Stability study of eighth-order iterative methods for solving nonlinear equations ☆

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Abstract

In this paper, we study the stability of the rational function associated to a known family of eighth-order iterative schemes on quadratic polynomials. The asymptotic behavior of the fixed points corresponding to the rational function is analyzed and the parameter space is shown, in which we find choices of the parameter for which there exists convergence to cycles or even chaotical behavior showing the complexity of the family. Moreover, some elements of the family with good stability properties are obtained.

Keywords: Nonlinear equations, iterative methods, stability, parameter space, basin of attraction.

1. Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \mathbb{C} \to \mathbb{C}$, gives rise to rational functions whose dynamics are not well-known. The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative algorithm is Newton's scheme. This case has been widely studied in the literature (see, for instance [1, 2]). The study of the dynamics of Newton's method has been extended to other one-point iterative schemes (see for example [3, 4, 5, 6]) and to multipoint iterative methods (see for example [7, 8, 9, 10, 11, 12]), for solving nonlinear equations.

From the numerical point of view, the dynamical properties of the rational function associated with an iterative method on polynomials give us important information about its stability and reliability on any nonlinear functions. In most of the mentioned papers, interesting dynamical planes, including periodical behavior and others anomalies, have been obtained. We are interested in the parameter planes associated to a family of iterative methods, which allow us to understand the behavior of the different elements of the family, helping us in the election of a particular one with good numerical properties.

In this work, the family under study is the class of three-step eighth-order iterative methods for solving nonlinear equations, introduced by Neta et al. in [13], denoted by HK8 and whose iterative expression is

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\ t_k &= y_k - \frac{f(y_k)}{f'(x_k)}\frac{f(x_k)+\beta f(y_k)}{f(x_k)+(\beta-2)f(y_k)}, \\ x_{k+1} &= t_k - \frac{H_3(t_k)}{f'(t_k)}, \end{aligned} \tag{1}$$

where β is an arbitrary complex parameter and

$$H_3(t_n) = f(x_n)+f'(x_n)\frac{(t_n-y_n)^2(t_n-x_n)}{(y_n-x_n)(x_n+2y_n-3t_n)}+f'(t_n)\frac{(t_n-y_n)(x_n-t_n)}{x_n+2y_n-3t_n}-f[x_n,y_n]\frac{(t_n-x_n)^3}{(y_n-x_n)(x_n+2y_n-3t_n)}$$

Let us observe that the first two steps of family (1) is King's class of iterative methods. In [13], the authors claim that the optimal value of parameter β for family (1) is $\beta = 3 - 2\sqrt{2}$; on the contrary, as it was proved in [12], the scheme for $\beta = 0$ is the most stable element of King's family and corresponds to well known Ostrowski's method; we will prove that, in case of family (1), the most stable scheme corresponds also to $\beta = 0$.

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In this paper, we are going to analyze the dynamics of this set of methods when they are applied to quadratic polynomials, characterizing the stability of all fixed points. As far as we know, there exist no previous dynamical analysis of eighth-order iterative schemes. The graphical tools used to obtain the parameter planes and the different dynamical planes have been introduced by Magreñán in [14] and Chicharro et al. in [15]. The qualitative behavior on quadratic polynomials can be extrapolated to any nonlinear equation, that is, an element of the family with bad numerical properties on quadratic polynomials will have the same bad behavior on other nonlinear equations. Although all the members of family (1) have order of convergence 8, from this study we are going to conclude which of these iterative schemes have good stability properties and which of them have bad numerical behavior.

Let us consider $p(z) = (z-a)(z-b)$, an arbitrary quadratic polynomial with roots a and b and denote by $HK8(z,\beta,a,b)$ the fixed point operator corresponding to the family (1) applied to $p(z)$. P. Blanchard, in [1], by considering the conjugacy map

$$M(z) = \frac{z-a}{z-b}, \qquad M^{-1}(z) = \frac{zb-a}{z-1}, \tag{2}$$

with the following properties:

$$M(\infty) = 1, \;\; M(a) = 0, \;\; M(b) = \infty,$$

proved that, for quadratic polynomials, Newton's operator is always conjugate to the rational map z^2. In an analogous way, as it was stated in [13], $HK8(z,\beta,a,b)$ is conjugated to operator $S(z,\beta)$,

$$S(z,\beta) = (M \circ HK8 \circ M^{-1})(z) = z^8 \frac{\left((1+z)^2 + \beta(2+z)\right)^2}{(1+z(2+\beta+z+2\beta z))^2}. \tag{3}$$

We observe that parameters a and b have been obviated in $S(z,\beta)$, as an effect of the Scaling Theorem that is verified by this iterative scheme.

We will study the general convergence of methods (1) for quadratic polynomials. To be more precise (see [16, 17]), a given method is generally convergent if the scheme converges to a root for almost every starting point and for almost every polynomial of a given degree.

Now, we are going to recall some dynamical concepts of complex dynamics (see [18]) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0, R(z_0), R^2(z_0), ..., R^n(z_0), ...\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_f \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_f) = z_f$. A *periodic point* z of period $p > 1$ is a point such that $R^p(z) = z$ and $R^k(z) \neq z$, for $k < p$. A *pre-periodic point* is a point z that is not periodic but there exists a $k > 0$ such that $R^k(z)$ is periodic. A *critical point* z^* is a point where the derivative of the rational function vanishes, $R'(z^*) = 0$. Moreover, a fixed point z_f is called *attractor* if $|R'(z_f)| < 1$, *superattractor* if $|R'(z_f)| = 0$, *repulsor* if $|R'(z_f)| > 1$ and *parabolic* if $|R'(z_f)| = 1$. So, a superattracting fixed point is also a critical point. The fixed points that are not associated to the roots of the polynomial $p(z)$ are called *strange fixed points*.

On the other hand, the *basin of attraction* of an attractor $\alpha \in \hat{\mathbb{C}}$ is defined as the set of starting points whose orbits tend to α:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \; : \; R^n(z_0) \to \alpha, \; n \to \infty\}.$$

The *Fatou set* of the rational function R, $\mathcal{F}(R)$ is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complementary set in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That is, the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The rest of the paper is organized as follows: in Section 2 we analyze the fixed points of the rational function associated to the fixed point operator of family (1) and their stability; in Section 3, the critical points of operator $S(z,\beta)$ and their corresponding parameter planes are studied. The dynamical behavior of family (3) is analyzed in Section 4, by using the associated parameter spaces. We finish the work with some remarks and conclusions.

2. Study of the strange fixed points

As we have seen, the eighth-order family of iterative methods (1), applied on the generic quadratic polynomial $p(z)$, and after Möbius transformation, gives rise to the rational function (3), depending on parameter β. It is clear that this rational function has 0 and ∞ as superattracting fixed points, but also different strange fixed points (that do not correspond to the roots of $p(z)$): $s_1(\beta) = 1$ and the roots of the high-degree equation, that will be denoted by $s_i(\beta)$, $i = 2, \ldots, 11$:

$$1 + z(5+2\beta) + z^2\left(11 + 10\beta + \beta^2\right) + 5z^3\left(3 + 4\beta + \beta^2\right) + z^4(4+3\beta)^2 + z^5(4+3\beta)^2$$

$$+z^6(4+3\beta)^2+5z^7\left(3+4\beta+\beta^2\right)+z^8\left(11+10\beta+\beta^2\right)+z^9(5+2\beta)+z^{10}=0.$$

Nevertheless, the complexity of the operator can be lower depending on the value of the parameter, as we can see in the following result.

Theorem 1. *The number of strange fixed points of operator $S(z,\beta)$ is eleven (including $s_1(\beta)=1$), except in the following cases:*

- *For $\beta=0$, there do not exist strange fixed points and the expression of the operator is*
$$S(z,\beta)=z^8.$$
- *When $\beta=-\frac{4}{3}$, there are nine strange fixed points, and*
$$S(z,\beta)=z^8\left(\frac{5+3z}{3+5z}\right)^2.$$
- *For $\beta=-\frac{28}{13}$, also there are nine strange fixed points, and*
$$S(z,\beta)=\frac{z^8\left(43+2z-13z^2\right)^2}{\left(-13+2z+43z^2\right)^2}.$$

In fact, the scheme corresponding to $\beta=0$ satisfies Cayley's Test (see [19]) and it is the only element of family (1) whose operator is always conjugate to rational map z^8.

In order to analyze the stability of each one of these strange fixed points, we define the stability function of each fixed point as $St_i(\beta)=|S'(s_i(\beta),\beta)|$, for each $i=1,2,\ldots,11$: if $St_i(\beta)<1$, then the strange fixed point $s_i(\beta)$ is attractive, parabolic in case of $St_i(\beta)=1$ and repulsive in other case. In these terms, we analyze both analytically and graphically the character of all these points in the complex plane, showing for complex values of β which is the value of the corresponding stability function. It is found that $s_1(\beta)=1$ is attractive for values of β in a circular region around $\beta=-2$; for this value, $St_1(-2)=0$ and $s_1(\beta)=1$ is superattractive. This is shown in the following result.

Theorem 2. *The character of the strange fixed point $s_1(\beta)=1$ is:*

i) If $\left|\beta+\frac{500}{247}\right|<\frac{32}{247}$, then $s_1(\beta)=1$ is an attractor and it is a superattractor if $\beta=-2$.

ii) When $\left|\beta+\frac{500}{247}\right|=\frac{32}{247}$, $s_1(\beta)=1$ is a parabolic point.

iii) If $\left|\beta+\frac{500}{247}\right|>\frac{32}{247}$, then $s_1(\beta)=1$ is a repulsor.

Proof. It is easy to prove that
$$St_1(\beta)=\left|\frac{16(2+\beta)}{4+3\beta}\right|.$$
So,
$$\left|\frac{16(2+\beta)}{4+3\beta}\right|\leq 1 \quad \text{is equivalent to} \quad 16\,|2+\beta|\leq|4+3\beta|.$$
Let us consider $\beta=a+ib$ an arbitrary complex number. Then,
$$16^2\left(4+4a+a^2+b^2\right)\leq 16+24a+9a^2+9b^2.$$
By simplifying
$$247a^2+247b^2+1000a+1008\leq 0,$$
that is,
$$\left(a+\frac{500}{247}\right)^2+b^2\leq\left(\frac{32}{247}\right)^2.$$
Therefore,
$$St_1(\beta)\leq 1 \text{ if and only if } \left|\beta+\frac{500}{247}\right|\leq\frac{32}{247}.$$
It is clear that $\beta=-2$ makes $s_1(\beta)$ be superattracting. Finally, if β satisfies $\left|\beta+\frac{500}{247}\right|>\frac{32}{247}$, then $|St_1(\beta)|>1$ and $s_1(\beta)=1$ is a repulsive point. ■

In terms of Theorem 2, the region of the complex plane where $\left|\beta+\frac{500}{247}\right|\leq\frac{32}{247}$, defines the stability region of $s_1(\beta)$, that is, includes the values of β that make attracting the strange fixed point $s_1(\beta)=1$. This region is showed in Figure 1.

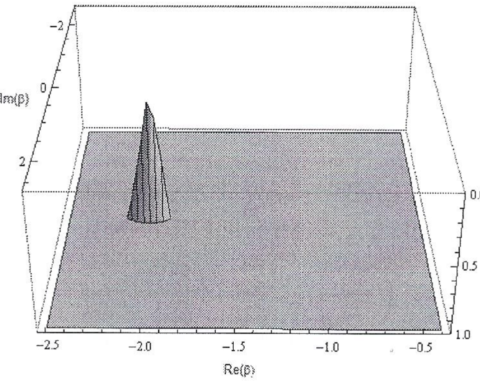
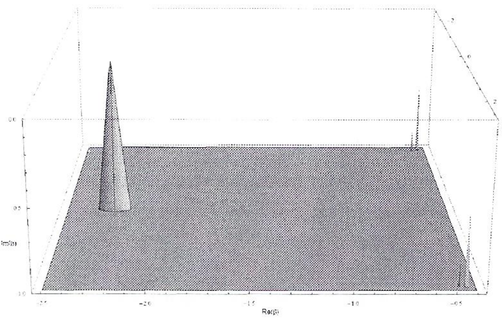
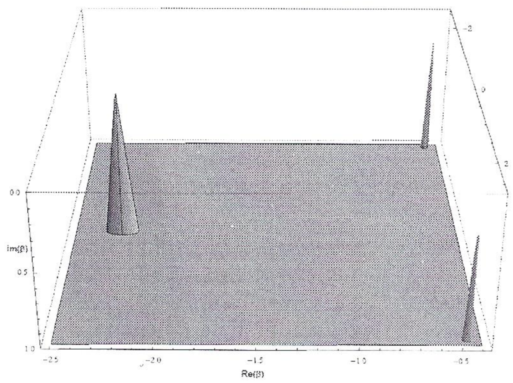


Figure 1: Stability function $St_1(\beta)$

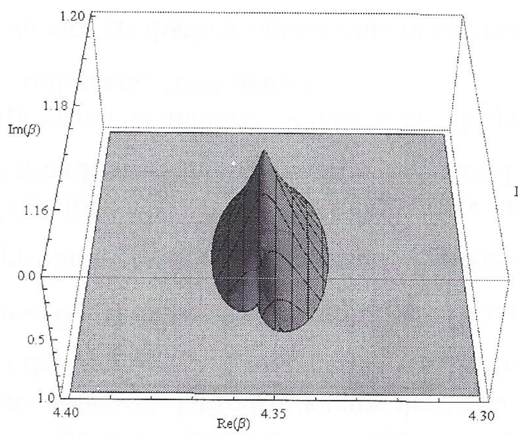
On the other hand, we have checked numerically (by using Mathematica 8.0 software) that strange fixed points $s_2(\beta)$ and $s_{11}(\beta)$ are repulsive for any value of parameter β. However, there exist some regions of the complex plane where the stability functions associated to other strange fixed points $s_3(\beta), \ldots, s_{10}(\beta)$ are lower than one, that is, where these strange fixed points are attractive or even superattractive.



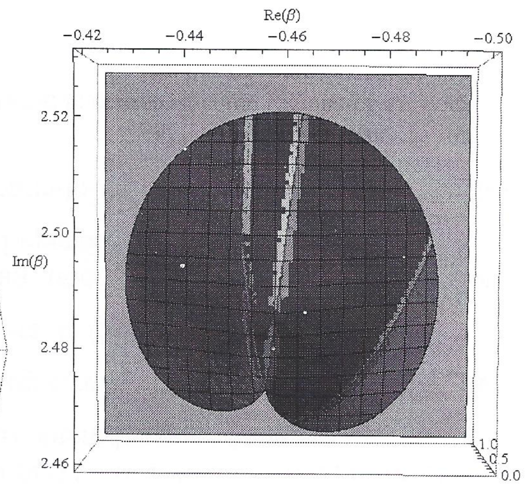
(a) $St_9(\beta)$



(b) $St_{10}(\beta)$



(c) $St_i(\beta), i = 3, 7, 8$



(d) $St_i(\beta), i = 6, 7, 8, 9, 10$

Figure 2: Common regions of stability of the strange fixed points.

In addition to $\beta = -2$, where $s_1(\beta)$ is superattracting, there exist other values of the parameter that make some strange fixed points become superattracting:

- For $\beta \approx -2.24571$, $s_9(\beta)$ and $s_{10}(\beta)$ become superattractors. In a disk surrounding this value of the parameter, the stability functions of these strange fixed points are lower than one, so both of them are attracting points in these region (see Figures 2a and 2b).
- When $\beta \approx 4.35299 \pm 1.17151i$, $s_3(\beta)$ and $s_8(\beta)$ are superattracting strange fixed points. These values of β are, respectively, inside two cardioidal areas which are the stability regions of $s_3(\beta)$, $s_7(\beta)$ and $s_8(\beta)$ (see Figure 2c).
- If $\beta \approx 0.427812$, $s_4(\beta)$ and $s_5(\beta)$ become superattractors. Around this point of the complex plane, a small cardioid is drawn by the stability functions of both strange fixed points: inside the cardioid, both are attractive and they are repulsive outside.
- For $\beta \approx 0.028512$, $s_4(\beta)$ and $s_5(\beta)$ are superattracting fixed points. Inside a small cardioid that contains $\beta \approx 0.028512$, both strange fixed points are attractive.
- When $\beta \approx -0.458294 \pm 2.48544i$, $s_8(\beta)$ and $s_{10}(\beta)$ are superattracting fixed points. These values of β are, respectively, inside two cardioids which correspond to the stability regions of $s_6(\beta)$, $s_7(\beta)$, $s_8(\beta)$, $s_9(\beta)$ and $s_{10}(\beta)$ (see Figure 2d).

In Figures 2c and 2d, different areas of the complex plane are shown where several strange fixed points are simultaneously attractive. In order to distinguish them, we have established the following code of colors:

$$\begin{array}{lll} St_3(\beta)\text{-magenta}, & St_6(\beta)\text{-red}, & St_7(\beta)\text{-yellow}, \\ St_8(\beta)\text{-green}, & St_9(\beta)\text{-purple}, & St_{10}(\beta)\text{-blue}. \end{array} \tag{4}$$

By combining the pictures of the stability regions of all these fixed points, we obtain in Figure 3 a particular view of the parameter plane, that will be defined and analyzed in the following section.

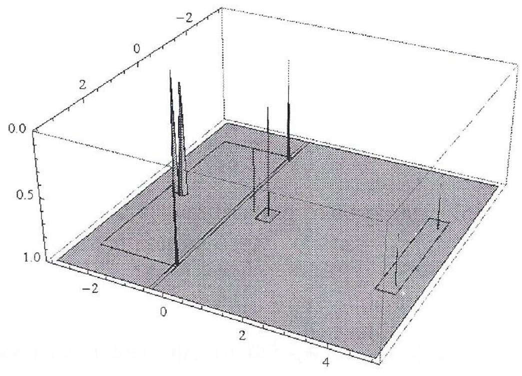


Figure 3: Stability function of all strange fixed points on the complex plane

3. Analysis of critical points

A classical result in complex dynamics (see [2]) establishes that there is at least one critical point associated with each invariant Fatou component. It is clear that $z = 0$ and $z = \infty$ (the image by the Möbius transformation of the roots of the polynomial), are critical points and give rise to their respective Fatou components, but there exist in the family some *free critical points*, that is, critical points not associated to the roots, some of them depending on the value of the parameter.

In order to calculate the critical points, we get the first derivative of $S(z,\beta)$,

$$S'(z,\beta) = \frac{4z^7(1+z)^2\left(2(1+z)^4 + 3\beta^3 z(2+z) + 4\beta(1+z)^2\left(2+z+z^2\right) + \beta^2\left(8+11z+16z^2+7z^3\right)\right)}{\left(1+(2+\beta)z+(1+2\beta)z^2\right)^3}$$

and solving the equation $S'(z,\beta) = 0$, we obtain the following result.

Theorem 3. *The critical points of operator $S(z,\beta)$ are 0, ∞, and the following free critical points:* $cr_1(\beta) = -1$,

$$cr_2(\beta) = \frac{1}{2}\left(-2 - \sqrt{-4+\beta}\sqrt{\beta} - \beta\right),$$

$$cr_3(\beta) = \frac{1}{2}\left(-2 + \sqrt{-4+\beta}\sqrt{\beta} - \beta\right),$$

$$cr_4(\beta) = \frac{-4 - 2\beta - 3\beta^2 - \sqrt{3}\sqrt{-16\beta - 12\beta^2 + 4\beta^3 + 3\beta^4}}{4(1+2\beta)}$$

and

$$cr_5(\beta) = \frac{-4 - 2\beta - 3\beta^2 + \sqrt{3}\sqrt{-16\beta - 12\beta^2 + 4\beta^3 + 3\beta^4}}{4(1+2\beta)}.$$

Some specific cases must be described:

a) If $\beta = 0$, there is no free critical points.

b) When $\beta = -2$, then $cr_2(\beta) = -cr_3(\beta) = -\sqrt{3}$ and $cr_4(\beta) = cr_5(\beta) = 1$ which is a superattracting strange fixed point.

c) For $\beta = -\frac{4}{3}$, the only free critical points are $cr_1(\beta) = -1$ and $cr_3(\beta) = -\frac{5}{3}$.

d) If $\beta = -\frac{1}{2}$, then $cr_1(\beta) = -1$ is not a critical point, $cr_2(\beta) = 0$, $cr_3(\beta) = -\frac{3}{2}$, $cr_4(\beta) = \frac{1}{108}\left(-107 - i\sqrt{215}\right)$ and $cr_5(\beta) = \frac{1}{108}\left(-107 + i\sqrt{215}\right)$.

e) When $\beta = 2$, there exist only three free critical points, as $cr_1(\beta) = cr_4(\beta) = cr_5(\beta) = -1$.

e) For $\beta = 4$, there are only two free critical points, as $cr_2(\beta) = cr_3(\beta) = cr_4(\beta) = -3$ and $cr_5(\beta)$ is not a critical point.

Let us note that $cr_2(\beta) = \frac{1}{cr_3(\beta)}$ and $cr_4(\beta) = \frac{1}{cr_5(\beta)}$ so there exist, at most, three independent free critical points.

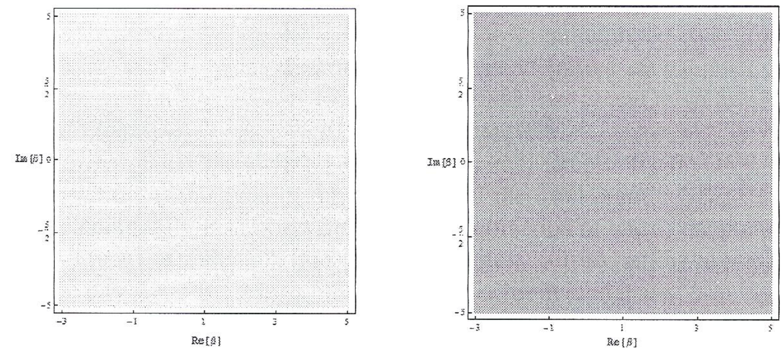
4. Parameter space

In order to deep in the dynamical behavior of operator $S(z,\beta)$, now we are going to study the parameter space associated to each independent free critical point. This task is developed by means of associating each point of the parameter plane with a complex value of parameter β, i.e., with an element of family. Specifically, we want to find regions of the parameter plane as much stable as possible, because in that regions we find the best members of the family in terms of stability.

As $cr_2(\beta) = \dfrac{1}{cr_3(\beta)}$ and $cr_4(\beta) = \dfrac{1}{cr_5(\beta)}$, their behavior will be complementary and so we only have to study one of each pair. Then, we will study $cr_1(\beta)$, $cr_2(\beta)$ and $cr_4(\beta)$. In Figures 4a, 4b and 5a the parameter planes associated to $cr_1(\beta)$, $cr_2(\beta)$ and $cr_4(\beta)$, respectively, are shown. Focussing the attention in the code color of parameter planes, we paint a point in cyan if the iteration of the method starting in an independent free critical point converges to the fixed point 0 (related to the root a), in magenta if it converges to ∞ (related to the root b) and in yellow if the iteration converges to 1 (related to ∞, that is, to divergence).

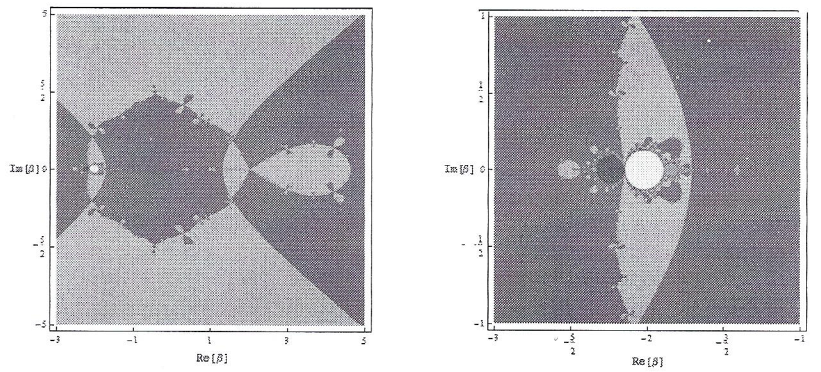
Let us observe that parameter planes shown in Figure 4 are extremely simple: the first one, corresponding to $cr_1(\beta) = -1$ appears fully painted in yellow as this point is the pre-image of the fixed point $s_1(\beta) = 1$; on the other hand, Figure 4b indicates that free critical point $cr_2(\beta)$ is always in the basin of attraction of one of the points associated to the roots of polynomial $p(z)$. So, we will focus our attention on the parameter plane corresponding to $cr_4(\beta)$. In it, the main problem resides in the zone where the parameter is close to -2. If we make a zoom in that area (see Figure 5b) we obtain a region of values of β for which the iteration of cr_4 converges to different cycles. We paint in light green the convergence to 2-cycles, in orange the convergence to 3-cycles, in dark blue to 4-cycles, in dark green to 5-cycles, dark yellow to 6-cycles, dark red to 7-cycles and in white the convergence to 8-cycles. The zones in black correspond to zones of convergence to other cycles.

In Figure 5b we observe two large disks (the yellow one is denoted by D_1 and the red one by D_2): D_1 corresponds to values of β for which $z = 1$ is attractive or superattractive (see Theorem 2) and D_2 is the region where strange fixed points $s_9(\beta)$ and $s_{10}(\beta)$ can be attractive or simultaneously superattractive. Indeed, there are bulbs of different colors surrounding D_1 and D_2; for these values of the parameter β, cycles of different periods appear.



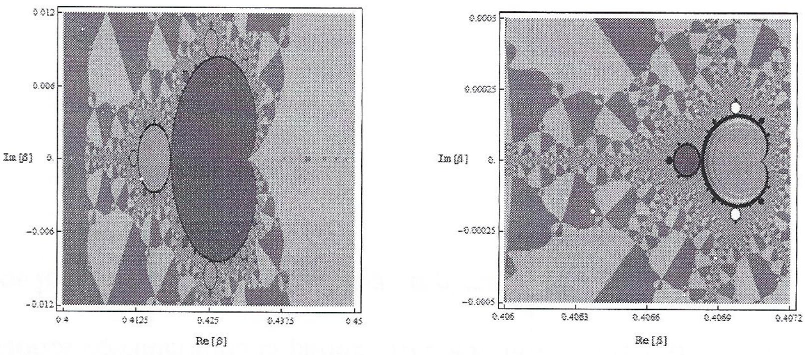
(a) $cr_1(\beta)$ (b) $cr_2(\beta)$

Figure 4: Parameter space associated to free critical points $cr_1(\beta)$ and $cr_2(\beta)$.



(a) $cr_4(\beta)$ (b) A detail on $cr_4(\beta)$

Figure 5: Parameter space associated to free critical point $cr_4(\beta)$ and a detail.



(a) A detail (b) A detail

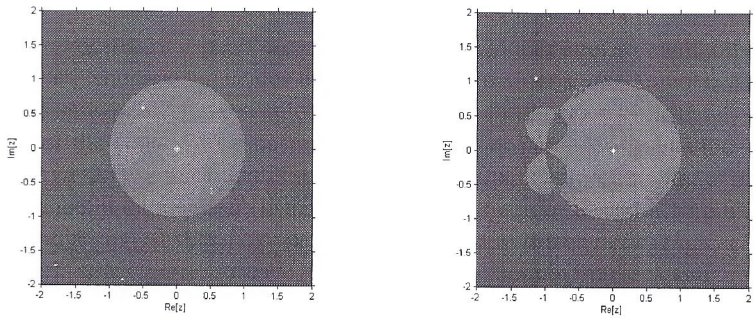
Figure 6: Parameter space associated to $cr_4(\beta)$ for values of β in which there are stability problems.

Moreover, in Figure 6 two details of Figure 5a can be seen, where Mandelbrot sets with different colors in the antennas of the parameter plane appear. The color indicates the convergence to strange fixed points (red color in Figure 6a) or the order of the different cycles shown (clear green color in Figure 6b as third period cycles).

For the representation of the convergence basins of every iterative procedure (dynamical planes) we have used the software described in [15]. We draw a mesh with four hundred points per axis; each point of the mesh is a different initial estimation which we introduce in the method. If the scheme reaches one of the attracting fixed points (being or not the roots of the original polynomial) in less than forty iterations, this point is drawn in different colors, depending on the fixed point that the iterative process converges to (orange and blue for 0 and ∞, respectively, and green, red, ... for strange fixed points). These attracting points are marked in the figures by white stars. The color will be more intense when the number of iterations is lower. Otherwise, if the method arrives at the maximum of iterations, the point will be drawn in black.

In Figure 7, the dynamical plane of the iterative methods for $\beta = 0$ (Figure 7a) and $\beta = 3 - 2\sqrt{2}$ (Figure 7b) appear.

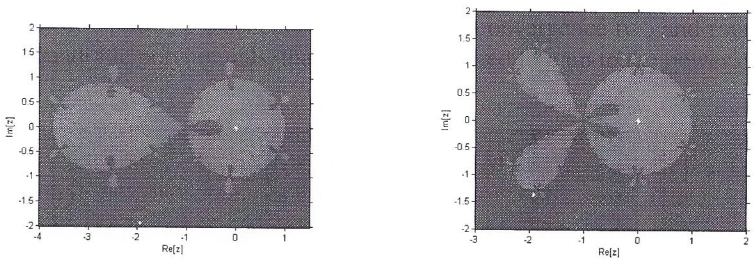
Both cases correspond to iterative schemes without stability problems. In spite of this, clearly $\beta = 0$ gives us a more stable scheme than the one of $\beta = 3 - 2\sqrt{2}$, as it is the only case that satisfies Cayley's test. This behavior is exactly the same as the one of Newton' or Ostrowski's schemes, with order of convergence two and four, respectively. All of them satisfy the Cayley Test on quadratic polynomials, that is, are conjugated to z up to the power of the order of convergence.



(a) $\beta = 0$ (b) $\beta = 3 - 2\sqrt{2}$

Figure 7: Dynamical planes for values of β in which there are not stability problems.

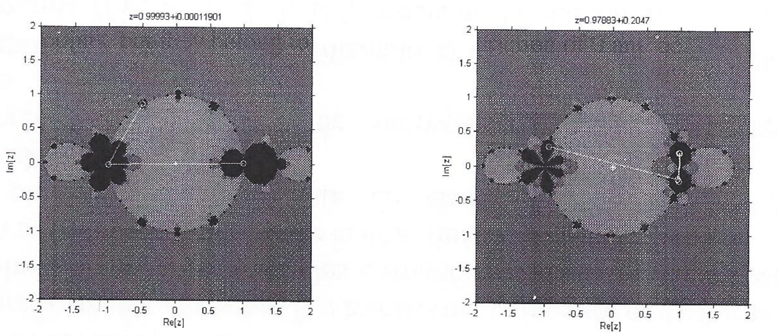
In Figure 8a, the dynamical plane of the iterative method corresponding to $\beta = 4$ is presented where there are only two free critical points (see Theorem 3). In Figure 8b, we see the stable behavior corresponding to $\beta = 2$. In this case, there are three free critical points, but they belong to the basins of attraction of 0 and ∞.



(a) $\beta = 4$ (b) $\beta = 2$

Figure 8: Dynamical planes for values of β in which there are less critical points.

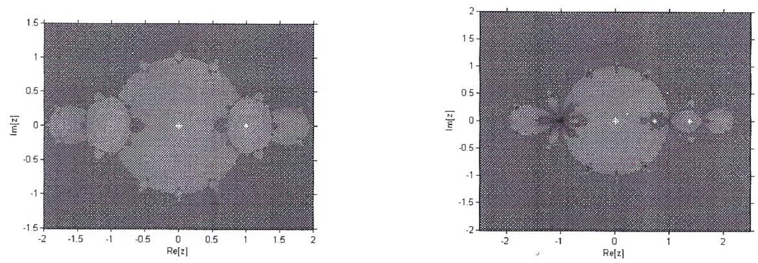
In Figure 9, different kinds of unstable behavior can be found: Figure 9a corresponds to $\beta = -\frac{28}{13}$; for this value of the parameter (see Theorem 2), $s_1(\beta) = 1$ is parabolic and then it can be found in the Julia set but it holds its own basin of attraction (in black in the figure). We have plotted in yellow the orbit of a point in this basin of attraction. In Figure 9b the dynamical plane associated to a value of β in the orange bulb at the right of D_1 (see Figure 6b) can be observed: in this case, the basin of attraction of a 2-periodic orbit appear and the trajectory of one point in this area is drawn in yellow.



(a) $\beta = -\frac{28}{13}$ (b) $\beta = -1.85$

Figure 9: Dynamical planes for values of β in which there are unstable behavior.

In Figure 10a, the dynamical plane of the iterative method corresponding to $\beta = -2$ (in disk D_1) is presented where there are only three free critical points (see Theorem 3); one of them coincides with $s_1(\beta) = 1$, so it is superattracting (see Theorem 2). In Figure 10b we see the dynamical planes associated to $\beta = -2.25$ (in disk D_1) for which strange fixed points $s_9(\beta)$ and $s_{10}(\beta)$ are attractive.



(a) $\beta = -2$ (b) $\beta = -2.25$

Figure 10: Dynamical planes for values of β in disks D_1 and D_2.

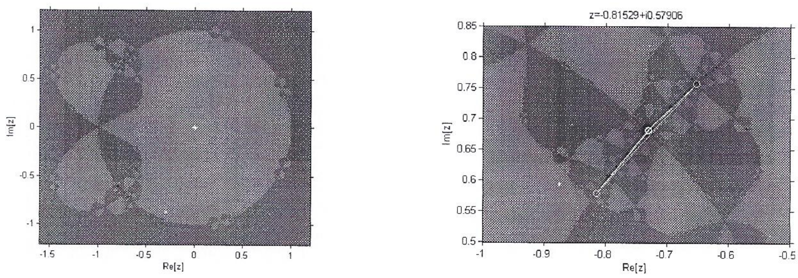
Finally, the dynamical plane for a particular value of $\beta = 0.407$ belonging to the green Mandelbrot set in Figure 6b, is shown. We can observe in Figure 11 the existence of four basins of attraction, two of them associated to the roots of $p(z)$ and the other to two attracting periodic orbits of period 3

$$\{-0.652774 - 0.757552i, -0.731625 - 0.681707i, -0.815288 - 0.579055i\}$$

and

$$\{-0.652774 + 0.757552i, -0.731625 + 0.681707i, -0.815288 + 0.579055i\}.$$

The second one is plotted in yellow at Figure 11b. This is an important fact, as Sharkovsky's theorem [2] states that the existence of orbits of period 3 guaranties orbits of any period.



(a) $\beta = 0.407$ (b) Detail on $\beta = 0.407$

Figure 11: Dynamical planes for values of β in a Mandelbrot set.

5. Conclusions

In this paper, we analyzed the dynamical behavior of family HK8 introduced in [13] in terms of convergence and stability. We have proved that there exist different values of the parameter that are not appropriate choices under the numerical point of view. Nevertheless, there are wide regions in parameter space whose corresponding iterative methods have a good numerical behavior, in terms of stability. Moreover, we have presented the parameter planes associated to free critical point $cr_4(\beta)$ and the dynamical planes for different members of the family in which it is shown there exist convergence to different cycles or to strange fixed points. Furthermore, we have seen that as the authors stated in [12] the case $\beta = 0$ is, also in the eighth-order family, the most stable member of the family due to Cayley's Test is satisfied.

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