A note on the Mackey-star topology on a dual Banach space

A. J. Guirao[*], V. Montesinos[†]

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Abstract

By using a result in R. B. Kirk, *A note on the Mackey topology for* $(C^b(X)^*, C^b(X))$, Pacific Journal of Mathematics, **45**, 2 (1973), 543–554, we show that there are separable Banach spaces such that their dual spaces, endowed with the Mackey-star topology, are not analytic. This solves a question raised in J. Kąkol, W. Kubiś, M. López-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Springer Verlag, 2012, and in J. Kąkol, M. López-Pellicer, *On realcompact topological vector spaces*, RACSAM **105** (2011), 39–70.

Let X be a Banach space. The topology Mackey-star (denoted μ^*) on X^* is the topology of the uniform convergence on the family of all (convex and balanced) weakly compact subsets of X (see, e.g., [Ko, §21.4], where this topology is denoted by $\mathfrak{T}_k(X)$). It is a simple consequence of the Grothendieck completeness criterium (see, e.g., [Gr, 2.§14]) that (X^*, μ^*) is always complete. A Banach space X is said to be *strongly weakly compactly generated* (SWCG, in short) (see [SW]) whenever there exists a weakly compact subset K_0 of X such that, for every weakly compact subset K of X and for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $K \subset nK_0 + \varepsilon B_X$. Examples of SWCG Banach spaces include the reflexive ones, the separable Schur spaces, and $L^1(\mu)$, where μ is a σ-finite measure (see, e.g., [SW] and [FMZ]). It is simple to prove that a Banach space X is SWCG if, and only if, (B_{X^*}, μ^*) is metrizable. This shows, in particular, that a separable Banach space X is SWCG if, and only if, (B_{X^*}, μ^*) is a Polish space.

A topological space is said to be *analytic* if it is the continuous image of a Polish space (alternatively, if it is the continuous image of the space $\mathbb{N}^{\mathbb{N}}$). Analytic spaces are clearly separable. Even more, they are hereditarily separable.

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[*]Universidad Politécnica de Valencia. Instituto de Matemática Pura y Aplicada. Camino de Vera, s/n, 46022 Valencia, Spain. Supported in part by MICINN and FEDER (project MTM2008-05396), by Fundación Séneca (project 08848/PI/08), by Generalitat Valenciana (GV/2010/036), and by Universidad Politécnica de Valencia (project PAID-06-09-2829). email: anguisa2@mat.upv.es

[†]Universidad Politécnica de Valencia. Instituto de Matemática Pura y Aplicada. Camino de Vera, s/n, 46022 Valencia, Spain. Supported in part by Project MICINN MTM2011-22417, Generalitat Valenciana (GV/2010/036), and by Universidad Politécnica de Valencia (project PAID-06-09-2829). email: vmontesinos@mat.upv.es.

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In [KLP, Prop. 18], and in [KKLP, Prop. 6.14], the following result is stated:

(*) *Let X be a SWCG Banach space. Then (X^*, μ^*) is analytic if, and only if, X is separable*.

In the notation above, the necessary condition follows from the fact that K_0, endowed with the restriction of the weak topology, is separable. That the condition is sufficient follows from the fact, that if X is separable and SWCG, then (B_{X^*}, μ^*) is a Polish space.
In [KLP, p. 61], and in [KKLP, p. 170], the authors raise the following question:

(Q) *Let X be a separable Banach space. Is it true that (X^*, μ^*) is an analytic space?*

We show here that the answer to this question is negative. For this purpose, it is enough to use the following result (note that a completely regular topological space T is isomorphic to a subspace of (X^*, w^*), where $(X, \|\cdot\|) := (C^b(T), \|\cdot\|_\infty)$, i.e., the space of all continuous and bounded real-valued functions on T, endowed with the supremum norm $\|\cdot\|_\infty$).

Theorem 1 (Kirk, [Kr]) *Let $(T, \mathcal{T})$ be a completely regular topological space. Then*
(i) *The topologies $\mathcal{T}$ and μ^* coincide on T if, and only if, $\mathcal{T}$ is discrete.*
(ii) *The space (T, μ^*) is totally disconnected.*
(iii) *If $\mathcal{T}$ is first countable, then μ^* on T is discrete.*

Consider now the separable Banach space $(X, \|\cdot\|) := (C(K), \|\cdot\|_\infty)$, where K is an uncountable compact metric space. According to Theorem 1, the space (K, μ^*) is discrete (and so it cannot be separable). It follows from the remark above that the space (X^*, μ^*) is not analytic. This solves in the negative question (Q). In view of the result (*) above, the space X is not SWCG.

A Banach space X has the property that (X^*, μ^*) is analytic if, and only if, (B_{X^*}, μ^*) is analytic. Indeed, every closed subspace of an analytic space is also analytic, so (B_{X^*}, μ^*) is analytic if (X^*, μ^*) is. In the other direction, it is enough to observe that a countable union of analytic subspaces of a topological space is analytic.
Note that (B_{X^*}, μ^*) is analytic if X is a separable Asplund Banach space. Indeed, the identity mapping $I : (B_{X^*}, \|\cdot\|) \to (B_{X^*}, \mu^*)$ is continuous, and $(B_{X^*}, \|\cdot\|)$ is a Polish space, since $(X^*, \|\cdot\|)$ is separable.

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