Solving random mixed heat problems: A random integral transform approach

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Abstract

This paper develops a random mean square Fourier transform approach to solve random partial differential heat problems with nonhomogeneous boundary value conditions. Random mean square operational rules for the random Fourier sine and cosine transforms are stated and illustrative examples are included.

Keywords: Random Fourier sine and cosine transforms; Random heat problem; Nonhomogeneous boundary value conditions; Mean square approach.

1. Introduction

The analysis of heat conduction involves modelling both temperature and heat flow. In practice, these quantities depend on a number of physical properties of the materials which often are not known from a deterministic point of view. Apart from uncertainties due to measurement errors needed to build physical models, the above comments motivate the consideration of random approaches to modelling heat conduction in heterogeneous medium. Differential equations have demonstrated to be powerful tools to model heat problems [1]. Hence, the consideration of uncertainty leads to random and stochastic differential equations. These kind of equations are distinctly different and they require completely different techniques for analysis and approximation. On the one hand, the uncertainty in stochastic differential equations is forced by an irregular stochastic process such a Wiener process or Brownian motion. Solving stochastic differential equations requires Itô or Stratonovich calculus [2–4]. On the other hand, random differential equations permit to consider other type of randomness in the input data (coefficients, forcing term and initial/boundary conditions) including exponential, beta or gaussian distributions, for instance. The so-called L_p-calculus constitutes an adequate framework to solve random differential equations [5, 6]. Alternative approaches include the so-called dishonest methods [7]; the random perturbation method that considers randomness through the perturbation of deterministic data [8]; Monte Carlo sampling consists of generating numerical values according to the distribution of the random inputs, then solving the governing differential equation, which becomes deterministic, and finally, estimate the required solution statistics, such as the mean and the variance [9]; finite difference methods [10, 11]; finite element methods [12]; homotopy transformation method [13, 14]; random transformation method [15]; and Fourier transformation methods [16].

In this paper we develop a random Fourier mean square transform method for solving heat problems which consider randomness into their formulation. The mean square approach developed for both, the ordinary and partial differential problems [6, 17–20], has two desirable properties. First, the mean square solution coincides with the one obtained in the deterministic case, that is, when the random data become deterministic. Secondly, if $X_n(t)$ represents an approximation of the exact solution, $X(t)$, in the mean square sense, then the expectation, $\mathrm{E}[X_n(t)]$, and the variance, $\mathrm{Var}[X_n(t)]$, will converge to the exact values, $\mathrm{E}[X(t)]$ and $\mathrm{Var}[X(t)]$, respectively, for each t, i.e.,

$$\lim_{n\to\infty} \mathrm{E}[X_n(t)] = \mathrm{E}[X(t)] \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}[X_n(t)] = \mathrm{Var}[X(t)]\,, \tag{1}$$

see Theorems 4.2.1 and 4.3.1. of [5].

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This paper, that may be considered as a continuation and generalization of [16], deals with random heat problems with nonhomogeneous boundary value conditions of the type

$$w_t(x,t) = L\, w_{xx}(x,t)\,, \quad x>0, \quad t>0\,, \tag{2}$$
$$w(0,t) = A\,, \qquad t>0, \tag{3}$$
$$w(x,0) = f(x;B)\,, \qquad x>0, \tag{4}$$

or

$$w_t(x,t) = L\, w_{xx}(x,t)\,, \quad x>0, \quad t>0\,, \tag{5}$$
$$w_x(0,t) = g(t;B)\,, \qquad t>0, \tag{6}$$
$$w(x,0) = f(x;A)\,, \qquad x>0, \tag{7}$$

where L is a positive random variable (r.v.) independent of r.v.'s A and B, all of them satisfying additional properties to be specified later. In the previous models, $f(x;B)$, $g(t;B)$ and $f(x;A)$ are stochastic processes (s.p.'s) described as functions that depend on a single r.v. The same results are available, but with more complicated notation, by considering functions with a finite degree of randomness (see comments quoted in [5, p.37]). Unlike to the finite medium random heat model, to the best of our knowledge there is a lack of reliable numerical answers to the solution of random heat problems in a infinite medium. This paper deals with the construction of reliable solutions of models (2)–(4) and (5)–(7) by extending to the random scenario the Fourier sine and cosine transforms.

This paper is organized as follows. Section 2 is devoted to introduce some preliminaries that will clarify both the understanding and reading of the paper as well as the presentation of results of next sections. Section 3 considers the random heat problem (2)–(4) with the so called third kind boundary conditions [1]. By using random Fourier sine transform and results of [16] a mean square solution of model (2)–(4) is explicitly constructed. In Section 4, the random heat problem (5)–(7) with second kind boundary conditions in the sense of [1] is treated. By decomposing the problem in two subproblems and considering the results of [16] and the application of the random Fourier cosine transform, an explicit mean square solution of model (5)–(7) is obtained. Illustrative numerical examples for both problems (2)–(4) and (5)–(7) are included in Sections 3 and 4, respectively. Section 5 is addressed to summarize the conclusions of the paper.

2. Preliminaries about mean square and mean fourth random calculus

For the sake of clarity, we begin this section by reviewing the main definitions and results belonging to the so-called L_p-calculus. In this paper, we are mainly interested in L_2 and L_4-calculus, which are usually referred to as mean square (m.s.) and mean fourth (m.f.) calculus (see [5, 6] for further details). Throughout this paper, the triplet $(\Omega, \mathcal{F}, \mathcal{P})$ will denote a probabilistic space.

We say that a real r.v. $X : \Omega \longrightarrow \mathbb{R}$ belongs to the set $L_p = L_p(\Omega, \mathcal{F}, \mathcal{P})$, $p \geq 1$, if the expectation of the r.v. $|X|^p$ is finite, i.e., $\mathrm{E}[|X|^p] < +\infty$. In such case, we say that X is a p-r.v. The following map

$$\begin{array}{cccl} \|X\|_p & : & L_p & \longrightarrow & [0,\infty[, \\ & & X & \hookrightarrow & (\mathrm{E}\,[\,|X|^p])^{1/p}\,, \end{array} \tag{8}$$

defines a norm in L_p, usually referred to as p-norm, in a such way that $(L_p, \|X\|_p)$ is a Banach space, [2, p.9].

The concept of convergence of a sequence of p-r.v.'s, say, $\{X_n : n \geq 0\} \in L_p$, to a r.v. $X \in L_p$, follows straightforwardly from the above definition of the p-norm

$$\lim_{n\to+\infty} \|X_n - X\|_p = 0.$$

The concept of random function or stochastic process, say $\{X(t) : t \in T\}$, where $T \subseteq \mathbb{R}$, in the space L_p is an extension of the one corresponding to a sequence of p-r.v.'s. We say that $\{X(t) : t \in T\}$ is a p-s.p. if, and only if, $X(t)$ is a p-r.v. for each $t \in T$. The definitions of continuity, differentiability and integrability of p-s.p.'s in the Banach space

$(L_p, \|\cdot\|_p)$ are the ones inferred by the p-norm. For instance, according to [5, p. 99], [21], a p-s.p. $\{X(t) : t \in \mathbb{R}\}$, is said to be L_p-locally integrable in $\mathbb{R}$ if, for all finite interval $[t_1, t_2] \subset \mathbb{R}$, the integral $\int_{t_1}^{t_2} X(t)\, dt$, exits in L_p and, it is L_p-absolutely integrable in $\mathbb{R}$, if $\int_{-\infty}^{+\infty} \|X(t)\|_p\, dt < +\infty$.

Let X be a r.v. in L_q, i.e., $\mathrm{E}[|X|^q] < \infty$, then by Lyapunov's inequality one gets $(\mathrm{E}[|X|^p])^{1/p} \le (\mathrm{E}[|X|^q])^{1/q}$, for $0 \le p \le q$, [22, p.157]. As a consequence, $L_q \subseteq L_p$, $0 \le p \le q$ and, moreover if $\{X_n : n \ge 0\}$ is q-th mean convergent to $X \in L_q$, then $\{X_n : n \ge 0\}$ is also p-th mean convergent to $X \in L_p$, [2, p.13]. In particular, m.f. convergence entails m.s. convergence. The following inequality

$$\|X\,Y\|_p \le \|X\|_{2p}\, \|Y\|_{2p}\,, \quad X, Y \in L_{2p},\ p \ge 1, \tag{9}$$

will play an important role in the subsequent development in the particular case that $p = 2$ which permits to relate m.s. and m.f. convergence, [23].

Next, we introduce a family of r.v.'s that have previously been used to solve some types of random differential equations (see [20] and references therein) and which will play an important role in the subsequent development. Let L be a r.v. such as its absolute statistical moments, $\mathrm{E}[|L|^n]$, behave as $O(H^n)$, i.e., there exist a non-negative integer n_0 and positive constants M and H such that

$$\mathrm{E}[|L|^n] \le MH^n, \quad \forall n \ge n_0\,. \tag{10}$$

Truncated r.v.'s constitute an important class of r.v.'s satisfying condition (10), see Remark 1 in [16].

Suppose that apart from (10) we assume that realizations of r.v. L have a positive lower bound $\ell_1 > 0$ such that

$$L(\omega) \ge \ell_1 > 0\,, \qquad \forall\, \omega \in \Omega\,, \tag{11}$$

then from the definition of expectation, it follows that

$$\mathrm{E}\left[\frac{1}{L^n}\right] \le \frac{1}{(\ell_1)^n}\,, \quad n \ge 0\,, \tag{12}$$

and from (12)

$$\left(\left\|e^{-\frac{x^2}{L}}\right\|_2\right)^2 = \mathrm{E}\left[e^{\frac{-2x^2}{L}}\right] = \sum_{n\ge 0} \frac{(-2x^2)^n\, \mathrm{E}\left[\frac{1}{L^n}\right]}{n!} \le \sum_{n\ge 0} \frac{\left(-\frac{2x^2}{\ell_1}\right)^n}{n!} = e^{-\frac{2x^2}{\ell_1}}\,, \quad \forall\, x \in \mathbb{R}. \tag{13}$$

Thus

$$\left\|e^{-\frac{x^2}{L}}\right\|_2 \le e^{-\frac{x^2}{\ell_1}}\,, \quad \forall\, x \in \mathbb{R}. \tag{14}$$

Note that in (13) the commutation between the expectation operation and the infinite sum is justified because of the m.s. convergence of the random series $\sum_{n\ge 0} \frac{(-2x^2/L)^n}{n!}$ and the application of property (1).

We close this section by computing the following random integral

$$\int_0^\infty e^{-t\,\xi^2 L} \cos(\xi\, x)\, d\xi = \frac{1}{2}\sqrt{\frac{\pi}{tL}}\, e^{-x^2/4tL}\,, \quad x > 0,\ t > 0\,, \tag{15}$$

that will be required later. Notice that this result follows from the application of Lemma 2 of [16], condition (10) and [21, p. 61].

2.1. Random Fourier Sine and Cosine Transforms' and their operational calculus

We define the *random Fourier sine* and *cosine* transforms of a 2-s.p. $\{u(x) : x > 0\}$ m.s. locally integrable in $[0, \infty[$, and m.s. absolutely integrable in $[0, \infty[$, i.e.,

$$\int_0^\infty \|u(x)\|_2\, dx < +\infty\,, \tag{16}$$

as the 2-s.p.'s

$$\mathfrak{F}_{\mathfrak{s}}\,[u(x)]\,(\xi) = F_{\mathfrak{s}}(\xi) = \int_0^{\infty} u(x)\sin(\xi\, x)\, dx\,, \qquad \xi > 0\,, \tag{17}$$

and

$$\mathfrak{F}_{\mathfrak{c}}\,[u(x)]\,(\xi) = F_{\mathfrak{c}}(\xi) = \int_0^{\infty} u(x)\cos(\xi\, x)\, dx\,, \qquad \xi > 0\,, \tag{18}$$

respectively. Note that from (16) both integrals appearing in (17) and (18) are convergent in L_2 and thus they are 2-s.p.'s well-defined. Analogously, the random Fourier sine and cosine transforms of a 4-s.p. $\{u(x) \,:\, x > 0\}$ m.f. locally integrable in $[0,\infty[$, and m.f. absolutely integrable in $[0,\infty[$ can be defined changing $\|\cdot\|_2$ by $\|\cdot\|_4$ in (16). Note that if the m.f. random Fourier sine and cosine transforms exist then the m.s. random Fourier sine and cosine transforms do and both coincide.

Although the *random Fourier exponential transform* is not going to be used here directly because the two problems under study (2)–(4) and (5)–(7) are stated in the positive real line $x > 0$, for the sake of convenience in the use of convolution properties, we introduce the definition of the *Fourier exponential transform* of a 2-s.p. $u(x)$, for $x \in \mathbb{R}$, m.s. locally and m.s. absolutely integrable by the formula

$$\mathfrak{F}[u(x)](\xi) = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{\infty} u(x)\, e^{-i\,\xi\, x}\, dx\,, \qquad \xi \in \mathbb{R}\,, \tag{19}$$

where $i = \sqrt{-1}$ denotes the imaginary unit. Given a 2-s.p. $\ell(x)$ m.s. locally and m.s. absolutely integrable in $[0,\infty[$ and let us denote by $\ell_p(x)$ its even extension to the real line, i.e., $\ell_p(-x) = \ell(x)$ for $x > 0$. Then, from definitions (18) and (19) it follows that

$$\mathfrak{F}[\ell_p(x)](\xi) = \sqrt{\frac{2}{\pi}}\, \mathfrak{F}_{\mathfrak{c}}[\ell(x)](\xi)\,, \qquad \xi > 0\,. \tag{20}$$

Let $r(x)$ and $s(x)$ be m.s. integrable s.p.'s defined in the real line, and note that the m.s. absolute integrability of $\int_{-\infty}^{\infty} r(x-\kappa)\, s(\kappa)\, d\kappa$ is guaranteed if $r(x)$ and $s(x)$ are s.p. m.f. absolutely integrable in $\mathbb{R}$, see (9) for $p = 2$. Hence, let $r(x)$ and $s(x)$ be s.p.'s m.f. locally and m.f. absolutely integrable in $\mathbb{R}$, then the *convolution s.p.* of r and s, denoted by $r * s$, is defined by the m.s. integral

$$(r * s)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r(x-\kappa)\, s(\kappa)\, d\kappa\,, \qquad x,\, \kappa \in \mathbb{R}. \tag{21}$$

Assume that $r(x)$, $s(x)$ satisfy

$$\int_{-\infty}^{\infty} \left(\|r(x)\|_4\right)^2\, dx < +\infty\,; \qquad \int_{-\infty}^{\infty} \left(\|s(x)\|_4\right)^2\, dx < +\infty\,. \tag{22}$$

Under hypothesis (22) and property (9) for $p = 2$ together with Cauchy-Schwarz inequality for deterministic real functions one gets

$$\int_{-\infty}^{\infty} \|r(x-\kappa)\, s(\kappa)\|_2\, d\kappa \le \int_{-\infty}^{\infty} \|r(x-\kappa)\|_4\, \|s(\kappa)\|_4\, d\kappa \le \left(\int_{-\infty}^{\infty} \left(\|r(x-\kappa)\|_4\right)^2\, d\kappa\right)^{1/2} \left(\int_{-\infty}^{\infty} \left(\|s(\kappa)\|_4\right)^2\, d\kappa\right)^{1/2} < +\infty\,.$$

Thus the convolution of two 4-s.p.'s $r(x)$ and $s(x)$ satisfying (22) is well-defined by a m.s. convergent integral. Taking into account the Fubini theorem in abstract normed spaces [24, p. 175], [21, sec. 1.85] and the proof of the Fourier exponential transform of convolution of real functions [25, chap. 7] it follows that if $r(x)$, $s(x)$ are m.f. continuous s.p.'s satisfying (22) and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|r(x-\kappa)\, s(\kappa)\|_2\, dx\, d\kappa < +\infty$, then

$$\mathfrak{F}[r * s] = \mathfrak{F}[r]\, \mathfrak{F}[s]\,. \tag{23}$$

Now we are going to give a convolution formula for the random Fourier cosine transform based on (23) and (20). Let $m(x)$ and $n(x)$ be m.f. continuous s.p.'s defined on $[0,\infty[$ and let $m_p(x)$, $n_p(x)$ be, respectively, their even extension s.p.'s on the real line. Assume that

$$\int_0^{\infty} \left(\|m(x)\|_4\right)^2\, dx < +\infty\,; \qquad \int_0^{\infty} \left(\|n(x)\|_4\right)^2\, dx < +\infty\,.$$

From (23) and (20), it follows that

$$\mathfrak{F}[(m_p * n_p)(x)](\xi) = \mathfrak{F}[m_p(x)](\xi)\, \mathfrak{F}[n_p(x)](\xi) = \frac{2}{\pi}\, \mathfrak{F}_{\mathfrak{c}}[m(x)](\xi)\, \mathfrak{F}_{\mathfrak{c}}[n(x)](\xi)\,, \qquad \xi > 0\,. \tag{24}$$

On the other hand, applying (20) on the even s.p. $(m_p * n_p)(x)$, one gets

$$\mathfrak{F}[(m_p * n_p)(x)](\xi) = \sqrt{\frac{2}{\pi}}\mathfrak{F}_{\mathfrak{c}}[(m_p * n_p)(x)](\xi)\,, \qquad \xi > 0\,. \tag{25}$$

Then, taking into account (24) and (25) it follows that

$$\mathfrak{F}_{\mathfrak{c}}[m(x)](\xi)\, \mathfrak{F}_{\mathfrak{c}}[n(x)](\xi) = \sqrt{\frac{\pi}{2}}\, \mathfrak{F}_{\mathfrak{c}}[(m_p * n_p)(x)](\xi)\,, \qquad \xi > 0\,, \tag{26}$$

where, it is easy to show (see [25, sec. 7.4] for the corresponding deterministic result) that

$$\left(m_p * n_p\right)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} m(\kappa)\, \{n(x+\kappa) + n(|x-\kappa|)\}\, d\kappa\,, \qquad x \geq 0\,. \tag{27}$$

Following the ideas of the deterministic inverse Fourier sine and cosine transforms, see [26, chap.2], we define the *random inverse Fourier sine* (and *cosine*) *transforms* of a 2-s.p. $F_{\mathfrak{s}}(\xi)$ (and $F_{\mathfrak{c}}(\xi)$) m.s. locally and m.s. absolutely integrable by the formulae

$$\begin{aligned} \mathfrak{F}_{\mathfrak{s}}{}^{-1}\left[F_{\mathfrak{s}}(\xi)\right](x) &= \frac{2}{\pi}\int_0^{\infty} F_{\mathfrak{s}}(\xi)\sin(\xi\, x)\, d\xi\,, \qquad x > 0\,, \\ \mathfrak{F}_{\mathfrak{c}}{}^{-1}\left[F_{\mathfrak{c}}(\xi)\right](x) &= \frac{2}{\pi}\int_0^{\infty} F_{\mathfrak{c}}(\xi)\cos(\xi\, x)\, d\xi\,, \qquad x > 0\,, \end{aligned} \tag{28}$$

respectively. This definition follows straighforwardly for 4-s.p. in the m.f. sense and, in this case, it coincides with the m.s. inverse Fourier sine and cosine transforms.

The following result contains some m.s. operational rules that will be used in Sections 3 and 4 to solve the random heat problems (2)–(4) and (5)–(7), see Theorem 3 of [16] for rules (i)–(iv). Note that (v) follows directly from a change of variable.

Theorem 1. *Let* $\{u(x) : x > 0\}$ *be a 2-stochastic process twice mean square differentiable with* $u''(x)$ *mean square locally integrable, and with* $u(x)$, $u'(x)$ *and* $u''(x)$ *mean square absolutely integrable in* $[0,\infty[$. *Then*

(i) $\mathfrak{F}_{\mathfrak{s}}\left[u'(x)\right](\xi) = -\xi\, \mathfrak{F}_{\mathfrak{c}}\left[u(x)\right](\xi)$, $\quad \xi > 0$.

(ii) $\mathfrak{F}_{\mathfrak{c}}\left[u'(x)\right](\xi) = -u(0) + \xi\, \mathfrak{F}_{\mathfrak{s}}\left[u(x)\right](\xi)$, $\quad \xi > 0$.

(iii) $\mathfrak{F}_{\mathfrak{s}}\left[u''(x)\right](\xi) = \xi\, u(0) - \xi^2\, \mathfrak{F}_{\mathfrak{s}}\left[u(x)\right](\xi)$, $\quad \xi > 0$.

(iv) $\mathfrak{F}_{\mathfrak{c}}\left[u''(x)\right](\xi) = -u'(0) - \xi^2\, \mathfrak{F}_{\mathfrak{c}}\left[u(x)\right](\xi)$, $\quad \xi > 0$.

(v) $\mathfrak{F}_{\mathfrak{c}}\left[u(ax)\right](\xi) = \frac{1}{a}\, \mathfrak{F}_{\mathfrak{c}}\left[u(x)\right]\left(\frac{\xi}{a}\right)$, $\quad \xi > 0,\; a > 0$.

For the sake of convenience for the subsequence development, we introduce the following examples.

Example 1. *Let* $t > 0$ *and assume that r.v.* L *has a positive lower bound satisfying condition* (11). *Then*

$$\mathfrak{F}_{\mathfrak{c}}\left[\frac{1}{\sqrt{\pi\, t\, L}}\, e^{-x^2/4tL}\right](\xi) = e^{-L\,t\,\xi^2}\,. \tag{29}$$

By linearity and applying Theorem 1-(v) with $u(x) = e^{-x^2/L}$ *and the constant* $a = \frac{1}{2\sqrt{t}}$, *we have*

$$\mathfrak{F}_{\mathfrak{c}}\left[\frac{1}{\sqrt{\pi\, t\, L}}\, e^{-x^2/4tL}\right](\xi) = \frac{1}{\sqrt{\pi\, t\, L}}\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/4tL}\right](\xi) = \frac{2\sqrt{t}}{\sqrt{\pi\, t\, L}}\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L}\right](2\sqrt{t}\,\xi) = \frac{2}{\sqrt{\pi\, L}}\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L}\right](2\sqrt{t}\,\xi)\,. \tag{30}$$

Note that $\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L}\right](\xi)$ *is a m.s. convergent s.p. because from* (14) *it follows that*

$$\int_0^\infty \|e^{-x^2/L}\cos(\xi\, x)\|_2\, dx \leq \int_0^\infty \|e^{-x^2/L}\|_2\, dx \leq \int_0^\infty e^{-x^2/\ell_1}\, dx < \infty\,.$$

Thus $\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L}\right](\xi)$ *is a well-defined 2-s.p. Now, we compute* $\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L}\right](\xi)$ *by using the exact value of its realizations* $\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L(\omega)}\right](\xi)$*,* $\omega\in\Omega$*. Given* $\omega\in\Omega$*, we wish to evaluate*

$$\mathfrak{F}_{\mathfrak{c}}\left[e^{-x^2/L(\omega)}\right](\xi) = \int_0^\infty e^{-x^2/L(\omega)}\cos(\xi\, x)\, dx\,. \tag{31}$$

By [21, p. 61] the real integral appearing in (31) *takes the value*

$$\int_0^\infty e^{-x^2/L(\omega)}\cos(\xi\, x)\, dx = \frac{1}{2}\sqrt{\pi\, L(\omega)}\, e^{-\xi^2 L(\omega)/4}\,, \qquad \omega\in\Omega\,. \tag{32}$$

From (30)*, and using* (31)–(32) *with* $2\sqrt{t}\,\xi$ *in place of* ξ *one gets* (29).

Example 2. *Let us assume that* L *is a positive 4-r.v. satisfying condition* (11)*. Let* $t>0$ *and* $x>0$ *and let us consider the s.p.*

$$q(x;L) = \frac{1}{\sqrt{\pi t L}}\, e^{-x^2/4tL}\,.$$

Next, we show that $q(x;L)$ *is m.f. continuos and*

$$\int_0^\infty \left(\|q(x;L)\|_4\right)^2\, dx < +\infty\,. \tag{33}$$

Let $x\in(x_0-\delta, x_0+\delta)$*,* x_0 *and* $\delta>0$ *such as* $x_0-\delta>0$*. Then, taking* $K = x_0+\delta>0$ *and using condition* (11)*, one gets*

$$\sum_{n\geq 0}\frac{1}{n!}\left\|\left(\frac{-x^2}{4tL}\right)^n\right\|_4 \leq \sum_{n\geq 0}\frac{K^{2n}}{n!}\frac{1}{(4t)^n}\left\|\frac{1}{L^n}\right\|_4 \leq \sum_{n\geq 0}\frac{K^{2n}}{n!}\frac{1}{(4t)^n}\frac{1}{(\ell_1)^n}\,. \tag{34}$$

Then, using D'Alembert test $q(x;L)$ *is a well-defined 4-s.p. and the m.f. locally uniform convergence guarantees the m.f. continuity of* $q(x;L)$ *because of the M-Weierstrass criterion. In order to prove* (33)*, note that*

$$\begin{aligned}
\left(\|q(x;L)\|_4\right)^4 &= \frac{1}{\sqrt{\pi\, t}}\mathrm{E}\left[\frac{e^{-x^2/t\,L}}{L^2}\right] = \frac{1}{\sqrt{\pi\, t}}\mathrm{E}\left[\sum_{n\geq 0}\frac{1}{L^{n+2}}\left(\frac{-x^2}{t}\right)^n\frac{1}{n!}\right] = \frac{1}{\sqrt{\pi\, t}}\sum_{n\geq 0}\mathrm{E}\left[\frac{1}{L^{n+2}}\right]\left(\frac{-x^2}{t}\right)^n\frac{1}{n!}\\
&\leq \frac{1}{\sqrt{\pi\, t}}\sum_{n\geq 0}\frac{1}{(\ell_1)^{n+2}}\left(\frac{-x^2}{t}\right)^n\frac{1}{n!}\\
&= \frac{1}{(\ell_1)^2\sqrt{\pi\, t}}e^{-x^2/\ell_1\, t},
\end{aligned} \tag{35}$$

where condition (11) *has been applied. Notice that following an analogous reasoning as it was shown in* (34)*, it is straightforward to prove the m.f. convergence, and hence the m.s. convergence, of the above random infinite series. Then, according to* (1) *the commutation between the expectation operator and the random infinite sum in* (35) *is legitimated. Therefore,*

$$\int_0^\infty \left(\|q(x;L)\|_4\right)^2\, dx \leq \frac{1}{\ell_1\sqrt[4]{\pi\, t}}\int_0^\infty e^{-x^2/2\ell_1\, t}\, dx < +\infty\,.$$

3. Random heat problem with third kind boundary condition

In this section we deal with the random heat problem for the temperature distribution $u(x,t)$ in a semi-infinite bar with zero temperature at the left-end $x = 0$ and random initial temperature:

$$
\begin{aligned}
u_t(x,t) &= L\, u_{xx}(x,t)\,, \quad x>0, \quad t>0\,, && (36)\\
u(0,t) &= 0\,, \qquad\qquad t>0, && (37)\\
u(x,0) &= f(x;B)\,, \qquad x>0, && (38)
\end{aligned}
$$

where L is a positive 4-r.v. satisfying certain properties to be specified later, and $f(x;B)$ is a m.s. locally and m.s. absolutely integrable s.p. We will suppose that L and B are independent r.v.'s. Assume that problem (36)–(38) admits a solution s.p. $u(x,t)$ m.s. locally and m.s. absolutely integrable, and let us denote

$$
\mathfrak{F}_{\mathfrak{s}}\left[u(\cdot,t)\right](\xi) = \mathcal{U}(t)(\xi)\,, \qquad \xi>0\,, \tag{39}
$$

the random Fourier sine transform of $u(x,t)$ regarded as a s.p. of the active variable $x>0$, for fixed $t>0$. By applying the random Fourier sine transform to both members of (36) and using Theorem 1-(iii), condition (37), the notation introduced in (39) and Lemma 2 of [16] it follows that

$$
\begin{aligned}
\mathfrak{F}_{\mathfrak{s}}\left[u_{xx}(\cdot,t)\right](\xi) &= \xi\, u(0,t) - \xi^2\, \mathfrak{F}_{\mathfrak{s}}\left[u(\cdot,t)\right](\xi) = -\xi^2 \mathcal{U}(t)(\xi)\,,\\
\mathfrak{F}_{\mathfrak{s}}\left[u_t(\cdot,t)\right](\xi) &= \frac{d}{dt}\left(\mathcal{U}(t)(\xi)\right)\,.
\end{aligned}
$$

By applying the random Fourier sine transform to (38) one gets

$$
\mathfrak{F}_{\mathfrak{s}}\left[u(\cdot,0)\right](\xi) = \mathcal{U}(0)(\xi) = \mathfrak{F}_{\mathfrak{s}}\left[f(\cdot\,;B)\right](\xi) = F_{\mathfrak{s}}(\xi;B)\,.
$$

Hence, the problem (36)–(38) is transformed into the random initial value problem for the variable t

$$
\left.
\begin{aligned}
\frac{d}{dt}\left(\mathcal{U}(t)(\xi)\right) &= -L\xi^2\, \mathcal{U}(t)(\xi)\,, \quad t>0,\\
\mathcal{U}(0)(\xi) &= F_{\mathfrak{s}}(\xi;B)\,.
\end{aligned}
\right\} \tag{40}
$$

Let us assume that $F_{\mathfrak{s}}(\xi;B)$ is a 4-s.p., and the moment generating function of r.v. $-L$, denoted by $\Phi_{-L}(t)$, verifies

$$
\Phi_{-L}(t) = \mathrm{E}[e^{-tL}] \text{ is locally bounded about } t=0\,. \tag{41}
$$

Then by Theorem 8 of [19], the solution s.p. of problem (40) is given by

$$
\mathcal{U}(t)(\xi) = F_{\mathfrak{s}}(\xi;B)\, e^{-t\xi^2 L}\,. \tag{42}
$$

By applying the random inverse Fourier sine transform, defined by (28), to both members of (42) one gets

$$
\begin{aligned}
u(x,t) &= \frac{2}{\pi}\int_0^\infty F_{\mathfrak{s}}(\xi;B)\, e^{-t\xi^2 L}\, \sin(\xi\, x)\; d\xi = \frac{2}{\pi}\int_0^\infty \left\{\int_0^\infty f(s;B)\sin(\xi\, s)\, ds\right\} e^{-t\xi^2 L}\, \sin(\xi\, x)\, d\xi\\
&= \frac{2}{\pi}\int_0^\infty \int_0^\infty f(s;B)\, e^{-t\xi^2 L}\, \sin(\xi\, s)\, \sin(\xi\, x)\, d\xi\, ds\,.
\end{aligned} \tag{43}
$$

Using the well-known trigonometric formula

$$
\sin a\, \sin b = \frac{1}{2}\left\{\cos(a-b) - \cos(a+b)\right\}\,,
$$

with $a = \xi\, s$, $b = \xi\, x$, we can rewrite (43) in the form

$$
u(x,t) = \frac{1}{\pi}\int_0^\infty \int_0^\infty f(s;B)\, e^{-t\xi^2 L}\, \left\{\cos(\xi(x-s)) - \cos(\xi(x+s))\right\}\, d\xi\, ds\,. \tag{44}
$$

From (15) one gets

$$\int_0^\infty e^{-t\xi^2 L} \cos(\xi(x-s))\, d\xi = \frac{1}{2}\sqrt{\frac{\pi}{tL}}\, e^{-(x-s)^2/4tL}, \quad x>0,\ t>0, \tag{45}$$

$$\int_0^\infty e^{-t\xi^2 L} \cos(\xi(x+s))\, d\xi = \frac{1}{2}\sqrt{\frac{\pi}{tL}}\, e^{-(x+s)^2/4tL}, \quad x>0,\ t>0. \tag{46}$$

Using Fubini theorem in abstract normed spaces [24, p. 175] and expressions (44)–(46), it follows that

$$\begin{aligned} u(x,t) &= \frac{1}{\pi}\int_0^\infty f(s;B)\left\{\int_0^\infty e^{-t\xi^2 L}\{\cos(\xi(x-s)) - \cos(\xi(x+s))\}\, d\xi\right\} ds \\ &= \frac{1}{2}\frac{1}{\sqrt{\pi t L}}\int_0^\infty f(s;B)\left(e^{-(x-s)^2/4tL} - e^{-(x+s)^2/4tL}\right) ds, \quad x>0,\ t>0. \end{aligned} \tag{47}$$

Summarizing, the following result has been established

Theorem 2. *Let us consider the random heat problem given by* (36)–(38) *where L is a positive 4-random variable satisfying* (10) *and* (41)*, and let* $f(x;B)$ *be mean fourth locally and mean fourth absolutely integrable stochastic process depending on random variable B. Let us assume that L and B are independent random variables. Then, the solution 2-stochastic process* $u(x,t)$ *of problem* (36)–(38) *is given by* (47).

Now, let us consider the problem treated in [16]:

$$v_t(x,t) = L\, v_{xx}(x,t), \quad x>0, \quad t>0, \tag{48}$$

$$v(0,t) = A, \qquad t>0, \tag{49}$$

$$v(x,0) = 0, \qquad x>0, \tag{50}$$

where A is a positive 4-r.v. independent of r.v. L which is assumed to satisfy properties of Theorem 2. By [16], a solution 2-s.p. of problem (48)–(50) is given by

$$v(x,t) = A\left(1 - \frac{1}{\sqrt{\pi t L}}\int_0^x e^{-r^2/4tL}\, dr\right).$$

Note that if $u(x,t)$ is a solution 2-s.p. of problem (36)–(38) and $v(x,t)$ is a solution 2-s.p. of problem (48)–(50), then by linearity,

$$w(x,t) = u(x,t) + v(x,t),$$

is a solution 2-s.p. of problem (2)–(4). Thus the following result is proved.

Corollary 1. Let A be a positive 4-random variable, and let L and $f(x;B)$ be a random variable and a stochastic process, respectively, both satisfying the conditions of Theorem 2. Suppose that A, B and L are mutually independent random variables. Then, a solution 2-stochastic process of problem (2)–(4) is given by

$$w(x,t) = A + \frac{1}{\sqrt{\pi t L}}\left\{\frac{1}{2}\int_0^\infty f(s;B)\left(e^{-(x-s)^2/4tL} - e^{-(x+s)^2/4tL}\right) ds - A\int_0^x e^{-r^2/4tL}\, dr\right\}, \quad x>0,\ t>0. \tag{51}$$

Using the independence of r.v.'s A, B and L, one computes the expectation and the variance functions of the solution 2-s.p. $w(x,t)$, given by (51), as closed expressions:

$$\mathrm{E}[w(x,t)] = \mathrm{E}[A]\left\{1 - \frac{1}{\sqrt{\pi t}}\int_0^x \mathrm{E}\left[\frac{1}{\sqrt{L}}\, e^{-r^2/4tL}\right] dr\right\} + \frac{1}{2\sqrt{\pi t}}\int_0^\infty \mathrm{E}[f(s;B)]\,\mathrm{E}\left[\frac{1}{\sqrt{L}}\left(e^{-(x-s)^2/4tL} - e^{-(x+s)^2/4tL}\right)\right] ds, \tag{52}$$

$$\mathrm{Var}\,[w(x,t)] = \mathrm{E}[(w(x,t))^2] - (\mathrm{E}[w(x,t)])^2\,, \tag{53}$$

where

$$\begin{aligned}
\mathrm{E}\left[(w(x,t))^2\right] &= \frac{1}{4\pi t}\int_0^\infty\int_0^\infty \mathrm{E}\,[f(s_1;B)f(s_2;B)]\,\mathrm{E}\left[\frac{1}{L}\left(e^{-(x-s_1)^2/4tL} - e^{-(x+s_1)^2/4tL}\right)\left(e^{-(x-s_2)^2/4tL} - e^{(x+s_2)^2/4tL}\right)\right] ds_1\, ds_2 \\
&+ \mathrm{E}\,[A]\left\{\frac{1}{\sqrt{\pi t}}\int_0^\infty \mathrm{E}\,[f(s;B)]\,\mathrm{E}\left[\frac{1}{\sqrt{L}}\left(e^{-(x-s)^2/4tL} - e^{-(x+s)^2/4tL}\right)\right] ds\right. \\
&- \left.\frac{1}{\pi t}\int_0^x\int_0^\infty \mathrm{E}\,[f(s;B)]\,\mathrm{E}\left[\frac{1}{L}\left(e^{-((x-s)^2+r^2)/4tL} - e^{-((x+s)^2+r^2)/4tL}\right)\right] ds\, dr\right\} \\
&+ \mathrm{E}\left[A^2\right]\left\{1 + \frac{1}{\pi t}\int_0^x\int_0^x \mathrm{E}\left[\frac{1}{L}e^{-(r_1^2+r_2^2)/4tL}\right] dr_1\, dr_2 - \frac{2}{\sqrt{\pi t}}\int_0^x \mathrm{E}\left[\frac{1}{\sqrt{L}}\,e^{-r^2/4tL}\right] dr\right\}.
\end{aligned} \tag{54}$$

Example 3. *Consider the problem* (2)–(4)*:*

$$\begin{aligned}
w_t(x,t) &= L\, w_{xx}(x,t)\,, \quad x>0, \quad t>0\,,\\
w(0,t) &= A\,, \qquad\qquad\quad t>0,\\
w(x,0) &= 100\, e^{-Bx}\,, \qquad x>0,
\end{aligned}$$

where the diffusion coefficient L has a beta distribution of parameters $\alpha = 2$ and $\beta = 1$, i.e., $L \sim Be\,(2,1)$; the temperature at the left-end $x = 0$ is described by the exponential r.v. $A \sim Exp\,(1)$, which is a positive 4-r.v.; and the initial temperature is modelled by the s.p. $f(x;B) = 100e^{-Bx}$ being B a uniform r.v., $B \sim Un\,(0.5,1)$. We assume that all r.v.'s, A, B and L are mutually independent.

Note that L is a positive 4-r.v. that satisfies (10)*, because it is a bounded r.v. and it also satisfies condition* (41) *since the moment generating function of r.v. $-L$:*

$$\Phi_{-L}(t) = \mathrm{E}[e^{-tL}] = \frac{2\,e^{-t}(-1+e^t-t)}{t^2} \;\xrightarrow{t\to 0}\; 1,$$

is locally bounded about $t = 0$.

For each $x \in]x_0-\delta, x_0+\delta[$, x_0 and $\delta > 0$ such as $x_0 - \delta > 0$, taking $K = x_0 + \delta > 0$ and using condition (10) *because B is a bounded r.v. (see Remark 1 in [16]), one gets*

$$100\sum_{n\geq 0}\frac{1}{n!}\,\|(-Bx)^n\|_4 \leq 100\sum_{n\geq 0}\frac{K^n}{n!}\,\|B^n\|_4 \leq 100\sqrt[4]{M}\sum_{n\geq 0}\frac{(KH)^n}{n!} = 100\sqrt[4]{M}e^{KH}\,. \tag{55}$$

Then, on account of the reasoning showed in the first part of Example 2 the m.f. continuity of 4-s.p. $f(x;B)$ is guaranteed and, as a consequence, $f(x;B)$ is m.f. locally integrable. Now, we need to show that $f(x;B)$ is m.f. absolutely integrable. For that, we also apply condition (10)

$$(\|f(x;B)\|_4)^4 = 100\,\mathrm{E}\left[e^{-4xB}\right] = 100\,\mathrm{E}\left[\sum_{n\geq 0}\frac{(-4xB)^n}{n!}\right] = 100\sum_{n\geq 0}\frac{\mathrm{E}\,[B^n]\,(-4x)^n}{n!} \leq 100\,M\sum_{n\geq 0}\frac{(-4xH)^n}{n!} = 100Me^{-4xH}. \tag{56}$$

Following the same reasoning showed in (55)*, it is easy to prove the m.f. convergence, and hence the m.s. convergence, of the above random infinite series. Then, by property* (1) *one justifies the commutation between the expectation operator and the random infinite sum in* (56)*. Therefore*

$$\int_0^\infty \|f(x;B)\|_4\; dx \leq \sqrt[4]{100\,M}\int_0^\infty e^{-xH}\, dx < +\infty. \tag{57}$$

Hence, the hypotheses of Corollary 1 are satisfied and expression given by (51) *is a solution 2-s.p.* $w(x,t)$ *of problem* (2)–(4)*. In Figures 1 and 2, we have plotted the values of the expectation and the standard deviation of temperature* $w(x,t)$ *on the spatial-time domain* $(x,t) \in]0,15] \times [0,20]$*, respectively. These plots have been performed taking into account expression* (52) *for the expectation, and expressions* (53)–(54) *for the standard deviation. Since these expressions involve improper integrals which, in general, cannot be computed exactly, the truncation of the intervals of integration has been required to keep feasible the computational burden. Notice that the m.s. convergence of the solution s.p.* (51) *together with properties given by* (1) *guarantee that these approximations will converge to the corresponding exact values of its expectation and its variance. For the sake of clarity, both plots have been made in two and three dimensions (2D and 3D). From these representations, we observe that the average temperature of the bar tends to stabilize at the value* $\mathrm{E}[A] = 1$ *as time goes on (see expression* (52)*) and, as a consequence, its variation, measured through standard deviation, decreases as time increases.*

4. Random heat problem with second kind boundary condition

Let us consider the auxiliary problem

$$
\begin{aligned}
u_t(x,t) &= L\, u_{xx}(x,t)\,, \quad x>0, \quad t>0\,, && (58)\\
u_x(0,t) &= 0\,, \qquad\qquad t>0, && (59)\\
u(x,0) &= f(x;A)\,, \qquad x>0, && (60)
\end{aligned}
$$

and note that if $u(x,t)$ is a solution 2-s.p. of (58)–(60), and $v(x,t)$ is a solution 2-s.p. of the problem

$$
\begin{aligned}
v_t(x,t) &= L\, v_{xx}(x,t)\,, \quad x>0, \quad t>0\,, && (61)\\
v_x(0,t) &= g(t;B)\,, \qquad t>0, && (62)\\
v(x,0) &= 0\,, \qquad\qquad x>0, && (63)
\end{aligned}
$$

then, by linearity

$$
w(x,t) = u(x,t) + v(x,t)\,, \tag{64}
$$

is a solution 2-s.p. of problem (5)–(7). As problem (61)–(63) was solved in [16], we focus our attention on problem (58)–(60). Assume L is a positive 4-r.v. with some additional properties to be specified later and $f(x;A)$ is a 2-s.p. depending on a single r.v. A. Let us assume that problem (58)–(60) admits a solution 2-s.p. $u(x,t)$ m.s. locally and m.s. absolutely integrable, and let us denote

$$
\mathfrak{F}_{\mathfrak{c}}\,[u(\cdot,t)]\,(\xi) = \mathcal{U}(t)(\xi)\,, \qquad \xi > 0\,, \tag{65}
$$

what means that $u(x,t)$ is regarded as a s.p. of the active variable x, for fixed $t>0$. By applying the random Fourier cosine transform to the right-hand side of equation (58) and using Theorem 1-(iv) together with condition (59) and (65), it follows that

$$
\mathfrak{F}_{\mathfrak{c}}\,[u_{xx}(\cdot,t)]\,(\xi) = -u_x(0,t) - \xi^2 \mathfrak{F}_{\mathfrak{c}}\,[u(\cdot,t)]\,(\xi) = -\xi^2\, \mathcal{U}(t)(\xi)\,. \tag{66}
$$

Applying the random Fourier cosine transform to the left-hand side of (58) and by Lemma 2 of [16], it follows that

$$
\mathfrak{F}_{\mathfrak{c}}\,[u_t(\cdot,t)]\,(\xi) = \frac{d}{dt}\left(\mathcal{U}(t)(\xi)\right)\,. \tag{67}
$$

Also from (60) one gets

$$
\mathfrak{F}_{\mathfrak{c}}\,[u(\cdot,0)]\,(\xi) = \mathcal{U}(0)(\xi) = \mathfrak{F}_{\mathfrak{c}}\,[f(\cdot\,;A)]\,(\xi) = F_{\mathfrak{c}}(\xi;A)\,. \tag{68}
$$

By linearity and (66)–(68) one gets the transformed random ordinary differential problem

$$
\left.
\begin{aligned}
\frac{d}{dt}\left(\mathcal{U}(t)(\xi)\right) &= -L\xi^2\,\mathcal{U}(t)(\xi)\,, \quad t>0,\\
\mathcal{U}(0)(\xi) &= F_{\mathfrak{c}}(\xi;A)
\end{aligned}
\right\}. \tag{69}
$$

Assuming that r.v. L satisfies (41) and $F_{\mathfrak{c}}(\xi; A)$ is a 4-s.p., by Theorem 8 of [19] the solution 2-s.p. of problem (69) is given by

$$\mathcal{U}(t)(\xi) = F_{\mathfrak{c}}(\xi; A)\, e^{-t\xi^2 L} . \tag{70}$$

Note that by Example 1, for each fixed $t > 0$,

$$e^{-t\xi^2 L} = \mathfrak{F}_{\mathfrak{c}}\left[q(x; L)\right](\xi) ; \qquad q(x; L) = \frac{1}{\sqrt{\pi t L}}\, e^{-x^2/4tL} . \tag{71}$$

Now in order to use the convolution property for the random Fourier cosine transform (see (19)–(27)) applied to the 4-s.p.'s $f(x; A)$ and $q(x; L)$, it is sufficient to assume that

$$f(x; A) \text{ is a m.f. continuous s.p.} \quad \text{and} \quad \int_0^\infty \left(\|f(x; A)\|_4\right)^2\, dx < +\infty . \tag{72}$$

Note that from Example 2 the 4-s.p. $q(x; L)$, defined by (71), also verifies conditions given by (72). Taking into account the previous exposition, from expressions (26)–(27) and (70)–(71) it follows that a solution 2-s.p of problem (58)–(60) is given by

$$\begin{aligned}
u(x,t) &= \mathfrak{F}_{\mathfrak{c}}{}^{-1}\left[\mathcal{U}(t)(\xi)\right](x) = \mathfrak{F}_{\mathfrak{c}}{}^{-1}\left[F_{\mathfrak{c}}(\xi; A)\, e^{-t\xi^2 L}\right](x) = \mathfrak{F}_{\mathfrak{c}}{}^{-1}\left[\mathfrak{F}_{\mathfrak{c}}\left[f(x; A)\right](\xi)\, \mathfrak{F}_{\mathfrak{c}}\left[q(x; L)\right](\xi)\right](x) \\
&= \mathfrak{F}_{\mathfrak{c}}{}^{-1}\left[\sqrt{\frac{\pi}{2}}\mathfrak{F}_{\mathfrak{c}}\left[(f * q)(x; A, L)\right](\xi)\right](x) = \sqrt{\frac{\pi}{2}}\,(f * q)(x; A, L) \\
&= \frac{1}{2}\int_0^\infty f(\kappa; A)\, \left\{q(x+\kappa; L) + q(|x-\kappa|; L)\right\}\, d\kappa \\
&= \frac{1}{2\sqrt{\pi t L}}\int_0^\infty f(\kappa; A)\left(e^{-(x+\kappa)^2/4tL} + e^{-(x-\kappa)^2/4tL}\right) d\kappa , \qquad x > 0,\ t > 0 .
\end{aligned} \tag{73}$$

Summarizing, the following result has been established

Theorem 3. *Let us consider the random heat problem given by* (58)–(60) *where L is a positive 4-random variable satisfying* (10)–(11) *and* (41). *Let $f(x; A)$ be mean fourth absolutely integrable stochastic process which depends on one single random variable A and verifies conditions given by* (72). *Suppose that A and L are independent random variables. Then, the solution 2-stochastic process $u(x, t)$ of problem* (58)–(60) *is given by* (73).

Finally, taking into account the solution 2-s.p. found in [16] for the subproblem (61)–(63) and by (64), one gets a solution 2-s.p. for the problem (5)–(7). Thus the following result is proved.

Corollary 2. Let $g(t; B)$ be a mean fourth continuous stochastic process depending on random variable B, and let L be a random variable and $f(x; A)$ be a stochastic process both satisfying the conditions of Theorem 3. Suppose that random variables A, B and L are mutually independent. Then, a solution 2-stochastic process of problem (5)–(7) is given by

$$w(x,t) = \frac{1}{2\sqrt{\pi t L}}\int_0^\infty f(\kappa; A)\left(e^{-(x+\kappa)^2/4tL} + e^{-(x-\kappa)^2/4tL}\right) d\kappa + 2\sqrt{\frac{L}{\pi}}\int_0^{\sqrt{t}} g(t-\nu^2; B)\, e^{-(x/2\nu\sqrt{L})^2}\, d\nu , \quad x > 0,\ t > 0 . \tag{74}$$

Using the independence of r.v.'s A, B and L, one computes the expectation and the variance functions of the solution 2-s.p. $w(x, t)$, given by (74), as closed expressions by

$$\begin{aligned}
\mathrm{E}\left[w(x,t)\right] &= \frac{1}{2\sqrt{\pi\, t}}\int_0^\infty \mathrm{E}\left[f(\kappa; A)\right]\mathrm{E}\left[\frac{1}{\sqrt{L}}\left(e^{-(x+\kappa)^2/4tL} + e^{-(x-\kappa)^2/4tL}\right)\right] d\kappa \\
&+ \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} \mathrm{E}\left[g(t-\nu^2; B)\right]\mathrm{E}\left[\sqrt{L}\, e^{-(x/2\nu\sqrt{L})^2}\right] d\nu , \qquad x > 0,\ t > 0 ,
\end{aligned} \tag{75}$$

and (53) being

$$
\begin{aligned}
\mathrm{E}\left[(w(x,t))^2\right] &= \frac{1}{4\pi t}\int_0^{\infty}\int_0^{\infty}\mathrm{E}\left[f(\kappa_1;A)\,f(\kappa_2;A)\right]\mathrm{E}\left[\frac{1}{L}\left(e^{-(x+\kappa_1)^2/4tL}+e^{-(x-\kappa_1)^2/4tL}\right)\left(e^{-(x+\kappa_2)^2/4tL}+e^{(x-\kappa_2)^2/4tL}\right)\right]d\kappa_1\,d\kappa_2 \\
&+ \frac{4}{\pi}\int_0^{\sqrt{t}}\int_0^{\sqrt{t}}\mathrm{E}\left[g(t-\nu_1^2;B)\,g(t-\nu_2^2;B)\right]\,\mathrm{E}\left[L\,e^{-(x/2\nu_1\nu_2\sqrt{L})^2(\nu_1^2+\nu_2^2)}\right]d\nu_1\,d\nu_2 \\
&+ \frac{2}{\pi\sqrt{t}}\int_0^{\infty}\int_0^{\sqrt{t}}\mathrm{E}\left[f(\kappa;A)\right]\mathrm{E}\left[g(t-\nu^2;B)\right]\,\mathrm{E}\left[e^{-((x+\kappa)^2\nu^2+x^2t)/4t\nu^2L}+e^{-((x-\kappa)^2\nu^2+x^2t)/4t\nu^2L}\right]d\nu\,d\kappa\,.
\end{aligned}
\tag{76}
$$

Example 4. *Consider the problem* (5)–(7)*:*

$$
\begin{aligned}
w_t(x,t) &= L\,w_{xx}(x,t)\,, \quad x>0, \quad t>0\,,\\
w_x(0,t) &= tB\,, \qquad\qquad t>0,\\
w(x,0) &= 50\,e^{-xA}\,, \qquad x>0,
\end{aligned}
$$

where the diffusion coefficient L follows a gamma distribution of parameters $\alpha = 2$ and $\beta = 1$ truncated on the interval $[0.5, 1]$*, i.e., $L \sim Trunc[Ga(2,1)]$; the spatial variation of the temperature at the left-end $x = 0$ is described by the s.p. $g(t;B) = t\,B$ where B is a gaussian r.v. of mean $\mu = 4$ and standard deviation $\sigma = 0.5$, i.e. $B \sim N(4; 0.5)$; the initial temperature is modelled by the s.p. $f(x;A) = 50\,e^{-xA}$ being A a beta r.v. of parameters $\alpha = 3$ and $\beta = 2$, that is, $A \sim Be\,(3,2)$. We assume that all r.v.'s, A, B and L are mutually independent.*

Note that L is a positive 4-r.v. verifying conditions (10)–(11) *and condition* (41) *since the moment generating function of r.v. $-L$:*

$$
\Phi_{-L}(t) = \mathrm{E}[e^{-tL}] = \frac{2.87295\,e^{-(1+t)}(-4+3\,e^{0.5(1+t)}-2\,t+t\,e^{0.5(1+t)})}{(1+t)^2} \quad \stackrel{t\to 0}{\longrightarrow} 1,
$$

is locally bounded about $t = 0$.

Since $\mathrm{E}\left[B^4\right] = 3(0.5)^4 < \infty$ (see [5, p.26]), $\|g(t;B) - g(s;B)\|_4 = \|B\|_4\,|t-s| \stackrel{t\to s}{\longrightarrow} 0$, then the 4-s.p. $g(t;B)$ is m.f. continuous.

Reasoning analogously as in Example 3 (see (56)–(57)*) taking the s.p. $f(x;A) = 50\,e^{-xA}$ and using condition* (10) *because A is a bounded r.v., it is proved that $f(x;A)$ is m.f. absolutely integrable. Furthermore, $f(x;A)$ verifies conditions given by* (72)*, that is, $f(x;A)$ is m.f. continuous (see the reasoning* (55)*) and*

$$
\int_0^{\infty}(\|f(x;A)\|_4)^2\,dx \le \sqrt{50\,\tilde{M}}\int_0^{\infty}e^{-2x\tilde{H}}\,dx < +\infty.
$$

Hence, the hypotheses of Corollary 2 are satisfied and expression given by (74) *is a solution 2-s.p. $w(x,t)$ of problem* (5)–(7)*. Figure 1 shows a two-dimensional plot of the expectation (plot (a)) and standard deviation (plot (b)) of $w(x,t)$ on the spatial-time domain $(x,t) \in]0,3] \times [0,30]$. As it was pointed out in the Example 3, again truncation of the intervals of integration for the computation of the expectation and variance has been required (see expressions* (75)–(76)*). For the sake of clarity, we also provide three-dimensional plots for these statistical moments in Figure 4.*

5. Conclusions

In this paper we have solved heat problems (2)–(4) and (5)–(7) which are formulated through random partial differential equations set in semi-infinite medium. To conduct the study we have extended the well-known deterministic sine and cosine Fourier integral transforms to the random scenario by taking advantage of mean square and mean fourth calculus. The provided examples illustrate the capability of the method to deal with other random partial equations formulated on unbounded domains to the positive spatial variable.

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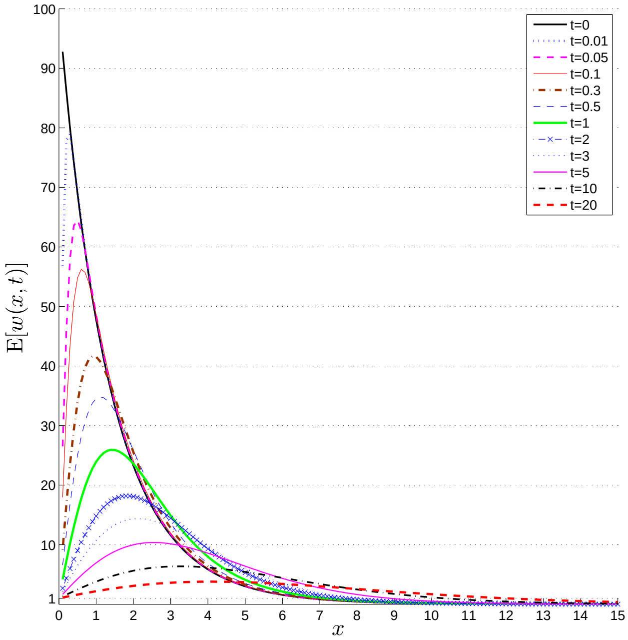
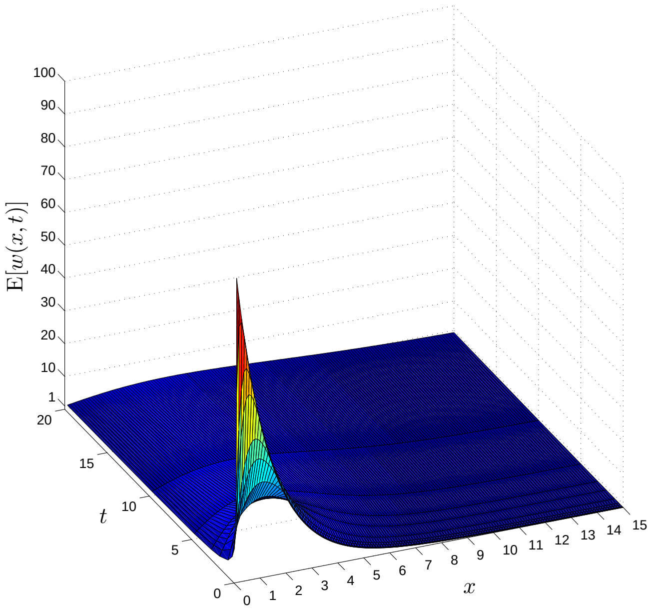
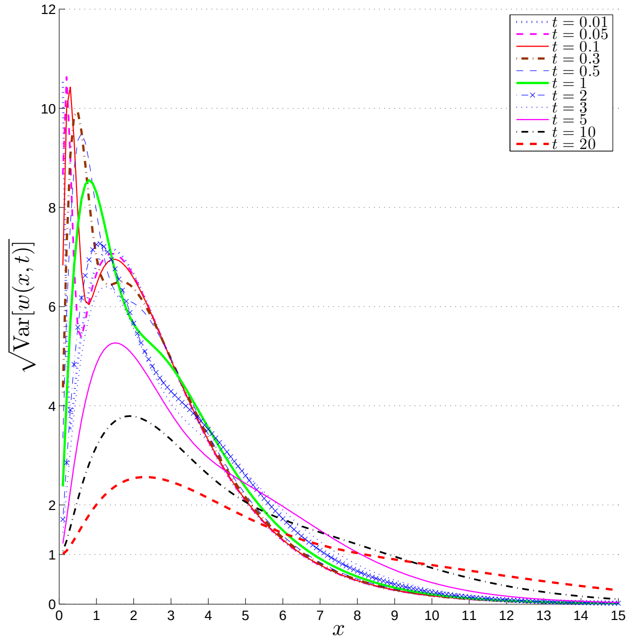


(a)

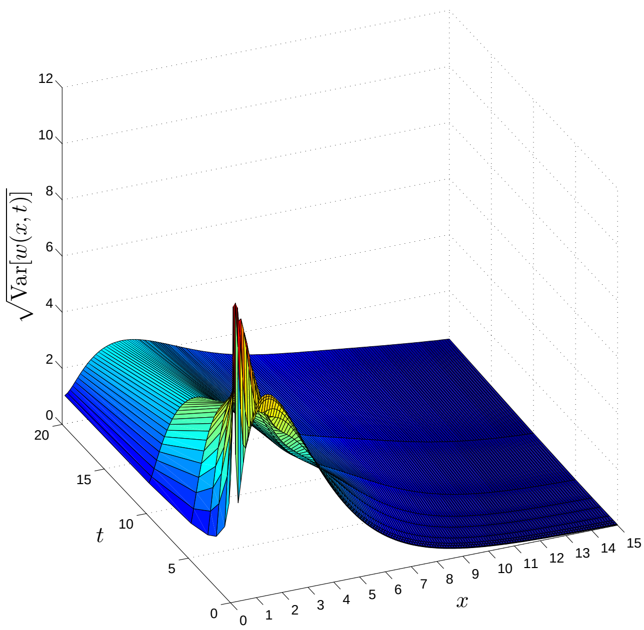


(b)

Figure 1: Approximations for the expectation $\mathrm{E}[w(x,t)]$ in 2D (plot (a)) and in 3D (plot (b)), on the spatial domain $x \in]0,15]$ for some selected values in the time interval $t \in [0,20]$ in the context of Example 3.

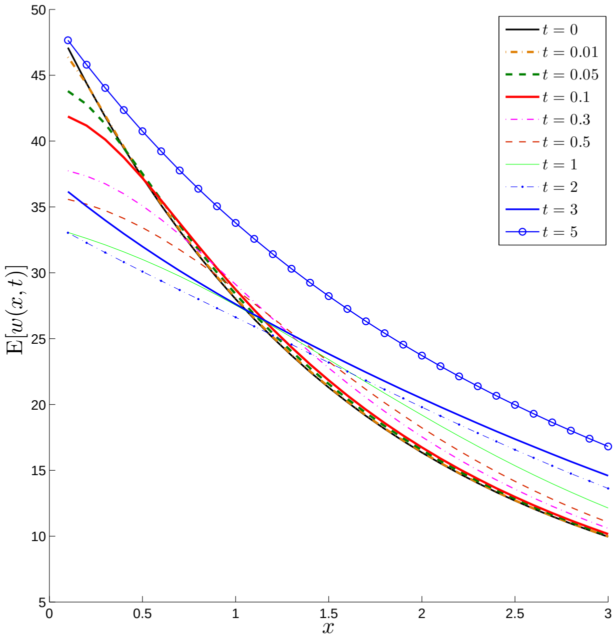


(a)

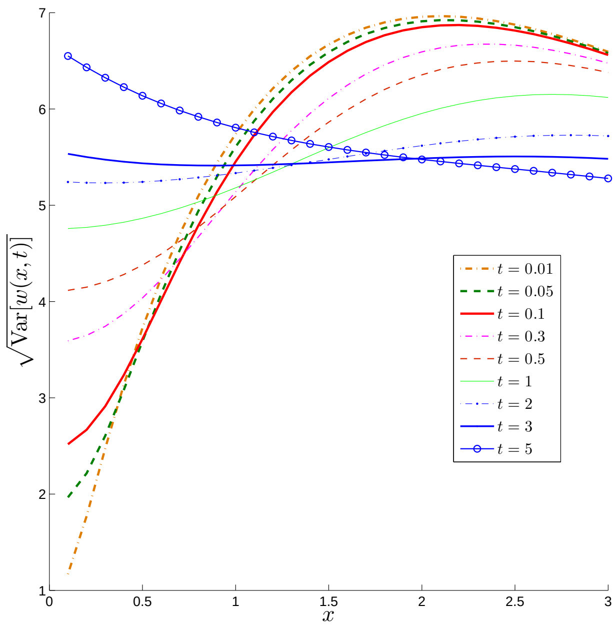


(b)

Figure 2: Approximations for the standard deviation $\sqrt{\mathrm{Var}\,[w(x,t)]}$ (plot (a)) in 2D and in 3D (plot (b)), on the spatial domain $x \in\,]0, 15]$ for some selected values in the time interval $t \in [0, 20]$ in the context of Example 3.

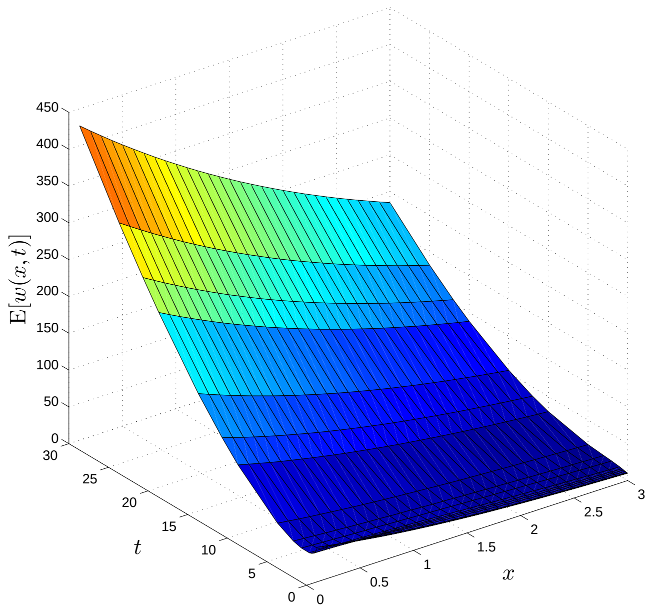


(a)

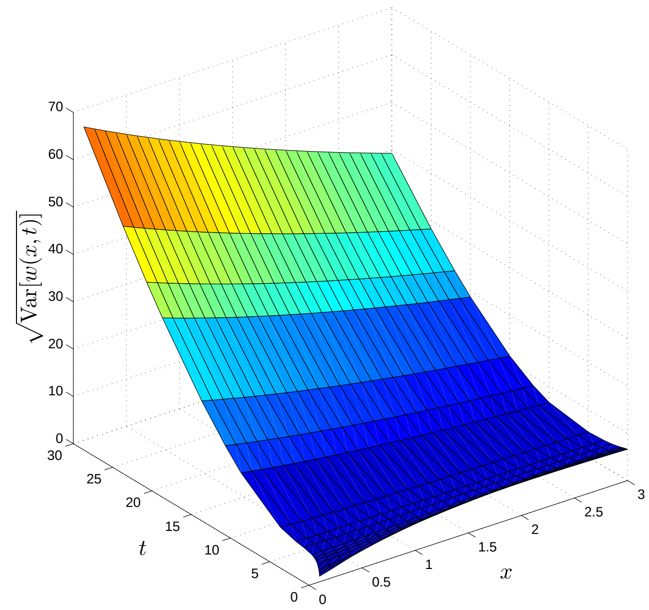


(b)

Figure 3: Two-dimensional approximations for the expectation $\mathrm{E}[w(x,t)]$ (plot (a)), and, the standard deviation $\sqrt{\mathrm{Var}[w(x,t)]}$ (plot (b)), on the spatial domain $x \in]0,3]$ for some selected values in the time interval $t \in [0,5]$ in the context of Example 4.



(a)



(b)

Figure 4: Three-dimensional approximations for the expectation $\mathrm{E}[w(x,t)]$ (plot (a)), and, the standard deviation $\sqrt{\mathrm{Var}[w(x,t)]}$ (plot (b)) on the spatial domain $x \in\,]0,3]$ in the time interval $t \in [0,30]$ in the context of Example 4.