined* α_{lk}*,* $k = 1, \dots, n_l$*,* $l = 1, \dots, p$*.*

Evidently, the definition with $p = 1$ refers to usual polytopic systems.

9.5.1 Robust stability conditions

Let us now generalise some well-known results on robust stability of polytopic dynamic systems (via parameter-dependent quadratic-in-the-state Lyapunov functions) to the previously defined multiaffine (multipolytopic) case.

Consider a uncertain multiaffine LTI system

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}_p} \alpha_{\mathbf{i}} A_{\mathbf{i}} x \tag{9.52}$$

where the additional assumption of $\alpha_{\mathbf{i}}$ being time-invariant is made.

Proposition 9.4 *The multiaffine LTI system* (9.52) *is robustly stable if there exists a collection of positive-definite matrices* $P_{\mathbf{j}}$*,* $j \in \mathbb{I}_p$ *such that:*

$$\Theta = \sum_{\mathbf{i} \in \mathbb{I}_p} \sum_{\mathbf{j} \in \mathbb{I}_p} \alpha_{\mathbf{i}} \alpha_{\mathbf{j}} x^T Q_{\mathbf{ij}} x > 0 \tag{9.53}$$

for all $x \neq 0$*, and for any value of the parameters* $\alpha_{\mathbf{i}}$*, where*

$$Q_{\mathbf{ij}} = -(A_{\mathbf{i}}^T P_{\mathbf{j}} + P_{\mathbf{j}} A_{\mathbf{i}}) \tag{9.54}$$

Proof: Considerer as a candidate Lyapunov function

$$V = \sum_{\mathbf{j} \in \mathbb{I}_p} \alpha_{\mathbf{j}} x^T P_{\mathbf{j}} x \tag{9.55}$$

then the system is stable if the conditions below hold for any value of the parameters $\alpha_{\mathbf{i}}$:

$$P_{\mathbf{j}} > 0 \quad \mathbf{j} \in \mathbb{I}_p \tag{9.56}$$

$$\dot{V} = \sum_{\mathbf{i} \in \mathbb{I}_p} \sum_{\mathbf{j} \in \mathbb{I}_p} \alpha_{\mathbf{i}} \alpha_{\mathbf{j}} x^T (A_{\mathbf{i}}^T P_{\mathbf{j}} + P_{\mathbf{j}} A_{\mathbf{i}}) x < 0 \tag{9.57}$$

□

Note that Q_{ij} is a symmetric matrix.

Of course, when the conditions are stated for $p = 1$, the well-known conditions for robust stability of polytopic systems (via affinely parameter dependent Lyapunov functions) are recovered.

The expression (9.53) is equivalent positivity condition that one in (9.37) and then theorem 9.1 can be applied in order to proof stability.

9.6 Examples

The following examples illustrates the effectivenes of the new stability conditions (Theorem 9.1) compared to the usual approach in literature. The first one applies this new conditions to a Takagi-Sugeno Fuzzy Model and the second applies it to a multiaffine Linear Time invariant system.

Example 9.6 *Consider a continuous fuzzy plant composed of the following four rules:*

R_{11}: **IF** x_1 is M_{11} **and** x_2 is M_{21} **THEN** $\dot{x} = A_{11}x + B_{11}u$
R_{12}: **IF** x_1 is M_{11} **and** x_2 is M_{22} **THEN** $\dot{x} = A_{12}x + B_{12}u$
R_{21}: **IF** x_1 is M_{12} **and** x_2 is M_{21} **THEN** $\dot{x} = A_{21}x + B_{21}u$
R_{22}: **IF** x_1 is M_{12} **and** x_2 is M_{22} **THEN** $\dot{x} = A_{22}x + B_{22}u$

where

$$A_{11} = \begin{pmatrix} 0.5 & -0.05 \\ 0 & -5 \end{pmatrix}, B_{11} = \begin{pmatrix} a \\ 0.1 \end{pmatrix}$$
$$A_{12} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}, B_{12} = \begin{pmatrix} 1 \\ 0.2 \end{pmatrix}$$
$$A_{21} = \begin{pmatrix} -1 & 0.1 \\ 0 & -2 \end{pmatrix}, B_{21} = \begin{pmatrix} 1 \\ 0.4 \end{pmatrix}$$
$$A_{22} = \begin{pmatrix} b & -0.01 \\ 0 & -3 \end{pmatrix}, B_{22} = \begin{pmatrix} 1 \\ 0.05 \end{pmatrix}$$

represented by the equations:

$$\dot{x} = \sum_{\mathbf{i} \in \mathbb{I}_2} \tilde{\mu}_{\mathbf{i}} (A_{\mathbf{i}} x + B_{\mathbf{i}} u) = \sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \mu_{1i_1} \mu_{2i_2} (A_{i_1 i_2} x + B_{i_1 i_2} u) \tag{9.58}$$

where $\mathbb{I}_2 = \{1,2\} \times \{1,2\}$. Membership functions $\{\mu_{11}, \mu_{12}\}$ and $\{\mu_{21}, \mu_{22}\}$ are supposed to be fuzzy partitions on the domain of x_1 and x_2 respectively. Hence, the system conforms to the definition of a rank-2 TPTS one. The shape of each of the four membership functions is arbitrary as long as $\mu_{11} = 1 - \mu_{12}$ and $\mu_{21} = 1 - \mu_{22}$.

A stabilising PDC controller with 4 rules is to be designed,

$$u = -\sum_{\mathbf{i}\in\mathbb{I}_2} \tilde{\mu}_{\mathbf{i}} F_{\mathbf{i}} x$$

The stabilization conditions expressed in the form (9.37) are obtained from (3.33) with $\alpha = 0$, resulting in:

$$Q_{\mathbf{ij}} = -ZA_{\mathbf{i}} - A_{\mathbf{i}}^T Z + B_{\mathbf{i}} N_{\mathbf{j}} + N_{\mathbf{j}}^T B_{\mathbf{j}} i^T \tag{9.59}$$

where $\mathbf{i},\mathbf{j} \in \mathbb{I}_2$, and Z, $N_{\mathbf{j}}$ are LMI decision variables. Z should be a symmetric positive-definite matrix, and the PDC controller is provided by $F_{\mathbf{j}} = N_{\mathbf{j}} Z^{-1}$.

The parameters a in B_{11}, and b in A_{22}, will take values in a prescribed grid, in order to check the feasibility of the associated fuzzy control synthesis problem under two different approaches.

Usual approach. A first possibility in order to design the above regulator would be considering the fuzzy system to be a four-rule standard one (unfolding), with $A_1 = A_{11}$, $A_2 = A_{12}$, $A_3 = A_{21}$ and $A_4 = A_{22}$, using a similar notation for B, generating Q_{ij}, $i, j = 1, \ldots, 4$.

This well-known approach has been compared to the one proposed in this work. Note that 16 Lyapunov matrices (9.59) are defined in both approaches, the only difference is how they are indexed (via two integer indices from 1 to 4, in the usual approach; via two rank-2 indices of size $\{1,2\} \times \{1,2\}$ in this work).

Proposed approach. Applying Theorem 9.1, expression (9.37) holds if there exist a rank-6 tensor X from which matrices $X_{\mathbf{i}k\mathbf{j}s}$ can be extracted so that $X_{\mathbf{i}k\mathbf{j}s} = X_{\mathbf{i}s\mathbf{j}k}^T$ for each $\mathbf{i},\mathbf{j} \in \mathbb{I}_1$, $k,s \in I_1$ ($\mathbb{I}_1 = I_1 = \{1,2\}$), and

$$X_{\mathbf{i}1\mathbf{j}1} \le Q_{\mathbf{i}1\mathbf{j}1}, \quad X_{\mathbf{i}2\mathbf{j}2} \le Q_{\mathbf{i}2\mathbf{j}2} \tag{9.60}$$

$$X_{\mathbf{i}1\mathbf{j}2} + X_{\mathbf{i}2\mathbf{j}1} \le Q_{\mathbf{i}1\mathbf{j}2} + Q_{\mathbf{i}2\mathbf{j}1} \tag{9.61}$$

$$Y_{\mathbf{ij}} = \begin{pmatrix} X_{\mathbf{i}1\mathbf{j}1} & X_{\mathbf{i}1\mathbf{j}2} \\ X_{\mathbf{i}2\mathbf{j}1} & X_{\mathbf{i}2\mathbf{j}2} \end{pmatrix} \tag{9.62}$$

$$\sum_{\mathbf{i}\in\mathbb{I}_1} \sum_{\mathbf{j}\in\mathbb{I}_1} \mu_{\mathbf{i}} \mu_{\mathbf{j}} \xi^T Y_{\mathbf{ij}} \xi > 0 \tag{9.63}$$

Then, regarding the positivity of $\sum_{\mathbf{i}\in\mathbb{I}_1} \sum_{\mathbf{j}\in\mathbb{I}_1} \mu_{\mathbf{i}} \mu_{\mathbf{j}} \xi^T Y_{\mathbf{ij}} \xi$ Theorem 3.5 is directly applied, because $\mathbf{i}$ and $\mathbf{j}$ are now one-dimensional indices: Theorem 3.5 requires the existence of matrices $W_{ij} = W_{ij}^T$ for each $i, j \in \{1,2\}$, such that

$$W_{11} \leq Y_{11}, W_{22} \leq Y_{22} \tag{9.64}$$

$$W_{12} + W_{21} \leq Y_{12} + Y_{21} \tag{9.65}$$

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \geq 0 \tag{9.66}$$

Note that the set of conditions (9.59) jointly with (9.60)–(9.62), (9.64)–(9.66) are LMIs.

Results. Figure 9.1 shows the values of a and b where a stabilising controller is found, based on either Theorem 3.5 (after unfolding) or Theorem 9.1, using a suitable LMI solver.

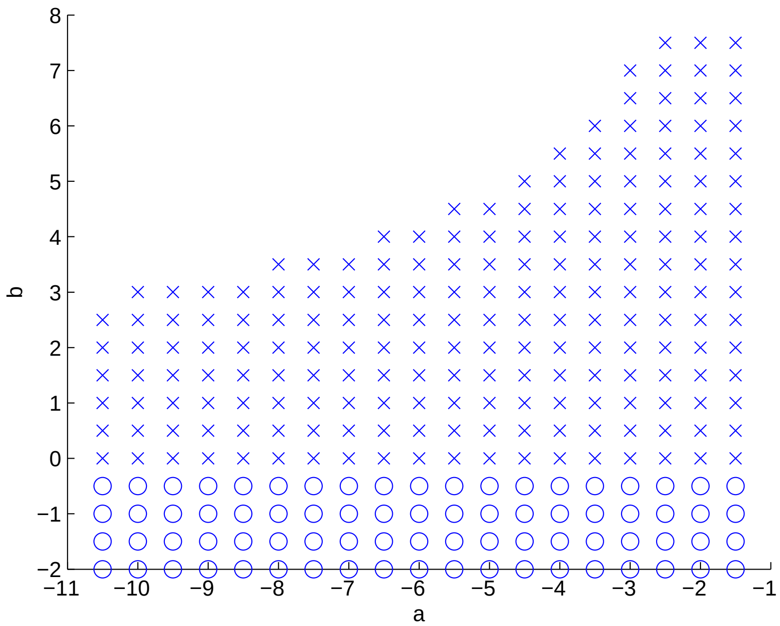


Figure 9.1: Parameter values for which feasible stabilising regulators are found: unfolding + Theorem 3.5 (○); Theorem 9.1(○, ×).

In this figure, the ○ mark indicates the existence of feasible stabilising regulators proved by Theorem 3.5 (and, of course, also by Theorem 9.1); the × mark indicates parameter values for which stabilizability is proved from Theorem 9.1, but not from Theorem 3.5. Hence, substantially better results are obtaining by exploiting the tensor-product structure of the four involved TS rules.

Similar results are obtained when the methodology is applied to the nonlinear system in Example 9.4 expressed as a rank-2 TPTS fuzzy system: the usual approach does not find a stabilising controller, whereas the one proposed in this work does.

Example 9.7 *Consider the system* (9.67) *below, with two uncertain time-invariant parameters* m *and* n *which are assumed to belong to a known interval,* i.e.*,* $m \in [m_1,\ m_2]$ *and* $n \in [n_1,\ n_2]$ *respectively.*

$$\dot{x} = \tilde{A}\ x = \begin{pmatrix} -\frac{1}{8}(1+3m\ n) & 0.3\ (n-m) \\ 0.3\ (m-n) & -1 \end{pmatrix} x \tag{9.67}$$

This system can be represented exactly taking the maximum and minimum posible values of m and n by the multiaffine expression

$$\dot{x} = \sum_{i=1}^{2}\sum_{j=1}^{2} \alpha_{1i}\alpha_{2j}A_{ij}x \tag{9.68}$$

for some α_{11}, α_{12}, α_{21}, α_{22} fulfilling $\alpha_{11}+\alpha_{12}=1$, $\alpha_{21}+\alpha_{22}=1$, $0 \le \alpha_{ij} \le 1$ and

$$A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & -1 \end{pmatrix}$$

where the values of matrix elements a_{ij}, b_{ij} are:

$$a_{ij} = -\frac{1}{8}(1+3m_i\ n_j), \quad b_{ij} = 0.3\ (n_j - m_i)$$

Indeed, the representation (9.68) is obtained by just replacing

$$m = \alpha_{11}m_1 + \alpha_{12}m_2 \quad n = \alpha_{21}n_1 + \alpha_{22}n_2$$

in (9.67) and carrying out straightforward algebraic manipulations. For instance, with m and n taking values in the interval $[-1, 1]$, the resulting vertex matrices are:

$$A_{11} = \begin{pmatrix} -0.5000 & 0 \\ 0 & -1 \end{pmatrix}$$
$$A_{12} = \begin{pmatrix} 0.2500 & -0.6 \\ 0.6 & -1 \end{pmatrix}$$
$$A_{21} = \begin{pmatrix} 0.2500 & 0.6 \\ -0.6 & -1 \end{pmatrix}$$
$$A_{22} = \begin{pmatrix} -0.500 & 0 \\ 0 & -1 \end{pmatrix}$$

The above multiaffine model is *not* equivalent to the polytopic one in the form:

$$\dot{x} = \sum_{i=1}^{4} \alpha_i A_i x \tag{9.69}$$

for any arbitrary set of $\alpha_i > 0$, such that $\sum_{i=1}^{4} \alpha_i = 1$, which describes the convex hull of

$$A_1 = A_{11}, A_2 = A_{12}, A_3 = A_{21}, A_4 = A_{22}$$

Instead, the multiaffine model is strictly included inside the mentioned convex hull.

Indeed, the matrix

$$A_u = \begin{pmatrix} 0.25 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.70}$$

with an unstable pole, is not described by any $m, n \in [-1, 1]$ in (9.67), but it belongs the polytopic form (9.69), as

$$A_u = 0.5A_2 + 0.5A_3$$

even if A_2 and A_3 are themselves stable.

To further illustrate this, Figure 9.2 shows the posibles values of the elements 11 and 12 of the matrix $\tilde{A}$, denoted as $\tilde{a}_{11}$ and $\tilde{a}_{12}$. In this figure, It is shown that the multiaffine model 9.9 is not convex in the space of values of the matrix $\tilde{A}$.

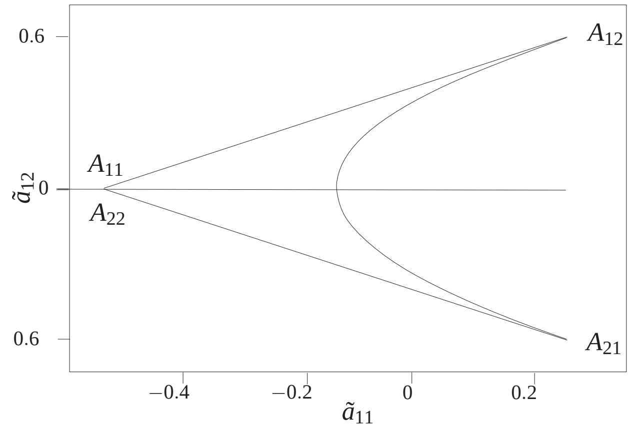


Figure 9.2: The multiaffine set of matrix coefficients in (9.67) lies inside the arrow-like shape.

The poles of the matrix $\tilde{A}$ can be computed symbolically. Obtained by using *Mathematica* 5.0 software, they are given by:

$$\frac{-45 - 15mn - \Delta}{80}$$

$$\frac{-45 - 15mn + \Delta}{80}$$

where

$$\Delta = \sqrt{1225 - 576m^2 + 102mn - 576n^2 + 225m^2n^2}$$

Then, with any $n, m \in [-1, 1]$, the system is stable (stability has been assessed by plotting the poles achieved in a dense enough grid covering the mentioned intervals; in fact, the range of values of m and n which stabilize the system appears in Figure 9.3).

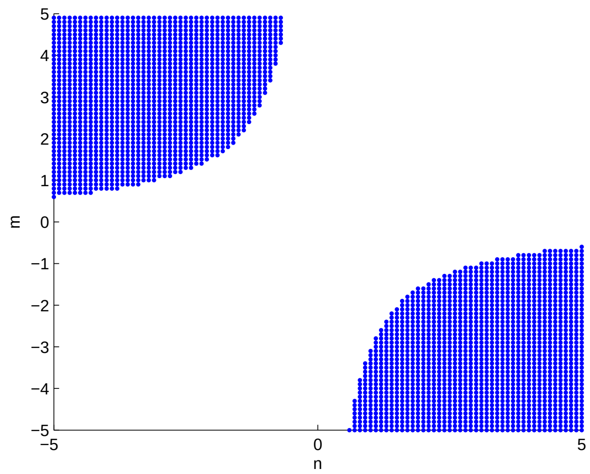


Figure 9.3: Values of n and m yielding a stable system (white).

However, the existence of unstable matrices (such as (9.70) above) in the polytopic convex hull makes any polytopic-based approach futile and, of course, a set of Lyapunov functions valid for all the convex hull is impossible to be found, no matter which technique were to be used (Gahinet et al., 1996; Feron, Apkarian, & Gahinet, 1996; Chesi, Garulli, Tesi, & Vicino, 2005; Oliveira & Peres, 2006).

Now, Theorem 9.1 is applied to the multiaffine parametric description in example 9.2 in order to look for a parameter-dependent Lyapunov function (9.55).

From the theorem, considering suitable Q_{ij} in the form (9.54) for the expressions below, the system is robustly stable if there exist symmetric matrices $X_{ik,jk}$ and matrices $X_{ik,js} = X^T_{is,jk}$ such that, from (9.41),

$$A_{11}^T P_{11} + P_{11}A_{11} + X_{11,11} \leq 0 \tag{9.71}$$
$$A_{12}^T P_{12} + P_{12}A_{12} + X_{12,12} \leq 0 \tag{9.72}$$
$$A_{21}^T P_{21} + P_{21}A_{21} + X_{21,21} \leq 0 \tag{9.73}$$
$$A_{22}^T P_{22} + P_{22}A_{22} + X_{22,22} \leq 0 \tag{9.74}$$
$$A_{11}^T P_{21} + P_{21}A_{11} + X_{11,21} \leq 0 \tag{9.75}$$
$$A_{12}^T P_{22} + P_{22}A_{12} + X_{12,22} \leq 0 \tag{9.76}$$
$$A_{21}^T P_{11} + P_{11}A_{21} + X_{21,11} \leq 0 \tag{9.77}$$
$$A_{22}^T P_{12} + P_{12}A_{22} + X_{22,12} \leq 0 \tag{9.78}$$

from (9.42)

$$A_{11}^T P_{12} + P_{12}A_{11} + A_{12}^T P_{11} + P_{11}A_{12} + \\ + X_{11,12} + X_{12,11} \leq 0 \tag{9.79}$$
$$A_{21}^T P_{22} + P_{22}A_{21} + A_{22}^T P_{21} + P_{21}A_{22} + \\ + X_{21,22} + X_{22,21} \leq 0 \tag{9.80}$$
$$A_{11}^T P_{22} + P_{11}A_{22} + A_{12}^T P_{21} + P_{11}A_{12} + \\ + X_{11,22} + X_{12,21} \leq 0 \tag{9.81}$$
$$A_{21}^T P_{12} + P_{12}A_{21} + A_{22}^T P_{11} + P_{11}A_{22} + \\ + X_{21,12} + X_{22,11} \leq 0 \tag{9.82}$$

and from (9.43), forming the following augmented symmetric matrices

$$Y_{11} = \begin{pmatrix} X_{11,11} & X_{11,12} \\ X_{12,12} & X_{12,12} \end{pmatrix} \tag{9.83}$$

$$Y_{12} = \begin{pmatrix} X_{11,21} & X_{11,22} \\ X_{12,22} & X_{12,22} \end{pmatrix} \tag{9.84}$$

$$Y_{21} = \begin{pmatrix} X_{21,11} & X_{21,12} \\ X_{22,12} & X_{22,12} \end{pmatrix} \tag{9.85}$$

$$Y_{22} = \begin{pmatrix} X_{21,21} & X_{21,22} \\ X_{22,22} & X_{22,22} \end{pmatrix} \tag{9.86}$$

robust stability is proved if the following reduced condition holds:

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \alpha_{1i}\alpha_{1j}\xi^T Y_{i,j}\xi^T > 0 \tag{9.87}$$

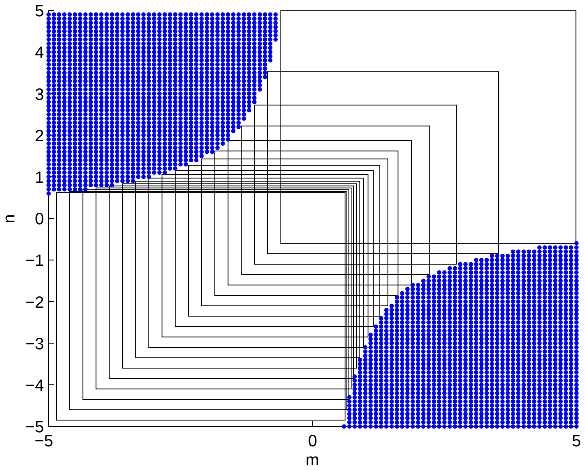


Figure 9.4: Different boxes for which robust stability of the multiaffine system can be proved.

This is a condition in which the indices in the summations are one-dimensional, compared to the original one with two-dimensional summations.

Thus, applying again the theorem 9.1 to (9.87) (*i.e.*, one iteration of the recursive procedure suggested in the previous section), the obtained conditions, introducing new auxiliary variables, are:

$$H_{11} \leq Y_{11} \quad H_{22} \leq Y_{22} \tag{9.88}$$

$$H_{12} + H_{21} \leq Y_{12} + Y_{21} \tag{9.89}$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} > 0 \tag{9.90}$$

where $H_{12} = H_{21}^T$, and H_{11}, H_{22} are symmetric decision matrices.

Then the inequalities (9.71)-(9.86), (9.88)-(9.90) form a set of LMI sufficient conditions than can be tested with the Matlab LMI Toolbox (Gahinet et al., 1995) obtaining a valid parameter-dependent Lyapunov function for this case. The decision variables are the matrices X_{ij}, P_{ij}, H_{ij} $(i, j \in \{1, 2\})$, where X_{ij} and P_{ij} are matrices of size 2×2 and H_{ij} are matrices of size 4×4.

Expanding the range of m and n to the interval $[-q, q]$ for both cases, the system has stable poles up to $q = \sqrt{3}$ (from the symbolic formula and Figure 9.3). The multiaffine parametric stability conditions being tested also keep feasible until that value of q so, in this particular example, they provide a tight result.

Of course, the multi-affine approach can only prove robust stability for boxes (or, with some modifications, for convex polytopes) in the parameter space. Other boxes of parameter values have also been tested with positive robust stability results (see Figure 9.4), so results seem to be non-conservative in this case (understanding, of course, that the non-convex robust stability zone in the above figure cannot be directly found by the algorithm). No such claim is made for other cases, of course.

9.7 Conclusions

This chapter has provided a generalization of double-fuzzy summation results in literature to multiple summations with a tensor-product structure. Such structure is indeed common in many fuzzy models allowing for less conservative results in fuzzy controller designs for such systems, as demonstrated in the numerical example. Although, for simplicity, the chosen example only considers stabilisation, the presented procedure applies to other more sophisticated performance/robustness requirements, by considering well-known different choices for $Q_{\mathbf{ij}}$.

This chapter has also presented robust stability results for multiaffine (multipolytopic) uncertain LTI systems by means of the use of a multiaffine parametric Lyapunov function. The results provide conditions which are less conservative than those that arise from only considering the polytopic convex hull, as such a hull comprises a larger set of systems, some of which could be unstable.

LMI conditions for computations are obtained via a recursive procedure involving the addition of artificial decision variables.

Part III

Conclusions

Chapter 10
Conclusions and future work

This thesis has dealt with the analysis of nonlinear systems via Takagi-Sugeno fuzzy models: new LMI conditions for stability and performance of a Takagi-Sugeno fuzzy model have been discussed. These conditions are presented in order to use the available membership function information in the linear matrix inequalities arising from well-known stability and performance problems. The proposed conditions improve over current literature, where Linear matrix inequalities methodologies do not include any information about the shape of the membership functions: stability is proved for any set of rules, with any membership function shape, as long as the linear vertex models are maintained. This is a source of conservatism that has been reduced in this work.

In this sense, this thesis has taken some steps towards improving the Lyapunov-based theory for nonlinear control via Takagi-Sugeno fuzzy systems. In these concluding remarks, we summarize the contributions and give some suggestions for future research.

Sufficient and asymptotical necessary results. Extension of Polya's Theorem

In Chapter 5, some results in previous literature have been improved, achieving less conservative sufficient conditions on positive-definiteness of fuzzy summations (related to stability and performance criteria in fuzzy control), with a extension of the Polya's theorem (Pólya & Szegö, 1928). We have shown the conditions are progressively less conservative as a complexity parameter n increases, becoming asymptotically exact. Bounds for n can be computed if a tolerance parameter is introduced. The number of conditions is polynomial in n; if decision variables are introduced, the number of them may be exponential in n. The achievable value of n in a particular fuzzy control problem depends on solver accuracy and available computing resources.

Local Stability

As discussed in Chapter 6, Some *local* stability results for TS models may be obtained in fuzzy systems via the knowledge of the membership functions, even when no feasible quadratic Lyapunov function can be found to prove *global* stability[1]. The stability is achieved in the largest sphere around $x = 0$ for which a quadratic Lyapunov function can be proven via LMIs), therefore the found sphere is part of a larger ellipsoidal guaranteed basin of attraction.

The methodology used is based on transformation of the membership functions by expressing them as a convex combination of some points in the membership space. These points are obtained from the knowledge of the maximum and minimum values of the memberships in the zone under study.

Overlap relaxed results

In order to relax stability and performance conditions for fuzzy control models with knowledge of membership function overlap, two theorems relaxing some LMI stability and performance conditions in (Tanaka & Wang, 2001; Kim & Lee, 2000; Liu & Zhang, 2003) have been presented in Chapter 7. The conditions consider a set of known bounds in the membership functions, which generalize the relaxations previously reported in literature for non-overlapping fuzzy sets. As a result, more freedom in guaranteeing control requirements is available.

The proposed technique may prove useful in fuzzy control applications: in fuzzy PDC control techniques, membership functions are assumed to be known so the required bounds may be easily obtained, and the conditions are computationally simple.

[1]The concept of global understood as for any membership function shape; sometimes, TS models are themselves a local model (sector nonlinearity approach), hence it will not be possible to prove anything outside the domain of definition of the original TS model.

Stability Analysis with uncertain membership functions

In most applications, we need to approximate the membership functions and in these cases, it is difficult to take into account the knowledge of the membership functions' shape. Chapter 8 considers arbitrary linear constraints in the shape of uncertain membership functions in a non-PDC fuzzy control setup. The proposed methodology applies to various stability and performance requirements in continuous and discrete systems. The gradual loss of performance from a "full-PDC" fuzzy controller to a "robust linear" one as uncertainty in the memberships increases is also shown.

Conditions for Tensor-product TS models

Tensor-product structures are common in many fuzzy models and the results in Chapter 9 allow for less conservative conditions in fuzzy controller designs for such systems. We also presented, as a corollary,robust stability results for multiaffine (multipolytopic) uncertain LTI systems by means of the use of a multiaffine parametric Lyapunov function. The results provide conditions which are less conservative than those that arise from only considering the polytopic convex hull, as such hull comprises a larger set of systems some of which could be unstable.

LMI conditions for computations are obtained via a recursive procedure involving the addition of artificial decision variables.

Applications of the results

The thesis has provided several methodologies in order to control nonlinear dynamical systems. The results are generic and have direct applications to basically any nonlinear system modelled via the sector-nonlinearity Takagi-Sugeno methodology discussed in Section 3.3. The results are specially useful when several nonlinearities yield a multi-dimensional tensor product system, where the methodology in Chapter 9 is able to relax the conditions significantly.

10.1 Future research lines

In this thesis, we have shown how LMI optimization can be used to analyze the stability of nonlinear dynamical systems. The results have direct applications, and can be extended in many promising ways. In this section, we point out some open problems and give ideas for future research.

- The thesis has presented many ways to reduce the conservationism of LMI stability and performance conditions. Most of them are not mutually exclusive, but the intensive use of all of them will increase the computational cost. Therefore further developments will compare the computational cost of the different methodologies and choose the best trade-off for different scenarios.
- Fuzzy systems can be embedded into the polynomial system class, for which sum of squares sufficient stability conditions can be stated and solved with recent tools, such as SOSTOOLS (Prajna et al., 2002), which translate some sum-of-squares (SOS) polynomial problems to linear matrix inequalities. So it would be interesting to apply the presented results to a more general class of polynomial fuzzy systems.
- There are several contributions where the complexity of the Lyapunov function is increased in order to relax the stability and performance conditions. In the thesis, the complexity of the LMI conditions is increased and most of the results are not mutually exclusive with other contributions in the literature, but there is not a criterium to choose the relative complexity of the Lyapunov function and the LMI conditions. It would be very useful to be able to compare between “more complex Lyapunov functions with simple conditions” and “simpler Lyapunov functions with complex conditions”.

Chapter 11

Published Contributions

Journal papers

- Sala, A. and Ariño, C. (2007). Assymptotically necessary and sufficient conditions for stability and performance in fuzzy control. *Fuzzy Sets and Systems*, 158, 2671-2686.

- Sala, A. and C. Ariño (2007). Relaxed stability and performance conditions for Takagi-Sugeno fuzzy systems with knowledge on membership function overlap. Systems, *Man and Cybernetics, Part B, IEEE Transactions on* 37(3), 727–732.

- Ariño, C. and Sala, A. (2007). Relaxed LMI conditions for closed loop fuzzy systems with tensor product structure. *Eng. Appl. Artif. Intell.*, 20, 1036-1046.

- Ariño, C.; Sala, A. and Navarro, J.L. Diseño de controladores en varios puntos de funcionamiento para una clase de modelos borrosos Takagi-Sugeno afines. Revista Iberoamericana de Automática e Informática Industrial,2007, 4, 98-105.

- Ariño, C. and Sala, A. Extensions to "Stability Analysis of Fuzzy Control Systems Subject to Uncertain Grades of Membership" *Man and Cybernetics, Part B, IEEE Transactions on* (in press)

- Sala, A. and Ariño. Relaxed stability and performance LMI conditions for Takagi-Sugeno fuzzy systems with polynomial constraints on membership function shapes. *Fuzzy Systems, IEEE Transactions* (submitted)

- Antonio Sala, Carlos Ariño. Reduciendo distancias entre el control borroso y el control no lineal: luces y sombras. *Revista Iberoamericana de Automática e Informática Industrial* (submitted)

Conference papers

- Ariño, C. and Sala, A. Design of Multiple-Parameterisation PDC Controllers via Relaxed Conditions for Multi-Dimensional Fuzzy Summations *Proc. FUZZ-IEEE 2007 IEEE International Conference on Fuzzy Systems*, London, 2007.

- Ariño, C. and Sala, A. Quadratic parameter-dependent stability of multiaffine systems *Proc. ECC '07 European Control Conference*, 2007.

- Ariño, C. and Sala, A. Tracking Control design for a class of affine MIMO Takagi-Sugeno models *Proc. ICINCO 2007 - 4th International Conference on Informatics in Control, Automation and Robotics*, 2007.

- Kruszewski, A., Sala, A., Guerra, T. and Arino, C. Sufficient and asymptotic necessary conditions for the stabilization of Takagi-Sugeno model *Adv. Fuzzy Neural Control, 2007, Proceedings of the 3rd IFAC Conference on*, 2007.

- Ariño, C. and Sala, A. Local Stability of Open- and Closed-Loop fuzzy systems *Proceedings' IEEE International CCA / CACSD / ISIC*, 2006.

- Ariño, C. and Sala, A. Determining Ellipsoidal Basins of Attraction of Fuzzy Systems. *Proceedings' ICINCO 2006 - Third International Conference on Informatics in Control, Automation and Robotics*, 2006.

- Sala, A.; Esparza, A.; Ariño, C. and Roig, J. V. Encoding Fuzzy diagnosis rules as optimisation problems *Proceedings' ICINCO 2006 - Third International Conference on Informatics in Control, Automation and Robotics*, 2006.

- Navarro, J.; Ariño, C. ; Sala, A. and Diez, J.L. A Saturation Based Interpolation Method for Fuzzy Systems *Proc. 16th IFAC World Congress*, 2005.

- Ariño, C.; Sala, A. and Navarro, J. Calibración en linea de un sensor de biomasa. *Proc. 26º Jornadas de Automática*, 2005.

- Ariño, C.; Sala, A. and Navarro, J. Controladores LMIs para sistemas Borrosos afines *Proc. 26º Jornadas de Automática*, 2005.

- Ariño, C. and Sala, A. An Alternative aggregation algorithm for fuzzy local models *Proc. IFAC'04 Workshop on advanced Fuzzy / Neural Control*, 2004.

References

Amato, F., Mattei, M., & Pironti, A. (1998). A note on quadratic stability of uncertain linear discrete-time systems. *Automatic Control, IEEE Transactions on*, *43*(2), 227–229.

Anderson, B. D. O., Kraus, F., Mansour, M., & Dasgupta, S. (1995). Easily testable sufficient conditions for the robust stability of systems with multilinear parameter dependence. *Automatica*, *31*(1), 25–40.

Antsaklis, P., & Mitchel, A. (1997). *Linear systems.* NY: McGraw-Hill.

Apkarian, P., & Adams, R. (1998). Advanced gain-scheduling techniques for uncertain systems. *IEEE Trans. Control Syst. Techn.*, *6*, 21–32.

Ariño, C., & Sala, A. (2006a). Determining ellipsoidal basins of attraction of fuzzy systems. In *Proceedings' ICINCO 2006 - Third International Conference on Informatics in Control, Automation and Robotics.*

Ariño, C., & Sala, A. (2006b). Local stability of open- and closed-loop fuzzy systems. In *Proceedings' IEEE International CCA/CACSD/ISIC.*

Arino, C., & Sala, A. (2007a). Design of multiple-parameterisation PDC controllers via relaxed conditions for multi-dimensional fuzzy summations. In *Fuzzy Systems Conference, 2007. FUZZ-IEEE 2007. IEEE International* (pp. 1–6).

Ariño, C., & Sala, A. (2007b). Quadratic parameter-dependent stability of multiaffine systems. In *Proc. ECC '07 European Control Conference 2007.*

Arino, C., & Sala, A. (2007c). Relaxed LMI conditions for closed-loop fuzzy systems with tensor-product structure. *Engineering Applications of Artificial Intelligence*, *20*(8), 1036–1046.

Assilian, S., & Mamdani, E. (1975). An experiment in Linguistic Synthesis with a Fuzzy Logic Controller. Internat. *J. Man-Machine Stud*, *7*, 1–13.

Babuska, R. (1998). *Fuzzy Modeling for Control.* Kluwer Academic Publishers Norwell, MA, USA.

Babuska, R., Fantuzzi, C., & Verbruggen, H. B. (1996). Improved inference for takagi-sugeno models. In *Proc. IEEE Conf. on Fuzzy Systems* (Vol. New Orleans, pp. 653–664).

Baranyi, P. (2004). TP model transformation as a way to LMI-based controller design. *Industrial Electronics, IEEE Transactions on*, *51*(2), 387–400.

Baranyi, P., Tikk, D., Yam, Y., & Patton, R. J. (2003). From differential equations to PDC controller design via numerical transformation. *Computers in Industry*, *51*(3), 281–297.

Bliman, P. (2004). A convex approach to robust stability for linear systems with uncertain scalar parameters. *SIAM Journal on Control and Optimization*, *42*(6), 2016–2042.

Bondia, J., Sala, A., Pico, J., & Sainz, M. A. (2006). Controller design under fuzzy pole-placement specifications: An interval arithmetic approach. *Fuzzy Systems, IEEE Transactions on*, *14*(6), 822–836.

Boyd, S., El Ghaoui, L., Feron, E., & Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory.* Philadelphia, USA: Ed. SIAM.

Boyd, S., & Vandenberghe, L. (2004). *Convex Optimization.* Cambridge University Press.

Calaÿore, G., & El Ghaoui, L. (2004). Ellipsoidal bounds for uncertain linear equations and dynamical systems. *Automatica*, *40*, 773-787.

Cannon, R. (2003). *Dynamics of Physical Systems.* Courier Dover Publications.

Chen, B., Tseng, C., & Uang, H. (2000). Mixed H 2/∞ fuzzy output feedback control design for nonlinear dynamic systems: an LMI approach. *Fuzzy Systems, IEEE Transactions on*, *8*(3), 249-265.

Chen, S.-S., Chang, Y.-C., Su, S.-F., Chung, S.-L., & Lee, T.-T. (2005). Robust static output-feedback stabilization for nonlinear discrete-time systems with time delay via fuzzy control approach. *Fuzzy Systems, IEEE Transactions on*, *13*(2), 263–272.

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2003). Homogeneous Lyapunov functions for systems with structured uncertainties. *Automatica*, *39*(6), 1027–1035.

Chesi, G., Garulli, A., Tesi, A., & Vicino, A. (2005). Polynomially parameter-dependent Lyapunov functions for robust stability of polytopic systems: an LMI approach. *Automatic Control, IEEE Transactions on*, *50*(3), 365–370.

Choi, D., & Park, P. (2003). H_∞ state-feedback controller design for discrete-time fuzzy systems using fuzzy weighting-dependent Lyapunov functions. *Fuzzy Systems, IEEE Transactions on*, *11*(2), 271-278.

De Lathauwer, L., De Moor, B., & Vandewalle, J. (2000). A Multilinear Singular Value Decomposition. *SIAM Journal on Matrix Analysis and Applications*, *21*, 1253.

Dettori, M., & Scherer, C. (2000). New robust stability and performance conditions based on parameter dependent multipliers. In *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on* (Vol. 5, pp. 4187–4192 vol.5).

Ebihara, Y., & Hagiwara, T. (2005). A dilated LMI approach to robust performance analysis of linear time-invariant uncertain systems. *Automatica*, *41*(11), 1933–1941.

Fang, C.-H., Liu, Y.-S., Kau, S.-W., Hong, L., & Lee, C.-H. (2006). A New LMI-Based Approach to Relaxed Quadratic Stabilization of T-S Fuzzy Control Systems. *IEEE Trans. Fuzzy Syst.*, *14*, 286-397.

Fantuzzi, C., & Rovatti, R. (1996). On the approximation capabilities of the homogeneous Takagi-Sugeno. In *Fuzzy Systems, 1996., Proceedings of the Fifth IEEE International Conference on* (Vol. 2, pp. 1067–1072 vol.2).

Feng, G. (2003). Controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions and bilinear matrix inequalities. *Fuzzy Systems, 2003. FUZZ'03. The 12th IEEE International Conference on*, *2*.

Feng, G. (2006). A Survey on Analysis and Design of Model-Based Fuzzy Control Systems. *IEEE TRANSACTIONS ON FUZZY SYSTEMS*, *14*(5), 676.

Feron, E., Apkarian, P., & Gahinet, P. (1996). Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. *Automatic Control, IEEE Transactions on*, *41*(7), 1041–1046.

Gahinet, P., & Apkarian, P. (1994). A linear matrix inequality approach to H_∞ control. *International Journal of Robust and Nonlinear Control*, *4*(4), 421–448.

Gahinet, P., Apkarian, P., & Chilali, M. (1996). Affine parameter-dependent Lyapunov functions and real parametric uncertainty. *Automatic Control, IEEE Transactions on*, *41*(3), 436–442.

Gahinet, P., Nemirovski, A., Laub, A., & Chilali, M. (1995). *LMI Control Toolbox.* Math Works.

Guerra, T., Kruszewski, A., Vermeiren, L., & Tirmant, H. (2006). Conditions of output stabilization for nonlinear models in the Takagi-Sugeno's form. *Fuzzy Sets and Systems*, *157*(9), 1248–1259.

Guerra, T., & Vermeiren, L. (2004). LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in the Takagi-Sugeno's form. *Automatica*, *10*, 823 - 829.

Hori, T., Tanaka, K., & Wang, H. (2002). A piecewise Takagi-Sugeno fuzzy model construction and relaxation of stability conditions. In *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on* (Vol. 2, pp. 2149–2150 vol.2).

Hsiao, F.-H., Hwang, J.-D., Chen, C.-W., & Tsai, Z.-R. (2005). Robust stabilization of nonlinear multiple time-delay large-scale systems via decentralized fuzzy control. *Fuzzy Systems, IEEE Transactions on*, *13*(1), 152–163.

Jankovic, M., Fontaine, D., & Kokotovic, P. (1996). TORA example: cascade- and passivity-based control designs. *Control Systems Technology, IEEE Transactions on*, *4*(3), 292–297.

Johansen, T. (2000). Computation of Lyapunov functions for smooth nonlinear systems using convex optimization. *Automatica*, *36*(11), 1617–1626.

Johansson, M. (1999). Piecewise quadratic stability of fuzzy systems. *IEEE Trans. Fuzzy Syst.*, *7*, 713-722.

Kawamoto, S. (1996). Nonlinear control and rigorous stability analysis based on fuzzy system for inverted pendulum. In *Fuzzy Systems, 1996., Proceedings of the Fifth IEEE International Conference on* (Vol. 2, pp. 1427–1432 vol.2).

Kawamoto, S., Tada, K., Ishigame, A., & Taniguchi, T. (1992). An approach to stability analysis of second order fuzzy systems. In *Fuzzy Systems, 1992., IEEE International Conference on* (pp. 1427–1434).

Khalil, H. (1996). *Nonlinear systems (2nd. ed.).* Upper Saddle River, New Jersey, USA: Ed. Prentice-Hall.

Kim, E., & Lee, H. (2000). New approaches to relaxed quadratic stability condition of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, *2*, 1571-1582.

Kruszewski, A., & Guerra, T. (2005). New Approaches for the Stabilization of Discrete Takagi-Sugeno Fuzzy Models. In *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on* (pp. 3255–3260).

Kruszewski, A., Sala, A., Guerra, T., & Arino, C. (2007). Sufficient and asymptotic necessary conditions for the stabilization of Takagi-Sugeno model. In *Adv. Fuzzy Neural Control, 2007, Proceedings of the 3rd IFAC Conference on.*

Kuschewski, J., Hui, S., & Zak, S. (1993). Application of feedforward neural networks to dynamical system identification and control. *Control Systems Technology, IEEE Transactions on*, *1*(1), 37–49.

Labiod, S., & Guerra, T. (2007). Adaptive fuzzy control of a class of SISO nonaffine nonlinear systems. *Fuzzy Sets and Systems*, *158*(10), 1126–1137.

Lam, H., & Leung, F. (2005). Stability analysis of fuzzy control systems subject to uncertain grades of membership. *Systems, Man and Cybernetics, Part B, IEEE Transactions on*, *35*(6), 1322–1325.

Lam, H., & Leung, F. (2007). Fuzzy controller with stability and performance rules for nonlinear systems. *Fuzzy Sets and Systems*, *158*(2), 147–163.

Li, J., Niemann, D., Wang, H., & Tanaka, K. (1999). Parallel distributed compensation for Takagi-Sugeno fuzzy models: multiobjective controller design. In *American Control Conference, 1999. Proceedings of the 1999* (Vol. 3, pp. 1832–1836 vol.3).

Liu, X., & Zhang, Q. (2003). New approaches to H_∞ controller designs based on fuzzy observers for T-S fuzzy systems via LMI. *Automatica*, *39*(9), 1571–1582.

Loera, J. A. de, & Santos, F. (1996). An effective version of Polya's theorem on positive definite forms. *Journal of Pure and Applied Algebra*, *108*(3), 231–240.

Loera, J. A. de, & Santos, F. (2001). Erratum to: "An effective version of Polya's theorem on positive definite forms" [J. Pure Appl. Algebra 108 (1996) 231-240]. *Journal of Pure and Applied Algebra*, *155*(2-3), 309–310.

Luenberger, D. (2003). *Linear and nonlinear programming* (2nd ed.). Springer.

Ma, X.-J., Sun, Z.-Q., & He, Y.-Y. (1998). Analysis and design of fuzzy controller and fuzzy observer. *Fuzzy Systems, IEEE Transactions on*, *6*(1), 41–51.

Nocedal, J., & Wright, S. (1999). Numerical Optimization.

Oliveira, M. d., Bernussou, J., & Geromel, J. (1999). A new discrete-time robust stability condition. *System & Control Letters*, *37*, 261-265.

Oliveira, R. C., & Peres, P. L. (2006). Lmi conditions for robust stability analysis based on polynomially parameter-dependent lyapunov functions. *Systems & Control Letters*, *55*(1), 52–61.

Pólya, G., & Szegö, G. (1928). Aufgaben und Lehrsätze aus der Analysis. *Bull. Amer. Math. Soc. 34 (1928), 233-234.*

Powers, V., & Reznick, B. (2001). A new bound for Polya's Theorem with applications to polynomials positive on polyhedra. *Journal of Pure and Applied Algebra*, *164*(1-2), 221–229.

Prajna, S., Papachristodoulou, A., & Parrilo, P. (2002). *Sostools: Sum of squares optimization toolbox for matlab – user's guide.*

Rugh, W. J., & Shamma, J. S. (2000). Research on gain scheduling. *Automatica*, *36*(10), 1401–1425.

Sala, A. (2007). Reducing the gap between fuzzy and nonlinear control. In *Adv. Fuzzy Neural Control, 2007, Proceedings of the 3rd IFAC Conference on.*

Sala, A., & Ariño, C. (2007a). Relaxed stability and performance conditions for Takagi-Sugeno fuzzy systems with knowledge on membership function overlap. *Systems, Man and Cybernetics, Part B, IEEE Transactions on*, *37*(3), 727–732.

Sala, A., & Arino, C. (2007b). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, *158*(24), 2671–2686.

Sala, A., Guerra, T. M., & Babuska, R. (2005). Perspectives of fuzzy systems and control. *Fuzzy Sets Syst.*, *156*, 432-444.

Scherer, C. (2005). Relaxations for robust linear matrix inequality problems with verifications for exactness. *SIAM Journal on Matrix Analysis and Applications*, *27*(2), 365–395.

Scherer, C., Gahinet, P., & Chilali, M. (1997). Multiobjective output-feedback control via LMI optimization. *Automatic Control, IEEE Transactions on*, *42*(7), 896–911.

Slotine, J.-J. E., & Li, W. (1991). *Applied nonlinear control.* Englewood Cliffs,New Jersey: Ed. Prentice Hall.

Spain, B. (2003). *Tensor Calculus: A Concise Course.* Dover Publications.

Sturm, J. (1999). Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones (Updated for Version 1.05). *Optimization Methods and Software*, 625–653.

Sugeno, M. (1992). Fuzzy control: Principles, practice, and perspectives. *Fuzzy Systems, 1992., IEEE International Conference on*, 109-.

Sugeno, M., & Kang, G. (1988). Structure identification of fuzzy model. *Fuzzy Sets and Systems*, *28*(1), 15–33.

Sugeno, M., & Kang, G. T. (1986). Fuzzy modeling and control of multilayer incinerator. *Fuzzy Sets Syst.*, *18*(3), 329–345.

Sugeno, M., & Nishida, M. (1985). Fuzzy control of model car. *Fuzzy Sets and Systems*, *16*(2), 103–113.

Takagi, T., & Sugeno, M. (1985). Fuzzy identification of systems and its applications to modelling and control. *IEEE Trans. on Systems, Man and Cybernetics*, *15(1)*, 116–132.

Tanaka, K., Hori, T., & Wang, H. (2003). A multiple lyapunov function approach to stabilization of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, *6*, 250-265.

Tanaka, K., Ikeda, T., & Wang, H. (1996). Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: quadratic stabilizability, H_∞ control theory, and linear matrix inequalities. *Fuzzy Systems, IEEE Transactions on*, *4*(1), 1–13.

Tanaka, K., Ikeda, T., & Wang, H. (1998). Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs. *IEEE Transactions on Fuzzy Systems*, *6*, 250-265.

Tanaka, K., & Sano, M. (1994). A robust stabilization problem of fuzzy control systems and its application to backing up control of a truck-trailer. *Fuzzy Systems, IEEE Transactions on*, *2*(2), 119–134.

Tanaka, K., & Sugeno, M. (1992). Stability analysis and design of fuzzy control systems. *Fuzzy Sets Syst.*, *45*(2), 135–156.

Tanaka, K., & Wang, H. (1997). Fuzzy regulators and fuzzy observers: a linear matrix inequality approach. In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on* (Vol. 2, pp. 1315–1320 vol.2).

Tanaka, K., & Wang, H. O. (2001). *Fuzzy control systems design and analysis.* New York, USA: Ed. John Wiley & Sons.

Teixeira, M., Assuncao, E., & Avellar, R. (2003). On relaxed LMI-based designs for fuzzy regulators and fuzzy observers. *Fuzzy Systems, IEEE Transactions on*, *11*(5), 613-623.

Temple, G. (2004). *Cartesian Tensors: An Introduction.* Dover Publications.

Tuan, H., Apkarian, P., Narikiyo, T., & Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. *Fuzzy Systems, IEEE Transactions on*, *9*(2), 324–332.

Wang, H., Li, J., Niemann, D., & Tanaka, K. (2000). T-S fuzzy model with linear rule consequence and PDC controller: a universal framework for nonlinear control systems. In *Fuzzy Systems, 2000. FUZZ IEEE 2000. The Ninth IEEE International Conference on* (Vol. 2, pp. 549–554 vol.2).

Wang, H., Tanaka, K., & Griffin, M. (1995). Parallel distributed compensation of nonlinear systems by Takagi-Sugeno fuzzy model. In *Fuzzy Systems, 1995. International Joint Conference of the Fourth IEEE International Conference on Fuzzy Systems and The Second International Fuzzy Engineering Symposium., Proceedings of 1995 IEEE International Conference on* (Vol. 2, pp. 531–538 vol.2).

Wolkowicz, H., & Stern, R. (1995). Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations. *SIAM Journal on Optimization*, *5*(2), 286–313.

Ying, H. (1998). Sufficient conditions on uniform approximation of multivariate functions by general Takagi-Sugeno fuzzy systems with linear rule consequent. *Systems, Man and Cybernetics, Part A, IEEE Transactions on*, *28*(4), 515–520.

Zadeh, L. (1973). Outline of a new approach to the analysis of complex systems and decision processes. *IEEE Transactions on Systems, Man, and Cybernetics*, *3*, 28–44.