Modified w-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces

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Abstract

In this paper we introduce the notion of modified w-distance (mw-distance) on a quasi-metric space which generalizes the concept of quasi-metric. We obtain a fixed point theorem for generalized contractions with respect to mw-distances on complete quasi-metric spaces.

Keywords: fixed point, generalized contraction, w-distance, mw-distance, complete quasi-metric space
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1. Introduction and preliminaries

In [12] Kada, Suzuki and Takahashi introduced the notion of w-distance on a metric space and improved the nonconvex minimization theorem of Takahashi [18], the Ekeland variational principle [8] and the Caristi-Kirk fixed point theorem [5], [6], among other results. Later Park [17] extended the notion of $w-$distance and generalized several results from [12] to quasi-metric spaces. Since then, the w-distance has been used in some directions in order to obtain fixed point results on complete metric and quasi-metric spaces ([1], [2], [3], [14], [15]).

In this paper we introduce a new notion of w-distance on a quasi-metric spaces which generalizes the concept of quasi-metric and we obtain a fixed point theorem for generalized contraction with respect to this new notion on complete quasi-metric spaces.

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ and ω will denote the set of real numbers, the set of non-negative real numbers, the set of positive integer numbers and the set of non-negative integer numbers, respectively. Our basic references for quasi-metric spaces are [10], [13] and [7].

A *quasi-pseudo-metric* on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Following the modern terminology, a quasi-pseudo-metric d on X satisfying (i') $d(x,y) = d(y,x) = 0$ if and only if $x = y$, is called a *quasi-metric* on X.

Each quasi-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ for all $x, y \in X$, is also a quasi-metric on X, called *conjugate quasi-metric*, and the function d^s defined by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ for all $x, y \in X$, is a metric on X.

There are a lot of completeness notions in quasi-metric spaces, all agreeing with the usual notion of completeness in the case metric (see e.g. [13]), each of them having its advantages and weaknesses. In this paper we shall use the following general notion.

A quasi-metric space (X,d) is called *complete* if every Cauchy sequence $(x_n)_{n\in\omega}$ in the metric space (X,d^s) converges with respect to the topology $\tau_{d^{-1}}$ (i.e., there exists $z \in X$ such that $d(x_n, z) \to 0$).

By an *asymmetric norm* on a real vector space X we mean a nonnegative real-valued function p on X such that for all $x, y \in X$ and $r \geq 0$: (i) $p(x) = p(-x) = 0 \Leftrightarrow x = 0$, (ii) $p(rx) = rp(x)$, and (iii) $p(x+y) \leq p(x) + p(y)$.

Each asymmetric norm p on a real vector space X induces a quasi-metric d_p on X defined by $d_p(x,y) = p(y-x)$.

2. *mw*-distances on a quasi-metric space

Let us recall the definitions of w-distance for metric and quasi-metric spaces.

Definition 1. *([12]) A* w***-distance*** *on a* ***metric space*** (X,d) *is a function* $q : X \times X \to \mathbb{R}^+$ *satisfying the following three conditions:*
(W1) $q(x,y) \leq q(x,z) + q(z,y)$, *for all* $x, y, z \in X$*;*
(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* (X, τ_d) *for all* $x \in X$*;*
(W3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $q(x,y) \leq \delta$ *and* $q(x,z) \leq \delta$ *then* $d(y,z) \leq \varepsilon$.

Definition 2. *([3], [17]) A* w***-distance*** *on a* ***quasi-metric space*** (X,d) *is a function* $q : X \times X \to \mathbb{R}^+$ *satisfying the following three conditions:*
(W1) $q(x,y) \leq q(x,z) + q(z,y)$, *for all* $x, y, z \in X$*;*
(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ *is lower semicontinuous on* $(X, \tau_{d^{-1}})$ *for all* $x \in X$*;*
(W3) for each $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $q(x,y) \leq \delta$ *and* $q(x,z) \leq \delta$ *then* $d(y,z) \leq \varepsilon$ *(and also* $d(z,y) \leq \varepsilon$*).*

Remark 1. It is clear that if d is a metric on X, d is a w-distance on the metric space (X,d). Unfortunately, if d is a quasi-metric on X, d is not necessarily a w-distance on the quasi-metric space (X,d) as we can see in the following paradigmatic examples.

Example 1. Let $(\mathbb{R}, d_S)$ the Sorgenfrey line, where d_S is the quasi-metric defined by $d_S(x,y) = y - x$ if $x \leq y$, and $d_S(x,y) = 1$ if $x > y$. Then, d_S does not satisfy condition $(W3)$. Indeed, taking $\varepsilon = 1/2$ and $\delta > 0$, then if $y = x+\delta/2$ and $z = x + \delta/3$, it satisfies that $d_S(x, y) = \delta/2 < \delta$, $d_S(x, z) = \delta/3 < \delta$, and $d_S(y, z) = 1 > \varepsilon$.

Example 2. Consider the quasi-metric space $(\mathbb{R}, d)$ where $d(x, y) = (y-x) \vee 0$. Then d is not w-distance, because the condition $(W3)$ does not hold. Indeed, given $\varepsilon > 0$, and $x, y, z \in \mathbb{R}$ such that $0 < z < y < x$ and $\varepsilon < y-z$, then for every $\delta > 0$ we have that $d(x, y) = d(x, z) = 0$ and $d(z, y) = (y - z) \vee 0 = y - z > \varepsilon$.

Motivated from above remark, we give the following definition:

Definition 3. *A* ***modified*** *$w-$**distance**(mw−distance, in short) on a* ***quasi-metric*** *space (X, d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:*
(W1) $q(x, y) \le q(x, z) + q(z, y)$ for all $x, y, z \in X$;
(W2) $q(x, \cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$;
(mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(y, x) \le \delta$ and $q(x, z) \le \delta$ then $d(y, z) \le \varepsilon$.

Remark 2. Note that every quasi-metric d on X is an mw-distance on the quasi-metric space (X, d).

Definition 4. *A* ***strong**-mw-distance on a* ***quasi-metric*** *space (X, d) is a mw-distance $q : X \times X \to \mathbb{R}^+$ satisfying the following condition:*
(mW2) $q(\cdot, x) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $x \in X$.

In the remainder of this section we give some examples of mw-distances.

Example 3. Let (X, d) be a quasi-metric space and let $c \in \mathbb{R}^+$. The function $q(x, y) = c$ is a strong-mw-distance on X.

Example 4. Let $(\mathbb{R}, d_S)$ the Sorgenfrey line (see Example 1). Then $q(x, y) = d_S(x, y)$ is a strong mw-distance on the quasi-metric space $(\mathbb{R}, d_S)$. Indeed, fix $y \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\lim_n d_S(x_n, x) = 0$. Then, given $0 < \varepsilon < 1$ there is $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, $d_S(x_n, x) = x - x_n < \varepsilon$ and $x_n \le x$.

If $x \le y$, then

$$d_S(x, y) - d_S(x_n, y) = (y - x) - (y - x_n) = x_n - x < 0 < \varepsilon.$$

If $y < x$, then there is $n_1 \in \mathbb{N}$, $n_1 \ge n_0$, such that for all $n \ge n_1$,

$$d_S(x_n, x) = x - x_n < x - y,$$

so that $y < x_n$. Then for all $n \ge n_1$

$$d_S(x, y) - d_S(x_n, y) = 1 - 1 = 0 < \varepsilon.$$

Therefore the function $q(\cdot, y)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ for all $y \in X$.

Example 5. Let $(X, \preceq, \|.\|)$ be a normed lattice. Denote by X^+ the positive cone of X, i.e., $X^+ := \{x \in X : \mathbf{0} \preceq x\}$, and we define the asymmetric norm on

X (see e.g. [9], Theorem 3.1) give by $\|.\|^+ : X \to \mathbb{R}^+$ as $\|x\|^+ = \|x \vee \mathbf{0}\|$ for all $x \in X$. Then the function d defined by $d(x,y) = \|y-x\|^+$ for all $x, y \in X$, is a quasi-metric on X. Hence (X^+, d_+) is a quasi-metric space, where d_+ denotes the restriction of d to X^+.

We show that the function q defined by $q(x,y) = \|y\|$ for all $x, y \in X^+$, is a mw-distance on (X^+, d_+). Indeed, condition $(W1)$ is trivially satisfied. Now fix $x \in X^+$ and let $(y_n)_{n\in\omega}$ be a sequence in X^+ such that $\lim d_+(y_n, y) = 0$ for some $y \in X^+$. Since

$$q(x,y) = \|y\| = \|y\|^+ = \|y - y_n + y_n\|^+ \leq$$

$$\leq \|y - y_n\|^+ + \|y_n\|^+ = d_+(y_n, y) + q(x, y_n) \quad \Rightarrow$$

$$\Rightarrow \quad q(x,y) - q(x,y_n) \leq d_+(y_n, y)$$

for all $n \in \omega$, we deduce that $q(x,\cdot)$ is lower semicontinuous for $(X^+, \tau_{(d_+)^{-1}})$, and condition $(W2)$ is satisfied.

On the other hand, choose $\varepsilon > 0$ and put $\delta = \varepsilon$. Suppose that there are $x, y, z \in X^+$ such that $q(y,x) = \|x\|^+ = \|x\| \leq \delta$ and $q(x,z) = \|z\|^+ = \|z\| \leq \delta$. Therefore

$$d_+(y,z) = \|z-y\|^+ \leq \|z\|^+ + \|-y\|^+ = \|z\|^+ \leq \delta = \varepsilon.$$

Consequently, the condition $(mW3)$ is also satisfied.

Finally, for every $z \in X^+$ we have that

$$q(y,z) - q(y_n, z) = \|z\| - \|z\| = 0 < \varepsilon.$$

Therefore, $q(\cdot, z)$ is lower semicontinuous function on $(X^+, \tau_{(d_+)^{-1}})$ and we conclude that q is a strong-mw-distance on (X^+, d_+).

Example 6. Consider the quasi-metric space $(\mathbb{R}, d)$ where $d(x,y) = (y-x) \vee 0$ (see Example 2). Then $q = d$ is a mw-distance but q is not strong-mw-distance, because the condition $(mW2)$ does not hold. Indeed, if we consider the sequence $\{n\}_{n\in\mathbb{N}}$, this sequence converges to zero in $(\mathbb{R}, d^{-1})$ because $d(n,0) = (-n) \vee 0 = 0$. Nevertheless, if $y > 0$ and $n > y$, then

$$d(0,y) - d(n,y) = (y \vee 0) - (y-n) \vee 0 = y - 0 = y.$$

Therefore, the function $d(\cdot, y) : \mathbb{R} \to \mathbb{R}^+$ is not lower semicontinuous on $(\mathbb{R}, \tau_{d^{-1}})$.

Now we give an example of an $mw-$distance q on a quasi-metric space (X,d) such that $q \neq d$ and q is not a $w-$distance.

Example 7. Let (X,p) be an asymmetric normed space. Let d_p the quasi-metric induced by p, namely $d_p(x,y) = p(y-x)$. Then $q : X \times X \to \mathbb{R}^+$ defined by

$$q(x,y) = p(-x) + p(y)$$

is an $mw-$distance on the quasi-metric space (X, d_p).

The condition $(W1)$ holds because

$$q(x,y) = p(-x) + p(y) \leq p(-x) + p(z) + p(-z) + p(y) = q(x,z) + q(z,y).$$

To prove condition $(W2)$ we take a sequence $(y_n)_{n\in\mathbb{N}} \subset X$ such that $(y_n)_{n\in\mathbb{N}}$ converges to y in $(X, \tau_{d_p^{-1}})$, i.e., $d_p(y_n, y)$ converges to zero. Then

$$q(x,y) - q(x,y_n) = p(-x) + p(y) - p(-x) - p(y_n)$$

$$= p(y) - p(y_n) \leq p(y - y_n) = d_p(y_n, y).$$

Hence $q(x,\cdot)$ is lower semicontinuous on $(X, \tau_{d_p^{-1}})$.

Finally, given $\varepsilon > 0$ put $\delta = \varepsilon/2$. Then if $q(x,y) < \delta$ and $q(y,z) < \delta$, we have that

$$d_p(x,z) \leq d_p(x,y) + d_p(y,z) = p(y-x) + p(z-y)$$

$$\leq p(y) + p(-x) + p(z) + p(-y) = q(x,y) + q(y,z) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore q satisfies the condition $(mW3)$.

In general, q is not a $w-$distance on (X,d). Indeed, taking $X = \mathbb{R}$ and $p(x) = x \vee 0$, we have that $q(x, -3\varepsilon) = 0 < \delta$, $q(x, -\varepsilon) = 0 < \delta$, for all $x \geq 0$ and for all $\delta > 0$. Nevertheless, $d(-3\varepsilon, -\varepsilon) = 2\varepsilon > \varepsilon$, for all $\varepsilon > 0$. So q does not satisfies $(W3)$.

The following example shows that there are w-distances which are not mw-distances.

Example 8. Let d be the usual metric on $\mathbb{R}$, that is, $d(x,y) = |y - x|$. It is easy to prove that the function $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by $q(x,y) = |y|$ is w-distance in the quasi-metric space (X,d). Nevertheless, q is not mw-distance in the quasi-metric space (X,d). Indeed, the condition $(mW3)$ does not hold because given $\varepsilon > 0$, then for every $\delta > 0$ we have that $q(2\varepsilon, 0) < \delta$, $q(0,0) < \delta$ and $d(2\varepsilon, 0) = 2\varepsilon > \varepsilon$.

3. A fixed point theorem involving mw-distances

Recently, Alegre, Marín and Romaguera [2] have obtained a fixed point theorem for generalized contractions with respect to w-distances on complete quasi-metric spaces from which they deduce w-distance versions of Boyd and Wong's fixed point theorem [4] and of Matkowski's fixed point theorem [16]. Its approach uses a kind of functions considered by Jachymski in [11, Corollary of Theorem 2] and that generalizes the notion of function of Meir-Keeler type [1].

Definition 5. *([2]) A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a* ***Jachymski function*** *if:*
(Ja1) $\phi(0) = 0$,
(Ja2) for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $t > 0$ with $\varepsilon < t < \varepsilon + \delta$, we have $\phi(t) \leq \varepsilon$.

Theorem 1. *([2, Theorem 2]) Let f be a self-map of a complete quasi-metric space (X, d). If there exist a w-distance q on (X, d) and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(fx, fy) \leq \phi(q(x, y)), \tag{1}$$

for all $x, y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z, z) = 0$.

Now we prove that Theorem 1 remains true if condition (1) is satisfied by a strong mw-distance on X.

Theorem 2. *Let f be a self-map of a complete quasi-metric space (X, d). If there exists a strong-mw-distance q on (X, d) and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(fx, fy) \leq \phi(q(x, y)), \tag{2}$$

for all $x, y \in X$, then f has a unique fixed point $z \in X$. Moreover $q(z, z) = 0$.

Proof. Fix $x_0 \in X$ and let $x_n = f^n x_0$ for each each $n \in \mathbb{N}$.

Let us first prove that $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d^s).

Let $a_n = q(x_n, x_{n+1})$ and $b_n = q(x_{n+1}, x_n)$ for all $n \in \omega$. Since

$$a_{n+1} = q(x_{n+1}, x_{n+2}) \leq \phi(q(x_n, x_{n+1})) \leq q(x_n, x_{n+1}) = a_n \tag{3}$$

and

$$b_{n+1} = q(x_{n+2}, x_{n+1}) \leq \phi(q(x_{n+1}, x_n)) \leq q(x_{n+1}, x_n) = b_n, \tag{4}$$

for all $n \in \omega$, then $(a_n)_{n\in\omega}$ converges to some $a \in \mathbb{R}^+$ and $(b_n)_{n\in\omega}$ converges to some $b \in \mathbb{R}^+$.

Now we prove that $a = b = 0$.

If there exists $n_0 \in \omega$ such that $a_{n_0} = 0$ then, by (3), $a_n = 0$ for all $n \geq n_0$. Therefore $a = 0$.

Suppose that $a_n \neq 0$, for all $n \in \omega$. This implies that $\phi(a_n) < a_n$, so that, by (3), $a_{n+1} < a_n$ for all $n \in \omega$. Then $a < a_n$ for all $n \in \omega$.

If we suppose that $a > 0$, by (Ja2), there exists $\delta = \delta(a)$ such that

$$a < t < a + \delta \quad \Rightarrow \quad \phi(t) \leq a.$$

Take $n_\delta \in \mathbb{N}$ such that $a_n < a + \delta$ for all $n \geq n_\delta$. Then $\phi(a_n) \leq a$, so that, by condition (3), $a_{n+1} \leq a$ for all $n \geq n_\delta$, a contradiction. Consequently $a = 0$.

In a similar way it is proved that $b = 0$.

Now choose an arbitrary $\varepsilon > 0$. Then there is $\delta \in (0, \varepsilon)$ for which $(mW3)$ holds and

$$\varepsilon < t < \varepsilon + \delta \ \Rightarrow \phi(t) \leq \varepsilon. \tag{5}$$

For $\delta/2 > 0$ there is $\mu \in (0, \delta/2)$ such that

$$\delta/2 < t < \delta/2 + \mu \Rightarrow \phi(t) \leq \delta/2 \tag{6}$$

Because $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$ there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

(C1) $a_n = q(x_n, x_{n+1}) < \mu/2$,
(C2) $b_n = q(x_{n+1}, x_n) < \mu/2$,
(C3) $q(x_n, x_n) \le q(x_n, x_{n+1}) + q(x_{n+1}, x_n) < \mu$,

for all $n \ge k_0$,

By using a similar technique to the one given by Jachymski in [11, Theorem 2] and [2] we shall prove, by induction, that for all $n \in \mathbb{N}$ and $k \ge k_0$ that

$$q(x_k, x_{k+n}) < \mu + \frac{\delta}{2} \tag{7}$$

and

$$q(x_{k+n}, x_k) < \mu + \frac{\delta}{2}. \tag{8}$$

Let $k \ge k_0$. Since $q(x_k, x_{k+1}) < \mu/2$, condition (7) follows for $n = 1$.

Suppose that (7) is true for $n \in \mathbb{N}$. We shall study two cases.

- Case 1: $q(x_k, x_{n+k}) > \delta/2$. Then we deduce from the induction hypothesis and (6) that $\phi(q(x_k, x_{n+k})) \le \delta/2$. Then

$$q(x_{k+1}, x_{n+k+1}) \le \phi(q(x_k, x_{n+k})) \le \delta/2$$

and by $(W1)$,

$$q(x_k, x_{n+k+1}) \le q(x_k, x_{k+1}) + q(x_{k+1}, x_{n+k+1}) < \frac{\mu}{2} + \frac{\delta}{2} < \mu + \frac{\delta}{2}.$$

- Case 2: $q(x_k, x_{n+k}) \le \delta/2$.

If $q(x_k, x_{n+k}) = 0$, we deduce that $q(x_{k+1}, x_{n+k+1}) = 0$ by (2). So, by $(W1)$,

$$q(x_k, x_{n+k+1}) \le q(x_k, x_{k+1}) + q(x_{k+1}, x_{n+k+1}) < \frac{\mu}{2} < \mu + \frac{\delta}{2}.$$

If $q(x_k, x_{n+k}) > 0$, we deduce that $\phi(q(x_k, x_{n+k})) < q(x_k, x_{n+k}) \le \delta/2$. Then

$$\begin{aligned} q(x_k, x_{n+k+1}) &\le q(x_k, x_{k+1}) + q(x_{k+1}, x_{n+k+1}) \le \\ &\le q(x_k, x_{k+1}) + \phi(q(x_k, x_{n+k})) < \frac{\mu}{2} + \frac{\delta}{2} < \mu + \frac{\delta}{2}. \end{aligned}$$

The inequality (8) can be proved similarly.

Now let $i, j \in \mathbb{N}$ with $j \ge i \ge k_0$. Then $i = n + k_0$ and $j = m + k_0$ for some $n, m \in \omega$, with $m \ge n$. Hence, by $(mW3)$ and (C3),

$$\left.\begin{aligned} q(x_{k_0}, x_j) = q(x_{k0}, x_{m+k0}) < \mu + \tfrac{\delta}{2} < \delta \\ q(x_i, x_{k_0}) = q(x_{n+k0}, x_{k_0}) < \mu + \tfrac{\delta}{2} < \delta \end{aligned}\right\} \Longrightarrow d(x_i, x_j) \le \varepsilon$$

and

$$\left.\begin{array}{l} q(x_{k_0}, x_i) = q(x_{k0}, x_{n+k0}) < \mu + \frac{\delta}{2} < \delta \\ q(x_j, x_{k_0}) = q(x_{m+k0}, x_{k_0}) < \mu + \frac{\delta}{2} < \delta \end{array}\right\} \Longrightarrow d(x_j, x_i) \leq \varepsilon.$$

Therefore $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d^s). Since (X, d) is complete, there exists $z \in X$ such that $d(x_n, z) \to 0$.

Now we shall prove that $q(x_n, z) \to 0$ and $q(x_n, fz) \to 0$.

Indeed, let $\varepsilon > 0$. By (7), there exist $\mu < \varepsilon/2$ and $n_0 \in \mathbb{N}$ such that

$$q(x_n, x_m) < \mu + \varepsilon/2$$

for $n \geq n_0$ and for all $m \geq n$.

Let $n \in \mathbb{N}$ such that $n \geq n_0$. Since $q(x_n, \cdot)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$ and $d(x_m, z) \to 0$, there exists $m_0 \in \mathbb{N}$, $m_0 \geq n_0$, such that

$$q(x_n, z) - q(x_n, x_m) < \epsilon,$$

for all $m \geq m_0$.

Therefore, if $n \geq n_0$ and $m \geq n$ then

$$q(x_n, z) < q(x_n, x_m) + \varepsilon < \mu + \varepsilon/2 + \varepsilon < 2\varepsilon,$$

so that $q(x_n, z) \to 0$.

Since

$$q(x_{n+1}, fz) \leq \phi(q(x_n, z)) \leq q(x_n, z),$$

we have that $q(x_n, fz) \to 0$.

Next we prove that $d(z, fz) = d(fz, z) = 0$.

By $(mW2)$, the function $q(\cdot, fz)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$. Then, since $d(x_n, z) \to 0$, we have that given $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then

$$q(z, fz) - q(x_n, fz) < \varepsilon,$$

implying

$$q(z, fz) < q(x_n, fz) + \varepsilon.$$

Therefore $q(z, fz) = 0$.

Since $q(x_n, z) \to 0$ and $q(z, fz) = 0$, by $(mW3)$, we have that $d(x_n, fz) \to 0$. Then, because $q(\cdot, z)$ is lower semicontinuous on $(X, \tau_{d^{-1}})$, given $\varepsilon > 0$ there exists $n_2 \in \mathbb{N}$ such that if $n \geq n_2$ then

$$q(fz, z) - q(x_n, z) < \varepsilon$$

that is

$$q(fz, z) < q(x_n, z) + \varepsilon.$$

Then $q(fz, z) = 0$. Taking to account $(W1)$, we have that $q(z, z) = q(fz, fz) = 0$. Therefore, by $(mW3)$, we obtain

$$d(z, fz) = d(fz, z) = 0.$$

Consequently $z = f(z)$, i.e., is a fixed point of f.

Finally, let $u \in X$ such that $u = fu$. If $q(u, z) > 0$, then

$$q(u, z) = q(fu, fz) \leq \phi(q(u, z)) < q(u, z),$$

a contradiction. So that $q(u, z) = 0$. In a similar way we obtain that $q(u, u) = 0$ and $q(z, z) = 0$. Therefore, by $(mW3)$, $d(u, z) = d(z, u) = 0$. Consequently $u = z$ and we conclude that z is the unique fixed point of f. □

Now we give an example where it is possible to apply Theorem 2 but not Theorem 1.

Example 9. Let $(\mathbb{R}, d_S)$ the Sorgenfrey line (see Example 1). $(\mathbb{R}, d_S)$ is complete because if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d_S^s), then there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for all $n \geq n_0$. Hence, $(x_n)_{n\in\mathbb{N}}$ converges in $(\mathbb{R}, d_S^{-1})$. Taking $q = d_S$, we have that q is a strong-mw-distance (see Example 4).

Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ defined by $fx = c$, for all $x \in \mathbb{R}$.

If we define $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) = \frac{t}{2}$, ϕ is a Jachymski function and $\phi(t) < t$ for all $t > 0$. Moreover, $q(fx, fy) = 0 \leq \phi(q(x, y))$ for all $x, y \in X$.

Therefore, all conditions of Theorem 2 are satisfied. In fact, $z = c$ is the unique fixed point of f. Nevertheless, q is not a w-distance (see Example 1), so we cannot apply Theorem 1.

The following example shows that in Theorem 2 the strong condition for the mw-distance cannot be omitted.

Example 10. Let $X = \{1/n : n \in \mathbb{N}\}$ and let d be the quasi-metric on X given by $d(x, x) = 0$, and $d(x, y) = x$. (X, d) is a complete quasi-metric space. Indeed, let $\{x_n\}$ be a Cauchy sequence in (X, d^s). If there exists $k \in \mathbb{N}$ such that $x_n = x_k$ for all $n \geq k$, obviously $\{x_n\}$ converges to x_k in (X, d^{-1}). If for all $n \in \mathbb{N}$ there exists $k_n \geq n$ such that $x_n \neq x_{k_n}$, then given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{k_n}) = x_n < \varepsilon$ for every $n \geq n_0$. Therefore $d(x_n, x) < \varepsilon$ for every $n \geq n_0$ and for every $x \in X$. So that $\{x_n\}$ converges to x in (X, d^{-1}).

The function $q(x, y) = d(x, y)$ is an $mw-$distance and it is not strong. Indeed, the sequence $\{1/n\}$ converges to 1 in (X, d^{-1}) but if $y \neq 1$, then $\lim_{n\to\infty}(q(1, y) - q(1/n, y)) = 1$. Hence, $q(\cdot, y)$ is not lower semicontinuous on $(X, \tau_{d^{-1}})$.

Let $f : X \to X$ given by $fx = x/3$ and let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ given by $\phi(t) = t/2$. Then ϕ is a Jachymski function such that $\phi(t) < t$, for all $t > 0$ and

$$q(fx, fy) = fx = x/3 < x/2 = \phi(x) = \phi(q(x, y)).$$

Nevertheless, f has not a fixed point in X.

The following example shows that Theorem 2 is not fulfilled if the hypothesis $\phi(t) < t$ for all $t > 0$ is replaced by the condition $\phi(t) \leq t$ for all $t > 0$.

Example 11. Let $X = \mathbb{R}^+$ and let d be the quasi-metric on X defined as $d(x, y) = (y - x) \vee 0$. Clearly (X, d) is complete (observe that $d(x_n, 0) = 0$ for all sequence $\{x_n\} \subset X$). Let q be the strong-mw-distance given by $q(x, y) = y$ for all $x, y \in X$ (see Example 5).

Let $f : X \to X$ defined by $fx = 0$ if $x \in [0, 1/2)$ and $fx = 1/2$ otherwise.

Now we define $\phi = f$. Then ϕ is a Jachymski function. Indeed, if $\varepsilon < 1/2$, taking $\delta > 0$ such that $\varepsilon + \delta < 1/2$ from $\varepsilon < t < \varepsilon + \delta$ it follows $\phi(t) = 0 \leq \varepsilon$. If $\varepsilon \geq 1/2$, then for all $\delta > 0$ from $\varepsilon < t < \varepsilon + \delta$ it follows $\phi(t) = 1/2 \leq \varepsilon$. Furthermore, $q(fx, fy) = fy = \phi(y) = \phi(q(x, y))$.

In this example the condition $\phi(t) < t$ is not satisfied for all $t > 0$ and f has two fixed points 0 and 1/2.

The next is an example where we can apply Theorem 2 for an appropriate strong mw-distance q on a complete quasi-metric space (X, d) but not for d.

Example 12. Let $(\mathbb{R}^+, d)$ the complete quasi-metric space of Example 11 and let q be the strong-mw-distance given by $q(x, y) = y$ for all $x, y \in X$.

Lef $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $fx = x/2$ if $x \geq 1$ and $f(x) = 0$ otherwise.

Now we define $\phi = f$. Then ϕ is a Jachymski function. Indeed, if $\varepsilon < 1$, taking $\delta > 0$ such that $\varepsilon + \delta < 1$ from $\varepsilon < t < \varepsilon + \delta$ it follows $\phi(t) = 0 \leq \varepsilon$. If $\varepsilon \geq 1$, taking $\delta = \varepsilon$ from $\varepsilon < t < \varepsilon + \delta$ it follows $\phi(t) = t/2 \leq \varepsilon$. Moreover,

$$q(fx, fy) = fy = \phi(y) = \phi(q(x, y)).$$

Therefore the conditions of Theorem 2 are satisfied. In fact $z = 0$ is the unique fixed point of f.

Nevertheless, the contraction condition of Theorem 2 is not satisfied for d. Indeed,

$$d(f\frac{1}{2}, f1) = d(0, 1/2) = 1/2 > 0 = \phi(d(\frac{1}{2}, 1)).$$

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