THE CESÀRO OPERATOR IN GROWTH BANACH SPACES OF ANALYTIC FUNCTIONS

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ABSTRACT. The Cesàro operator C, when acting in the classical growth Banach spaces $A^{-\gamma}$ and $A_0^{-\gamma}$, for $\gamma > 0$, of analytic functions on $\mathbb{D}$, is investigated. Based on a detailed knowledge of their spectra (due to A. Aleman and A.-M. Persson) we are able to determine the norms of these operators precisely. It is then possible to characterize the mean ergodic and related properties of C acting in these spaces. In addition, we determine the largest Banach space of analytic functions on $\mathbb{D}$ which C maps into $A^{-\gamma}$ (resp. into $A_0^{-\gamma}$); this *optimal domain* space always contains $A^{-\gamma}$ (resp. $A_0^{-\gamma}$) as a *proper* subspace.

1. INTRODUCTION

Let $H(\mathbb{D})$ denote the Fréchet space of all holomorphic functions $f\colon \mathbb{D} \to \mathbb{C}$ equipped with the topology of uniform convergence on the compact subsets of the open unit disc $\mathbb{D}$. The classical Cesàro operator C is given by

$$f \mapsto \mathsf{C}(f)\colon z \mapsto \frac{1}{z}\int_0^z \frac{f(\zeta)}{1-\zeta}d\zeta, \quad z \in \mathbb{D}\setminus\{0\}, \quad \text{and} \quad \mathsf{C}(f)(0) = f(0), \tag{1.1}$$

for $f \in H(\mathbb{D})$. It is a Fréchet space isomorphism of $H(\mathbb{D})$ onto itself. In terms of the Taylor coefficients $\widehat{f}(n) := \frac{f^{(n)}(0)}{n!}$, for $n \in \mathbb{N}_0$, of functions $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n \in H(\mathbb{D})$ one has the description

$$\mathsf{C}(f)(z) = \sum_{n=0}^\infty \Big(\frac{1}{n+1}\sum_{k=0}^n \widehat{f}(k)\Big) z^n, \quad z \in \mathbb{D}.$$

A vector space $X \subseteq H(\mathbb{D})$ is called a *Banach space of analytic functions on* $\mathbb{D}$ if it is a Banach space relative to a norm for which the natural inclusion of X into $H(\mathbb{D})$ is continuous. Since evaluation at points of $\mathbb{D}$ are continuous linear functionals on $H(\mathbb{D})$, this is equivalent to each evaluation functional $\delta_z : f \mapsto f(z)$ at a point $z \in \mathbb{D}$ being an element of the dual Banach space X^* of X. In this case, the Cesàro operator C is said to *act in* X if it maps X into itself, i.e. $\mathsf{C}(X) \subseteq X$; by the closed graph theorem C is necessarily continuous. It is known that C acts in many classical Banach spaces of

Key words and phrases. Cesàro operator, growth Banach space of analytic functions, mean ergodic operator, optimal domain.

Mathematics Subject Classification 2010: Primary 47B38; Secondary 46E15, 47A10, 47A16, 47A35.

analytic functions on $\mathbb{D}$; for instance, the Hardy spaces $H^p(\mathbb{D}), 1 \leq p < \infty$, the Bergman and the Dirichlet spaces, etc. (see [5, 6, 8, 9, 31] and the references therein). It also acts in certain weighted Banach spaces of analytic functions on $\mathbb{D}$, such as weighted Hardy spaces $H^p(w), 1 < p < \infty$, and others, [4, 7, 15, 28]. On the other hand, C *fails* to act in $H^\infty(\mathbb{D})$ since $\mathsf{C}(\mathbf{1})(z) = (1/z)\log(1/(1-z))$, for $z \in \mathbb{D}$; see [16] for an investigation of C on $H^\infty(\mathbb{D})$. Once C is known to act in X, it is desirable to identify its spectrum $\sigma(\mathsf{C})$ and its point spectrum $\sigma_{pt}(\mathsf{C})$ as these often influence various operator theoretic properties of C (e.g. power boundedness, mean ergodicity, linear dynamics, decomposability, etc).

The aim of this note is to investigate various properties of the Cesàro operator C related to its action in the classical growth Banach space $A^{-\gamma}$, for each $\gamma > 0$, consisting of those analytic functions on $\mathbb{D}$ specified by

$$A^{-\gamma} := \{f \in H(\mathbb{D})\colon \|f\|_{-\gamma} := \sup_{z\in\mathbb{D}}(1-|z|)^\gamma |f(z)| < \infty\}$$

and its (proper) closed subspace

$$A_0^{-\gamma} := \{f \in H(\mathbb{D})\colon \lim_{|z|\to 1^-}(1-|z|)^\gamma |f(z)| = 0\ \},$$

equipped with the norm

$$\|f\|_{-\gamma} := \sup_{z\in\mathbb{D}}(1-|z|)^\gamma |f(z)|, \qquad f \in A^{-\gamma}.$$

Since $(1-|z|)^\gamma \leq (1-|z|^2)^\gamma \leq 2^\gamma(1-|z|)^\gamma$, for $z \in \mathbb{D}$, these are the same spaces (with an equivalent norm) as those treated in [21, Ch. 4].

The space $A_0^{-\gamma}$ coincides with the closure of the polynomials in $A^{-\gamma}$, [30, Lemma 3], and point evaluations on $\mathbb{D}$ belong to both $(A_0^{-\gamma})^*$ and $(A^{-\gamma})^*$, [30, Lemma 1]. Moreover, the bidual $(A_0^{-\gamma})^{**} = A^{-\gamma}$ for all $\gamma > 0$, [29], [30, Theorem 2]. Shields and Williams, [30], proved that $A^{-\gamma}$ is isomorphic to ℓ^∞ and that $A_0^{-\gamma}$ is isomorphic to c_0, for each $\gamma > 0$; see also Theorem 1.1 in the paper by Lusky, [25]. The growth Banach spaces $A^{-\gamma}$ and $A_0^{-\gamma}$ play an important role in connection with the interpolation and sampling of holomorphic functions, [21, Ch. 4 & 5]. They are particular examples of weighted Banach spaces H_v^∞ and H_v^0 of holomorphic functions on $\mathbb{D}$, which have been investigated by several authors since the work of Shields and Williams, [30]; see, for example, [11, 12, 13, 25] and the references therein. Observe, for each pair $0 < \mu_1 < \mu_2$, that $A^{-\mu_1} \subsetneq A_0^{-\mu_2}$.

Recently, Aleman and Persson have made an extensive investigation of various properties (including the spectrum) of generalized Cesàro operators acting in a large class of Banach spaces of analytic functions on $\mathbb{D}$, [7], [28]. In particular, their results apply to the classical Cesàro operator C given by (1.1) when acting in the growth spaces $A^{-\gamma}, A_0^{-\gamma}$ for $\gamma > 0$. For this setting our additional results will complement and extend their work. For instance, they show that C acts in both $A^{-\gamma}$ (where we denote it by C_γ) and in $A_0^{-\gamma}$ (where we denote it by $\mathsf{C}_{\gamma,0}$). This follows from Theorem 4.1 of

[7] with $g(z) = -\log(1-z)$, for $z \in \mathbb{D}$, but no quantitative estimates of the operator norms $\|\mathsf{C}_\gamma\|$ and $\|\mathsf{C}_{\gamma,0}\|$ are given. We show in Theorem 2.3, that

$$\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1, \text{ for } \gamma \geq 1, \text{ and } \|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1/\gamma^n, \text{ for } 0 < \gamma < 1, \tag{1.2}$$

for each $n \in \mathbb{N}$. Crucial for the proof of Theorem 2.3 is knowing the precise nature of the spectra $\sigma(\mathsf{C}_\gamma)$ and $\sigma(\mathsf{C}_{\gamma,0})$, which have been completely determined by Aleman and Persson, [7, 28]. This knowledge of the spectra, combined with (1.2), makes it possible to determine the mean ergodic properties of C_γ and $\mathsf{C}_{\gamma,0}$. Namely, it turns out that both C_γ and $\mathsf{C}_{\gamma,0}$ fail to be mean ergodic when $0 < \gamma \leq 1$ but, both operators are (even uniformly) mean ergodic for all $\gamma > 1$; see Theorem 2.5 below. Somewhat surprisingly, the ranges $\mathrm{Im}(\mathrm{I} - \mathsf{C}_\gamma) := \{(\mathrm{I} - \mathsf{C}_\gamma)(\mathrm{f}) : \mathrm{f} \in \mathrm{A}^{-\gamma}\}$, resp. $\mathrm{Im}(\mathrm{I} - \mathsf{C}_{\gamma,0})$, are *closed* in $A^{-\gamma}$, resp. $A_0^{-\gamma}$, for all positive $\gamma \neq 1$ but, not for $\gamma = 1$; see Theorem 2.5. None of the Cesàro operators C_γ, $C_{\gamma,0}$, for $\gamma > 0$, are supercyclic and hence, they also fail to be hypercyclic (cf. Proposition 2.7).

The final section is devoted to determining the *optimal domain* (a concept introduced in [15]) for both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ for all $\gamma > 0$. That is, we are able to identify the *largest* Banach space of analytic functions on $\mathbb{D}$, denoted by $[\mathsf{C}, A^{-\gamma}]$, (resp. by $[\mathsf{C}, A_0^{-\gamma}]$), which C maps continuously into $A^{-\gamma}$ (resp. into $A_0^{-\gamma}$). More specifically, it is shown in Theorem 3.1 that

$$[\mathsf{C}, A^{-\gamma}] = \{f \in H(\mathbb{D}) \ : \ f(z)/(1-z) \in A^{-(\gamma+1)}\}, \qquad \gamma > 0,$$

equipped with the norm $\|f\| = \|f(z)/(1-z)\|_{-(\gamma+1)}$. A similar description is available for $[\mathsf{C}, A_0^{-\gamma}]$; see Theorem 3.2. An important feature is that the containment $A^{-\gamma} \subseteq [\mathsf{C}, A^{-\gamma}]$ is *proper*, that is, $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ has a *genuine* extension $\mathsf{C} : [\mathsf{C}, A^{-\gamma}] \to A^{-\gamma}$ to its optimal domain space, as is the case for $\mathsf{C}_{\gamma,0}$ acting in $A_0^{-\gamma}$. As a consequence, it is shown that the nested family of spaces $A^{-\gamma} \subsetneqq A^{-\beta}$ (resp. $A_0^{-\gamma} \subsetneqq A_0^{-\beta}$), for $0 < \gamma < \beta$, has the property that C fails to map *every* larger space $A^{-\beta}$ (resp. $A_0^{-\beta}$) into $A^{-\gamma}$ (resp. into $A_0^{-\gamma}$). In particular, $A^{-\beta} \not\subseteq [\mathsf{C}, A^{-\gamma}]$ and $A_0^{-\beta} \not\subseteq [\mathsf{C}, A_0^{-\gamma}]$ for every $\beta > \gamma$. On the other hand, the Banach space of analytic functions $[\mathsf{C}, A_0^{-\gamma}]$ is *non-comparable* with $A^{-\gamma}$ (i.e., $A^{-\gamma} \not\subseteq [\mathsf{C}, A_0^{-\gamma}]$ and $[\mathsf{C}, A_0^{-\gamma}] \not\subseteq A^{-\gamma}$) and, akin to $A^{-\gamma}$, is also a *proper* subspace of the optimal domain space $[\mathsf{C}, A^{-\gamma}]$; see Proposition 3.6.

Our notation for concepts from functional analysis and operator theory is standard and we refer to [18, 26, 27], for example. For Banach spaces of analytic functions see [17, 21], for mean ergodic operators [18, 23], and for linear dynamics [10, 19].

2. Mean ergodic Properties of C_γ and $\mathsf{C}_{\gamma,0}$

As noted above, $A^{-\gamma}$ is canonically isomorphic to the bidual $(A_0^{-\gamma})^{**}$ of the Banach space $A_0^{-\gamma}$. In terms of this biduality the operator $\mathsf{C}_\gamma \colon A^{-\gamma} \to A^{-\gamma}$ is the bidual operator $(\mathsf{C}_{\gamma,0})^{**}$ of $\mathsf{C}_{\gamma,0} \colon A_0^{-\gamma} \to A_0^{-\gamma}$.

We recall the following result of Aleman and Persson (see [28, Theorem 4.1] and [7, Theorems 4.1, 5.1 and Corollaries 2.1, 5.1], that will be used on several occasions.

Theorem 2.1. *Let* $\gamma > 0$. *The Cesàro operator* $\mathsf{C}_{\gamma,0}\colon A_0^{-\gamma} \to A_0^{-\gamma}$ *has the following properties.*

(i) $\sigma_{pt}(\mathsf{C}_{\gamma,0}) = \{\frac{1}{m}\colon m \in \mathbb{N},\ m < \gamma\}$.

(ii) $\sigma(\mathsf{C}_{\gamma,0}) = \sigma_{pt}(\mathsf{C}_{\gamma,0}) \cup \left\{\lambda \in \mathbb{C}\colon \left|\lambda - \frac{1}{2\gamma}\right| \leq \frac{1}{2\gamma}\right\}$.

(iii) *If* $\left|\lambda - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}$ *(equivalently* $\mathrm{Re}\left(\frac{1}{\lambda}\right) > \gamma$*), then* $\mathrm{Im}(\lambda I - \mathsf{C}_{\gamma,0})$ *is a closed subspace of* $A_0^{-\gamma}$ *and has codimension* 1.

Moreover, the Cesàro operator $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ *satisfies*

(iv) $\sigma_{pt}(\mathsf{C}_\gamma) = \{\frac{1}{m}\colon m \in \mathbb{N},\ m \leq \gamma\}$, *and*

(v) $\sigma(\mathsf{C}_\gamma) = \sigma(\mathsf{C}_{\gamma,0})$.

As an immediate consequence we have the following result.

Corollary 2.2. *For* $\gamma > 0$, *neither of the operators* $\mathsf{C}_\gamma, \mathsf{C}_{\gamma,0}$ *is weakly compact.*

Proof. Fix $\gamma > 0$. As noted in Section 1 there exists a Banach space isomorphism Φ from c_0 onto $A_0^{-\gamma}$. Suppose that $\mathsf{C}_{\gamma,0}$ is weakly compact. Then $\widetilde{\mathsf{C}} := \Phi^{-1}\mathsf{C}_{\gamma,0}\Phi$ is weakly compact from c_0 into c_0 and hence, $\widetilde{\mathsf{C}}$ is also compact, [26, Ex. 3.54(b), p.347]. Accordingly, $\mathsf{C}_{\gamma,0} = \Phi\,\widetilde{\mathsf{C}}\Phi^{-1}$ is compact. But, this is impossible as $\sigma(\mathsf{C}_{\gamma,0})$ is an uncountable set; see Theorem 2.1(ii). Hence, $\mathsf{C}_{\gamma,0}$ cannot be weakly compact.

By Gantmacher's Theorem, [18, Ch. VI, Sect. 4, Theorem 8], also $\mathsf{C}_\gamma = (\mathsf{C}_{\gamma,0})^{**}$ cannot be weakly compact. □

The continuity of C_γ and $\mathsf{C}_{\gamma,0}$ as established in [7, Theorem 4.1] gives no quantitative estimate for their operator norm. So, the following result is of some interest.

Theorem 2.3. (i) *Let* $\gamma \geq 1$. *Then* $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1$ *for all* $n \in \mathbb{N}$.

(ii) *Let* $0 < \gamma < 1$. *Then* $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| = 1/\gamma^n$ *for all* $n \in \mathbb{N}$.

Proof. Let $\gamma > 0$. Fix $f \in A^{-\gamma}$ with $||f||_\gamma \leq 1$. Then $|f(\xi)| \leq (1-|\xi|)^{-\gamma}$ for $\xi \in \mathbb{D}$. For $z \in \mathbb{D}\setminus\{0\}$, it follows from (1.1) and the previous inequality for $|f|$ that

$$\begin{aligned}
|\mathsf{C}_\gamma(f)(z)| &= \frac{1}{|z|}\left|\int_0^1 \frac{f(tz)}{1-tz} z\,dt\right| \leq \int_0^1 \frac{|f(tz)|}{|1-tz|}dt \leq \int_0^1 \frac{|f(tz)|}{1-t\,|z|}dt \\
&\leq \int_0^1 \frac{dt}{(1-t\,|z|)^{\gamma+1}} = \left[\frac{1}{\gamma\,|z|}(1-t\,|z|)^{-\gamma}\right]_{t=0}^{t=1} \\
&= \frac{1}{\gamma\,|z|}\left(\frac{1}{(1-|z|)^\gamma} - 1\right) = \frac{1}{(1-|z|)^\gamma}\frac{1-(1-|z|)^\gamma}{\gamma\,|z|}.
\end{aligned}$$

The non-negative function $\phi(s) := \frac{1-(1-s)^\gamma}{s}$ for $s \in (0,1]$ and $\phi(0) := \gamma$ is continuous as $\lim_{s\to 0^+} \frac{1-(1-s)^\gamma}{s} = \gamma$. Set $M_\gamma := \sup_{s\in[0,1]} \phi(s)$. Since $|\mathsf{C}_\gamma(f)(0)| = |f(0)| \le 1$, we have $1 \le \sup_{z\in\mathbb{D}} (1-|z|)^\gamma\, |\mathsf{C}_\gamma(f)(z)| \le \max\{1, \frac{M_\gamma}{\gamma}\}$. Hence,

$$1 \le \|\mathsf{C}_\gamma\| \le \max\Big\{1, \frac{M_\gamma}{\gamma}\Big\}, \qquad \gamma > 0. \tag{2.1}$$

(i) Let now $\gamma \ge 1$. The claim is that $\frac{M_\gamma}{\gamma} \le 1$, which then implies that $||\mathsf{C}_\gamma|| = 1$. To verify the claim it suffices to show that the function $\psi(s) := (1-s)^\gamma - 1 + \gamma s$, for $s \in [0,1]$, is non-negative. Observe that $\psi(0) = 0$ and $\psi(1) = (\gamma-1) \ge 0$. If $\gamma = 1$, then $\psi(s) = 0$ for all $s \in [0,1]$ and we are done. In case $\gamma > 1$ we have $\psi'(s) = -\gamma(1-s)^{\gamma-1} + \gamma = \gamma\,[1-(1-s)^{\gamma-1}] \ge 0$ and so ψ is increasing. In particular, $\psi(s) \ge \psi(0) = 0$ for $s \in [0,1]$, as required.

Clearly, $\|\mathsf{C}_\gamma\| \le 1$ implies that $\|\mathsf{C}_\gamma^n\| \le \|\mathsf{C}_\gamma\|^n \le 1$ for $n \in \mathbb{N}$. On the other hand, Theorem 2.1 shows that $1 \in \sigma(C_\gamma)$ and so, by the spectral mapping theorem, $1 \in \sigma(C_\gamma^n)$. In particular, the spectral radius $r(C_\gamma^n) \ge 1$, for $n \in \mathbb{N}$, and so $1 \le r(\mathsf{C}_\gamma^n) \le \|\mathsf{C}_\gamma^n\|$ for $n \in \mathbb{N}$. This completes the proof of (i) for C_γ. The statement for $\mathsf{C}_{\gamma,0}$ now follows from $\|\mathsf{C}_{\gamma,0}^n\| = \|(C_{\gamma,0}^n)^{**}\| = \|\mathsf{C}_\gamma^n\|$ for $n \in \mathbb{N}$.

(ii) Let $0 < \gamma < 1$. It follows from Theorem 2.1 (as $\sigma_{pt}(C_{\gamma,0}) = \emptyset$ in this case), that $\sigma(C_{\gamma,0}) = \Big\{\lambda \in \mathbb{C}\colon \Big|\lambda - \frac{1}{2\gamma}\Big| \le \frac{1}{2\gamma}\Big\}$. In particular, $[0, \frac{1}{\gamma}] \subseteq \sigma(C_{\gamma,0})$. By the spectral mapping theorem $[0, \frac{1}{\gamma^n}] \subseteq \sigma(C_{\gamma,0}^n)$ for each $n \in \mathbb{N}$, and so $1/\gamma^n \le r(C_{\gamma,0}^n) \le \|C_{\gamma,0}^n\| = \|C_\gamma^n\|$ for each $n \in \mathbb{N}$.

To establish the reverse inequality we show, because of $0 < \gamma < 1$, that $M_\gamma \le 1$. Indeed, for $0 < s < 1$ we have $(1-s) < (1-s)^\gamma$, i.e., $1-(1-s)^\gamma < s$, and hence, $0 < \phi(s) < 1$. Since $\phi(0) = \gamma < 1$ and $\phi(1) = 1$, it is clear that $M_\gamma \le 1$ and so $\frac{M_\gamma}{\gamma} \le \frac{1}{\gamma}$. It follows from (2.1) that $\|C_\gamma\| \le \frac{1}{\gamma}$. Hence, $\|\mathsf{C}_\gamma^n\| = \|\mathsf{C}_{\gamma,0}^n\| \le \frac{1}{\gamma^n}$ for all $n \in \mathbb{N}$, and the proof is complete. □

For a Banach space X, denote by $\mathcal{L}(X)$ the space of all bounded linear operators from X into itself. Recall that an operator $T \in \mathcal{L}(X)$ is *mean ergodic* if its sequence of Cesàro averages

$$T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m, \quad n \in \mathbb{N}, \tag{2.2}$$

converges to some operator $P \in \mathcal{L}(X)$ in the strong operator topology τ_s, i.e., $\lim_{n\to\infty} T_{[n]}x = Px$ for each $x \in X$, [18, Ch.VIII]. It follows from (2.2) that $\frac{T^n}{n} = T_{[n]} - \frac{n-1}{n} T_{[n-1]}$, for $n \ge 2$. Hence, τ_s-$\lim_{n\to\infty} \frac{T^n}{n} = 0$ whenever T is mean ergodic and, in particular, $\sup_n \frac{\|T^n\|}{n} < \infty$. According to [18, VIII Corollary 5.2, p.662], when T is mean ergodic one has the direct decomposition

$$X = \mathrm{Ker}(I-T) \oplus \overline{\mathrm{Im}(I-T)}. \tag{2.3}$$

A Banach space operator $T \in \mathcal{L}(X)$ is called *uniformly mean ergodic* if there exists $P \in \mathcal{L}(X)$ such that $\lim_{n\to\infty} \|T_{[n]} - P\| = 0$. It is then immediate that necessarily $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$. A result of M.Lin, [24], states that a Banach space operator $T \in \mathcal{L}(X)$ satisfying $\lim_{n\to\infty} \frac{\|T^n\|}{n} = 0$ is uniformly mean ergodic if and only if $\mathrm{Im}(I-T)$ is a *closed* subspace of X.

An operator $T \in \mathcal{L}(X)$ is called *power bounded* if $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$. Since $T_{[n]}(I-T) = \frac{1}{n}(T - T^{n+1})$ for $n \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} T_{[n]}x = 0, \qquad x \in \mathrm{Im}(I-T), \tag{2.4}$$

whenever T is power bounded.

Lemma 2.4. (i) *Define $\varphi \in H(\mathbb{D})$ via $\varphi(z) := 1/(1-z)$ for $z \in \mathbb{D}$. The Cesàro operator $\mathsf{C} : H(\mathbb{D}) \to H(\mathbb{D})$ satisfies $\mathrm{Ker}(I-\mathsf{C}) = \mathrm{span}\{\varphi\}$ and*

$$\mathrm{Im}(I-\mathsf{C}) = \{h \in H(\mathbb{D}) : h(0) = 0\} = \mathrm{Ker}(\delta_0),$$

with $\delta_0 \in H(\mathbb{D})^*$. In particular, $\mathrm{Im}(I-C)$ is closed in $H(\mathbb{D})$.

(ii) *Let X be any Banach space of analytic functions on $\mathbb{D}$ which contains the constant functions, is continuously included in $H(\mathbb{D})$ and such that $\mathsf{C} : X \to X$ is continuous. If $\mathsf{C} : X \to X$ is mean ergodic, then $\varphi \in X$ and $\mathrm{Ker}(I-C) = \mathrm{span}\{\varphi\}$.*

Proof. (i) This is proved in [28, Section 2]; see (2.4) and (2.5) on page 1184 with $\lambda = 1$.

(ii) Assume that $\mathsf{C} : X \to X$ is continuous and mean ergodic. Let C_X denote the restriction of C to X. Then

$$X = \mathrm{Ker}(I-\mathsf{C}_X) \oplus \overline{\mathrm{Im}(I-\mathsf{C}_X)}. \tag{2.5}$$

If $\varphi \notin X$, then $\mathrm{Ker}(I-\mathsf{C}_X) = \{0\}$ by part (i). Since point evaluations on $\mathbb{D}$ belong to X^* (see Section 1), it follows from (2.5) that $X \subseteq \{h \in H(\mathbb{D}) : h(0) = 0\}$. This is a contradiction, since the constant function $\mathbf{1}$ belongs to X. Hence, $\varphi \in X$. Since $\mathrm{Ker}(I-\mathsf{C}_X) \subseteq \mathrm{Ker}(I-\mathsf{C})$, it follows from part (i) that $\mathrm{Ker}(I-\mathsf{C}_X) = \mathrm{span}\{\varphi\}$. $\square$

We can now establish the main result of this section.

Theorem 2.5. (i) *Let $0 < \gamma < 1$. Both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ fail to be power bounded and are not mean ergodic. Moreover,*

$$\mathrm{Ker}(I-\mathsf{C}_\gamma) = \mathrm{Ker}(I-\mathsf{C}_{\gamma,0}) = \{0\},$$

and $\mathrm{Im}(I-\mathsf{C}_\gamma)$ (resp. $\mathrm{Im}(I-\mathsf{C}_{\gamma,0})$) is a proper closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).

(ii) *Both of the operators C_1 and $\mathsf{C}_{1,0}$ are power bounded but not mean ergodic. Moreover, $\mathrm{Im}(I-\mathsf{C}_1)$ (resp. $\mathrm{Im}(I-\mathsf{C}_{1,0})$) is not a closed subspace of $A^{-\gamma}$ (resp. of $A_0^{-\gamma}$).*

(iii) *Let $\gamma > 1$. Both of the operators C_γ and $\mathsf{C}_{\gamma,0}$ are power bounded and uniformly mean ergodic. Moreover, $\mathrm{Im}(I-\mathsf{C}_\gamma)$ (resp. $\mathrm{Im}(I-\mathsf{C}_{\gamma,0})$) is a* *proper closed subspace of* $A^{-\gamma}$ *(resp. of* $A_0^{-\gamma}$*). In addition,*

$$\mathrm{Im}(I-\mathsf{C}_\gamma)=\{h\in A^{-\gamma}: h(0)=0\}. \tag{2.6}$$

Moreover, with $\varphi(z):=1/(1-z)$*, for* $z\in\mathbb{D}$*, the linear projection operator* $P_\gamma: A^{-\gamma}\to A^{-\gamma}$ *given by*

$$P_\gamma(f):=f(0)\varphi, \qquad f\in A^{-\gamma},$$

is continuous and satisfies $\lim_{n\to\infty}(\mathsf{C}_\gamma)_{[n]}=P_\gamma$ *in the operator norm.*

Proof. Clearly the function $\varphi\in A_0^{-\gamma}\subseteq A^{-\gamma}$ if $\gamma>1$. Moreover, $\varphi\in A^{-1}\setminus A_0^{-1}$ and $\varphi\notin A^{-\gamma}$ in case $0<\gamma<1$.

(i) For $0<\gamma<1$ it is clear from Theorem 2.3(ii) that both C_γ and $\mathsf{C}_{\gamma,0}$ are not power bounded. Neither are they mean ergodic because $\sup_n\|\mathsf{C}_\gamma^n\|/n=\sup_n\|\mathsf{C}_{\gamma,0}^n\|/n=\infty$; see the discussion after (2.2). Since $\mathrm{Ker}(I-\mathsf{C})=\mathrm{span}\{\varphi\}$ in $H(\mathbb{D})$, by Lemma 2.4(i) and because $\varphi\notin A^{-\gamma}$, we can conclude that $\mathrm{Ker}(I-\mathsf{C}_\gamma)=\mathrm{Ker}(I-\mathsf{C}_{\gamma,0})=\{0\}$.

Since $|1-\frac{1}{2\gamma}|<\frac{1}{2\gamma}$, we can apply Theorem 2.1(iii) to deduce that $\mathrm{Im}(I-\mathsf{C}_{\gamma,0})$ is a proper closed subspace of $A_0^{-\gamma}$. Moreover, $(I-\mathsf{C}_\gamma)\in\mathcal{L}(A^{-\gamma})$ is the bidual operator of $I-\mathsf{C}_{\gamma,0}$ and so it follows from [2, Proposition 2.1] that $\mathrm{Im}(I-\mathsf{C}_\gamma)$ is also closed in $A^{-\gamma}$. It is a proper subspace of $A^{-\gamma}$ because every $h\in\mathrm{Im}(I-\mathsf{C}_\gamma)$ satisfies $\mathsf{C}_\gamma(h)(0)=0$; see (1.1).

(ii) Both C_1 and $\mathsf{C}_{1,0}$ are power bounded by Theorem 2.3(i). Since $\varphi\notin A_0^{-1}$, it follows from Lemma 2.4(ii), with $X=A_0^{-1}$, that $\mathsf{C}_{1,0}$ is not mean ergodic in A_0^{-1}. By [2, Proposition 2.2], C_1 also fails to be mean ergodic.

Assume that $\mathrm{Im}(I-\mathsf{C}_1)$ is closed in A^{-1}. Since $\lim_{n\to\infty}\|\mathsf{C}_1^n\|/n=0$, it follows from Lin's theorem, [24], that C_1 is uniformly mean ergodic. But, we just argued that C_1 is not even mean ergodic; contradiction! The argument for $\mathsf{C}_{1,0}$ is the same.

(iii) Let $\gamma>1$. By Theorem 2.3(i) both C_γ and $\mathsf{C}_{\gamma,0}$ are power bounded. In order to verify (2.6), it is clear that $\mathrm{Im}(I-\mathsf{C}_\gamma)\subseteq\{h\in A^{-\gamma}\,:\,h(0)=0\}$.

For the reverse inclusion, fix $h\in A^{-\gamma}$ satisfying $h(0)=0$. Theorem 2.1 reveals that $1\in\rho(\mathsf{C}_\gamma)$. By the calculations on p.1184 of [28], with $\lambda=1$, there is a unique $f\in H(\mathbb{D})$ satisfying $f(0)=0$ and $(I-\mathsf{C})f=h$. In fact, f is given by [28, formula (2.7)]:

$$f(z)=h(z)+\frac{1}{1-z}\int_0^z\frac{h(\zeta)}{\zeta}, \qquad z\in\mathbb{D}.$$

In order to conclude that $f\in A^{-\gamma}$, we first show that the function $H(z):=h(z)/z$, for $z\in\mathbb{D}\setminus\{0\}$ and $H(0):=h'(0)$, belongs to $A^{-\gamma}$. Recalling that $h(0)=0$, it is clear that $H\in H(\mathbb{D})$. If $\frac12\le|z|<1$, then $|H(z)|\le2|h(z)|$. Hence, $(1-|z|)^\gamma|H(z)|\le2\|h\|_{-\gamma}$. On the other hand, for $|z|\le\frac12$ we can apply the maximum modulus principle to $H\in H(\mathbb{D})$ to conclude that

$$|H(z)|\le\max_{|\zeta|=1/2}|H(\zeta)|=2\max_{|\zeta|=1/2}|h(\zeta)|=2^{\gamma+1}\max_{|\zeta|=1/2}(1-|\zeta|)^\gamma|h(\zeta)|\le2^{\gamma+1}\|h\|_{-\gamma}.$$

Summarizing, $\|H\|_{-\gamma}=\sup_{z\in\mathbb{D}}(1-|z|)^\gamma|H(z)|\le2^{\gamma+1}\|h\|_{-\gamma}$, i.e., $H\in A^{-\gamma}$.

Consider $G(z) := \int_0^z H(\zeta)\, d\zeta$, for $z \in \mathbb{D}$. Clearly $G'(z) = H(z)$ for $z \in \mathbb{D}$, with $H \in A^{-\gamma}$. By a classical result of Hardy and Littlewood (consider $p = \infty$ and $\beta := \gamma$ in [17, Theorem 5.5]), the function $G \in A^{-(\gamma-1)} = A^{-\gamma+1}$ (recall $(\gamma - 1) > 0$)). Therefore, for each $z \in \mathbb{D}$, we have for φG that

$$(1-|z|)^\gamma |\varphi(z)G(z)| = (1-|z|)^\gamma \Big| \frac{1}{1-z} \int_0^z H(\zeta)\, d\zeta \Big|$$
$$\leq (1-|z|)^{\gamma-1} \Big| \int_0^z H(\zeta)\, d\zeta \Big| \leq \|G\|_{-\gamma+1},$$

that is, $\varphi G \in A^{-\gamma}$. Hence, also $f = h + \varphi G \in A^{-\gamma}$. This establishes that $\mathrm{Im}(I - \mathsf{C}_\gamma))$ is closed in $A^{-\gamma}$. Since C_γ is power bounded, a theorem of Lin [24] implies that C_γ is uniformly mean ergodic.

By [2, Proposition 2.1] it follows that also $\mathrm{Im}(I - \mathsf{C}_{\gamma,0}))$ is closed in $A_0^{-\gamma}$. So, $\mathsf{C}_{\gamma,0}$ is uniformly mean ergodic, again as a consequence of Lin's theorem.

Lemma 2.4(ii), with $X = A^{-\gamma}$, and the mean ergodicity of C_γ yield that $\varphi \in A^{-\gamma}$. Since $\delta_0 \in (A^{-\gamma})^*$, the formula

$$P_\gamma(f) := f(0)\varphi = \langle f, \delta_0 \rangle \varphi, \qquad f \in A^{-\gamma},$$

implies that P_γ is a continuous projection in $A^{-\gamma}$.

For each $f \in A^{-\gamma}$ we have

$$(\mathsf{C}_\gamma)_{[n]}(f) = (\mathsf{C}_\gamma)_{[n]}(f - f(0)) + f(0)(\mathsf{C}_\gamma)_{[n]}(\mathbf{1}), \qquad n \in \mathbb{N}. \tag{2.7}$$

On the other hand, $\varphi \in A^{-\gamma}$ satisfies $(\mathsf{C}_\gamma)_{[n]}(\varphi) = \varphi$ for all $n \in \mathbb{N}$; see Lemma 2.4(ii) with $X = A^{-\gamma}$. According to (2.6) the function $h(z) := z\varphi(z)$ for $z \in \mathbb{D}$ belongs to $\mathrm{Im}(I - \mathsf{C}_\gamma)$ and hence, the sequence

$$(\mathsf{C}_\gamma)_{[n]}(\mathbf{1}) = (\mathsf{C}_\gamma)_{[n]}(\varphi - h) = \varphi - (\mathsf{C}_\gamma)_{[n]}(h), \qquad n \in \mathbb{N},$$

converges to φ in $A^{-\gamma}$ for $n \to \infty$; see (2.4). It then follows from (2.7) and the fact that $\lim_{n\to\infty}(\mathsf{C}_\gamma)_{[n]}(f - f(0)) = 0$ in $A^{-\gamma}$ via (2.4) (since $(f - f(0)) \in \mathrm{Im}(I - \mathsf{C}_\gamma)$ by (2.6)) that

$$\lim_{n\to\infty} (\mathsf{C}_\gamma)_{[n]}(f) = f(0)\varphi = P_\gamma(f).$$

This establishes that τ_s- $\lim_{n\to\infty}(\mathsf{C}_\gamma)_{[n]} = P_\gamma$. But, the uniform mean ergodicity of C_γ means that $\{(\mathsf{C}_\gamma)_{[n]}\}_{n\in\mathbb{N}}$ is a convergent sequence for the operator norm and hence, its operator norm limit must also be P_γ. □

Remark 2.6. (i) Let $\gamma \geq 1$. It follows from Theorem 2.1 that the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ satisfies $\sigma(\mathsf{C}_\gamma) \cap \partial\mathbb{D} = \{1\} = \sigma(\mathsf{C}_{\gamma,0}) \cap \partial\mathbb{D}$. Since both C_γ and $\mathsf{C}_{\gamma,0}$ are power bounded, it follows from Theorem 1 and the Remark on p.317 of [22] that

$$\lim_{n\to\infty} \left\| \mathsf{C}_\gamma^{n+1} - \mathsf{C}_\gamma^n \right\| = 0 = \lim_{n\to\infty} \left\| \mathsf{C}_{\gamma,0}^{n+1} - \mathsf{C}_{\gamma,0}^n \right\|.$$

(ii) Let $\gamma > 1$. Theorem 2.1 shows that $\lambda = 1$ is an isolated singularity of the resolvent map of both C_γ and $\mathsf{C}_{\gamma,0}$. Since both C_γ and $\mathsf{C}_{\gamma,0}$ are uniformly mean ergodic, it follows that 1 is actually a simple pole of the resolvent map, [23, Theorem 2.7, p.90].

(iii) For $\gamma \geq 1$ the point $1 \in \sigma(\mathsf{C}_\gamma) = \sigma(\mathsf{C}_{\gamma,0})$ and so the spectral mapping theorem applied to the polynomial $p_n(z) := \frac{1}{n}\sum_{m=1}^n z^m$ (i.e., $(\mathsf{C}_\gamma)_{[n]} = p_n(\mathsf{C}_\gamma)$) yields that $1 \in \sigma((\mathsf{C}_\gamma)_{[n]})$, for $n \in \mathbb{N}$. Hence,

$$1 \leq r\left((\mathsf{C}_\gamma)_{[n]}\right) \leq \|(\mathsf{C}_\gamma)_{[n]}\|, \qquad n \in \mathbb{N}.$$

Then Theorem 2.3(i) and the formula $(\mathsf{C}_\gamma)_{[n]} = \frac{1}{n}\sum_{m=1}^n \mathsf{C}_\gamma^m$ imply that

$$\left\|(\mathsf{C}_\gamma)_{[n]}\right\| = \left\|(\mathsf{C}_{\gamma,0})_{[n]}\right\| = 1, \qquad n \in \mathbb{N}.$$

For $0 < \gamma < 1$ the point $\frac{1}{\gamma} \in \sigma(\mathsf{C}_\gamma)$ and so again the spectral mapping theorem yields that $p_n(\frac{1}{\gamma}) = \frac{1}{n}\sum_{m=1}^n \frac{1}{\gamma^m} \in \sigma((\mathsf{C}_\gamma)_{[n]})$ for $n \in \mathbb{N}$. In particular,

$$\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\gamma^m} \leq r\left((\mathsf{C}_\gamma)_{[n]}\right) = r\left((\mathsf{C}_{\gamma,0})_{[n]}\right), \qquad n \in \mathbb{N}.$$

It then follows from Theorem 2.3(ii) that

$$\left\|(\mathsf{C}_{\gamma,0})_{[n]}\right\| = \left\|(\mathsf{C}_\gamma)_{[n]}\right\| = \frac{1}{n}\sum_{m=1}^{n}\frac{1}{\gamma^m} = \frac{(\frac{1}{\gamma^n}) - 1}{(n+1)(1-\gamma)}, \qquad n \in \mathbb{N}.$$

In particular, for $0 < \gamma < 1$, not only do C_γ and $\mathsf{C}_{\gamma,0}$ fail to be power bounded but, their Cesàro averages $\{(\mathsf{C}_\gamma)_{[n]}\}_{n=1}^\infty$ and $\{(\mathsf{C}_{\gamma,0})_{[n]}\}_{n=1}^\infty$ are also unbounded sequences in $\mathcal{L}(A^{-\gamma})$ and $\mathcal{L}(A_0^{-\gamma})$, respectively. □

Concerning the dynamics of C recall that an operator $T \in \mathcal{L}(X)$, with X a separable Fréchet space, is called *hypercyclic* if there exists $x \in X$ such that the orbit $\{T^n x\colon n \in \mathbb{N}_0\}$ is dense in X. If, for some $z \in X$, the projective orbit $\{\lambda T^n z\colon \lambda \in \mathbb{C},\ n \in \mathbb{N}_0\}$ is dense in X, then T is called *supercyclic*. Clearly, hypercyclicity implies supercyclicity. It is proved in [3] that C is not supercyclic on $H(\mathbb{D})$. Since the image of a dense subset of $A_0^{-\gamma}$ under the natural inclusion map into $H(\mathbb{D})$ is dense in $H(\mathbb{D})$, we have the following consequence.

Proposition 2.7. *The Cesàro operator* $\mathsf{C}_{\gamma,0}$ *is not supercyclic and hence, also not hypercyclic, in each space* $A_0^{-\gamma}$*, for* $\gamma > 0$.

3. Optimal extension of C_γ and of $\mathsf{C}_{\gamma,0}$

The *optimal domain* $[\mathsf{C}, H^p(\mathbb{D})]$ of the Cesàro operator $C : H^p(\mathbb{D}) \to H^p(\mathbb{D})$, $1 \leq p < \infty$, was introduced and thoroughly investigated in [15, Section 3]. The definition given there is rather general and can be applied to C when it acts in any Banach space of analytic functions X on $\mathbb{D}$. Namely,

$$[\mathsf{C}, X] := \{f \in H(\mathbb{D}) \ : \ \mathsf{C}(f) \in X\},$$

which is a Banach space for the norm

$$\|f\|_{[\mathsf{C},X]} := \|\mathsf{C}(f)\|_X, \qquad f \in [\mathsf{C}, X], \tag{3.1}$$

as of a consequence of $\mathsf{C} : H(\mathbb{D}) \to H(\mathbb{D})$ being a topological Fréchet space isomorphism. When point evaluations on $\mathbb{D}$ belong to $[\mathsf{C}, X]^*$, then $[\mathsf{C}, X]$ is actually a Banach space of analytic functions on $\mathbb{D}$ and C maps $[\mathsf{C}, X]$ isometrically onto X. If, in addition, C acts in X, then $X \subseteq [\mathsf{C}, X]$ and the natural inclusion map is continuous. Most important is that $[\mathsf{C}, X]$ is the *largest* of all Banach spaces of analytic functions Y on $\mathbb{D}$ that C maps continuously into X; the argument is analogous to [15, Remark 3.1]. To describe more concretely which functions from $H(\mathbb{D})$ are members of $[\mathsf{C}, X]$ may not be easy in general: for $X = H^p(\mathbb{D})$ this is based on properties of the Littlewood-Paley g-function [15, Proposition 3.2 & Corollary 3.3]. To express the norm (3.1) in a more explicit way, if possible, would also be an advantage.

The aim of this section it to investigate the optimal domain spaces $[\mathsf{C}, A^{-\gamma}]$ and $[\mathsf{C}, A_0^{-\gamma}]$ for $\gamma > 0$. Clearly $[\mathsf{C}, A_0^{-\gamma}] \subseteq [\mathsf{C}, A^{-\gamma}]$. Moreover, with continuous inclusions, we clearly have

$$A^{-\gamma} \subseteq [\mathsf{C}, A^{-\gamma}] \subsetneqq H(\mathbb{D}) \quad \text{and} \quad A_0^{-\gamma} \subseteq [\mathsf{C}, A_0^{-\gamma}] \subsetneqq H(\mathbb{D}),$$

due to the closed graph theorem and the fact that point evaluations on $\mathbb{D}$ belong to $[\mathsf{C}, A^{-\gamma}]^*$ hence, also to $[\mathsf{C}, A_0^{-\gamma}]^*$. To verify this latter claim, fix $z_0 \in \mathbb{D}$ and choose $0 < r < 1$ with $|z_0| < r$. For $f \in H(\mathbb{D})$ observe that $f(z) = (1 - z)(z\mathsf{C}(f)(z))'$, for $z \in \mathbb{D}$, which is routine to verify. Given $f \in [\mathsf{C}, A^{-\gamma}]$, the previous identity and the Cauchy integral formula yield

$$|\langle f, \delta_{z_0}\rangle| = |f(z_0)| = |1 - z_0| \cdot \left|\frac{1}{2\pi i}\int_{|\zeta|=r} \frac{\zeta\mathsf{C}(f)(\zeta)}{(\zeta - z_0)^2} d\zeta\right| \leq$$

$$\frac{|1 - z_0|r^2}{(r - |z_0|)^2(1 - r)^\gamma} \sup_{|\zeta|=r}(1 - |\zeta|)^\gamma |\mathsf{C}(f)(\zeta)| \leq \frac{|1 - z_0|r^2}{(r - |z_0|)^2(1 - r)^\gamma} \|\mathsf{C}(f)\|_{-\gamma}.$$

Accordingly, $|f(z_0)| \leq \frac{|1-z_0|r^2}{(r-|z_0|)^2(1-r)^\gamma}\|f\|_{[\mathsf{C},A^{-\gamma}]}$ and so each evaluation functional $\delta_z : f \to f(z)$ is continuous on both $[\mathsf{C}, A^{-\gamma}]$ and $[\mathsf{C}, A_0^{-\gamma}]$, for each $z \in \mathbb{D}$.

Theorem 3.1. *Let $\gamma > 0$ and $\varphi(z) := 1/(1 - z)$ for $z \in \mathbb{D}$. The optimal domain $[\mathsf{C}, A^{-\gamma}]$ of $\mathsf{C}_\gamma : A^{-\gamma} \to A^{-\gamma}$ is isometrically isomorphic to $A^{-\gamma}$ and is given by*

$$[\mathsf{C}, A^{-\gamma}] = \{f \in H(\mathbb{D}) \ : \ f\varphi \in A^{-(\gamma+1)}\}. \tag{3.2}$$

Moreover, the norm $\|.\|_{[\mathsf{C},A^{-\gamma}]}$ is equivalent to the norm $f \to \|f\varphi\|_{-(\gamma+1)}$ and the containment $A^{-\gamma} \subseteq [\mathsf{C}, A^{-\gamma}]$ is proper.

Proof. That $[\mathsf{C}, A^{-\gamma}]$ is isometrically isomorphic to $A^{-\gamma}$ follows from the facts that $\mathsf{C} : [\mathsf{C}, A^{-\gamma}] \to A^{-\gamma}$ is both injective (as C is bijective on $H(\mathbb{D})$) and surjective (by the definition of $[\mathsf{C}, A^{-\gamma}]$), and that it is an isometry (via (3.1)).

To verify (3.2) observe, for $f \in H(\mathbb{D})$, that $\mathsf{C}(f) \in A^{-\gamma}$ if and only if the function $z \mapsto \int_0^z f(\zeta)/(1 - \zeta)d\zeta \in A^{-\gamma}$. This follows from the argument in the proof of Theorem 2.5(iii) above, where it is shown that if $h \in A^{-\gamma}$

satisfies $h(0) = 0$, then also $H(z) := \frac{h(z)}{z} \in A^{-\gamma}$ (the converse is clear from $h(z) = zH(z)$). Note that this part of the argument in the proof of Theorem 2.5(iii) did not require $\gamma > 1$ (which was being assumed there). By a result of Hardy and Littlewood, [17, Theorem 5.5], this in turn is equivalent to $f\varphi \in A^{-(\gamma+1)}$.

To verify the equivalence of the norms $\|.\|_{[\mathsf{C},A^{-\gamma}]}$ and $f \to \|f\varphi\|_{-(\gamma+1)}$ we proceed as follows. First, the space $E := \{f \in H(\mathbb{D}) \ : \ f\varphi \in A^{-(\gamma+1)}\}$ endowed with the norm $\|f\varphi\|_{-(\gamma+1)}$, for $f \in E$, is a Banach space which is isomorphic to $A^{-(\gamma+1)}$. On the other hand, since $\mathsf{C} : [\mathsf{C}, A^{-\gamma}] \to A^{-\gamma}$ is continuous we have, for some constant $K > 0$, that

$$\|f\varphi\|_{-(\gamma+1)} \leq \|f\|_{-\gamma} \leq K\|f\|_{[\mathsf{C},A^{-\gamma}]}, \qquad f \in [\mathsf{C}, A^{-\gamma}].$$

Therefore the identity map $I : [\mathsf{C}, A^{-\gamma}] \to E$ is a continuous bijection between Banach spaces. By the open mapping theorem its inverse is also continuous. This implies that the two norms are equivalent in $[\mathsf{C}, A^{-\gamma}]$.

Finally, the function $g(z) := (1-z)/(1+z)^{\gamma+1}$ for $z \in \mathbb{D}$ satisfies $\|g\varphi\|_{-(\gamma+1)} \leq 1$. i.e., $g \in [\mathsf{C}, A^{-\gamma}]$. However, $g \notin A^{-\gamma}$ since

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\gamma}|1-z|}{|1+z|^{\gamma+1}} \geq \sup_{s\in[0,1]} \frac{1+s}{1-s} = \infty.$$

This shows that the containment $A^{-\gamma} \subsetneq [\mathsf{C}, A^{-\gamma}]$ is proper. □

Theorem 3.2. *Let $\gamma > 0$ and $\varphi(z) := 1/(1-z)$ for $z \in \mathbb{D}$. The optimal domain $[\mathsf{C}, A_0^{-\gamma}]$ of $\mathsf{C}_{\gamma,0} : A_0^{-\gamma} \to A_0^{-\gamma}$ is isometrically isomorphic to $A_0^{-\gamma}$ and is given by*

$$[\mathsf{C}, A_0^{-\gamma}] = \left\{f \in H(\mathbb{D}) \ : \ f\varphi \in A_0^{-(\gamma+1)}\right\}. \tag{3.3}$$

Moreover, the norm $\|\cdot\|_{[\mathsf{C},A_0^{-\gamma}]}$ is equivalent to the norm $f \mapsto \|f\varphi\|_{-(\gamma+1)}$ and the containment $A_0^{-\gamma} \subsetneq [\mathsf{C}, A_0^{-\gamma}]$ is proper.

Proof. The Banach space $[\mathsf{C}, A_0^{-\gamma}]$ is isometrically isomorphic to $A_0^{-\gamma}$; adapt the argument from the beginning of the proof of Theorem 3.1.

To see that $A_0^{-\gamma} \subsetneq [\mathsf{C}, A_0^{-\gamma}]$ is proper assume, on the contrary, that $A_0^{-\gamma} = [\mathsf{C}, A_0^{-\gamma}]$. Then $\mathsf{C}_{\gamma,0} : [\mathsf{C}, A_0^{-\gamma}] = A_0^{-\gamma} \to A_0^{-\gamma}$ is a Banach space isomorphism and so $0 \notin \sigma(\mathsf{C}_{\gamma,0})$; contradiction to Theorem 2.1(ii).

To establish (3.3) is similar to the proof of (3.2) in Theorem 3.1. It is enough to keep in mind the following two facts.

Fact 1. Let $f \in H(\mathbb{D})$ satisfy $f(0) = 0$. Then $f \in A_0^{-\gamma}$ if and only if $f(z)/z \in A_0^{-\gamma}$.

Fact 2. A function $f \in H(\mathbb{D})$ belongs to $A_0^{-\gamma}$ if and only if $f' \in A_0^{-(\gamma+1)}$.

Indeed, by [17, Theorem 5.5] the differentiation operator $D : A^{-\gamma} \to A^{-(\gamma+1)}$, given by $D(f) := f'$, and the integration operator $J : A^{-(\gamma+1)} \to A^{-\gamma}$, given by $J(f)(z) := \int_0^z f(\zeta)d\zeta$, for $z \in \mathbb{D}$, are continuous (see also [20, Theorem 2.1(a) & Proposition 2.2(a)]). Since the space of polynomials $\mathcal{P}$ is dense in $A_0^{-\beta}$ for each $\beta > 0$ and $\mathcal{P}$ is invariant for both D and J, it follows that their restrictions $D : A_0^{-\gamma} \to A_0^{-(\gamma+1)}$ and $J : A_0^{-(\gamma+1)} \to A_0^{-\gamma}$ are continuous. This implies Fact 2.

Now that (3.3) is established, one can argue as in the proof of Theorem 3.1 to show that the stated norms are equivalent. $\square$

Remark 3.3. (i) As noted in Section 1, for every $\gamma > 0$ the Banach space $A^{-\gamma}$ (resp. $A_0^{-\gamma}$) is isomorphic to ℓ^∞ (resp. c_0). In particular, $A^{-\gamma}$ is non-separable whereas $A_0^{-\gamma}$ is separable. Also, $A_0^{-\gamma}$ is not weakly sequentially complete and hence, neither is $A^{-\gamma}$ (as it contains $A_0^{-\gamma}$ as a closed subspace). Since both c_0 and ℓ^∞ fail to have the Radon-Nikodym property, the same is true for $A^{-\gamma}$ and $A_0^{-\gamma}$. And so on. Since $A^{-\gamma}$ (resp. $A_0^{-\gamma}$) is isomorphic to $[\mathsf{C}, A^{-\gamma}]$ (resp. to $[\mathsf{C}, A_0^{-\gamma}]$), we can conclude that the optimal domain spaces $[\mathsf{C}, A^{-\gamma}]$ and $[\mathsf{C}, A_0^{-\gamma}]$ inherit such properties as those mentioned above (and others) from $A^{-\gamma}$ and $A_0^{-\gamma}$, respectively.

(ii) Let $f, g \in H(\mathbb{D})$ satisfy $|g(z)| \le |f(z)|$ for $z \in \mathbb{D}$. If $f \in A^{-\gamma}$ (resp. $f \in A_0^{-\gamma}$), for $\gamma > 0$, then it is routine to check that also $g \in A^{-\gamma}$ (resp. $g \in A_0^{-\gamma}$). It follows from (3.2) (resp. (3.3)) that this useful property carries over to the optimal domain space $[\mathsf{C}, A^{-\gamma}]$ (resp. $[\mathsf{C}, A_0^{-\gamma}]$).

(iii) According to [17, p.92] there exists $g \in H^\infty(\mathbb{D})$ such that g' fails to have boundary values at a.e. point in $\partial\mathbb{D}$. Since $H^\infty(\mathbb{D}) \subseteq A_0^{-\gamma}$, for every $\gamma > 0$, and the differentiation operator $D : A_0^{-\gamma} \to A_0^{-(\gamma+1)}$ is continuous, it follows that $g' \in A_0^{-(\gamma+1)}$. Accordingly, for every $\beta > 1$ there exists a function in $A_0^{-\beta}$ which fails to have a.e. boundary values. Since $A_0^{-\beta} \subseteq A^{-\beta} \subseteq [\mathsf{C}, A^{-\beta}]$ and $A_0^{-\beta} \subseteq [\mathsf{C}, A_0^{-\beta}]$ we can conclude, for every $\beta > 1$, that there exist functions in the spaces $A_0^{-\beta}, A^{-\beta}, [\mathsf{C}, A_0^{-\beta}]$ and $[\mathsf{C}, A^{-\beta}]$ which fail to have a.e. boundary values. $\square$

For $\gamma > 0$, the following result shows that the largest growth space $A^{-\beta}$ that is contained in $[\mathsf{C}, A^{-\gamma}]$ is $A^{-\gamma}$. The same is true for $A_0^{-\gamma}$ and $[\mathsf{C}, A_0^{-\gamma}]$.

Proposition 3.4. *Let $\gamma > 0$. For each $\beta > \gamma$, the space $A^{-\beta} \not\subseteq [\mathsf{C}, A^{-\gamma}]$ and the space $A_0^{-\beta} \not\subseteq [\mathsf{C}, A_0^{-\gamma}]$.*

Proof. That $A^{-\beta} \not\subseteq [\mathsf{C}, A^{-\gamma}]$ is a direct consequence of Theorem 3.1 and the fact that the function $f(z) := 1/(1-z)^\beta \in A^{-\beta} \setminus [\mathsf{C}, A^{-\gamma}]$ because $f(z)/(1-z) \notin A^{-(\gamma+1)}$.

Suppose that $A_0^{-\beta} \subseteq [\mathsf{C}, A_0^{-\gamma}]$. Then $\mathsf{C}(A_0^{-\beta}) \subseteq A_0^{-\gamma}$. Using that $A_0^{-\gamma} \subseteq A_0^{-\beta}$ continuously and a closed graph argument it follows that the natural inclusion map Φ for $A_0^{-\beta} \subseteq [\mathsf{C}, A_0^{-\gamma}]$ is continuous. Hence, also $\mathsf{C} : A_0^{-\beta} \to A_0^{-\gamma}$ is continuous (being the composition of Φ and $\mathsf{C} : [\mathsf{C}, A_0^{-\gamma}] \to A_0^{-\gamma}$). Passing to the bidual yields that $\mathsf{C} : A^{-\beta} \to A^{-\gamma}$ is continuous, i.e., $A^{-\beta} \subseteq [\mathsf{C}, A^{-\gamma}]$, which is a contradiction to the previous paragraph. So, $A_0^{-\beta} \not\subseteq [\mathsf{C}, A_0^{-\gamma}]$. $\square$

Given $\gamma > 0$, Proposition 3.4 raises the question of whether there are any weighted Banach spaces of analytic functions X on $\mathbb{D}$ which satisfy $A^{-\gamma} \subseteq X \subseteq [\mathsf{C}, A^{-\gamma}]$.

A weight $v : \mathbb{D} \to [0, \infty)$ is called *essential* if it is radial (i.e., $v(z) = v(|z|)$ for $z \in \mathbb{D}$), continuous, satisfies that $v(r)$ decreases to 0 as r approaches 1, and there exists a constant $d > 0$ such that for each $z_0 \in \mathbb{D}$ there is $f_0 \in H(\mathbb{D})$ such that $|f_0(z_0)| \geq d/v(z_0)$ and $v(z)|f_0(z)| \leq 1$ for each $z \in \mathbb{D}$ (i.e., $\|f_0\|_v \leq 1$ in the notation of (3.4) below). It is established in [14] that a radial continuous weight $v : \mathbb{D} \to (0, \infty)$ which is decreasing to 0 in $[0, 1)$ as r approaches 1 is essential if and only if there exists a weight $w : \mathbb{D} \to (0, \infty)$ satisfying the same assumptions as for v such that the function $r \mapsto -\log(w(e^r))$ is convex and, for some constant $c > 1$, the weight w satisfies

$$\frac{1}{c}\, w(r) \leq v(r) \leq c w(r), \qquad r \in [0, 1).$$

In other words, radial continuous weights in $\mathbb{D}$ that are essential are those which are equivalent to "logarithmic convex weights". All the weights $v_\gamma(z) = (1 - |z|)^\gamma$, for $\gamma > 0$, considered in this paper are essential. For further examples and properties of essential weights we refer to [12].

Given an essential weight v, the *weighted space associated with* v is defined by

$$H_v^\infty := \{f \in H(\mathbb{D}) \mid \|f\|_v := \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty\}; \tag{3.4}$$

it is a Banach space of analytic functions on $\mathbb{D}$ for the norm $\|.\|_v$. According to [30, Lemma 1], point evaluations on $\mathbb{D}$ belong to $(H_v^\infty)^*$.

Proposition 3.5. *Let $\gamma > 0$ and v be an essential weight on $\mathbb{D}$. If it is the case that $A^{-\gamma} \subseteq H_v^\infty \subseteq [\mathsf{C}, A^{-\gamma}]$, then there exists $b > 0$ such that*

$$(1/b)(1 - |z|)^\gamma \leq v(z) \leq b(1 - |z|)^\gamma, \qquad z \in \mathbb{D}. \tag{3.5}$$

In particular, $A^{-\gamma} = H_v^\infty$, both algebraically and topologically.

Proof. The assumption that $A^{-\gamma} \subseteq H_v^\infty \subseteq [\mathsf{C}, A^{-\gamma}]$ and the closed graph theorem imply that both inclusions are continuous. In particular, there is $M > 0$ such that $\|f\|_v \leq M\|f\|_{-\gamma}$ for each $f \in A^{-\gamma}$ and, by Theorem 3.1, $\|g/(1 - z)\|_{-(\gamma+1)} \leq K\|g\|_v$ for each $g \in H_v^\infty$ (and some $K > 0$).

Given an arbitrary point $u \in \mathbb{D} \setminus \{0\}$, we set $f_0(z) := 1/(1 - z|u|u^{-1})^\gamma$, for $z \in \mathbb{D}$. For $u = 0$, define $f_0(z) = 1$, for $z \in \mathbb{D}$. Then $\|f_0\|_{-\gamma} = 1$ and $f_0(u) = 1/(1 - |u|)^\gamma$. Moreover, $v(u)|f_0(u)| \leq \|f_0\|_v \leq M$. This implies that $v(u) \leq M(1 - |u|)^\gamma$ for each $u \in \mathbb{D}$.

Now fix $w \in \mathbb{D}$ arbitrary. Since the weight v is essential, there is $d > 0$ (independent of w) and $g_0 \in H(\mathbb{D})$ with $\|g_0\|_v \leq 1$ (i.e., $g_0 \in H_v^\infty$) such that $|g_0(w)| \geq d/v(w)$. Observe that

$$\|g_0/(1 - z)\|_{-(\gamma+1)} = \sup_{z \in \mathbb{D}} (1 - |z|)^{\gamma+1} \frac{|g_0(z)|}{|1 - z|} \leq K\|g_0\|_v \leq K.$$

Accordingly,

$$\frac{d}{v(w)}\frac{(1-|w|)^{\gamma+1}}{|1-w|} \le (1-|w|)^{\gamma+1}\frac{|g_0(w)|}{|1-w|} \le \sup_{z\in\mathbb{D}}(1-|z|)^{\gamma+1}\frac{|g_0(z)|}{|1-z|} \le K$$

from which it follows that

$$\frac{(1-|w|)^{\gamma+1}}{|1-w|} \le \frac{K}{d}v(w), \qquad w \in \mathbb{D}.$$

In particular, $(1-r)^\gamma \le (K/d)v(r)$ for $r \in [0,1)$. Setting $b := \max\{M, K/d\}$, yields (3.5). □

Proposition 3.6. *Let* $\gamma > 0$.

(i) *The containment* $A^{-\gamma} \subseteq [\mathsf{C}, A^{-\gamma}]$ *is proper.*

(ii) $[\mathsf{C}, A_0^{-\gamma}]$ *is a proper, closed subspace of* $[\mathsf{C}, A^{-\gamma}]$.

(iii) *The spaces* $A^{-\gamma}$ *and* $[\mathsf{C}, A_0^{-\gamma}]$ *are non-comparable. That is,*

$$A^{-\gamma} \nsubseteq [\mathsf{C}, A_0^{-\gamma}] \quad \textit{and} \quad [\mathsf{C}, A_0^{-\gamma}] \nsubseteq A^{-\gamma}.$$

Proof. (i) Theorem 3.1 shows that $A^{-\gamma}$ is *properly* contained in $[\mathsf{C}, A^{-\gamma}]$.

(ii) Since $A_0^{-\gamma} \subseteq A^{-\gamma}$, it is clear that $[\mathsf{C}, A_0^{-\gamma}] \subseteq [\mathsf{C}, A^{-\gamma}]$. Moreover, $[\mathsf{C}, A_0^{-\gamma}]$ has the same norm as that in $[\mathsf{C}, A^{-\gamma}]$; see (3.1). Since $A_0^{-\gamma}$ is a closed subspace of $A^{-\gamma}$, it is routine to verify that $[\mathsf{C}, A_0^{-\gamma}]$ is closed in $[\mathsf{C}, A^{-\gamma}]$. It was noted in the proof of Theorem 3.1 that the function $g(z) := (1-z)/(1+z)^{\gamma+1}$ for $z \in \mathbb{D}$ belongs to $[\mathsf{C}, A^{-\gamma}]$. However,

$$(1-|z|)^{\gamma+1}\Big|\frac{g(z)}{1-z}\Big| = \Big(\frac{1-|z|}{|1+z|}\Big)^{\gamma+1}, \qquad z \in \mathbb{D},$$

has the value 1 for every real $z \in (-1, 0]$ and so, via (3.3), $g \notin [\mathsf{C}, A_0^{-\gamma}]$. Hence, the containment $[\mathsf{C}, A_0^{-\gamma}] \subseteq [\mathsf{C}, A^{-\gamma}]$ is *proper*.

(iii) We first show that $A^{-\gamma} \nsubseteq [\mathsf{C}, A_0^{-\gamma}]$. On the contrary, if $A^{-\gamma} \subseteq [\mathsf{C}, A_0^{-\gamma}]$, then it follows that $\mathsf{C}(A^{-\gamma}) \subseteq A_0^{-\gamma}$. Since $(A_0^{-\gamma})^{**} = A^{-\gamma}$ and $(\mathsf{C}_{\gamma,0})^{**} = \mathsf{C}_\gamma$, we can conclude that $(\mathsf{C}_{\gamma,0})^{**}((A_0^{-\gamma})^{**}) \subseteq A_0^{-\gamma}$. But, this contradicts the fact that $\mathsf{C}_{\gamma,0}$ is not weakly compact; see Corollary 2.2 above and [18, Ch. VI, Sect. 4, Theorem 2]. Accordingly, $A^{-\gamma} \nsubseteq [\mathsf{C}, A_0^{-\gamma}]$.

To show that $[\mathsf{C}, A_0^{-\gamma}] \nsubseteq A^{-\gamma}$ assume, on the contrary, that $[\mathsf{C}, A_0^{-\gamma}] \subseteq A^{-\gamma}$. Via part (i), to achieve a contradiction it suffices to deduce that $[\mathsf{C}, A^{-\gamma}] \subseteq A^{-\gamma}$.

By the closed graph theorem the inclusion $[\mathsf{C}, A_0^{-\gamma}] \subseteq A^{-\gamma}$ is continuous. Hence, there exists $M > 0$ such that

$$\|h\|_{-\gamma} \le M\|h\|_{[\mathsf{C},A^{-\gamma}]}, \qquad h \in [\mathsf{C}, A_0^{-\gamma}],$$

after noting that the spaces $[\mathsf{C}, A_0^{-\gamma}]$ and $[\mathsf{C}, A^{-\gamma}]$ have the same norm. Fix $f \in [\mathsf{C}, A^{-\gamma}]$, in which case $\mathsf{C}(f) \in A^{-\gamma}$. According to Proposition 1.2 of [11] there exists a sequence of polynomials $\{p_j\}_{j\in\mathbb{N}} \subseteq A_0^{-\gamma} \subseteq [\mathsf{C}, A_0^{-\gamma}]$ which converges to $\mathsf{C}(f)$ in $H(\mathbb{D})$ and satisfies $\|p_j\|_{-\gamma} \le \|\mathsf{C}(f)\|_{-\gamma}$ for $j \in \mathbb{N}$. Then $\{\mathsf{C}^{-1}(p_j)\}_{j\in\mathbb{N}}$ converges to f in $H(\mathbb{D})$ with $\{\mathsf{C}^{-1}(p_j)\}_{j\in\mathbb{N}} \subseteq [\mathsf{C}, A_0^{-\gamma}] \subseteq A^{-\gamma}$

(where the last containment is being assumed). Moreover, $\{\mathsf{C}^{-1}(p_j)\}_{j\in\mathbb{N}}$ is a bounded set in $A^{-\gamma}$ because

$$\|\mathsf{C}^{-1}(p_j)\|_{-\gamma} \leq M\|\mathsf{C}^{-1}(p_j)\|_{[\mathsf{C},A^{-\gamma}]} = M\|p_j\|_{-\gamma} \leq M\|C(f)\|_{-\gamma}, \qquad j \in \mathbb{N}.$$

Since closed balls of $A^{-\gamma}$ are compact sets in the Montel space $H(\mathbb{D})$, it follows that $f \in A^{-\gamma}$. This shows that $[\mathsf{C}, A^{-\gamma}] \subseteq A^{-\gamma}$ which yields the required contradiction. □

Proposition 3.6 shows that the Banach spaces of analytic functions $A^{-\gamma}$ and $[\mathsf{C}, A_0^{-\gamma}]$ are two *non-comparable, proper* vector subspaces of the optimal domain space $[\mathsf{C}, A^{-\gamma}]$, that is, C maps both of these spaces into $A^{-\gamma}$.

Acknowledgements. The research of the first two authors was partially supported by the projects MTM2013-43540-P and GVA Prometeo II/2013/013.

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