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Probability measure monad on the category of ultrametric spaces

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Abstract. The set of all probability measures with compact support on an ultrametric space can be endowed with a natural ultrametric. We show that the functor of probability measures with finite supports (respectively compact supports) forms a monad in the category of ultrametric spaces (respectively complete ultrametric spaces) and nonexpanding maps. It is also proven that the G-symmetric power functor has an extension onto the Kleisli category of the probability measure monad.

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1. Introduction

The space $P(X)$ of probability measures with compact supports on a metric space X can be endowed with different topologies. One of them is that induced by the Hutchinson metric ([6]). More precisely, if (X, d) is a metric space then the Hutchinson metric $\tilde{d}$, on the set PX of probability measures with compact support is defined as follows:

$$\tilde{d}(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi\colon X \to \mathbb{R} \text{ is 1-Lipschitz}\}.$$

In this note we show that the functor P of probability measures with compact supports determines a monad in the category UMET of ultrametric spaces and nonexpanding maps. This functor was first defined in [8] (see also [4]) and since then turned out to be a natural tool in the metric approach to programming language semantics.

We prove that the G-symmetric power functor has an extension onto the Kleisli category of the probability measure monad (i.e. the category of ultrametric spaces and nonexpanding measure-valued maps).

Note that the probability measure monad on the category of compact Hausdorff spaces was investigated by different authors; see, e.g. [7]. The category of algebras of this monad is known to be isomorphic to the category of compact convex sets and affine continuous maps. Therefore, establishing a monad structure for the probability measure functor in the category UMET allows us to introduce a counterpart of the notion of convexity for ultrametric spaces.

It is known that the space $P(X)$ is complete if so is an ultrametric space X (see [4]). We denote by CUMET the full subcategory of UMET, whose objects are complete ultrametric spaces. We show that, for the category CUMET, one can find counterparts of the mentioned results.

2. Monads and Kleisli categories

We provide some basic definitions concerning monads; see, e.g. [2] for details.

If T is an endofunctor in a category $\mathcal{C}$, by T^n we denote the nth iteration of T. If $\eta\colon 1_{\mathcal{C}} \to T$ and $\mu\colon T^2 \to T$ are natural transformations, then $\mathbb{T} = (T, \eta, \mu)$ is called a *monad* on the category $\mathcal{C}$ if the diagrams

$$\begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[d, "T\eta"'] \arrow[dr, "1_T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu_T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd}$$

commute.

Then η is called the *unity* and μ the *multiplication* of $\mathbb{T}$.

Let $\mathbb{T} = (T, \eta, \mu)$, $\mathbb{T}' = (T', \eta', \mu')$ be two monads in a category $\mathcal{C}$. We say that a natural transformation $\alpha\colon T \to T'$ is a *morphism* of $\mathbb{T}$ into $\mathbb{T}'$ if $\alpha\eta = \eta'$ and $\mu'\alpha_{T'}T\alpha = \alpha\mu$. If all the components of α are monomorphisms, we say that $\mathbb{T}$ is a *submonad* of $\mathbb{T}'$.

For an arbitrary monad $\mathbb{T} = (T, \eta, \mu)$ in $\mathcal{C}$ a pair (X, f), where $f\colon TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-*algebra* if the following commute :

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "f"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "Tf"'] & TX \arrow[d, "f"] \\ TX \arrow[r, "f"] & X \end{tikzcd}$$

A morphism $\varphi\colon X \to Y$ is called *a map of algebras* $(X, f) \to (Y, g)$ if and only if the diagram

$$\begin{tikzcd} TX \arrow[r, "T\varphi"] \arrow[d, "f"'] & TY \arrow[d, "g"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

commutes. The $\mathbb{T}$-algebras and their morphisms form the category, which is denoted by $\mathcal{C}^{\mathbb{T}}$.

The *Kleisli category*, $\mathcal{C}_{\mathbb{T}}$, of a monad $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{C}$ is defined as follows. The objects of $\mathcal{C}_{\mathbb{T}}$ are those of $\mathcal{C}$ and the set of morphisms $\mathcal{C}_{\mathbb{T}}(X, Y)$

coincides with $\mathcal{C}(X,Y)$. The composition $g*f$ of $f \in \mathcal{C}_{\mathbb{T}}(X,Y)$ and $g \in \mathcal{C}_{\mathbb{T}}(Y,Z)$ is defined as $g * f = \mu_Z \circ Tg \circ f$.

The following criterion for existence of extensions of functors in $\mathcal{C}$ onto the Kleisli category is given in [2].

Proposition 2.1. *A functor $F\colon \mathcal{C} \to \mathcal{C}$ admits an extension onto the category $\mathcal{C}_{\mathbb{T}}$ for a monad $\mathbb{T} = (T, \eta, \mu)$ in a category $\mathcal{C}$ if and only if there exists a natural transformation $\xi\colon FT \to TF$ such that*

(1) $\xi \circ F\eta = \eta_F$;
(2) $\mu_F \circ T\xi \circ \xi_T = \xi \circ F\mu$.

3. Metric on the set of probability measures

Recall that a metric d on a set X is said to be an *ultrametric* if the following strong triangle inequality holds:

$$d(x,y) \le \max\{d(x,z), d(z,y)\}$$

for all $x, y, z \in X$.

Let (X,d) be an ultrametric space. Given $y \in X$, we denote by $O_r(y)$ the open r-ball centered at y. If $A \subset X$, then $O_r(A) = \cup\{O_r(y) \mid y \in A\}$.

By $\exp X$ we denote the set of all nonempty compact subsets in X endowed with the Hausdorff metric:

$$d_H(A,B) = \inf\{\varepsilon > 0 \mid A \subset O_\varepsilon(B),\ B \subset O_\varepsilon(A)\}.$$

For a continuous map $f\colon X \to Y$ the map $\exp f\colon \exp X \to \exp Y$ is defined as $\exp f(A) = f(A)$. It is well-known that $\exp f$ is a nonexpanding map if so is f. Thus, we obtain a functor exp in the category UMET (the hyperspace functor). We denote by $s_X\colon X \to \exp X$ the singleton map, $s_X(x) = \{x\}$. It is well-known that s_X is an isometric embedding. By $u_X\colon \exp^2(X) \to \exp(X)$ we denote the union map. It is well-known that u_X is well-defined and nonexpanding and that $u = (u_X)$ is a natural transformation of $\exp^2$ into exp. The triple $\mathbb{H} = (\exp, s, u)$ is a monad on the category UMET (the hyperspace monad).

By $P(X)$ we denote the set of all probability measures with compact support on X. Following [4], we endow $P(X)$ with the following metric, $\hat{d}$,

$$\hat{d}(\mu,\nu) = \inf\{\varepsilon > 0 \mid \mu(O_\varepsilon(x)) = \nu(O_\varepsilon(x)) \text{ for every } x \in X\}.$$

Proposition 3.1. *The identity map $1_X\colon (PX, \hat{d}) \to (PX, \tilde{d})$ is continuous.*

Proof. Suppose that $\hat{d}(\mu,\nu) < \varepsilon$, then $\mu(O_\varepsilon(x)) = \mu(O_\varepsilon(x))$ for every $x \in X$. Given a 1-Lipschitz function φ on X, choose a disjoint cover of the set $\operatorname{supp}(\mu) \cup \operatorname{supp}(\nu)$ by closed balls, $O_\varepsilon(x_1)$,..., $O_\varepsilon(x_m)$. Let $\psi\colon X \to \mathbb{R}$ be a function defined by the condition $\psi|O_\varepsilon(x_i) = \varphi(x_i)$, $i = 1, \dots, n$. Then

$$|\mu(\varphi) - \nu(\varphi)| \le |\mu(\varphi) - \mu(\psi)| + |\mu(\psi) - \nu(\psi)| + |\nu(\psi) - \nu(\varphi)| \le 2\varepsilon,$$

because

$$|\mu(\psi)-\nu(\psi)| = \left|\sum_{i=1}^{m} \psi(x_i)(\mu(O_\varepsilon(x_i)) - \nu(O_\varepsilon(x_i)))\right| = 0.$$

This implies that the map 1_X is 2-Lipschitz. □

For any $x \in X$, by δ_x we denote the Dirac measure concentrated at x.

Proposition 3.2. *The set $P_\omega(X)$ of all measures with finite supports is dense in $P(X)$.*

Proof. Let $\varepsilon > 0$. Given $\mu \in P(X)$, decompose the set $\mathrm{supp}(\mu)$ into the union of disjoint balls of radius ε, $\mathrm{supp}(\mu) = B_1 \cup \dots \cup B_k$. Put $\mu' = \sum_{i=1}^{k} \mu(B_i)\delta_{x_i}$, where x_i is an arbitrary point of B_i, $i = 1, \dots, k$. It follows from the construction that $\hat{d}(\mu, \mu') \le \varepsilon$. □

One can also prove that the set of probability measures with finite supports from a dense subset in X is dense in $P(X)$.

It is known [8] that the construction P is functorial on the category UMET of ultrametric spaces and nonexpanding maps. Recall that P^i denotes the ith iteration of P.

Let $M \in P^2(X)$. We suppose that the set $\mathrm{supp}_{P(X)}(M)$ is finite (i.e., $M \in P_\omega(P(X))$ and, moreover, for every $\mu \in \mathrm{supp}_{P(X)}(M)$ the set $\mathrm{supp}(\mu)$ is also finite. Denote the set of all metrics described by the above condition, by D. As already remarked, D is dense in $P^2(X)$.

Thus every $M \in D$ can be represented as follows:

$$M = \sum_i \alpha_i \delta_{\mu_i}, \text{ where } \mu_i = \sum_j \beta_{ij}\delta_{x_{ij}},\ x_{ij} \in X.$$

We then define $\psi_X(M) = \sum_{i,j} \alpha_i \beta_{ij} \delta_{x_{ij}}$.

In other words, if $M = \sum_i \alpha_i \delta_{\mu_i}$, then $\psi_X(M) = \sum_i \alpha_i \mu_i$.

Proposition 3.3. *The map $\psi_X \colon D \to P(X)$ is nonexpanding.*

Proof. Suppose that $M, M' \in D$, $M = \sum_i \alpha_i \delta_{\mu_i}$, $M' = \sum_i \alpha'_i \delta_{\mu_i}$, and $\hat{\hat{d}}(M, M') < \varepsilon$. Let $\hat{B}_1 \sqcup \hat{B}_2 \sqcup \dots \hat{\sqcup} B_k$ be a decomposition of $\mathrm{supp}_{P(X)}(M) \cup \mathrm{supp}_{P(X)}(M')$ into the sum of disjoint balls in $P(X)$ of radius ε. Then $M(\hat{B}_i) = M'(\hat{B}_i)$, $i = 1, \dots, k$. Let $M|\hat{B}_i = \sum_{p=1}^{j_i} \alpha_{ip}\delta_{\mu_{ip}}$, $M'|\hat{B}_i = \sum_{p=1}^{j_i} \alpha'_{ip}\delta_{\mu_{ip}}$. Then $\sum_{p=1}^{j_i} \alpha_{ip} = \sum_{p=1}^{j_i} \alpha'_{ip}$.

Note that, for every $p, l \in \{1, \dots, j_i\}$, we have $\hat{d}(\mu_{ip}, \mu_{il}) < \varepsilon$. This implies that, for any $x \in X$, we have $\mu_{ip}(O_\varepsilon(x)) = \mu_{il}(O_\varepsilon(x))$.

Let $x \in X$. Then

$$\begin{aligned}\psi_X(M)(O_\varepsilon(x)) &= \sum_{i=1}^{k}\sum_{p=1}^{j_i} \alpha_{ip}\mu_{ip}(O_\varepsilon(x)) \\ &= \sum_{i=1}^{k}\sum_{p=1}^{j_i} \alpha'_{ip}\mu_{ip}(O_\varepsilon(x)) \\ &= \psi_X(M)(O_\varepsilon(x)).\end{aligned}$$

Therefore $\hat{d}(\psi_X(M), \psi_X(M')) < \varepsilon$.

□

Proposition 3.3 allows us to define the map $\psi_X \colon P^2(X) \to P(X)$ as a unique continuous extension of the above map defined on D. Clearly, this extension is also nonexpanding.

The proof of the following proposition is obvious.

Proposition 3.4. *The map $\eta_X \colon X \to P(X)$, $\eta_X(x) = \delta_x$, is an isometric embedding.*

Theorem 3.5. *The triple $\mathbb{P} = (P, \eta, \psi)$ is a monad on the category* UMET.

Proof. Since the measures with finite supports are dense in the spaces of probability measures, one has to establish the identities from the definition of monad for such measures. But this is essentially verified in, e.g., [7]. □

The support map $\mathrm{supp} = \mathrm{supp}_X \colon P(X) \to \exp X$ is nonexpanding (see, e.g. [8]) for any ultrametric space X and therefore a morphism in UMET. It is easy to see that supp is a natural transformation of the probability measure functor P into the hyperspace functor exp.

Proposition 3.6. *The natural transformation* supp *is a morphism of the monad $\mathbb{P}$ to the monad $\mathbb{H}$.*

Proof. It is obvious that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\eta_X"'] \arrow[dr, "s_X"] & \\ PX \arrow[rr, "\mathrm{supp}"'] & & \exp X \end{tikzcd}$$

is commutative. One has to show that the diagram

$$\begin{tikzcd} P^2X \arrow[r, "\exp(\mathrm{supp})\,\mathrm{supp}_{PX}"] \arrow[d, "\psi_X"'] & \exp^2 X \arrow[d, "\cup"] \\ PX \arrow[r, "\mathrm{supp}"'] & \exp X \end{tikzcd}$$

is also commutative. Let $M \in P_\omega(P_\omega X)$, $M = \sum_{i=1}^k \alpha_i\delta_{\mu_i}$, where $\mu_i = \sum_{j=1}^{l_i} \beta_{ij}\delta_{x_{ij}}$, $x_{ij} \in X$. Without loss of generality, one may assume that $\alpha_i \neq 0$ and $\beta_{ij} \neq 0$, for all i, j. Then $\psi_X(M) = \sum_{i=1}^k \sum_{j=1}^{l_i} \alpha_i\beta_{ij}\delta_{x_{ij}}$ and

$$\begin{aligned}\operatorname{supp}(\psi_X(M)) =& \{x_{ij} \mid 1 \le i \le k,\ 1 \le j \le l_i\}\\ =& \cup\{\{x_{ij} \mid 1 \le j \le l_i\} \mid i = 1, \dots, k\}\\ =& \cup \exp(\operatorname{supp})\operatorname{supp}_{P(X)}(M).\end{aligned}$$

Since the set $P_\omega(P_\omega X)$ is dense in $P^2(X)$, we are done. □

Recall that, for every ultrametric spaces X and Y, and every $\mu \in P(X)$, $\nu \in P(Y)$, by $\mu \otimes \nu$ we denote the *product* of μ and ν.

Proposition 3.7. *Let X_i, $i = 1, \dots, n$, be metric spaces. The natural map*

$$PX_1 \times \dots \times PX_n \to P(X_1 \times \dots \times X_n),\ (m_1, \dots, m_n) \mapsto m_1 \otimes \dots \otimes m_n,$$

is nonexpanding.

Proof. Suppose that $c > 0$, $(m_1, \dots, m_n), (m_1', \dots, m_n') \in PX_1 \times \dots \times PX_n$, and $\hat{d}(m_i, m_i') < c$ for every $i = 1, \dots, n$. Then for every $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$ we have

$$\begin{aligned}&(m_1 \otimes \dots \otimes m_n)(O_c(x_1, \dots, x_n)) = m_1(O_c(x_1)) \times \dots \times m_n(O_c(x_n))\\ =& m_1'(O_c(x_1)) \times \dots \times m_n'(O_c(x_n)) = (m_1' \otimes \dots \otimes m_n')(O_c(x_1, \dots, x_n)),\end{aligned}$$

whence the result follows. □

4. G-symmetric power functor

Let S_n denote the permutation group of the set $\{1, 2, \dots, n\}$. Any subgroup G of S_n acts on the nth power X^n of a space X by permutation of coordinates. Let $SP^n_G(X)$ denote the orbit space of this action. By $[x_1, \dots, x_n]$ (or briefly $[x_i]$) we denote the orbit containing $(x_1, \dots, x_n) \in X^n$.

If (X, d) is a metric space, we endow $SP^n_G(X)$ by the following metric, $\tilde{d}$,

$$\tilde{d}([x_i], [y_i]) = \min\{\max\{d(x_i, y_{\sigma(i)}) \mid i = 1, \dots, n\} \mid \sigma \in G\}.$$

Proposition 4.1. *The space $(SP^n_G(X), \tilde{d})$ is an ultrametric space.*

Proof. One has only to check that $\tilde{d}([x_i], [y_i]) \le \max\{\tilde{d}([x_i], [z_i]), \tilde{d}([z_i], [y_i])\}$ for any $[x_i], [y_i], [z_i] \in SP^n_G(X)$. There exist $\sigma, \tau \in G$ such that

$$\begin{aligned}\tilde{d}([x_i], [z_i]) &= \max\{d(x_i, z_{\sigma(i)}) \mid i = 1, \dots, n\},\\ \tilde{d}([z_i], [y_i]) &= \max\{d(z_i, y_{\tau(i)}) \mid i = 1, \dots, n\}.\end{aligned}$$

Then

$$\begin{aligned}&\tilde{d}([x_i],[y_i] \leq \max\{d(x_i, y_{\tau(\sigma(i))}) \mid i=1,\dots,n\}\\ =&\max\{\max\{d(x_i, z_{\sigma(i)}), d(z_{\sigma(i)}, y_{\tau(\sigma(i))})\} \mid i=1,\dots,n\}\\ =&\max\{\max\{d(x_i, z_{\sigma(i)}) \mid i=1,\dots,n\}, \max\{d(z_{\sigma(i)}, y_{\tau(\sigma(i))}) \mid i=1,\dots,n\}\}\\ =&\max\{\tilde{d}([x_i],[z_i]), \tilde{d}([z_i],[y_i])\}.\end{aligned}$$

□

One can easily verify that, for any nonexpanding map $f\colon X \to Y$ of ultrametric spaces, the map $SP^n_G(f)$ is nonexpanding as well. Thus, we have defined the G-symmetric power functor SP^n_G in the category UMET.

The proof of the following simple proposition is left to the reader.

Proposition 4.2. *The natural map*

$$\pi_G = \pi_G X\colon X^n \to SP^n_G(X),\ \ \pi_G(x_1,\dots,x_n) = [x_1,\dots,x_n],$$

is nonexpanding.

Theorem 4.3. *The G-symmetric power functor admits an extension onto the Kleisli category of the monad $\mathbb{P}$.*

Proof. We apply Proposition 2.1. Similarly as in [9] define the map $\xi_X : SP^n_G PX \to PSP^n_G X$ by the formula

$$\xi_X[\mu_1,\dots,\mu_n] = \frac{1}{|G|}\sum_{\sigma\in G} P\pi_G X(\mu_1\otimes\dots\otimes\mu_n).$$

We are going to show that ξ_X is a nonexpanding map. Given

$$[\mu_1,\dots,\mu_n],[\nu_1,\dots,\nu_n] \in SP^n_G PX$$

such that $\hat{d}([\mu_1,\dots,\mu_n],[\nu_1,\dots,\nu_n]) \leq \varepsilon$, without loss of generality, we may assume that $\hat{d}(\mu_i,\nu_i) \leq \varepsilon$ for all i. Given $(x_1,\dots,x_n) \in X^n$, we see that $O_\varepsilon(x_1,\dots,x_n) = \prod_{i=1}^n O_\varepsilon(x_i)$. Then

$$\begin{aligned}(\mu_{\sigma(1)}\otimes\dots\otimes\mu_{\sigma(n)})(O_\varepsilon(x_1,\dots,x_n)) &= \prod_{i=1}^n \mu_{\sigma(i)}(O_\varepsilon(x_i))\\ =\prod_{i=1}^n \nu_{\sigma(i)}(O_\varepsilon(x_i)) &= (\nu_{\sigma(1)}\otimes\dots\otimes\nu_{\sigma(n)})(O_\varepsilon(x_1,\dots,x_n)),\end{aligned}$$

whence $\hat{d}(\mu_{\sigma(1)}\otimes\dots\otimes\mu_{\sigma(n)}, \nu_{\sigma(1)}\otimes\dots\otimes\nu_{\sigma(n)}) \leq \varepsilon$.

Since the map $\pi_G X$ is nonexpanding, so is $P\pi_G X$ and we see that

$$\hat{d}(P\pi_G X(\mu_{\sigma(1)}\otimes\dots\otimes\mu_{\sigma(n)}), P\pi_G X(\nu_{\sigma(1)}\otimes\dots\otimes\nu_{\sigma(n)})) \leq \varepsilon.$$

Since $P\pi_G X(\nu_{\sigma(1)}\otimes\dots\otimes\nu_{\sigma(n)}) = P\pi_G X(\nu_1\otimes\dots\otimes\nu_n)$, we are done.

The proof of [9, Theorem] can be rewritten verbally in order to demonstrate that the natural transformation ξ satisfies the conditions of Proposition 2.1. Therefore, SP^n_G can be extended over the category $\mathrm{UMET}_{\mathbb{P}}$. □

5. Category CUMET

It is known (see, e.g., [4] and [1]), that the spaces $\exp(X)$ and $P(X)$ are complete if so is an ultrametric space X. This allows us to consider exp and P as endofunctors in the category CUMET.

Proposition 5.1. *Let (X, d) be a complete ultrametric space. Then the space $(SP^n_G(X), \tilde{d})$ is complete as well.*

Proof. Let $([x^{(j)}_1, \dots, x^{(j)}_n])_{j=1}^{\infty}$ be a Cauchy sequence in $SP^n_G(X)$. Without loss of generality, one may assume that

$$d(x^{(j)}_i, x^{(k)}_i) \leq \tilde{d}([x^{(j)}_1, \dots, x^{(j)}_n], [x^{(k)}_1, \dots, x^{(k)}_n]),$$

for all $i = 1, \dots, n$ and all j, k. Let $x_i = \lim_{j\to\infty} x^{(j)}_i$. Then, clearly,

$$[x_1, \dots, x_n] = \lim_{j\to\infty}[x^{(j)}_1, \dots, x^{(j)}_n]$$

and we are done. □

Therefore, the G-symmetric power functor can also be regarded as an endofunctor in CUMET. Clearly, the results of the previous sections can be extended over the case of the category CUMET.

6. Remarks and open questions

6.1. The monad $\mathbb{P}$ in the category UMET has its counterpart in topological categories, in particular, in the category COMP of compact Hausdorff spaces and continuous maps. It is known [7] that the category of algebras of the latter monad is isomorphic to the category of compact convex sets (in locally convex spaces) and continuous affine maps.

Problem 6.1. *Characterize the category of $\mathbb{P}$-algebras.*

Note that no finite ultrametric space of cardinality ≥ 2 can be endowed with a structure of $\mathbb{P}$-algebra. Indeed, let X be such a space and $Y = \{x_1, \dots, x_k\} \subset X$, $x_i \neq x_j$, whenever $i \neq j$, be the set of points that realize the minimal mutual distance, which is supposed to be equal to $c > 0$). Let $f\colon X \to X$ be a homeomorphism whose set of fixed points precisely $X \setminus Y$. Suppose now that (X, ξ) is a $\mathbb{P}$-algebra, then $\mu = \sum_{i=1}^{k} \frac{1}{k}\delta_{x_i}$ is a fixed point of the map $P(f)$ and $f\xi(\mu) = \xi P(f)(\mu) = \xi(\mu)$ whence $\xi(\mu)$ is a fixed point of f. Therefore, $\xi(\mu) \in X \setminus Y$ (thus $Y \neq \varnothing$). We see that $\hat{d}(\delta_{x_1}, \mu) = c$ while $d(\xi(\delta_{x_1}), \xi(\mu)) = d(x_1, \xi(\mu)) > c$.

Problem 6.2. *Is the space of p-adic numbers a $\mathbb{P}$-algebra?*

One can easily see that the functor P_{f} of P of probability measures with finite supports is the functorial part of a submonad $\mathbb{P}_{\mathrm{f}}$ of the monad $\mathbb{P}$. Note that the results concerning the monad $\mathbb{P}$ on the category UMET have their natural counterparts also for the monad $\mathbb{P}_{\mathrm{f}}$.

6.2. A metric space (X, d) is said to be *uniformly disconnected* [3], [5] if there exists $c \in (0, 1)$ such that, for every natural n and every $x_0, x_1, \ldots, x_n \in X$, we have

$$cd(x_0, x_n) \leq \max\{d(x_{i-1}, x_i) \mid i = 1, \ldots, n\}.$$

The uniformly disconnected spaces and Lipschitz maps form a category. We leave as an open question that of extension of the results of this note over this category.

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