Probabilistic solution of the homogeneous Riccati differential equation: A case-study by using linearization and transformation techniques

M.-C. Casabán[a], J.-C. Cortés[a], A. Navarro-Quiles[a], J.-V. Romero[a], M.-D. Roselló[a,*], R.-J. Villanueva[a]

[a] *Instituto Universitario de Matemática Multidisciplinar, Universitat Politècnica de València, Camino de Vera s/n, 46022, Valencia, Spain*

Abstract

This paper deals with the determination of the first probability density function of the solution stochastic process to the homogeneous Riccati differential equation taking advantage of both linearization and Random Variable Transformation techniques. The study is split in all possible casuistries regarding the deterministic/random character of the involved input parameters. An illustrative example is provided for each one of the considered cases.

Keywords: random Riccati differential equations, random variable transformation technique, first probability density function

1. Introduction and motivation

Numerous physical and social phenomena involve the study of uncertainty due not only to measurement errors required to conduct the analysis of such phenomena but also the inherent complexity associated to their own nature. The consideration of randomness leads to two main types of differential equations, namely, stochastic differential equations (s.d.e.'s) and random differential equations (r.d.e.'s). These two classes of differential equations are different in the manner the uncertainty is considered and, as a consequence, completely different approaches for solving, analysing and approximating are required. On the one hand, in dealing with s.d.e.'s, uncertainty is forced by an irregular stochastic process such as a Brownian motion. When possible, s.d.e.'s are solved by taking advantage of a special stochastic calculus usually referred to as Itô-Stratonovic calculus, otherwise numerical techniques are developed [1, 2, 3]. On the other hand, r.d.e.'s constitute a natural generalization of their deterministic counterpart since random effects are directly manifested through input parameters (coefficients, source terms and initial/boundary conditions) which are assumed to be random variables (r.v.'s) and/or stochastic processes (s.p.'s).

*Corresponding author

Email addresses: macabar@imm.upv.es (M.-C. Casabán), jccortes@imm.upv.es (J.-C. Cortés), annaqui@doctor.upv.es (A. Navarro-Quiles), jvromero@imm.upv.es (J.-V. Romero), drosello@imm.upv.es (M.-D. Roselló), rjvillan@imm.upv.es (R.-J. Villanueva)

An important advantage of r.d.e.'s with respect to s.d.e.'s is that wider range of probability distributions for the inputs are allowed. This includes standard distributions such as beta, gaussian, exponential, etc., as well as many other *ad hoc* distributions like the ones built using copulas [4]. Solving r.d.e.'s require the application of the so-called L_p-calculus, [5, 6]. A number of numerical and analytical methods, which extend their deterministic counterpart, have been proposed to deal with r.d.e.'s including numerical schemes [7], spectral methods [8], Fröbenius series [9], etc. Throughout this paper only r.d.e.'s will be considered. We point out that a common approach to approximate the solutions of s.d.e.'s and r.d.e.'s, is Monte Carlo sampling [10]. Although widely used due to easy implementation, the main drawback of Monte Carlo method is its slow convergence rate, $O(1/\sqrt{M})$ being M the number of simulations. In addition, Monte Carlo technique only provides numerical approximations of solution s.p. in spite of an exact representation could exist.

Solving a r.d.e. means not only to compute, exact or approximately, its solution s.p., say $X(t)$, but also its main statistical functions such as the mean and variance. However, in order to have a full statistical description of the solution in every time instant t, the determination of the first probability density function (1-p.d.f.) is required. The Random Variable Transformation (R.V.T.) technique constitutes a powerful tool to calculate the p.d.f. of a r.v. which comes from the mapping of other r.v. whose p.d.f. is known [11, 12]. In the context of r.d.e.'s, R.V.T. technique has been used to compute the 1-p.d.f. of the solution s.p. of both ordinary and partial differential equations, see for example [13, 14, 15] and references therein.

The aim of this paper is to compute the 1-p.d.f. of the solution s.p. of the following random initial value problem (i.v.p.) based on an homogeneous Riccati-type differential equation

$$\left.\begin{array}{rcl} \dot{X}(t) & = & CX(t) + D(X(t))^2, \quad t \geq 0\,, \\ X(0) & = & X_0\,, \end{array}\right\} \tag{1}$$

where all the input parameters X_0, C and D are assumed to be absolutely continuous r.v.'s defined on a common probability space, $(\Omega, \mathfrak{F}, \mathbb{P})$. Their p.d.f.'s will be denoted by $f_{X_0}(x_0)$, $f_C(c)$, and $f_D(d)$, respectively. Hereinafter, $\mathcal{D}(X_0)$, $\mathcal{D}(C)$ and $\mathcal{D}(D)$, will represent their respective domains. For the sake of generality, statistical dependence among the input r.v.'s X_0, D and C will be also considered. In such case, $f_{X_0,D}(x_0, d)$, $f_{X_0,C}(x_0, c)$, $f_{D,C}(d, c)$ and $f_{X_0,D,C}(x_0, d, c)$, will denote the joint p.d.f.'s of the random vectors (X_0, D), (X_0, C), (D, C) and (X_0, D, C), respectively. Since input parameters can be deterministic or random, in the following we will distinguish them by writing deterministic variables by lower cases and r.v.'s by upper cases. In this way, if the nonlinear coefficient in (1) is deterministic, then it will be denoted as d, whereas D will mean that it is a r.v.

In order to determine the 1-p.d.f. of the solution s.p. of i.v.p. (1), we will take advantage of the results recently established by some of the authors in [16], where a comprehensive study to compute the 1-p.d.f. of the linear random i.v.p.

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & AZ(t) + B, \quad t \geq t_0\,, \\ Z(t_0) & = & Z_0\,, \end{array}\right\} \tag{2}$$

is provided. With this aim, notice that making the change of variable

$$Z(t) = \frac{1}{X(t)}\,, \tag{3}$$

the nonlinear i.v.p. (1) can be transformed into the linear i.v.p. (2), using the following identification of the random inputs

$$Z_0 = \frac{1}{X_0}\,, \quad B = -D\,, \quad A = -C\,, \tag{4}$$

and taking $t_0 = 0$. In this manner, all the results obtained in [16] are available.

In order to facilitate the comparison regarding the notation as well as the casuistries considered in [16] for the i.v.p. (2) with respect to the one to be used for the i.v.p. (1), an identification between both problems is shown in Table 1.

It is important to underline that Cases I.1–I.3, corresponding to the situation where nonlinear coefficient $D = 0$ with probability 1, i.e., $\mathbb{P}\left[\{\omega \in \Omega : D(\omega) = 0\}\right] = 1$, will be omitted in our subsequent analysis since it was already studied in reference [16]. Specifically, it corresponds to the random homogeneous linear differential equation given in the i.v.p. (2) taking $B = 0$ with probability 1, i.e., $\mathbb{P}\left[\{\omega \in \Omega : B(\omega) = 0\}\right] = 1$.

The study of i.v.p. (1) has interest by itself from a mathematical standpoint since it constitutes the extension of the homogeneous Riccati differential equation to the random scenario. In addition, this differential equation arises frequently in important applications to classical control problems, as decoupling techniques for both analytic and numerical study of boundary value problems [17, 18], and also, for instance, in dealing with SI-type epidemiological models [19]. Therefore, its generalization to the random framework can be very useful in order to develop more accurate models that consider the uncertainty usually involved in real phenomena. We want to point out that in the stochastic context, some of the authors have dealt with random Riccati differential equations [20]. In that paper, coefficients are assumed to be analytic s.p.'s and taking advantage of L_p-calculus approximate solutions for the mean and the variance of the solution s.p. are constructed. However, in that contribution none information about the 1-p.d.f. of the solution s.p. is provided.

This paper is organized as follows. In Section 2 a number of results coming from specializations of the Random Variable Transformation method that will be required throughout the paper are established. Sections 3 and 4 are devoted to compute explicit expressions for the 1-p.d.f. of the solution s.p. of i.v.p. (1) in the Cases II.1–II.3 and Cases III.1–III.7, respectively. Examples in each one of these cases are included to illustrate the theoretical results. Conclusions are drawn in Section 5.

2. Auxiliary results

In this section we will establish several results that will be required throughout this paper. They are specializations of scalar and multi-dimensional versions of R.V.T. technique which can be found in [16, Eq. (3)] and [16, Theorem 4], respectively.

Proposition 1 (R.V.T. technique: inverse transformation). *Let U be an absolutely continuous real r.v. defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, with p.d.f. $f_U(u)$. Assume that $U(\omega) \neq 0$ for all $\omega \in \Omega$ and, let us denote by $\mathcal{D}(U)$ the domain of r.v. U, where*

$$\mathcal{D}(U) = I_u^- \cup I_u^+\,, \quad \begin{cases} I_u^- = \{u = U(\omega) \in \mathbb{R} \,:\, u < 0\,,\ \omega \in \Omega\}\,, \\ I_u^+ = \{u = U(\omega) \in \mathbb{R} \,:\, 0 < u\,,\ \omega \in \Omega\}\,. \end{cases}$$

	I.V.P.(2)				I.V.P.(1)		
	CASE	random	deterministic		CASE	random	deterministic
$\mathbb{P}\left[\{\omega \in \Omega : B(\omega) = 0\}\right] = 1$	I.1	Z_0	a	$\mathbb{P}\left[\{\omega \in \Omega : D(\omega) = 0\}\right] = 1$	I.1	X_0	c
	I.2	A	z_0		I.2	C	x_0
	I.3	(Z_0, A)	—		I.3	(X_0, C)	—
$\mathbb{P}\left[\{\omega \in \Omega : A(\omega) = 0\}\right] = 1$	II.1	Z_0	b	$\mathbb{P}\left[\{\omega \in \Omega : C(\omega) = 0\}\right] = 1$	II.1	X_0	d
	II.2	B	z_0		II.2	D	x_0
	II.3	(Z_0, B)	—		II.3	(X_0, D)	—
$\mathbb{P}\left[\{\omega \in \Omega : A(\omega) \neq 0,\ B(\omega) \neq 0\}\right] = 1$	III.1	Z_0	(b, a)	$\mathbb{P}\left[\{\omega \in \Omega : D(\omega) \neq 0,\ C(\omega) \neq 0\}\right] = 1$	III.1	X_0	(d, c)
	III.2	B	(z_0, a)		III.2	D	(x_0, c)
	III.3	A	(z_0, b)		III.3	C	(x_0, d)
	III.4	(Z_0, B)	a		III.4	(X_0, D)	c
	III.5	(Z_0, A)	b		III.5	(X_0, C)	d
	III.6	(B, A)	z_0		III.6	(D, C)	x_0
	III.7	(Z_0, B, A)	—		III.7	(X_0, D, C)	—

Table 1: List of different cases in which i.v.p. (1) is split to conduct the study and identification for the notation used regarding the involved deterministic/random inputs in i.v.p.'s (2) and (1).

Then, the p.d.f. $f_V(v)$ of the inverse transformation $V = \frac{1}{U}$ is given by

$$f_V(v) = \frac{1}{v^2} f_U\left(\frac{1}{v}\right), \qquad v \in \mathcal{D}(V) = I_v^- \cup I_v^+, \quad \begin{cases} I_v^- = \{v = V(\omega) \in \mathbb{R} \,:\, v < 0\,,\ \omega \in \Omega\}\,, \\ I_v^+ = \{v = V(\omega) \in \mathbb{R} \,:\, v > 0\,,\ \omega \in \Omega\}\,. \end{cases} \tag{5}$$

Proof. Let us consider the mapping $v = r(u) = \frac{1}{u}$. Notice that r is strictly monotone over the intervals $-\infty < u < 0$ and $0 < u < +\infty$. Hence, its inverse mapping exists and is given by $u = s(v) = \frac{1}{v}$, being its derivative $s'(v) = -\frac{1}{v^2}$. Then, by applying [16, Eq.(3)] with the identification $X = U$ and $Y = V$ in each subinterval, the expression (5) is straightforwardly obtained. The determination of the domain $\mathcal{D}(V)$ follows easily since the transformation $r(u)$ is decreasing monotone in each subinterval. ⊠

Proposition 2 (R.V.T. technique: opposite transformation). *Let U be an absolutely continuous real r.v. defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, with p.d.f. $f_U(u)$. Let us denote by $\mathcal{D}(U) = \{u = U(\omega) \in \mathbb{R} : u_1 < u < u_2\}$ the domain of r.v. U. Then, the p.d.f. $f_V(v)$ of the opposite transformation $V = -U$ is given by*

$$f_V(v) = f_U(-v)\,, \qquad \mathcal{D}(V) = \{v = V(\omega) \in \mathbb{R} : -u_2 < v < -u_1\}\,. \tag{6}$$

Proof. Let us consider the mapping $v = r(u) = -u$. Notice that r is strictly monotone over $\mathbb{R}$. Hence, its inverse mapping exists and is given by $u = s(v) = -v$, being its derivative $s'(v) = -1$. Then, by applying [16, Eq.(3)] with the identification $X = U$ and $Y = V$, the expression (6) is straightforwardly obtained. The determination of the domain $\mathcal{D}(V)$ follows easily since the transformation $r(u)$ is decreasing monotone in $\mathbb{R}$. ⊠

Proposition 3 (R.V.T. technique: inverse-opposite transformation). *Let $\mathbf{U} = (U_1, U_2)$ be an absolutely continuous real random vector defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, with joint p.d.f. $f_{\mathbf{U}}(u_1, u_2)$. Assume that $U_1(\omega) \neq 0$ for all $\omega \in \Omega$ and, let us denote by $\mathcal{D}(U_1)$ the domain of r.v. U_1, where*

$$\mathcal{D}(U_1) = I_{u_1}^- \cup I_{u_1}^+\,, \quad \begin{cases} I_{u_1}^- = \{u_1 = U_1(\omega) \in \mathbb{R} \,:\, u_1 < 0\,,\ \omega \in \Omega\}\,, \\ I_{u_1}^+ = \{u_1 = U_1(\omega) \in \mathbb{R} \,:\, 0 < u_1\,,\ \omega \in \Omega\}\,. \end{cases}$$

Let us denote by $\mathcal{D}(U_2) = \{u_2 = U_2(\omega) \in \mathbb{R} : u_{2,1} < u_2 < u_{2,2}\}$ the domain of r.v. U_2. Notice that the domain $\mathcal{D}(\mathbf{U})$ of the random vector $\mathbf{U}$ is given by $\mathcal{D}(\mathbf{U}) = \mathcal{D}(U_1) \times \mathcal{D}(U_2)$.

Then, the joint p.d.f. $f_{\mathbf{V}}(v_1, v_2)$ of the inverse-opposite transformation $\mathbf{V} = (V_1, V_2) = (\frac{1}{U_1}, -U_2)$ is given by

$$f_{\mathbf{V}}(v_1, v_2) = \frac{1}{(v_1)^2} f_{\mathbf{U}}\left(\frac{1}{v_1}, -v_2\right), \qquad (v_1, v_2) \in \mathcal{D}(\mathbf{V}) = \mathcal{D}(V_1) \times \mathcal{D}(V_2)\,, \tag{7}$$

where

$$\mathcal{D}(V_1) = I_{v_1}^- \cup I_{v_1}^+, \quad \begin{cases} I_{v_1}^- = \{v_1 = V_1(\omega) \in \mathbb{R} \,:\, v_1 < 0\,,\ \omega \in \Omega\}\,, \\ I_{v_1}^+ = \{v_1 = V_1(\omega) \in \mathbb{R} \,:\, v_1 > 0\,,\ \omega \in \Omega\}\,, \end{cases}$$

$$\mathcal{D}(V_2) = \{v_2 = V_2(\omega) \in \mathbb{R} : -u_{2,2} < v_2 < -u_{2,1}\}\,.$$

Proof. Let us consider the two-dimensional transformation $(v_1, v_2) = \mathbf{r}(u_1, u_2) = (1/u_1, -u_2)$. Notice that its inverse mapping is given by $(u_1, u_2) = \mathbf{s}(v_1, v_2) = (1/v_1, -v_2)$, being its Jacobian

$$J_2 = \det \begin{pmatrix} -\frac{1}{(v_1)^2} & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{(v_1)^2} \neq 0 \, .$$

Then, by applying [16, Theorem 4] for $n = 2$ and the identification $X_i = U_i$, $Y_i = V_i$, $i = 1, 2$, the expression (7) is straightforwardly obtained. The determination of the domain $\mathcal{D}(\mathbf{V})$ follows easily from Propositions 1 and 2. ⊠

Proposition 4 (R.V.T. technique: opposite-opposite transformation). *Let* $\mathbf{U} = (U_1, U_2)$ *be an absolutely continuous real random vector defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*, with joint p.d.f.* $f_{\mathbf{U}}(u_1, u_2)$*. Let us denote by* $\mathcal{D}(U_i) = \{u_i = U_i(\omega) \in \mathbb{R} : u_{i,1} < u_i < u_{i,2}\}$ *the domain of r.v.* U_i*,* $i = 1, 2$*. Notice that the domain* $\mathcal{D}(\mathbf{U})$ *of the random vector* $\mathbf{U}$ *is given by* $\mathcal{D}(\mathbf{U}) = \mathcal{D}(U_1) \times \mathcal{D}(U_2)$*.*

Then, the joint p.d.f. $f_{\mathbf{V}}(v_1, v_2)$ *of the opposite-opposite transformation* $\mathbf{V} = (V_1, V_2) = (-U_1, -U_2)$ *is given by*

$$f_{\mathbf{V}}(v_1, v_2) = f_{\mathbf{U}}(-v_1, -v_2) \, , \qquad (v_1, v_2) \in \mathcal{D}(\mathbf{V}) = \mathcal{D}(V_1) \times \mathcal{D}(V_2) \, , \tag{8}$$

where

$$\mathcal{D}(V_i) = \{v_i = V_i(\omega) \in \mathbb{R} : -u_{i,2} < v_i < -u_{i,1}\} \, , \quad i = 1, 2 \, .$$

Proof. Let us consider the two-dimensional transformation $(v_1, v_2) = \mathbf{r}(u_1, u_2) = (-u_1, -u_2)$. Notice that its inverse mapping is given by $(u_1, u_2) = \mathbf{s}(v_1, v_2) = (-v_1, -v_2)$, being its Jacobian

$$J_2 = \det \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1 \neq 0 \, .$$

Then, by applying [16, Theorem 4] for $n = 2$ and the identification $X_i = U_i$, $Y_i = V_i$, $i = 1, 2$, the expression (8) follows straightforwardly. The determination of the domain $\mathcal{D}(\mathbf{V})$ is directly obtained from Proposition 2. ⊠

Proposition 5 (R.V.T. technique: inverse-opposite-opposite transformation). *Let* $\mathbf{U} = (U_1, U_2, U_3)$ *be an absolutely continuous real random vector defined on a probability space* $(\Omega, \mathfrak{F}, \mathbb{P})$*, with joint p.d.f.* $f_{\mathbf{U}}(u_1, u_2, u_3)$*. Assume that* $U_1(\omega) \neq 0$ *for all* $\omega \in \Omega$*, and let us denote by* $\mathcal{D}(U_1)$ *the domain of r.v.* U_1*, where*

$$\mathcal{D}(U_1) = I_{u_1}^{-} \cup I_{u_1}^{+} \, , \quad \begin{cases} I_{u_1}^{-} = \{u_1 = U_1(\omega) \in \mathbb{R} \, : \, u_1 < 0 \, , \, \omega \in \Omega\} \, , \\ I_{u_1}^{+} = \{u_1 = U_1(\omega) \in \mathbb{R} \, : \, 0 < u_1 \, , \, \omega \in \Omega\} \, , \end{cases}$$

and by $\mathcal{D}(U_i) = \{u_i = U_i(\omega) \in \mathbb{R} : u_{i,1} < u_i < u_{i,2}\}$ *the domain of r.v.* U_i*,* $i = 2, 3$*. Notice that the domain* $\mathcal{D}(\mathbf{U})$ *of the random vector* $\mathbf{U}$ *is given by* $\mathcal{D}(\mathbf{U}) = \mathcal{D}(U_1) \times \mathcal{D}(U_2) \times \mathcal{D}(U_3)$*.*

Then, the joint p.d.f. $f_{\mathbf{V}}(v_1, v_2, v_3)$ *of the inverse-opposite-opposite transformation* $\mathbf{V} = (V_1, V_2, V_3) = (1/U_1, -U_2, -U_3)$ *is given by*

$$f_{\mathbf{V}}(v_1, v_2, v_3) = \frac{1}{(v_1)^2} f_{\mathbf{U}}(1/v_1, -v_2, -v_3) \, , \qquad (v_1, v_2, v_3) \in \mathcal{D}(\mathbf{V}) = \mathcal{D}(V_1) \times \mathcal{D}(V_2) \times \mathcal{D}(V_3) \, , \tag{9}$$

where

$$\mathcal{D}(V_1) = I_{v_1}^- \cup I_{v_1}^+, \quad \begin{cases} I_{v_1}^- = \{v_1 = V_1(\omega) \in \mathbb{R} \, : \, v_1 < 0 \,, \; \omega \in \Omega\} \,, \\ I_{v_1}^+ = \{v_1 = V_1(\omega) \in \mathbb{R} \, : \, v_1 > 0 \,, \; \omega \in \Omega\} \,, \end{cases}$$

$$\mathcal{D}(V_i) = \{v_i = V_i(\omega) \in \mathbb{R} : -u_{i,2} < v_i < -u_{i,1}\}, \quad i = 2, 3 \,.$$

Proof. Let us consider the three-dimensional transformation $(v_1, v_2, v_3) = \mathbf{r}(u_1, u_2, u_3) = (1/u_1, -u_2, -u_3)$. Notice that its inverse mapping is given by $(u_1, u_2, u_3) = \mathbf{s}(v_1, v_2, v_3) = (1/v_1, -v_2, -v_3)$, being its Jacobian

$$J_3 = \det \begin{pmatrix} -\frac{1}{(v_1)^2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -\frac{1}{(v_1)^2} \neq 0 \,.$$

Then, by applying [16, Theorem 4] for $n = 3$ and the identification $X_i = U_i$, $Y_i = V_i$, $i = 1, 2, 3$, the expression (9) follows straightforwardly. The determination of the domain $\mathcal{D}(\mathbf{V})$ is directly obtained from Propositions 1 and 2. ⊠

3. Solving the Cases II.1–II.3

This section is addressed to compute the 1-p.d.f., $f_1(x, t)$, of the solution s.p. of i.v.p. (1) in each one of the Cases II.1-II.3 collected in Table 1. Thus, throughout this section the deterministic parameter c that appears into the problem (1) is assumed to be null, $c = 0$. As it was pointed out in Section 1, to conduct our analysis we will take advantage of results obtained in Cases II.1-II.3 studied in [16] (see i.v.p. (2) in Table 1).

3.1. Case II.1: X_0 is a random variable

Notice that regarding problem (1), we are assuming implicitly that $d \in \mathbb{R} - \{0\}$ and X_0 is a r.v. with p.d.f. $f_{X_0}(x_0)$. In accordance with Table 1 and (4), this situation corresponds to the following particular case of linear i.v.p. (2)

$$\left.\begin{aligned} \dot{Z}(t) &= b \,, \\ Z(0) &= Z_0 \,, \end{aligned}\right\} \quad Z_0 = \frac{1}{X_0}, \quad b = -d. \tag{10}$$

Now, we fix $t \geq 0$ and apply [16, Eq. (59)] in order to compute the p.d.f. of the solution s.p. of i.v.p. (10) evaluated at that t, since the randomness character of X_0 is transferred to Z_0

$$f_Z(z) = f_{Z_0}(z - b\,t) \,. \tag{11}$$

Note that for the sake of clarity, we have used the notation $f_Z(z)$ instead of $f_1(z, t)$ since the time variable t has been fixed, so $Z = Z(t)$ is a r.v. rather than a s.p.

In order to express (11) in terms of the data, we take into account (10) and apply Proposition 1 to $U = X_0$, $V = Z_0$. This yields

$$f_Z(z) = f_{Z_0}(z + d\,t) = \frac{1}{(z + d\,t)^2} f_{X_0}\left(\frac{1}{z + d\,t}\right).$$

Considering (3) which establishes the relationship between the solutions of i.v.p.'s (1) and (2), $X(t) = 1/Z(t)$, and applying Proposition 1 to $U = Z$ and $V = X$, with $Z = Z(t)$ and $X = X(t)$, for each $t \geq 0$, one gets

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \frac{1}{x^2} \frac{1}{\left(\frac{1}{x} + d\,t\right)^2} f_{X_0}\left(\frac{1}{\frac{1}{x} + d\,t}\right).$$

Since $t \geq 0$ is arbitrary, this expression represents the 1-p.d.f. of the solution s.p. $X(t)$ of the i.v.p. (1)

$$f_1(x,t) = \frac{1}{(1 + d\,t\,x)^2} f_{X_0}\left(\frac{x}{1 + d\,t\,x}\right), \quad t \geq 0. \tag{12}$$

Although the domains of the 1-p.d.f.'s that will be determined throughout this paper could be specified in the same way they were done in [16], now we will omit them because their specification become cumbersome. For instance, in the context of the current Case II.1, if $\mathcal{D}(X_0)$ denotes the domain of the r.v. X_0, then the domain of the 1-p.d.f. (12) can be determined by imposing that

$$\frac{x}{1 + d\,t\,x} \in \mathcal{D}(X_0)\,. \tag{13}$$

We illustrate this issue in the following example, where the domain of the 1-p.d.f. will be completely determined.

Example 1. *Let us assume that $d = -1$ and X_0 has an exponential distribution of parameter $\lambda = 1$, i.e., $X_0 \sim Exp(1)$. Then, in accordance with* (12) *the* 1*-p.d.f. to the solution s.p. $X(t) = X(t, \omega)$, $\omega \in \Omega$, of i.v.p.* (1) *is given by*

$$f_1(x,t) = \frac{1}{(1 - tx)^2} \mathrm{e}^{-\frac{x}{1-tx}}, \qquad t > 0, \quad 0 < x < \frac{1}{t}. \tag{14}$$

For the full specification of the domain, observe that as $X_0 \sim Exp(1)$ and $t > 0$ then, in accordance with (13) *we impose*

$$\frac{x}{1 - tx} > 0 \Longleftrightarrow x < \frac{1}{t}.$$

It is easy to check that $\int_0^{1/t} f_1(t,x)\mathrm{d}x = 1$. In Figure 1, $f_1(x,t)$ is represented for different values of t. Important statistical information associated to the solution s.p. $X(t)$ can be determined from its 1*-p.d.f., such as, the mean, $\mathbb{E}[X(t)]$, and the variance, $\mathbb{V}[X(t)]$. Taking into account* (14)*, the expectation is given by*

$$\mathbb{E}[X(t)] = \int_{-\infty}^{\infty} x f_1(x,t)\mathrm{d}x = \int_0^{1/t} \frac{x}{(1 - tx)^2} \mathrm{e}^{-\frac{x}{1-tx}} \mathrm{d}x = \frac{t - \mathrm{e}^{\frac{1}{t}} \int_{1/t}^{\infty} \mathrm{e}^{-\xi}/\xi \,\mathrm{d}\xi}{t^2}.$$

In order to determine $\mathbb{V}[X(t)]$, first we need to compute

$$\mathbb{E}\left[(X(t))^2\right] = \int_0^{1/t} x^2 f_1(x,t)\mathrm{d}x = \frac{t(1+t) - \mathrm{e}^{\frac{1}{t}}(1+2t) \int_{1/t}^{\infty} \mathrm{e}^{-\xi}/\xi \,\mathrm{d}\xi}{t^4}.$$

Therefore

$$\mathbb{V}\left[X(t)\right] = \mathbb{E}\left[(X(t))^2\right] - \left(\mathbb{E}\left[X(t)\right]\right)^2 = \frac{-\mathrm{e}^{2/t}\left(\int_{1/t}^{\infty} \mathrm{e}^{-\xi}/\xi \,\mathrm{d}\xi\right)^2 - \mathrm{e}^{1/t} \int_{1/t}^{\infty} \mathrm{e}^{-\xi}/\xi \,\mathrm{d}\xi + t}{t^4}.$$

Plots for the mean and the variance are shown in Figure 2. Notice that these plots are in agreement with the plot of the 1*-p.d.f.* $f_1(x,t)$*. Indeed, as the mean tends to stabilize as t increases, hence the variance goes to zero and the shape of* $f_1(x,t)$ *becomes leptokurtic.*

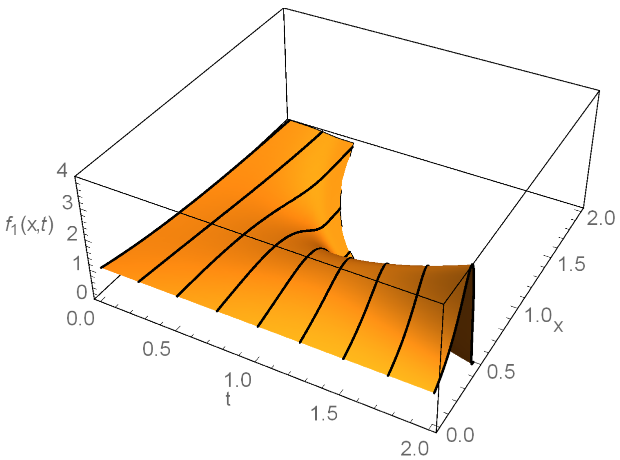

Figure 1: Plot of the 1-p.d.f. $f_1(x,t)$ given by (14) in the Example 1 at different values of $t = \{0, 0.25, 0.5, 0.75, \ldots, 2\}$ (corresponding to the solid lines) with $X_0 \sim \text{Exp}(\lambda = 1)$ and $d = -1$.

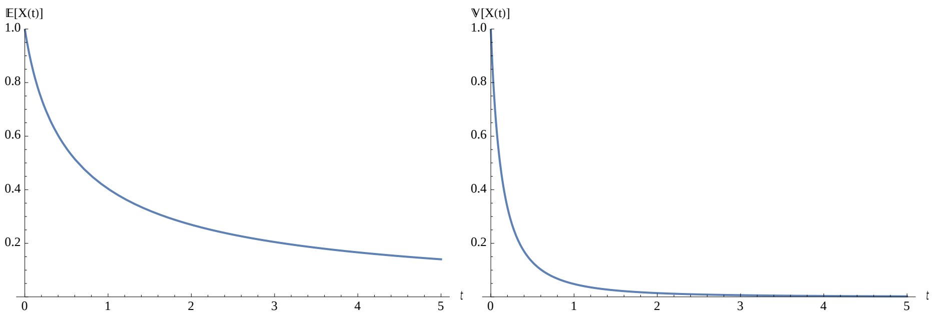

Figure 2: Plot of the expectation (left) and the variance (right) of the solution s.p. in the Example 1.

3.2. *Case II.2: D is a random variable*

Let us assume that nonlinear coefficient D is a r.v. with p.d.f. $f_D(d)$ and the initial condition is a deterministic constant x_0. In agreement with Table 1 and (4), it corresponds to i.v.p. (2)

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & B\,, \\ Z(0) & = & z_0\,, \end{array}\right\} \quad z_0 = \frac{1}{x_0}\,, \qquad B = -D\,. \tag{15}$$

For $t > 0$ fixed, according to [16, Eq. (64)] the p.d.f. of the solution s.p. of i.v.p. (15) evaluated at that t is given by

$$f_Z(z) = \frac{1}{t} f_B\left(\frac{z - z_0}{t}\right).$$

Next, we represent the above expression in terms of the p.d.f. of r.v. D taking into account (15) and Proposition 2 to $U = D$, $V = B$

$$f_Z(z) = \frac{1}{t} f_B\left(\frac{z - 1/x_0}{t}\right) = \frac{1}{t} f_D\left(\frac{1 - z x_0}{x_0 t}\right). \tag{16}$$

Finally, taking into account that $X(t) = 1/Z(t)$, applying (16) and Proposition 1 to $U = Z$ and $V = X$, with $Z = Z(t)$ and $X = X(t)$, for each $t > 0$, one follows

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \frac{1}{x^2}\frac{1}{t} f_D\left(\frac{1 - \frac{1}{x}x_0}{x_0 t}\right) = \frac{1}{x^2\, t} f_D\left(\frac{x - x_0}{x x_0 t}\right).$$

Therefore, in this case the 1-p.d.f. of the solution s.p. $X(t)$ of the i.v.p. (1) is given by

$$f_1(x,t) = \frac{1}{x^2\, t} f_D\left(\frac{x - x_0}{x x_0 t}\right), \quad t > 0. \tag{17}$$

If $t = 0$, $X(0) = x_0$ and then

$$f_1(x,0) = \delta(x - x_0), \quad -\infty < x < \infty,$$

where $\delta(\cdot)$ denotes the Dirac delta function.

Example 2. *Let us take $x_0 = 1$ and D a standard gaussian r.v., $D \sim N(0;1)$. According to* (17) *the* 1*-p.d.f. to the solution s.p. $X(t)$, of i.v.p.* (1) *is given by*

$$f_1(x,t) = \frac{\mathrm{e}^{-\frac{(x-1)^2}{2t^2x^2}}}{\sqrt{2\pi}\, t x^2}. \tag{18}$$

In Figure 3 a plot of $f_1(x,t)$ is shown. One observes that the variability of the solution decreases as t goes on and the 1*-p.d.f. concentrates about $x = 0$.*

3.3. Case II.3: (X_0, D) is a random vector

In this context, we assume that both, the initial condition X_0, and the nonlinear coefficient D, are r.v.'s with joint p.d.f. $f_{X_0,D}(x_0,d)$. As it is listed in Table 1 and considering the identification (4), this case corresponds to the following specialization of i.v.p. (2)

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & B\,, \\ Z(0) & = & Z_0\,, \end{array}\right\} \quad Z_0 = \frac{1}{X_0}, \quad B = -D. \tag{19}$$

Let us fix $t > 0$, according to [16, Eq. (74)] the p.d.f. of the solution s.p. of i.v.p. (19) evaluated at that t is given by

$$f_Z(z) = \frac{1}{t}\int_{\mathcal{D}(Z_0)} f_{Z_0,B}\left(\xi, \frac{z-\xi}{t}\right) \mathrm{d}\xi\,, \tag{20}$$

where $\mathcal{D}(Z_0)$ denotes the domain of r.v. $Z_0 = 1/X_0$. Now, we apply Proposition 3 to $U_1 = X_0$, $U_2 = D$, $V_1 = Z_0$ and $V_2 = B$ to express (20) in terms of the joint p.d.f. $f_{X_0,D}(x_0,d)$.

$$f_Z(z) = \frac{1}{t}\int_{\mathcal{D}(1/X_0)} \frac{1}{\xi^2} f_{X_0,D}\left(\frac{1}{\xi}, \frac{\xi - z}{t}\right) \mathrm{d}\xi\,.$$

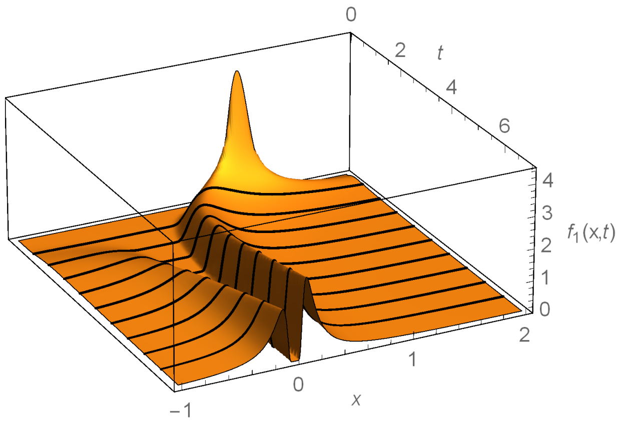

Figure 3: Plot of the 1-p.d.f. $f_1(x,t)$ given by (18) in the Example 2 at different values of $t = \{0, 0.25, 0.5, 0.75, \ldots, 2\}$ (corresponding to the solid lines) with $x_0 = 1$ and $D \sim \mathrm{N}(0; 1)$.

For each $t > 0$, by (3) $X = 1/Z$ and, applying Proposition 1 one gets

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \frac{1}{x^2\, t} \int_{\mathcal{D}(1/X_0)} \frac{1}{\xi^2} f_{X_0,D}\left(\frac{1}{\xi}, \frac{x\xi - 1}{tx}\right) \mathrm{d}\xi.$$

Therefore, in this case the 1-p.d.f. of the solution s.p. $X(t)$ of the i.v.p. (1) is given by

$$f_1(x,t) = \frac{1}{x^2\, t} \int_{\mathcal{D}(1/X_0)} \frac{1}{\xi^2} f_{X_0,D}\left(\frac{1}{\xi}, \frac{x\xi - 1}{tx}\right) \mathrm{d}\xi, \quad t > 0\,.$$

In accordance with (5), the domain $\mathcal{D}(1/X_0)$ can be easily computed from $\mathcal{D}(X_0)$, which is assumed to be known.

If $t = 0$, as $X(0) = X_0$ the 1-p.d.f. is just the marginal p.d.f. of the joint p.d.f. $f_{X_0,D}(x_0, d)$, hence

$$f_1(x,0) = \int_{\mathcal{D}(D)} f_{X_0,D}(x_0, d)\, \mathrm{d}d.$$

Example 3. *Let us assume that the joint p.d.f. of the random vector* (X_0, D) *is given by*

$$f_{X_0,D}(x_0,d) = \begin{cases} \frac{1}{4} + \frac{1}{4}(x_0)^3 d - \frac{1}{4} x_0 d^3 & \text{if} \quad -1 \le x_0 \le 1, \ -1 \le d \le 1, \\ 0 & \text{otherwise}. \end{cases} \tag{21}$$

A plot of $f_1(x,t)$ *is depicted in Figure 4. From it, we see that for each* t *the probability of the solution s.p.* $X(t)$ *distributes symmetrically about* $x = 0$ *becoming leptokurtic as* t *increases.*

4. Solving the Cases III.1–III.7

This section is devoted to provide explicit formulas for the 1-p.d.f., $f_1(x,t)$, of the solution s.p. of i.v.p. (1) in each one of the Cases III.1–III.7 listed in Table 1. Notice that in contrast to

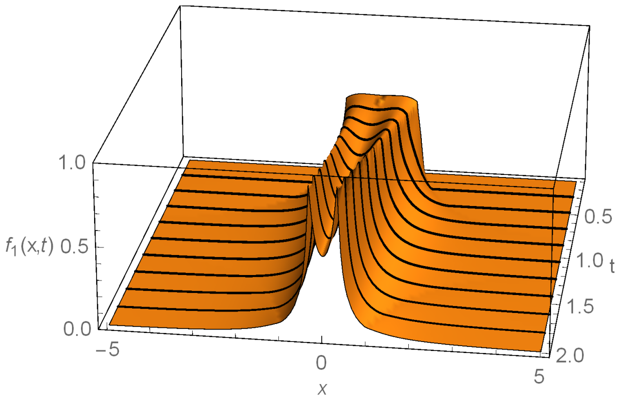

Figure 4: Plot of the 1-p.d.f. $f_1(x,t)$ in the Example 3 at different values of $t = \{0, 0.25, 0.5, 0.75, \ldots, 2\}$ (corresponding to the solid lines) in the case that (X_0, D) has the joint p.d.f. given by (21).

what was assumed when analyzing Cases II.1–II.3, throughout this section the linear coefficient C can be either deterministic and different from zero, or a r.v. As it was indicated previously, in the first case it will be denoted by c and in the latter as C.

4.1. *Case III.1: X_0 is a random variable*

Let $f_{X_0}(x_0)$ be the p.d.f. of r.v. X_0, $c \in \mathbb{R} - \{0\}$ and $d \in \mathbb{R}$. According to Table 1 and (4), it corresponds to the following particular case of i.v.p. (2)

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & aZ(t) + b\,, \\ Z(0) & = & Z_0\,, \end{array}\right\} \quad Z_0 = \frac{1}{X_0}, \quad b = -d, \quad a = -c\,. \tag{22}$$

Let us fix $t \geq 0$, then by [16, Eq. (84)] the p.d.f. of the solution s.p. of i.v.p. (22) evaluated at that t is given by

$$f_Z(z) = \mathrm{e}^{-at} f_{Z_0}\left(\mathrm{e}^{-at}\left(z + \frac{b}{a}\right) - \frac{b}{a}\right).$$

Now, taking into account (22) and Proposition 1 to $U = X_0$, $V = Z_0$ we can express $f_Z(z)$ as follows

$$f_Z(z) = \frac{\mathrm{e}^{ct}}{\left(\mathrm{e}^{ct}\left(z + \frac{d}{c}\right) - \frac{d}{c}\right)^2} f_{X_0}\left(\frac{1}{\mathrm{e}^{ct}\left(z + \frac{d}{c}\right) - \frac{d}{c}}\right) = \frac{c^2 \mathrm{e}^{ct}}{(\mathrm{e}^{ct}(zc + d) - d)^2} f_{X_0}\left(\frac{c}{\mathrm{e}^{ct}(zc + d) - d}\right).$$

Following the same argument exhibited in the previous cases, for each $t \geq 0$, this p.d.f. can be expressed as a function of the r.v. $X = 1/Z$ by applying Proposition 1, this yields

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \frac{c^2 \mathrm{e}^{ct}}{(\mathrm{e}^{ct}(c + dx) - dx)^2} f_{X_0}\left(\frac{cx}{\mathrm{e}^{ct}(c + dx) - dx}\right).$$

Summarizing, the 1-p.d.f. of the solution s.p. $X(t)$ of the i.v.p. (1) is given by

$$f_1(x,t) = \frac{c^2 \mathrm{e}^{ct}}{(\mathrm{e}^{ct}(c + dx) - dx)^2} f_{X_0}\left(\frac{cx}{\mathrm{e}^{ct}(c + dx) - dx}\right), \quad t \geq 0\,. \tag{23}$$

Example 4. *Let $c = 1/2$, $d = -1$ and X_0 a standard gaussian r.v., $X_0 \sim N(0;1)$ be the input parameters of i.v.p.* (1)*. According to* (23)*, the* 1*-p.d.f. to the solution s.p. $X(t)$ to this i.v.p. is given by*

$$f_1(x,t) = \frac{e^{\frac{t}{2}-\frac{x^2}{8\left(e^{t/2}\left(\frac{1}{2}-x\right)+x\right)^2}}}{4\sqrt{2\pi}\left(e^{t/2}\left(\frac{1}{2}-x\right)+x\right)^2}. \tag{24}$$

Figure 5 shows the plot of $f_1(x,t)$. From this representation, one observes that the variability of the 1*-p.d.f. reduces as t increases.*

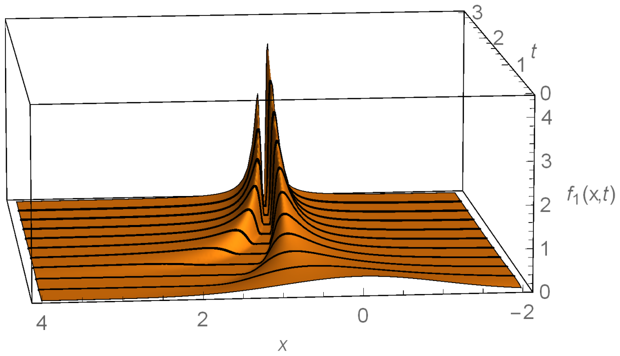

Figure 5: Plot of the 1-p.d.f. $f_1(x,t)$ given by (24) in the Example 4 at different values of $t = \{0, 0.25, 0.5, 0.75, \dots, 2\}$ (corresponding to the solid lines) with $c = 1/2$, $d = -1$ and $X_0 \sim N(0;1)$.

4.2. *Case III.2: D is a random variable*

Let $f_D(d)$ be the p.d.f. of r.v. D and let us assume that both the initial condition x_0 and the linear coefficient c are deterministic constants. Taking into account Table 1 and (4), this case corresponds to the following particularization of i.v.p. (2)

$$\left.\begin{array}{rcl}\dot{Z}(t) &=& aZ(t)+B\,,\\ Z(0) &=& z_0\,,\end{array}\right\} \quad z_0 = \frac{1}{x_0}, \quad B = -D, \quad a = -c\,. \tag{25}$$

For each $t > 0$ fixed, by applying [16, Eq. (92)] the following expression for the p.d.f. of the solution s.p. of i.v.p. (25) evaluated at that t is obtained

$$f_Z(z) = \frac{a}{e^{at}-1} f_B\left(\frac{a(z - z_0 e^{at})}{e^{at}-1}\right).$$

This p.d.f. can be expressed in terms of the data x_0, D and c by considering (25), and applying Proposition 2 to $U = D$ and $V = B$, this yields

$$f_Z(z) = \frac{c}{1-e^{-ct}} f_D\left(\frac{-c(1 - z x_0 e^{ct})}{x_0(1-e^{ct})}\right).$$

Finally, taking into account that $X = 1/Z$ and applying Proposition 1, the 1-p.d.f. of the solution s.p. $X(t)$ of the i.v.p. (1) is obtained as follows

$$f_1(x,t) = \frac{c}{x^2(1-e^{-ct})} f_D\left(\frac{-c(x - x_0 e^{ct})}{x x_0(1-e^{ct})}\right), \quad t > 0\,. \tag{26}$$

For $t = 0$, as $X(0) = x_0$ one gets

$$f_1(x, 0) = \delta(x - x_0), \quad -\infty < x < \infty. \tag{27}$$

Example 5. *Let us consider the i.v.p.* (1) *with* $x_0 = 1$, $c = -1$ *and* X_0 *a gamma r.v. of parameters* $\alpha = 4$ *and* $\beta = 2$, $X_0 \sim Ga(4; 2)$. *In Figure 5, the* 1*-p.d.f. of the solution s.p. given by* (26) *is plotted. One observes that the probability density concentrates about* $x = 0$ *as t increases.*

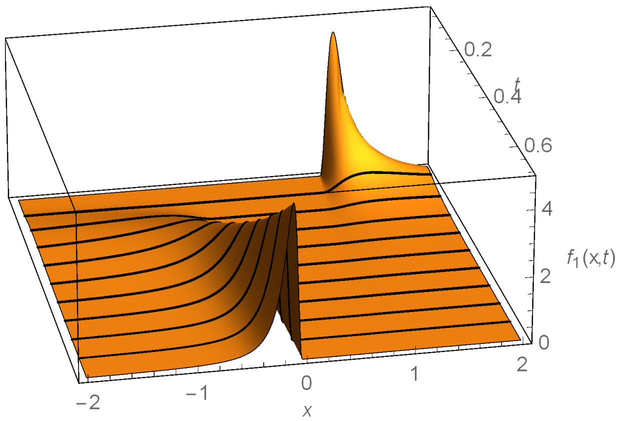

Figure 6: Plot of the 1-p.d.f. $f_1(x, t)$ given by (26) in the Example 5 at different values of $t = \{0, 0.25, 0.5, 0.75, \ldots, 2\}$ (corresponding to the solid lines) with $x_0 = 1$, $c = -1$ and $D \sim \text{Ga}(4; 2)$.

4.3. Case III.3: C is a random variable

So far the computation of the 1-p.d.f. of the solution s.p. of the nonlinear i.v.p. (1) has relied on the application of the results previously established for the linear i.v.p. (2) by some of the authors in [16]. In fact, notice that we have taken advantage of the explicit expression of the 1-p.d.f. of the solution of i.v.p. (2) in each one of the Cases II.1-II.3 and Cases III.1-III.2 to obtain a closed expression of the 1-p.d.f. in the corresponding cases to i.v.p. (1). Unfortunately, this strategy is not feasible when the single random input in (1) is the coefficient C because of the complexity of the approximate expression to the 1-p.d.f. of the underlying linear i.v.p. (2) (see Case III.3 of [16]). To overcome this drawback, we will apply the same strategy we used in the Case III.3 of [16], but directly on the closed expression of the solution s.p. of the nonlinear i.v.p. (1), which in the current case is given by

$$X(t) = \frac{C x_0 e^{Ct}}{C + d x_0 - d x_0 e^{Ct}}. \tag{28}$$

In order to apply R.V.T. technique, for each $t \geq 0$, first from (28) we define the mapping $r(C) = (C x_0 e^{Ct})/(C + d x_0 - d x_0 e^{Ct})$. As it is not possible to isolate the r.v. C to determine the inverse mapping, say s of r, we approximate s using the Lagrange-Bürmann theorem which permits to calculate the inverse mapping of an analytic function. This approximation comes from the truncation of an infinite series (see [16, Th.19]). As can be checked in detail in the analysis of the Case III.1 studied in [16], the 1-p.d.f. of the solution s.p. (28) can be represented as follows

$$f_1(x,t) = \sum_{j=1}^{k} f_C\left(s_{j,N_j}\right)\left|\frac{\mathrm{d}s_{j,N_j}(x)}{\mathrm{d}x}\right|, \tag{29}$$

where $f_C(c)$ represents the p.d.f. of r.v. C, k denotes the number of subintervals in which the domain of r.v. C must be split to guarantee that the mapping r is monotone and s_{j,N_j} is the approximation of the inverse mapping s on the subinterval j, using truncation of order N_j, $1 \le j \le k$.

Example 6. *Let us assume that* $x_0 = 1$, $d = 1$ *and* $C \sim Be(\alpha = 2; \beta = 3)$. *In Figure 7, we have plotted the* 1*-p.d.f. of the solution s.p. of i.v.p.* (1) *using Lagrange-Bürmann theorem and* (29) *for different values of* t. *To carry out these computations the domain of the r.v.* C *has been split into* $k = 1$ *piece. We observe that the variance of the solution s.p.* $X(t)$ *increases as* t *goes on.*

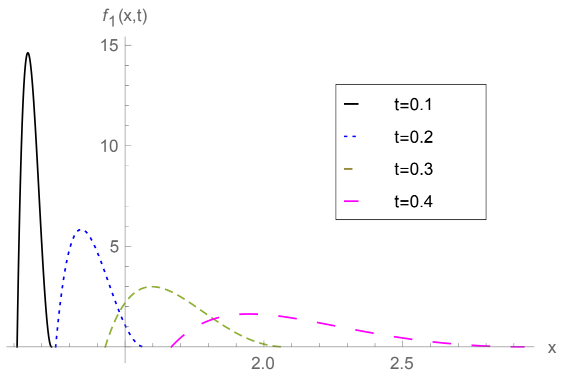

Figure 7: Plot of the 1-p.d.f. $f_1(x,t)$ in the Example 6 at different values of $t = \{0.1, 0.2, 0.3, 0.4\}$ with $x_0 = 1$, $d = 1$ and $C \sim \mathrm{Be}(\alpha = 2; \beta = 3)$.

4.4. *Case III.4:* (X_0, D) *is a random vector*

Now, the initial condition X_0 and the nonlinear coefficient D are assumed to be r.v.'s whose joint p.d.f. is denoted by $f_{X_0,D}(x_0,d)$, whereas the parameter c is deterministic. In agreement to Table 1 and (4), this corresponds to the following particular case of i.v.p. (2)

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & aZ(t) + B\,, \\ Z(0) & = & Z_0\,, \end{array}\right\} \quad Z_0 = \frac{1}{X_0}, \quad B = -D, \quad a = -c\,. \tag{30}$$

Let $t > 0$ be fixed, then applying [16, Eq. (110)] we obtain the p.d.f. of the solution s.p. of i.v.p. (30) evaluated at that t

$$f_Z(z) = \int_{\mathcal{D}(Z_1)} f_{Z_0,B}\left(\xi \mathrm{e}^{-at}, \frac{a(z-\xi)}{\mathrm{e}^{at}-1}\right)\frac{a\mathrm{e}^{-at}}{\mathrm{e}^{at}-1}\mathrm{d}\xi = \int_{\mathcal{D}(Z_1)} f_{Z_0,B}\left(\xi \mathrm{e}^{ct}, \frac{-c(z-\xi)}{\mathrm{e}^{-ct}-1}\right)\frac{c\mathrm{e}^{ct}}{1-\mathrm{e}^{-ct}}\mathrm{d}\xi\,,$$

where $Z_1 = \mathrm{e}^{at} Z_0$. We represent $f_Z(z)$ in terms of (X_0, D) taking into account (30) and applying Proposition 3 to $U_1 = X_0$, $U_2 = D$, $V_1 = Z_0$ and $V_2 = B$

$$f_Z(z) = \frac{c}{\mathrm{e}^{ct}-1} \int_{\mathcal{D}(Z_1)} f_{X_0,D}\left(\frac{1}{\xi \mathrm{e}^{ct}}, \frac{c(z-\xi)}{\mathrm{e}^{-ct}-1}\right) \frac{1}{\xi^2} \mathrm{d}\xi \,.$$

By (3), $X(t) = 1/Z(t)$ for each $t > 0$, then denoting $X = X(t)$ and $Z = Z(t)$ the application of Proposition 1 yields

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \frac{1}{x^2} \frac{c}{\mathrm{e}^{ct}-1} \int_{\mathcal{D}(Z_1)} f_{X_0,D}\left(\frac{1}{\xi \mathrm{e}^{ct}}, \frac{c(1-\xi x)}{x(\mathrm{e}^{-ct}-1)}\right) \frac{1}{\xi^2} \mathrm{d}\xi \,.$$

Finally, taking into account that $Z_1 = \mathrm{e}^{at} Z_0 = 1/(\mathrm{e}^{ct} X_0)$, the 1-p.d.f. of the solution s.p. $X(t)$ of the i.v.p. (1) is given by

$$f_1(x,t) = \frac{c}{x^2(\mathrm{e}^{ct}-1)} \int_{\mathcal{D}\left(\frac{1}{\mathrm{e}^{ct} X_0}\right)} f_{X_0,D}\left(\frac{1}{\xi \mathrm{e}^{ct}}, \frac{c(1-\xi x)}{x(\mathrm{e}^{-ct}-1)}\right) \frac{1}{\xi^2} \mathrm{d}\xi, \quad t > 0.$$

If $t = 0$, as $X(0) = X_0$, the 1-p.d.f. of $X(t)$ is the D-marginal p.d.f. of $f_{X_0,D}(x_0, d)$

$$f_1(x,0) = \int_{\mathcal{D}(D)} f_{X_0,D}(x_0, d)\, \mathrm{d}d.$$

Example 7. *Let us assume that $c = -1$ and the joint p.d.f. of the random vector (X_0, D) is a bivariate gaussian distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$ given by*

$$\boldsymbol{\mu} = (\mu_{X_0}, \mu_D) = (1, 0), \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{X_0}^2 & \rho_{X_0,D}\sigma_{X_0}\sigma_D \\ \rho_{X_0,D}\sigma_{X_0}\sigma_D & \sigma_D^2 \end{pmatrix}, \ \sigma_{X_0} = \sigma_D = 1/10, \ \rho_{X_0,D} = 1/2. \tag{31}$$

Figure 8 shows a piece of surface which defines the 1-p.d.f. As in previous cases, $f_1(x,t)$ has less variability as t increases.

4.5. *Case III.5: (X_0, C) is a random vector*

Let us denote by $f_{X_0,C}(x_0, c)$ the joint p.d.f. of random vector (X_0, C) and let us assume that the parameter d is a deterministic constant. In this context according to Table 1 and (4), the i.v.p. (2) writes

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & AZ(t) + b\,, \\ Z(0) & = & Z_0\,, \end{array}\right\} \quad Z_0 = \frac{1}{X_0}, \quad b = -d, \quad A = -C. \tag{32}$$

Let us fix $t \geq 0$, then applying [16, Eq. (126)] the p.d.f. of the solution s.p. of i.v.p. (32) evaluated at that t can be written as

$$f_Z(z) = \int_{\mathcal{D}(Z_2)} \frac{|b|}{\xi^2} \mathrm{e}^{\frac{b}{\xi}t} f_{Z_0,A}\left(z\mathrm{e}^{\frac{b}{\xi}t} + \xi\left(1 - \mathrm{e}^{\frac{b}{\xi}t}\right), \frac{-b}{\xi}\right) \mathrm{d}\xi = \int_{\mathcal{D}(Z_2)} \frac{d}{\xi^2} \mathrm{e}^{-\frac{d}{\xi}t} f_{Z_0,A}\left(z\mathrm{e}^{-\frac{d}{\xi}t} + \xi\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right), \frac{d}{\xi}\right) \mathrm{d}\xi,$$

where $Z_2 = -b/A$. $f_Z(z)$ can be represented in terms of (X_0, C) by applying Proposition 3 to $U_1 = X_0$, $U_2 = C$, $V_1 = Z_0$ and $V_2 = A$ as follows

$$f_Z(z) = \int_{\mathcal{D}(Z_2)} \frac{d}{\xi^2} \mathrm{e}^{-\frac{d}{\xi}t} f_{X_0,C}\left(\frac{1}{z\mathrm{e}^{-\frac{d}{\xi}t} + \xi\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right)}, -\frac{d}{\xi}\right) \frac{1}{\left(z\mathrm{e}^{-\frac{d}{\xi}t} + \xi\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right)\right)^2} \mathrm{d}\xi.$$

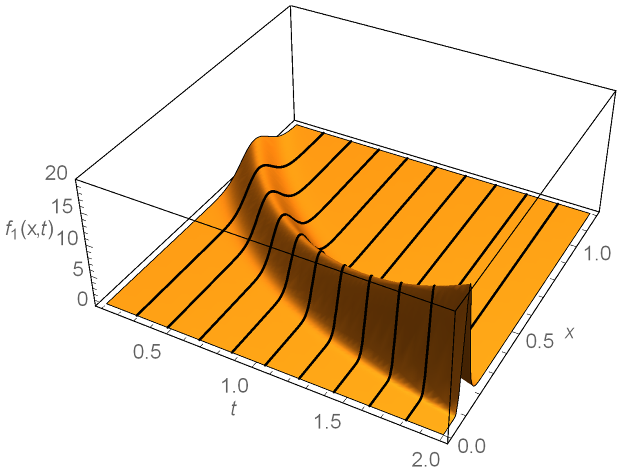

Figure 8: Plot of the 1-p.d.f. $f_1(x,t)$ in the Example 7 at different values of $t = \{0, 0.25, 0.5, 0.75, \ldots, 2\}$ (corresponding to the solid lines) in the case that $c = -1$ and X_0 and D are correlated r.v.'s according to a bivariate gaussian distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$ given by (31).

Taking into account that $X(t) = 1/Z(t)$ for each $t \geq 0$, $f_Z(z)$ can be represented in terms of X applying Proposition 1

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \int_{\mathcal{D}(Z_2)} \frac{d}{\xi^2} \mathrm{e}^{-\frac{d}{\xi}t} f_{X_0,C}\left(\frac{x}{\mathrm{e}^{-\frac{d}{\xi}t} + \xi x\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right)}, -\frac{d}{\xi}\right) \frac{1}{\left(\mathrm{e}^{-\frac{d}{\xi}t} + \xi x\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right)\right)^2} \mathrm{d}\xi.$$

As $Z_2 = -b/A = -d/C$, the domain of the above integral can be expressed in terms of the data. Hence, the 1-pd.f. of the solution s.p. $X(t)$ of the i.v.p. (1) is given by

$$f_1(x,t) = \int_{\mathcal{D}(-d/C)} \frac{d}{\xi^2} \mathrm{e}^{-\frac{d}{\xi}t} f_{X_0,C}\left(\frac{x}{\mathrm{e}^{-\frac{d}{\xi}t} + \xi x\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right)}, -\frac{d}{\xi}\right) \frac{1}{\left(\mathrm{e}^{-\frac{d}{\xi}t} + \xi x\left(1 - \mathrm{e}^{-\frac{d}{\xi}t}\right)\right)^2} \mathrm{d}\xi. \tag{33}$$

Example 8. *Let us take $d = 1$ and $(X_0, C) \sim N(\boldsymbol{\mu}; \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are defined by* (31). *In Figure 9 a plot of the* 1*-p.d.f. $f_1(x,t)$ given by* (33) *is shown. From it, we observe that the variance of the solution s.p. of the corresponding i.v.p.* (1) *increases as t does.*

4.6. Case III.6: (D, C) is a random vector

Throughout this section, $f_{D,C}(d,c)$ will denote the p.d.f. of random vector (D, C) and the initial condition will be assumed to be a deterministic constant x_0. Notice that, in accordance with Table 1 and (4), now we are dealing with the following specialization of i.v.p. (2)

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & AZ(t) + B\,, \\ Z(0) & = & z_0\,, \end{array}\right\} \quad z_0 = \frac{1}{x_0}, \quad B = -D, \quad A = -C\,. \tag{34}$$

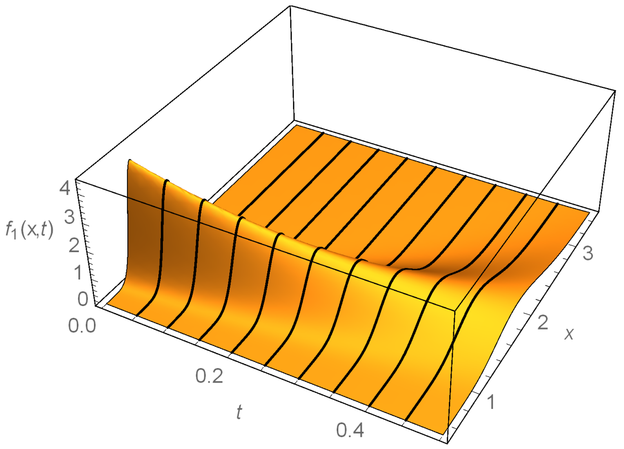

Figure 9: Plot of the 1-p.d.f. $f_1(x,t)$ in the Example 8 at different values of $t = \{0, 0.25, 0.5, 0.75, \ldots, 2\}$ (corresponding to the solid lines) in the case that $d = 1$ and X_0 and C are correlated r.v.'s according to a bivariate gaussian distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$ given by (31).

Let $t > 0$ be fixed, then applying [16, Eq. (140)] the p.d.f. of the solution s.p. of i.v.p. (34) evaluated at that t is given by

$$\begin{aligned} f_Z(z) &= \frac{z_0}{t^2}\int_{\mathcal{D}(Z_2)} f_{B,A}\left(\frac{z_0(z-\xi)}{t}\frac{\ln(\xi)-\ln(z_0)}{\xi - z_0}, \frac{\ln(\xi)-\ln(z_0)}{t}\right)\frac{1}{\xi}\left|\frac{\ln(\xi)-\ln(z_0)}{\xi - z_0}\right| \mathrm{d}\xi \\ &= \frac{1}{x_0 t^2}\int_{\mathcal{D}(Z_2)} f_{B,A}\left(\frac{z-\xi}{t}\frac{\ln(\xi)+\ln(x_0)}{\xi x_0 - 1}, \frac{\ln(\xi)+\ln(x_0)}{t}\right)\frac{|x_0|}{\xi}\left|\frac{\ln(\xi)+\ln(x_0)}{\xi x_0 - 1}\right| \mathrm{d}\xi, \end{aligned}$$

where $Z_2 = z_0 \mathrm{e}^{At}$. This p.d.f. $f_Z(t)$ can be expressed in terms of the random vector (D, C) by applying Proposition 4 to $U_1 = D$, $U_2 = C$, $V_1 = B$ and $V_2 = A$,

$$f_Z(z) = \frac{1}{x_0 t^2}\int_{\mathcal{D}(Z_2)} f_{D,C}\left(\frac{z-\xi}{t}\frac{\ln(\xi)+\ln(x_0)}{1-\xi x_0}, -\frac{\ln(\xi)+\ln(x_0)}{t}\right)\frac{|x_0|}{\xi}\left|\frac{\ln(\xi)+\ln(x_0)}{\xi x_0 - 1}\right| \mathrm{d}\xi.$$

Now, by applying Proposition 1 to $X = 1/Z$, $f_Z(z)$ is represented in terms of X

$$f_X(x) = \frac{1}{x^2} f_Z\left(\frac{1}{x}\right) = \frac{1}{x^2}\frac{1}{x_0 t^2}\int_{\mathcal{D}(Z_2)} f_{D,C}\left(\frac{1-\xi x}{xt}\frac{\ln(\xi)+\ln(x_0)}{1-\xi x_0}, -\frac{\ln(\xi)+\ln(x_0)}{t}\right)\frac{|x_0|}{\xi}\left|\frac{\ln(\xi)+\ln(x_0)}{\xi x_0 - 1}\right| \mathrm{d}\xi.$$

As $Z_2 = z_0 \mathrm{e}^{At} = 1/(x_0 \mathrm{e}^{Ct})$, the 1-p.d.f. of the solution s.p. $X(t)$ to the i.v.p. (1) is given by

$$f_1(x,t) = \frac{1}{x^2 x_0 t^2}\int_{\mathcal{D}\left(1/(x_0 \mathrm{e}^{Ct})\right)} f_{D,C}\left(\frac{1-\xi x}{xt}\frac{\ln(\xi)+\ln(x_0)}{1-\xi x_0}, -\frac{\ln(\xi)+\ln(x_0)}{t}\right)\frac{|x_0|}{\xi}\left|\frac{\ln(\xi)+\ln(x_0)}{\xi x_0 - 1}\right| \mathrm{d}\xi. \tag{35}$$

If $t = 0$, as $X(0) = x_0$ one gets

$$f_1(x,0) = \delta(x - x_0), \qquad -\infty < x < \infty\,.$$

Example 9. *Let us assume that $x_0 = 1$ and the joint p.d.f. of the random vector (D, C) is given by*

$$f_{D,C}(d,c) = \begin{cases} \frac{2}{3}(2 - d - c + 2dc) & \textit{if} \quad 0 \le d \le 1,\ 0 \le c \le 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{36}$$

For the sake of clarity in the presentation, in Figure 10 the 1*-p.d.f. $f_1(x,t)$ given by* (35) *is shown for different values of t. From it, one infers that the variability of the solution s.p. of the i.v.p.* (1) *tends to increases as time t goes on.*

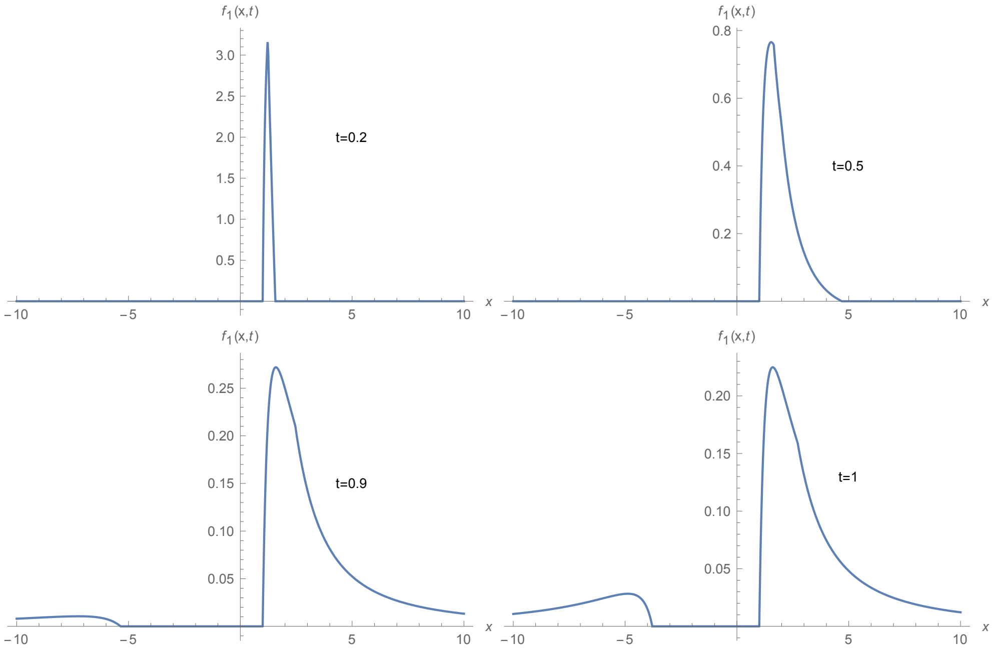

Figure 10: Plot of the 1-p.d.f. $f_1(x,t)$ in the Example 9 at different values of t in the case that $x_0 = 1$ and (D, C) has the joint p.d.f. given by (36).

4.7. *Case III.7: (X_0, D, C) is a random vector*

In this last case, we deal with the i.v.p. (2) assuming that all the inputs (X_0, D, C) are r.v.'s whose joint p.d.f. is $f_{X_0,D,C}(x_0, d, c)$. Taking into account Table 1 and (4), this corresponds to

$$\left.\begin{array}{rcl} \dot{Z}(t) & = & AZ(t) + B\,, \\ Z(0) & = & Z_0\,, \end{array}\right\} \quad Z_0 = \frac{1}{X_0}, \quad B = -D, \quad A = -C\,. \tag{37}$$

Let $t > 0$ be fixed, then applying [16, Eq. (157)] the p.d.f. of the solution s.p. of i.v.p. (37) evaluated at that t is given by

$$f_Z(z) = \int_{\mathcal{D}(Z_3)} \int_{\mathcal{D}(Z_2)} f_{Z_0,B,A}\left(-\frac{(z-\xi-\eta)\eta}{\xi}, -\frac{\eta}{t}\ln\left(\frac{-\xi}{\eta}\right), \frac{1}{t}\ln\left(\frac{-\xi}{\eta}\right)\right) \frac{|\eta|}{\xi^2}\frac{1}{t^2}\left|\ln\left(-\frac{\xi}{\eta}\right)\right| \mathrm{d}\xi\, \mathrm{d}\eta\,,$$

where $Z_2 = \mathrm{e}^{At}B/A$ and $Z_3 = -B/A$. Now, we will express $f_Z(z)$ as a function of (X_0, D, C) by applying Proposition 5 to $U_1 = X_0$, $U_2 = D$, $U_3 = C$, $V_1 = Z_0$, $V_2 = B$ and $V_3 = A$,

$$f_Z(z) = \int_{\mathcal{D}(Z_3)} \int_{\mathcal{D}(Z_2)} f_{X_0,D,C}\left(-\frac{\xi}{(z-\xi-\eta)\eta}, \frac{\eta}{t}\ln\left(-\frac{\xi}{\eta}\right), -\frac{1}{t}\ln\left(-\frac{\xi}{\eta}\right)\right) \frac{|\eta|}{\eta^2} \frac{1}{(z-\xi-\eta)^2 t^2} \left|\ln\left(-\frac{\xi}{\eta}\right)\right| \mathrm{d}\xi\, \mathrm{d}\eta \,. \tag{38}$$

In order to represent (38) as a function of X, we apply Proposition 1 taking into account that $X = 1/Z$

$$\begin{aligned} f_X(x) &= \tfrac{1}{x^2} f_Z\left(\tfrac{1}{x}\right) \\ &= \int_{\mathcal{D}(Z_3)} \int_{\mathcal{D}(Z_2)} f_{X_0,D,C}\left(-\frac{x\xi}{(1-x\xi-x\eta)\eta}, \frac{\eta}{t}\ln\left(-\frac{\xi}{\eta}\right), -\frac{1}{t}\ln\left(-\frac{\xi}{\eta}\right)\right) \frac{|\eta|}{\eta^2} \frac{1}{(1-x\xi-x\eta)^2 t^2} \left|\ln\left(-\frac{\xi}{\eta}\right)\right| \mathrm{d}\xi\, \mathrm{d}\eta \,. \end{aligned}$$

As $Z_2 = \mathrm{e}^{At}B/A = D/(C\mathrm{e}^{Ct})$ and $Z_3 = -B/A = -D/C$, the 1-p.d.f. of the solution s.p. $X(t)$ to the i.v.p. (1) is given by

$$f_1(x,t) = \int_{\mathcal{D}(-D/C)} \int_{\mathcal{D}(D/(C\mathrm{e}^{Ct}))} f_{X_0,D,C}\left(-\frac{x\xi}{(1-x\xi-x\eta)\eta}, \frac{\eta}{t}\ln\left(-\frac{\xi}{\eta}\right), -\frac{1}{t}\ln\left(-\frac{\xi}{\eta}\right)\right) \frac{|\eta|}{\eta^2} \frac{1}{(1-x\xi-x\eta)^2 t^2} \left|\ln\left(-\frac{\xi}{\eta}\right)\right| \mathrm{d}\xi\, \mathrm{d}\eta \,. \tag{39}$$

If $t = 0$, as $X(0) = X_0$, the 1-p.d.f. of $X(t)$ is the (D, C)-marginal p.d.f. of $f_{X_0,D,C}(x_0, d, c)$

$$f_1(x, 0) = \int_{\mathcal{D}(C)} \int_{\mathcal{D}(D)} f_{X_0,D,C}(x_0, d, c)\, \mathrm{d}d\, \mathrm{d}c \,.$$

Example 10. *Let us assume that the random vector (X_0, D, C) has a multivariate gaussian distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$ defined as follows*

$$\boldsymbol{\mu} = (\mu_{X_0}, \mu_D, \mu_C) = (1, 1, 1), \qquad \boldsymbol{\Sigma} = \frac{1}{10}\begin{pmatrix} 4 & 4 & 4 \\ 1 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \tag{40}$$

Figure 11 shows the 1*-p.d.f. $f_1(x,t)$ given by* (39) *at different values of t. From it, one observes that the variability of the solution s.p. of the i.v.p.* (1) *reduces as t increases.*

5. Conclusions

In this paper we have shown that the Random Variable Transformation method together with linearization techniques can be used successfully to obtain explicit formulas for the first probability density function of the solution stochastic process of nonlinear random differential equations. The study has been conducted through the homogeneous Riccati differential equation although it opens the possibility to be extended to other significant types of nonlinear continuous models. The usefulness of applying both techniques to deal with these class of problems has been shown through a number of illustrative examples.

Acknowledgements

This work has been partially supported by the Ministerio de Economía y Competitividad grants MTM2013-41765-P and TRA2012-36932.

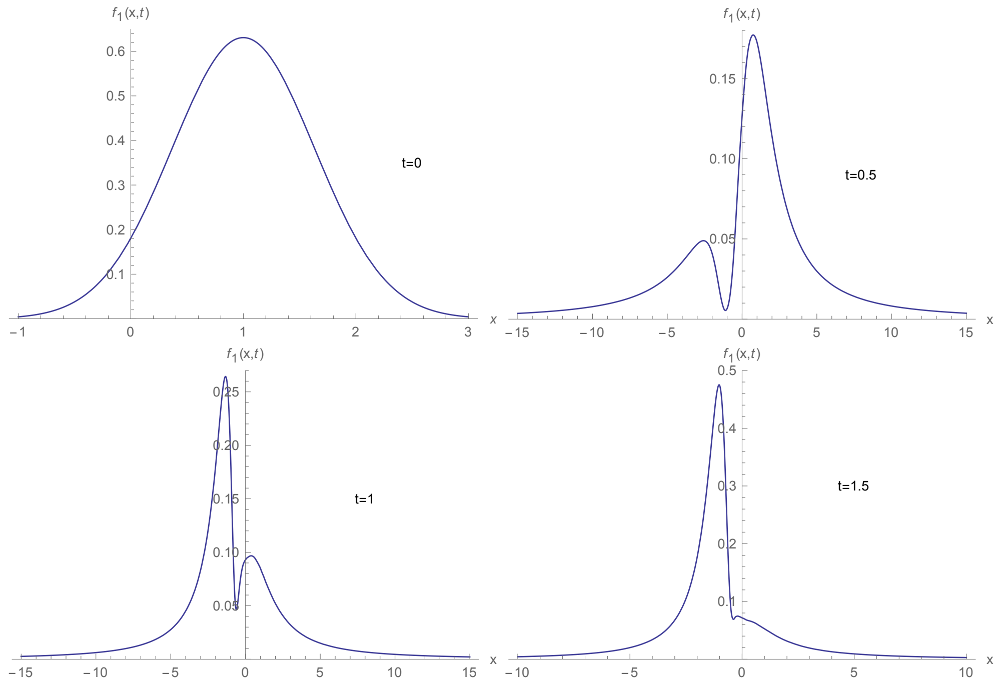

Figure 11: Plot of the 1-p.d.f. $f_1(x,t)$ in the Example 10 at different values of t in the case that (X_0, D, C) has a trivariate gaussian distribution with mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}$ given by (40).

Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

References

[1] B. Øksendal, Stochastic Differential Equations: An Introduction with Applications, 6th Edition, Springer, New York, 2010.

[2] P. E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, 3rd Edition, Vol. 23 of Applications of Mathematics: Stochastic Modelling and Applied Probability, Springer, New York, 1999.

[3] E. Allen, Modeling with Itô Stochastic Differential Equations, Springer, New York, 2007.

[4] R. B. Nelsen, An Introduction to Copulas, Springer, New York, 1999.

[5] T. T. Soong, Random Differential Equations in Science and Engineering, Academic Press, New York, 1973.

[6] L. Villafuerte, C. Braumann, J. C. Cortés, L. Jódar, Random differential operational calculus: Theory and applications, Computers and Mathematics with Applications 59 (1) (2010) 115–125. doi:10.1016/j.camwa.2009.08.061.

[7] J. C. Cortés, L. Villafuerte, L. Jódar, Numerical solution of random differential initial value problems: Multistep methods, Mathematical Methods in the Applied Science 34 (1) (2011) 63–75. doi:10.1002/mma.1331.

[8] R. G. Ghanem, P. D. Spanos, Stochastic Finite Elements: A Spectral Approach, Courier Dover Publ., New York, 2003.

[9] G. Calbo, J. C. Cortés, L. Jódar, Random Hermite differential equations: Mean square power series solutions and statistical properties, Applied Mathematics and Computation 218 (7) (2011) 3654–3666. doi:10.1016/j.amc.2011.09.008.

[10] D. P. Kroese, T. Taimre, Z. I. Botev, Handbook of Monte Carlo Methods, Wiley, New Jersey, 2011.

[11] G. Casella, R. Berger, Statistical Inference, Brooks/Cole, New York, 2002.

[12] A. Hussein, M. M. Selim, On the generalization of probabilistic transformation method, Applied Mathematics and Computation 190 (15) (2007) 1284–1289. doi:10.1016/j.amc.2011.12.088.

[13] M. A. El-Tawil, The approximate solutions of some stochastic differential equations using transformations, Applied Mathematics and Computation 164 (1) (2005) 167–178. doi:10.1016/j.amc.2004.04.062.

[14] A. Hussein, M. M. Selim, Solution of the stochastic radiative transfer equation with Rayleigh scattering using RVT technique, Applied Mathematics and Computation 218 (13) (2012) 7193–7203. doi:10.1016/j.amc.2011.12.088.

[15] L. T. Santos, F. A. Dorini, M. C. C. Cunha, The probability density function to the random linear transport equation, Applied Mathematics and Computation 216 (5) (2010) 1524–1530. doi:10.16/j.amc.2010.03.001.

[16] M. C. Casabán, J. C. Cortés, J. V. Romero, M. D. Roselló, Determining the first probability density function of linear random initial value problems by the Random Variable Transformation (RVT) technique: A comprehensive study, Abstract and Applied Analysis 2014–ID248512 (2014) 1–25. doi:10.1155/2013/248512.

[17] W. T. Reid, Riccati Differential Equations, Academic Press, New York, 1972.

[18] U. M. Ascher, R. M. Mattheij, R. D. Russell, Numerical Solution of Boundary Value Problems for Ordinary Differential Equations, Englewood Cliffs: Prentice Hall, New Jersey, 1988.

[19] F. Brauer, C. Castillo-Chávez, Mathematical Models in Population Biology and Epidemiology, Springer, New York, 2001.

[20] J. C. Cortés, L. Jódar, R. Company, L. Villafuerte, Solving Riccati time-dependent models with random quadratic coefficient, Applied Mathematics Letters 24 (12) (2011) 2193–2196. doi:10.1016/j.aml.2011.06.024.