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Additional Information

On fuzzy ψ-contractive sequences and fixed point theorems

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Abstract

In this paper we give a fixed point theorem in the context of fuzzy metric spaces in the sense of George and Veeramani. As a consequence of our result we obtain a fixed point theorem due to D. Mihet and generalize a fixed point theorem due to D. Wardowski. Also, we answer in a positive way to a question posed by D. Wardowski, and solve partially an open question on Cauchyness and contractivity.

Keywords: Fuzzy metric space, Fuzzy contractive mapping, Fixed point
2010 MSC: 54A40, 54D35, 54E50

1. Introduction

In 1975, Kramosil and Michalek [6] gave a notion of fuzzy metric space (KM-fuzzy metric space along the paper), which was modified later by George and Veeramani [1] (fuzzy metric space along the paper). Since then, many authors have contributed to the study of these concepts of fuzzy metric. One of the most important topics of research in this field has been the fixed point theory. The first attempt to extend the well-known Banach contraction theorem to KM-fuzzy metrics was done by Grabiec in [2]. Later, Gregori

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and Sapena [5] gave another notion of fuzzy contractive mapping and studied its applicability to fixed point theory in both contexts of fuzzy metrics above mentioned. In their study, the authors needed to demand additional conditions to the completeness of the fuzzy metric in order to obtain a fixed point theorem, which constitutes a significant difference with the classical theory. So, in [5] it was formulated the question ($Q1$): Is a fuzzy contractive sequence a Cauchy sequence (in the sense of George and Veeramani)?) D. Mihet showed that the answer to this question in the context of KM-fuzzy metric spaces is negative [7, Remark 3.1]. Later, this notion of fuzzy contractive mapping and others appeared in the literature were generalized by D. Mihet in [7] introducing the concept of fuzzy ψ-contractive mapping and he obtained a fixed point theorem for the class of complete non-Archimedean KM-fuzzy metrics.

Recently, D. Wardowski [9] has provided a new contribution to the study of fixed point theory in fuzzy metric spaces. In [9], the author introduced the concept of fuzzy $\mathcal{H}$-contractive mappings (Definition 2.9), which constitutes a generalization of the concept given by V. Gregori and A. Sapena, and he obtained the next fixed point theorem for complete fuzzy metric spaces in the sense of George and Veeramani.

Theorem 1.1. *(Wardowski [9]). Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a fuzzy $\mathcal{H}$-contractive mapping with respect to $\eta \in \mathcal{H}$ such that:*

(a) $\Pi_{i=1}^{k} M(x, f(x), t_i) \neq 0$*, for all* $x \in X$*,* $k \in \mathbb{N}$ *and any sequence* $\{t_i\} \subset]0, \infty[$*,* $t_i \searrow 0$*;*

(b) $r * s > 0 \Rightarrow \eta(r * s) \leq \eta(r) + \eta(s)$*, for all* $r, s \in \{M(x, f(x), t) : x \in X, t > 0\}$*;*

(c) $\{\eta(M(x, f(x), t_i)) : i \in \mathbb{N}\}$ *is bounded for all* $x \in X$ *and any sequence* $\{t_i\} \subset]0, \infty[$*,* $t_i \searrow 0$*.*

Then f has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ the sequence $\{f^n(x_0)\}$ converges to x^*.

In [9], Wardowski proposed the question ($Q2$): Is it possible to omit condition (a) in the last theorem?

Notice that, V. Gregori and J. J. Miñana [3] have shown recently that the class of fuzzy $\mathcal{H}$-contractive mappings is included in the class of fuzzy ψ-contractive mappings.

In this paper we answer in affirmative way the question $(Q1)$ for the (more general) class of fuzzy ψ-contractive mappings when M is strong (Lemma 3.12) or M satisfies $\bigwedge_{t>0} M(x,y,t) > 0$ for each $x,y \in X$ (Corollary 3.8). Then, we state our fuzzy fixed point theorem (Theorem 3.3). As a consequence we answer in affirmative way the question $(Q2)$ and, moreover, we show that the condition (b) in the above theorem can also be omitted (Corollary 3.6). Also, as a consequence of our Lemma 3.12 we deduce a fixed point theorem due to D. Mihet (Theorem 3.13).

The structure of the paper is as follows. After a preliminaries' section, we give our main results in Section 3, which we have mentioned in the last paragraph.

2. Preliminaries

Definition 2.1. (George and Veeramani [1]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ (or simply M) is a *fuzzy metric* on X.

The next definition of KM-fuzzy metric space is the reformulation due to Grabiec of the original definition of Kramosil and Michalek [6], which is commonly used by several authors.

Definition 2.2. (Grabiec [2]). A KM-fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty[$ that satisfies (GV3) and (GV4), and (GV1), (GV2), (GV5) are replaced by (KM1), (KM2), (KM5), respectively, below:

(KM1) $M(x, y, 0) = 0$;

(KM2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;

(KM5) $M(x,y,\_) : [0,\infty[\to [0,1]$ is left continuous.

Remark 2.3. *$M(x,y,\_)$ is non-decreasing for all $x, y \in X$.*

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x,\epsilon,t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x,\epsilon,t) = \{y \in X : M(x,y,t) > 1-\epsilon\}$ for all $x \in X$, $\epsilon \in]0,1[$ and $t > 0$.

Let (X,d) be a metric space and let M_d a function on $X \times X \times]0,\infty[$ defined by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [1], and M_d is called the *standard fuzzy metric* induced by d. The topology τ_{M_d} coincides with the topology $\tau(d)$ on X deduced from d.

Definition 2.4. (Gregori and Romaguera [4]). A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x,y,t)$ is constant. In this case we write $M(x,y)$ instead of $M(x,y,t)$.

Definition 2.5. Let $(X,M,*)$ be a fuzzy metric space. The fuzzy metric M (or the fuzzy metric space $(X,M,*)$) is said to be *strong* (non-Archimedean) if it satisfies for each $x, y, z \in X$ and each $t > 0$

$$(GV4') \quad M(x,z,t) \geq M(x,y,t) * M(y,z,t)$$

Proposition 2.6. *(George and Veeramani [1]). A sequence $\{x_n\}$ in X converges to x if and only if $\lim_n M(x_n,x,t) = 1$, for all $t > 0$.*

Definition 2.7. (George and Veeramani [1]). A sequence $\{x_n\}$ in a fuzzy metric space $(X,M,*)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0,1[$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\epsilon$ for all $n, m \geq n_0$ or, equivalently, $\lim_{n,m} M(x_n,x_m,t) = 1$ for all $t > 0$. X is said to be *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

Definition 2.8. (Mihet [7]). Let Ψ be the class of all mappings $\psi:]0,1] \to]0,1]$ such that ψ is continuous, non-decreasing and $\psi(t) > t$ for all $t \in]0,1[$. Let $\psi \in \Psi$. A mapping $f : X \to X$ is said to be fuzzy ψ-contractive mapping if:

$$M(f(x), f(y), t) \geq \psi(M(x, y, t)) \text{ for all } x, y \in X \text{ and } t > 0. \tag{1}$$

A sequence $\{x_n\}$ in X is said to be *fuzzy ψ-contractive sequence* if satisfies

$$M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t)) \text{ for all } n \in \mathbb{N} \text{ and } t > 0. \tag{2}$$

Definition 2.9. (Wardowski [9]). Denote by $\mathcal{H}$ the family of mappings $\eta :]0,1] \to [0,\infty[$ satisfying the following two conditions:

(H1) η transforms $]0,1]$ onto $[0,\infty[$;

(H2) η is strictly decreasing.

A mapping $f : X \to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta \in \mathcal{H}$ if there exists $k \in]0,1[$ satisfying the following condition:

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)) \tag{3}$$

for all $x, y \in X$ and $t > 0$.

The authors in [3] gave the next proposition, which relates the class of fuzzy ψ-contrative mappings and the class of fuzzy $\mathcal{H}$-contractive mappings.

Proposition 2.10. *(Gregori and Miñana [3]). The class of fuzzy $\mathcal{H}$-contractive mappings are included in the class of fuzzy ψ-contrative mappings.*

The author in [8] generalizes the concept of fuzzy $\mathcal{H}$-contractive removing the requirement that f to be onto as we see in the next definition. We denote this larger class of fuzzy contractive mappings by fuzzy $\mathcal{H}_w$-contractive mappings.

Definition 2.11. (Mihet [8]). Let $\mathcal{H}_w$ be the family of all continuous, strictly decreasing mappings $\eta :]0,1] \to [0,\infty[$, with $\eta(1) = 0$ and let $(X, M, *)$ be a fuzzy metric space. A mapping $f : X \to X$ is said to be fuzzy $\mathcal{H}_w$-contractive with respect to $\eta \in \mathcal{H}_w$ if there exists $k \in]0,1[$ satisfying

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)) \tag{4}$$

for all $x, y \in X$ and $t > 0$.

Theorem 2.12. *(Mihet [8]). Let $*_g$ be the strict t-norm generated by the mapping $g \in \mathcal{H}_w$ (see [8]). Let $(X, M, *)$ be a complete KM-fuzzy metric space with $* \geq *_g$. Then every fuzzy $\mathcal{H}_w$-contractive mapping $f : X \to X$ for g has a fixed point, provided $\lim_{t\to 0^+} M(x, f(x), t) > 0$ for some $x \in X$ ($\lim_{t\to 0^+}$ denotes the one-sided limit as t approaches 0 from the right).*

Remark 2.13. *In both original definitions of fuzzy ψ-contractive and $\mathcal{H}_w$-contractive mapping, respectively, the domain of definition of ψ and η is $[0, 1]$, since the author made his study for KM-fuzzy metric spaces. Now, a purpose of our paper is to answer a question posed by Wardowsi, who made his study in fuzzy metric spaces. Then, in Definition 2.8 and 2.11, respectively, we have changed the mentioned domain by $(0, 1]$, since in a fuzzy metric space $M(x, y, t) > 0$ for each $x, y \in X$ and each $t > 0$, and so it is not necessary to define ψ and η in 0. Now, our results also can be established for KM-fuzzy metric spaces. (Indeed, in the proofs of the mentioned theorems, it does not play any role the fact that $M(x, y, t)$ could take the value 0 for some $x, y \in X$ and $t > 0$.)*

3. The results

We begin this section with the next two lemmas under the above terminology.

Lemma 3.1. *If $\psi \in \Psi$, then $\lim_n \psi^n(t) = 1$ for each $t \in]0, 1]$.*

Proof. We will make this proof by contradiction.

Suppose that there exists $t_0 \in]0, 1]$ such that $\lim_n \psi^n(t_0) \neq 1$. Note that $\psi^{n+1}(t_0) > \psi^n(t_0)$ for each $n \in \mathbb{N}$. Then, the sequence $\{\psi^n(t_0)\}_n$ converges in $]0, 1]$, since it is strictly increasing.

Suppose that $\lim_n \psi^n(t_0) = l$ for some $l \in]0, 1[$. Then, for each $n \in \mathbb{N}$ we have that $\psi^n(t_0) \leq l$. So, $\psi(\psi^{n-1}(t_0)) \leq l$ and by continuity of ψ we have that $l \geq \psi(\lim_n \psi^{n-1}(t_0)) = \psi(l) > l$, a contradiction. □

Lemma 3.2. *Let $(X, M, *)$ be a fuzzy metric space and let $\{x_n\}$ be a fuzzy ψ-contractive sequence in X. If $\bigwedge_{t>0} M(x_0, x_1, t) > 0$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a fuzzy ψ-contractive sequence in X and suppose that $\bigwedge_{t>0} M(x_0, x_1, t) = a > 0$.

We will see that $\lim_n \left(\bigwedge_{t>0} M(x_n, x_{n+1}, t)\right) = 1$. For it, first we will prove by induction on n, that

$$\bigwedge_{t>0} M(x_n, x_{n+1}, t) \geq \psi^n(a), \text{ for each } n \in \mathbb{N}. \tag{5}$$

Since $\{x_n\}$ is a fuzzy ψ-contractive sequence, for each $t > 0$ we have that $M(x_1, x_2, t) \geq \psi(M(x_0, x_1, t)) \geq \psi(a)$. Then, $\bigwedge_{t>0} M(x_1, x_2, t) \geq \psi(a)$.

Suppose that $\bigwedge_{t>0} M(x_n, x_{n+1}, t) \geq \psi^n(a)$ for some $n \in \mathbb{N}$ and we will see that the inequality holds for $n + 1$.

As above, since $\{x_n\}$ is a fuzzy ψ-contractive sequence, for each $t > 0$ we have that $M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t)) \geq \psi\left(\bigwedge_{t>0} M(x_n, x_{n+1}, t)\right) \geq \psi(\psi^n(a))$. (The second inequality is a consequence that $M(x, y, \_)$ is non-decreasing for each $x, y \in X$ and ψ is increasing, and the last one is obtained by the induction hypothesis.) Then, $\bigwedge_{t>0} M(x_{n+1}, x_{n+2}, t) \geq \psi^{n+1}(a)$.

Thus, $\bigwedge_{t>0} M(x_n, x_{n+1}, t) \geq \psi^n(a)$ for each $n \in \mathbb{N}$ and so taking limit as n tends to infinity, by Lemma 3.1 we have that

$$\lim_n \left(\bigwedge_{t>0} M(x_n, x_{n+1}, t)\right) \geq \lim_n \psi^n(a) = 1.$$

Therefore, $\lim_n \left(\bigwedge_{t>0} M(x_n, x_{n+1}, t)\right) = 1$.

Now, we will show that $\{x_n\}$ is a Cauchy sequence by contradiction.

Suppose that $\{x_n\}$ is not Cauchy. Then, there exist $\epsilon \in]0, 1[$ and $t_0 > 0$ such that for each $k \in \mathbb{N}$ we can find $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and $M(x_{m(k)}, x_{n(k)}, t_0) \leq 1 - \epsilon$.

Fix $k \in \mathbb{N}$. Then we can find $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and $M(x_{m(k)}, x_{n(k)}, t_0) \leq 1 - \epsilon$. Given $n(k)$, we choice $m_n(k)$ as the least integer such that $m_n(k) > n(k)$ and $M(x_{m_n(k)}, x_{n(k)}, t_0) \leq 1 - \epsilon$. Then, $M(x_{m_n(k)-1}, x_{n(k)}, t_0) > 1 - \epsilon$. We will prove that $\lim_k M(x_{m_n(k)}, x_{n(k)}, t_0) = 1 - \epsilon$.

For each $k \in \mathbb{N}$ and each $\delta \in]0, t_0[$ we have that

$$1-\epsilon \geq M(x_{m_n(k)}, x_{n(k)}, t_0) \geq M(x_{m_n(k)}, x_{m_n(k)-1}, \delta) * M(x_{m_n(k)-1}, x_{n(k)}, t_0 - \delta).$$

Then, for each $k \in \mathbb{N}$

$$1-\epsilon \geq M(x_{m_n(k)}, x_{n(k)}, t_0) \geq \lim_{\delta \to 0} \left(M(x_{m_n(k)}, x_{m_n(k)-1}, \delta) * M(x_{m_n(k)-1}, x_{n(k)}, t_0 - \delta)\right) =$$

$$\left(\lim_{\delta\to 0} M(x_{m_n(k)}, x_{m_n(k)-1}, \delta)\right) * \left(\lim_{\delta\to 0} M(x_{m_n(k)-1}, x_{n(k)}, t_0 - \delta)\right) =$$

$$\left(\bigwedge_{t>0} M(x_{m_n(k)}, x_{m_n(k)-1}, t)\right) * M(x_{m_n(k)-1}, x_{n(k)}, t_0).$$

The first equality has been obtained by continuity of $*$ and the second equality is consequence that $M(x, y, \_)$ is non-decreasing for each $x, y \in X$ and continuous.

Therefore,

$$\limsup_k M(x_{m_n(k)}, x_{n(k)}, t_0) \leq 1 - \epsilon,$$

and

$$\liminf_k M(x_{m_n(k)}, x_{n(k)}, t_0) \geq$$

$$\lim_k \left(\left(\bigwedge_{t>0} M(x_{m_n(k)}, x_{m_n(k)-1}, t)\right) * M(x_{m_n(k)-1}, x_{n(k)}, t_0)\right) =$$

$$\left(\lim_k \left(\bigwedge_{t>0} M(x_{m_n(k)}, x_{m_n(k)-1}, t)\right)\right) * \left(\lim_k M(x_{m_n(k)-1}, x_{n(k)}, t_0)\right) \geq$$

$$1 * (1 - \epsilon) = 1 - \epsilon.$$

Then, $1 - \epsilon \geq \limsup_k M(x_{m_n(k)}, x_{n(k)}, t_0) \geq \liminf_k M(x_{m_n(k)}, x_{n(k)}, t_0) \geq 1 - \epsilon$. Thus, $\lim_k M(x_{m_n(k)}, x_{n(k)}, t_0) = 1 - \epsilon$.

On the other hand, for each $k \in \mathbb{N}$ and each $\delta \in]0, t_0/2[$ we have that

$$M(x_{m_n(k)}, x_{n(k)}, t_0) \geq$$

$$M(x_{m_n(k)}, x_{m_n(k)+1}, \delta) * M(x_{m_n(k)+1}, x_{n(k)+1}, t_0 - 2\delta) * M(x_{n(k)+1}, x_{n(k)}, \delta) \geq$$

$$M(x_{m_n(k)}, x_{m_n(k)+1}, \delta) * \psi\left(M(x_{m_n(k)}, x_{n(k)}, t_0 - 2\delta)\right) * M(x_{n(k)+1}, x_{n(k)}, \delta).$$

Taking limit as δ tends to 0, in a similar way that before, we have that for each $k \in \mathbb{N}$

$$M(x_{m_n(k)}, x_{n(k)}, t_0) \geq$$

$$\left(\bigwedge_{t>0} M(x_{m_n(k)}, x_{m_n(k)+1}, t)\right) * \psi\left(M(x_{m_n(k)}, x_{n(k)}, t_0)\right) * \left(\bigwedge_{t>0} M(x_{n(k)+1}, x_{n(k)}, t)\right).$$

Letting $k \to \infty$, by continuity of $*$ and continuity of ψ we have that

$$1 - \epsilon = \lim_k M(x_{m_n(k)}, x_{n(k)}, t_0) \geq$$

$$\left(\lim_k \left(\bigwedge_{t>0} M(x_{m_n(k)}, x_{m_n(k)+1}, t)\right)\right) * \psi\left(\lim_k M(x_{m_n(k)}, x_{n(k)}, t_0)\right) *$$

$$* \left(\left(\lim_k \bigwedge_{t>0} M(x_{n(k)+1}, x_{n(k)}, t)\right)\right) = 1 * \psi(1-\epsilon) * 1 = \psi(1-\epsilon) > 1-\epsilon,$$

a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

□

The next theorem gives a characterization of the class of fuzzy ψ-contractive mappings with a unique fixed point in a complete fuzzy metric space.

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a fuzzy ψ-contractive mapping. Then,*

f has a unique fixed point if and only if there exists $x \in X$ such that $\bigwedge_{t>0} M(x, f(x), t) > 0$.

Proof. Suppose that f has a unique fixed point, then there exists $x \in X$ such that $f(x) = x$. Thus, $M(x, f(x), t) = M(x, x, t) = 1$ for each $t > 0$ and so $\bigwedge_{t>0} M(x, f(x), t) = 1$.

Conversely, suppose that there exists $x \in X$ such that $\bigwedge_{t>0} M(x, f(x), t) > 0$. Take $x_0 = x$ and consider $x_n = f^n(x)$ for each $n \geq 1$. Then,

$$M(x_{n+1}, x_{n+2}, t) = M(f(x_n), f(x_{n+1}), t) \geq \psi\left(M(x_n, x_{n+1}, t)\right).$$

Thus, $\{x_n\}$ is a fuzzy ψ-contractive sequence. Further, $\bigwedge_{t>0} M(x_0, x_1, t) = \bigwedge_{t>0} M(x, f(x), t) > 0$.

By the above lemma $\{x_n\}$ is a Cauchy sequence and since $(X, M, *)$ is complete, there exists $y \in X$ such that $\lim_n M(x_n, y, t) = 1$ for each $t > 0$.

On the other hand, $M(f(y), x_{n+1}, t) \geq \psi(M(y, x_n, t))$ for each $n \in \mathbb{N}$ and each $t > 0$. Then, $M(f(y), y, t) = \lim_n M(f(y), x_{n+1}, t) \geq \lim_n \psi(M(y, x_n, t)) = 1$ for each $t > 0$. Thus, y is a fixed point of f. By Proposition 3.1 in [7] we have that this fixed point is unique.

□

In the next, we will show that the fixed point theorems of Wardowski [9, Theorem 3.2] and Mihet [8, Theorem 2.4] can be obtained, without any extra condition on the t-norm, as a consequence of our main theorem. First, in the next proposition we will see the relationship between our characterization in Theorem 3.3 and the conditions given by Mihet and Wardowski, respectively.

Proposition 3.4. *Let $(X, M, *)$ be a fuzzy metric space, $\eta \in \mathcal{H}$ and let $f : X \to X$ be a mapping. Given $x \in X$, they are equivalent:*

(i) $\bigwedge_{t>0} M(x, f(x), t) > 0$.

(ii) $\lim_{t\to 0^+} M(x, f(x), t) > 0$.

(iii) $\{\eta(M(x, f(x), t_i)) : i \in \mathbb{N}\}$ *is bounded for any sequence* $\{t_i\} \subset (0, \infty)$ *converging to* 0 *(for the usual topology of* $\mathbb{R}$*).*

Proof. Obviously, (i) and (ii) are equivalent, since $M(x, f(x), \_)$ is non-decreasing. Also, it is obvious that $(iii) \Rightarrow (ii)$. To conclude the proof we will see that $(i) \Rightarrow (iii)$.

Suppose that $\bigwedge_{t>0} M(x, f(x), t) > 0$ and consider a sequence $\{t_i\} \subset (0, \infty)$ converging to 0. Let $a = \bigwedge_{t>0} M(x, f(x), t) \in]0, 1[$.

Then, $M(x, f(x), t_i) \geq a$ for each $i \in \mathbb{N}$. Since η is strictly decreasing, we have that $\eta(M(x, f(x), t_i)) \leq \eta(a)$ for each $i \in \mathbb{N}$. On the other hand, $\eta(a) < \infty$, since η is strictly decreasing and it transforms $]0, 1]$ onto $[0, \infty[$. Therefore, $\sup\{\eta(M(x, f(x), t_i)) : i \in \mathbb{N}\} \leq \eta(a) < \infty$.

□

Next, we will see that the class of fuzzy $\mathcal{H}_w$-contractive mappings are also included in the class of fuzzy ψ-contractive mappings.

Proposition 3.5. *Every fuzzy $\mathcal{H}_w$-contractive mapping is a fuzzy ψ-contractive mapping.*

Proof. Let $(X, M, *)$ be a fuzzy metric space. Suppose that $f : X \to X$ is $\mathcal{H}_w$-contractive with respect to $\eta \in \mathcal{H}_w$. Then there exists $k \in]0, 1[$ such that for all $x, y \in X$ and for all $t > 0$ we have that

$$\eta(M(f(x), f(y), t)) \leq k\eta(M(x, y, t)). \tag{6}$$

If η is onto then f is a fuzzy $\mathcal{H}$-contractive mapping and so by Proposition 2.10 f is fuzzy ψ-contractive.

Suppose that η is not onto. Since η is continuous and strictly decreasing, there exists $a \in]0, \infty[$ such that $\sup\{\eta(t) : t \in]0, 1]\} = a$. Then, $\eta(]0, 1]) = [0, a[$ and $\eta^{-1} : [0, a[\to]0, 1]$ is now well-defined and obviously it is a continuous bijection. Let $\psi :]0, 1] \to]0, 1]$, where $\psi(t) = \eta^{-1}(k\eta(t))$ for each $t \in]0, 1]$. We will see that $\psi \in \varPsi$.

Obviously, ψ is continuous.

Let $s, t \in]0, 1]$, with $s < t$. Since η is strictly decreasing we have that $k\eta(s) > k\eta(t)$. Further, η^{-1} is also strictly decreasing and so, $\eta^{-1}(k\eta(s)) < \eta^{-1}(k\eta(t))$. Thus, $\psi(s) < \psi(t)$ and so ψ is increasing.

We will see now that $\psi(t) > t$ for each $t \in]0,1[$. Contrary, suppose that $\psi(t_0) \leq t_0$ for some $t_0 \in]0,1[$. Then, $t_0 \geq \eta^{-1}(k\eta(t_0))$ and so $\eta(t_0) \leq k\eta(t_0)$, a contradiction, since $k \in]0,1[$. Therefore, $\psi \in \Psi$.

Finally, we will see that f is a fuzzy ψ-contraction.

By definition of ψ and by (6) we have

$$M(f(x), f(y), t) = \eta^{-1}(\eta(M(f(x), f(y), t))) \geq$$

$$\eta^{-1}(k \cdot \eta(M(x, y, t)) = \psi(M(x, y, t)).$$

□

A consequence of the last proposition and Theorem 3.3 is the next corollary, which constitutes a generalization in two senses of the theorem given by Wardoswki [9, Theorem 3.2]. Notice that, this corollary is formulated without any restriction on the t-norm and, also, it is established for the class of fuzzy $\mathcal{H}_w$-contractive mappings. The next corollary also generalizes a result given by Mihet [8, Theorem 2.4].

Corollary 3.6. *Let $(X, M, *)$ be a complete fuzzy metric space and let $f : X \to X$ be a fuzzy $\mathcal{H}_w$-contractive mapping. If $\bigwedge_{t>0} M(x, f(x), t) > 0$ for each $x \in X$, then f has a unique fixed point $x^* \in X$ and for each $x \in X$ the sequence $\{f^n(x)\}$ converges to x^*.*

Proof. Suppose that f is a fuzzy $\mathcal{H}_w$-contractive mapping. By the last proposition, f is a fuzzy ψ-contractive mapping. Then, by Theorem 3.3 f has a unique fixed point $x^* \in X$.

Let $x \in X$ and consider the sequence $\{f^n(x)\}$. Imitating the proof of Theorem 3.3 one can verify that $\{f^n(x)\}$ converges to x^*.

□

Remark 3.7. *Taking into account Proposition 3.4, the last corollary constitutes a positive answer to the question posed by Wardowski [9, Conclusions], since $\mathcal{H} \subseteq \mathcal{H}_w$. Further, this last corollary shows that condition (b) in Wardowski's theorem can also be omitted.*

The condition demanded in Theorem 3.3 (there exists $x \in X$ such that $\bigwedge_{t>0} M(x, f(x), t) > 0$) involves the self-mapping in which is studied the existence of a fixed point. One would wish that in the study of existence of a fixed point of a self-mapping did not appear any restriction on the self-mapping, since the expression of it could be complex. Indeed, commonly the conditions in a fixed point theorem are given on the space of definition of the self-mapping. Next, we give two results in which the conditions demanded do not involve the self-mapping. They are immediate taking into account Lemma 3.2 and Theorem 3.3.

Corollary 3.8. *Let $(X, M, *)$ be a fuzzy metric space such that $\bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$. Then, every fuzzy ψ-contractive sequence is a Cauchy sequence.*

Corollary 3.9. *Let $(X, M, *)$ be a complete fuzzy metric space such that $\bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$ and let $f : X \to X$ be a fuzzy ψ-contractive mapping. Then, f has a unique fixed point.*

Remark 3.10. *Notice that all stationary fuzzy metric M on X satisfies the condition $M(x, y) = \bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$.*

In the next example we see non-stationary fuzzy metrics satisfying this condition.

Example 3.11. (i) Consider the fuzzy metric space $(]0, \infty[, M, \cdot)$ where M is given by $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$. Then, $\bigwedge_{t>0} M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}} > 0$ for each $x, y \in]0, \infty[$.

(ii) Let (X, d) be a metric space. Define on $X^2 \times]0, \infty[$ the fuzzy set $M(x, y, t) = \frac{t+1}{t+1+d(x,y)}$. It is easy to verify that $(X, M, \cdot)$ is a fuzzy metric space and $\bigwedge_{t>0} M(x, y, t) = \frac{1}{1+d(x,y)} > 0$ for each $x, y \in X$. (Notice that the standard fuzzy metric $(X, M_d, \cdot)$ does not satisfy the above condition. Indeed, $\bigwedge_{t>0} M_d(x, y, t) = \bigwedge_{t>0} \frac{t}{t+d(x,y)} = 0$ for each $x, y \in X$, with $x \neq y$.)

As a consequence of Corollary 3.8 we obtain the next lemma.

Lemma 3.12. *Let $(X, M, *)$ be a strong fuzzy metric space. Then, every fuzzy ψ-contractive sequence is a Cauchy sequence.*

Proof. Let $\{x_n\}$ be a fuzzy ψ-contractive sequence. We will show that $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$.

Fix $t > 0$ and consider the stationary fuzzy metric given by $M_t(x, y) = M(x, y, t)$ for each $x, y \in X$. Since M_t is stationary, by Remark 3.10 the condition of Corollary 3.8 is fulfilled.

On the other hand, $\{x_n\}$ is a fuzzy ψ-contractive sequence for M_t. Indeed,

$$M_t(x_{n+1}, x_{n+2}) = M(x_{n+1}, x_{n+2}, t) \geq \psi(M(x_n, x_{n+1}, t) = \psi(M_t(x_n, x_{n+1}))$$

for all $n \in \mathbb{N}$.

Then, by Corollary 3.8 we have that $\{x_n\}$ is M_t-Cauchy, i.e. $\lim_{n,m} M_t(x_n, x_m) = 1$ and so $\lim_{n,m} M(x_n, x_m, t) = \lim_{n,m} M_t(x_n, x_m) = 1$.

Taking into account that $t > 0$ is arbitrary, then $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$ and so $\{x_n\}$ is M-Cauchy.

$\square$

Now, we will use the last lemma for obtaining the fixed point theorem of D. Mihet [7, Theorem 3.1] in a fuzzy metric space (in the sense of George and Veeramani).

Theorem 3.13. *Let $(X, M, *)$ be a complete strong (non-Archimedean) fuzzy metric space and let $f : X \to X$ be a fuzzy ψ-contractive mapping. Then, f has a unique fixed point.*

Proof. Let $x_0 \in X$ and consider $x_n = f^n(x_0)$ for each $n \geq 1$. It is easy to verify that $\{x_n\}$ is a fuzzy ψ-contractive sequence. By the above lemma $\{x_n\}$ is a Cauchy sequence and so it is convergent.

By the same way that the end of the proof of Theorem 3.3 one can show that f has a unique fixed point.

$\square$

Remark 3.14. *We have seen that the standard fuzzy metric M_d does not satisfy the condition $\bigwedge_{t>0} M_d(x, y, t) > 0$ for each $x, y \in X$ and so Corollary 3.9 cannot be applied on it. Now, Theorem 3.13 can be applied on M_d since, as it is well-known, M_d is strong.*

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References

[1] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994) 395-399.

[2] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems **27** (1989) 385-389.

[3] V. Gregori, J.J. Miñana, *Some remarks on fuzzy contractive mappings*, Fuzzy Sets and Systems **251** (2014) 101-013.

[4] V. Gregori, S. Romaguera, *Characterizing completable fuzzy metric spaces*, Fuzzy Sets and Systems **144** (2004) 411-420.

[5] V. Gregori, A. Sapena, *On fixed-point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems **125** (2002) 245-252.

[6] I. Kramosil, J. Michalek, *Fuzzy metrics and statistical metric spaces*, Kybernetika **11** (1975) 326-334.

[7] D. Mihet, *Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces*, Fuzzy Sets and Systems **159** (2008) 739-744.

[8] D. Mihet, *A note on fuzzy contractive mappings in fuzzy metric spaces*, Fuzzy Sets and Systems **251** (2014) 83-91.

[9] D. Wardowski, *Fuzzy contractive mappings and fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems **222** (2013) 108-114.