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A note on a result of Guo and Isaacs about p -supersolubility of finite groups

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Abstract

In this note, global information about a finite group is obtained by assuming that certain subgroups of some given order are S-semipermutable. Recall that a subgroup H of a finite group G is said to be S-semipermutable if H permutes with all Sylow subgroups of G of order coprime to $|H|$. We prove that for a fixed prime p , a given Sylow p -subgroup P of a finite group G , and a power d of p dividing $|G|$ such that $1 \leq d < |P|$, if $H \cap O^p(G)$ is S-semipermutable in $O^p(G)$ for all normal subgroups H of P with $|H| = d$, then either G is p -supersoluble or else $|P \cap O^p(G)| > d$. This extends the main result of Guo and Isaacs in *Arch. Math. (Basel)*, 105, 215–222 (2015). We derive some theorems that extend some known results concerning S-semipermutable subgroups.

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1 Introduction

All groups mentioned are implicitly assumed to be finite.

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Recall that two subgroups H and K of a group G are said to *permute* if $HK = KH$, that is, HK is a subgroup of G . A subgroup H of a group G is said to be *S-permutable* ([5], see also [1, Section 1.2]) in G if H permutes with all Sylow subgroups of G , and is said to be *S-semipermutable* in G ([9]) if H permutes with all Sylow q -subgroups of G for the primes q not dividing $|H|$.

Skiba, in his seminal paper [8], introduced the following subgroup embedding property: a subgroup H of a group G is said to be *weakly S-permutable* in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the largest S -permutable subgroup of G contained in H . This embedding property of subgroups has a strong structural impact and generalises many other known properties.

Recently, Guo and Isaacs considered in [2] the condition $U \cap H \trianglelefteq U$, with $U = O^p(G)$, for a subgroup H of order d , where $d > 1$ is a power of p such that d divides $|G|$ and p is a prime that we hold fixed. This condition is less restrictive than weak S -permutability [2, Lemma A] and allows them to prove the following result.

Theorem 1 ([2, Theorem B]). *Let $P \in \text{Syl}_p(G)$ and let d be a power of p such that $1 \leq d < |P|$. Let $U = O^p(G)$, and assume that $H \cap U \trianglelefteq U$ for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either G is p -supersoluble, or else $|P \cap U| > d$.*

An interesting idea of [2] is that in the hypothesis of the theorem, only the normal subgroups of order d are considered, not necessarily the set of all subgroups of order d , at the drawback of obtaining as a conclusion either p -supersolubility or a restriction on the order of the Sylow p -subgroup of $O^p(G)$.

We prove an analogous result, but instead of assuming that all subgroups $H \cap U$ are normal in U , we assume that all of them are S -semipermutable in U . Our starting point is the following observation. Let A be a p -subgroup of a group G . By [1, Lemma 1.2.16], $A \cap O^p(G) \trianglelefteq O^p(G)$ if and only if $A \cap O^p(G)$ is S -permutable in G . If A is contained in $O^p(G)$, and q is a prime different from p , then every Sylow q -subgroup of G is contained in $O^p(G)$. Hence A is S -semipermutable in G if and only if A is S -semipermutable in $O^p(G)$.

We prove:

Theorem 2. *Let $P \in \text{Syl}_p(G)$ and let d be a power of p such that $1 \leq d < |P|$. Assume that $H \cap O^p(G)$ is S -semipermutable in G for all subgroups $H \trianglelefteq P$ with $|H| = d$. Then either G is p -supersoluble, or else $|P \cap O^p(G)| > d$.*

We present some applications of Theorem 2. They concern the structure of a group G in which the subgroups $H \cap O^p(G)$ are S -semipermutable in G for

all subgroups H of a fixed order d of a given Sylow p -subgroup of G , and can be regarded as S-semipermutable versions of [2, Corollary C and Corollary E]. They can be also considered as an improvement of the following result.

Theorem 3 ([7, Theorem 3.3]). *Let $P \in \text{Syl}_p(G)$ and let d be a power of p such that $p \leq d < |P|$. Assume that every subgroup of P with order d and all cyclic subgroups of P of order 4 (if $d = 2$ and P is not abelian) are S-semipermutable in G . Then G is p -supersoluble.*

We consider first the case when d is a prime.

Theorem 4. *Let $P \in \text{Syl}_p(G)$. Suppose that $H \cap O^p(G)$ is S-semipermutable in G for all subgroups $H \leq P$ with H cyclic of order p or 4 (if $p = d = 2$ and P is not abelian). Then G is p -supersoluble.*

Our last result concerns the case when $p < d < |P|$.

Theorem 5. *Let $P \in \text{Syl}_p(G)$ and let d be a power of p such that $p < d < |P|$. Suppose that $H \cap O^p(G)$ is S-semipermutable in G for all subgroups $H \leq P$ with $|H| = d$. Then G is p -supersoluble.*

We mention that Corollary C and Corollary E of [2] by Guo and Isaacs are immediate consequences of Theorems 4 and 5. Note that the proof of Corollary E in [2] is incomplete.

2 Proofs

Proof of Theorem 2. Assume the result is not true and let G be a counterexample of least order. Write $U = O^p(G)$, $N = P \cap U$. Then $|N| \leq d$ and G is not p -supersoluble. In particular, $N \neq 1$ and $d \geq p$. Write $\mathcal{H} = \{H \leq P \mid |H| = d\}$. By hypothesis, $H \cap U$ is S-semipermutable in G for each $H \in \mathcal{H}$.

Step 1. $O_{p'}(G) = 1$.

Write $V = O_{p'}(G)$ and assume that $V \neq 1$. Consider the factor group $\bar{G} = G/V$. Let \bar{H} be a normal subgroup of \bar{P} of order d . Then there is $H \in \mathcal{H}$ such that $\bar{H} = HV/V$. Since $H \cap U$ is S-semipermutable in G , we have $\bar{H} \cap \bar{U} = (H \cap U)V/V$ is S-semipermutable in G . Since $|\bar{U} \cap \bar{P}| \leq d$, it follows that \bar{G} is p -supersoluble by the minimal choice of G . Hence G is p -supersoluble and this is contradiction. Thus $V = 1$, as required.

Step 2. N is S-semipermutable in G .

Since N is normal in P of order at most d , there is $H \in \mathcal{H}$ such that $N \leq H \leq P$. Then $N = H \cap U$ is S-semipermutable in G .

Step 3. Let Y be a maximal subgroup of P . Then $Y \cap U = Y \cap N$ is S -semipermutable in G .

Since $|N| \leq d$, there is an $H \in \mathcal{H}$ such that $Y \cap N \leq Y \cap U \leq H \leq Y$. Then $Y \cap N = Y \cap N \cap H = H \cap N = H \cap P \cap U = H \cap U$. Thus $Y \cap N = Y \cap U = H \cap U$ is S -semipermutable in G .

Step 4. Let $\langle N^G \rangle$ be the normal closure of N in G . Then $G/\langle N^G \rangle$ is p -nilpotent. In particular, G is p -soluble.

Since $\langle N^G \rangle \leq U$ and $|U : \langle N^G \rangle|$ divides the p' -number $|U : N|$, we have that $U/\langle N^G \rangle$ is a p' -group. Moreover, G/U is a p -group. It follows that $G/\langle N^G \rangle$ is p -nilpotent. On the other hand, we know that $G = PU$ and, by Step 2, N permutes with each Sylow q -subgroup of U for every prime $q \neq p$. By [4, Theorem A], the normal closure $\langle N^G \rangle$ of N in G is soluble. We conclude that G is p -soluble.

Step 5. A final contradiction.

Assume that $d = p$. Then N is a normal Sylow p -subgroup of U by Steps 1 and 4. Hence G is p -supersoluble, against the choice of G . Thus $d > p$. Let T be a minimal normal subgroup of G contained in U . By Steps 1 and 4, $T \leq N$. Thus $|T| \leq d$. Suppose that $|T| < d$. We argue that G/T satisfies the hypotheses of the theorem. Clearly, $1 \leq d/|T| < |P/T|$. Let H/T be a normal subgroup of P/T of order $d/|T|$. Then $H \in \mathcal{H}$. It follows that $H/T \cap O^p(G/T) = H/T \cap U/T = (H \cap U)/T$, which is S -semipermutable in G/T . This shows that G/T satisfies the hypotheses of the theorem, as claimed. Since $|P/T \cap U/T| = |(P \cap U)/T| \leq d/|T|$, it follows that G/T is p -supersoluble by minimality of G . By [3, Kapitel VI, Satz 8.6], we may suppose that $T \not\leq \Phi(P)$. Let Y be a maximal subgroup of P such that $T \not\leq Y$. By Step 3, $Y \cap N$ is S -semipermutable in G , and so $(Y \cap N)Q$ is a subgroup of G for every Sylow q -subgroup Q of G with $q \neq p$. Since T is a normal p -subgroup of G , it follows that $T \cap (Y \cap N)Q = T \cap Y \cap N = T \cap Y$, which is normal in $(Y \cap N)Q$. Then $T \cap Y$ is normalised by Q . It follows that $T \cap Y$ is normalised by U . Since $T \cap Y$ is a normal subgroup of P , we have that $T \cap Y \trianglelefteq G$. This implies that $T \cap Y = 1$, $|T| = p$, and so G is p -supersoluble, which is a contradiction.

Let $|T| = d$. Then $T = N$. By Step 4, G/T is p -nilpotent. Since the class of all p -nilpotent groups is a saturated formation by [3, Kapitel VI, Beispiele 7.6], T is a proper subgroup of P which is not contained in $\Phi(G)$. Let Y be a maximal subgroup of P such that $T \not\leq Y$. By Step 3, $N \cap Y = T \cap Y$ is S -semipermutable in G . Arguing as in the previous paragraph, we obtain that $Y \cap T = Y \cap N$ is normal in G . This implies that $T \cap Y = 1$ and $|T| = p$, against our assumption $|T| = d > p$. This final contradiction completes the proof. \square

Proof of Theorem 4. We proceed by induction on $|G|$. Write $U = O^p(G)$, $U_p = P \cap U$. We may suppose that $O_{p'}(G) = 1$. By Theorem 2, we may suppose that $|U_p| > p$. If $G = U$, then each subgroup of order p or 4 is S-semipermutable in G . Applying Theorem 3, we get that G is p -supersoluble. Therefore, we may assume that U is a proper subgroup of G . Clearly $U = O^p(U)$ and every subgroup of order p or 4 of U_p is S-semipermutable in U by [1, Lemma 1.2.7]. By the induction hypothesis, U is p -supersoluble. Since $O_{p'}(U) = 1$, it follows that U_p is normal in U by [1, Lemma 1.2.16]. Hence U_p is a normal subgroup of G . Assume that $p = 2$. Then U is 2-nilpotent and so it is a 2-group. This implies that G is a 2-group. Therefore we may suppose that $p > 2$. Then every chief factor of G below U_p is cyclic, by [6, Theorem 3.3]. Since G/U_p has a normal Hall p' -subgroup, we conclude that G is p -supersoluble. \square

The proof of Theorem 5 depends on the following lemmas. The first one is the S-semipermutable version of [2, Corollary C].

Lemma 6. *Let $P \in \text{Syl}_p(G)$ with $|P| > p$. Suppose that, for every maximal subgroup H of P , $H \cap O^p(G)$ is S-semipermutable in G . Then G is p -supersoluble.*

Proof. With a contradiction in mind, assume that G is not p -supersoluble. Write $U = O^p(G)$. By Theorem 2, $P \cap U = P$, that is, $P \leq U$, and so $G = U$. This means that every maximal subgroup of P is S-semipermutable in G . Applying [7, Theorem 3.2], we conclude that G is p -supersoluble. \square

Lemma 7. *Let $P \in \text{Syl}_p(G)$ and write $U = O^p(G)$. Suppose that $U_p = P \cap U$ is a normal p -subgroup of G and that d is a power of p such that $p < d < |U_p|$. Suppose also that H is S-permutable in G for all subgroups $H \leq U_p$ with $|H| = d$. Then G is p -supersoluble.*

Proof. We proceed by induction on $|G|$. We may suppose that $O_{p'}(G) = 1$. If $U = G$, we can apply Theorem 3 to conclude that G is p -supersoluble. Therefore we may assume that U is a proper subgroup of G . Since $U = O^p(U)$ and every subgroup of order d of U_p is S-permutable in U by [1, Lemma 1.2.7], we have that U is p -supersoluble. If $p = 2$, we can argue as in Theorem 4 to conclude that G is a 2-group. Hence we may assume $p > 2$.

Let T be a minimal normal subgroup of G contained in U_p . Assume that $|T| > d$. Let H be a normal subgroup of P such that $H \leq T$ and $|H| = d$. Since H is S-permutable in G , we have that $U \leq N_G(H)$ by [1, Lemma 1.2.16]. Therefore $G = UP \leq N_G(H)$, that is, H is a normal subgroup of G . Since T is a minimal normal subgroup of G , we conclude that $H = 1$, against our

assumption $|H| = d > p$. Therefore $|T| \leq d$. We focus now on G/T and prove that it is p -supersoluble.

Assume that $|T| < d$. Then H/T is S-permutable in G/T for each $H/T \leq U_p/T$ with $|H/T| = d/|T|$. If $d/|T| > p$, then G/T is p -supersoluble by induction. If $d/|T| = p$, the conclusion follows from Theorem 4.

Assume now that $|T| = d$. Therefore we may assume that p is odd. Bearing in mind Lemma 6, we may suppose that $|U_p| > p|T| = pd$. Let K be a subgroup of U_p such that $|K/T| = p$. Since T is non-cyclic, there is a maximal subgroup L of K such that $K = TL$. Then L is of order d and so it is S-permutable in G . By [1, Lemma 1.2.7], K/T is S-permutable in G/T . By Theorem 4, G/T is p -supersoluble.

Since G/T is p -supersoluble and the class of all p -supersoluble groups is a saturated formation by [3, Kapitel VI, Hilfssatz 8.3], we may assume that T is the unique minimal normal subgroup of G contained in U_p and $\Phi(G) \cap U_p = 1$. By [3, Kapitel III, Satz 4.5], U_p is a direct product of some minimal normal subgroups of G . Thus $|U_p| = |T|$, against the hypothesis $|T| \leq d < |U_p|$. This final contradiction completes the proof. \square

Proof of Theorem 5. We may suppose that $O_{p'}(G) = 1$. By Theorem 2, we may suppose that $|P \cap U| > p$. If $G = U$, then each subgroup of order d is S-semipermutable in G . By Theorem 3, the conclusion follows. Suppose that $U < G$. By [1, Lemma 1.2.7], the hypotheses of the theorem hold in U . By induction, U is p -supersoluble. Since $O_{p'}(U) = 1$, we can apply [1, Lemma 1.2.16] to conclude that $U_p = U \cap P$ is a normal subgroup of G . Therefore all subgroups of order d are S-permutable in G by [7, Lemma 2.2]. Applying Lemma 7, we conclude that G is p -supersoluble. \square

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