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Additional Information

THE MIXED GENERAL ROUTING POLYHEDRON

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Abstract

In Arc Routing Problems, ARPs, the aim is to find on a graph a minimum cost traversal satisfying some conditions related to the links of the graph. Due to restrictions to traverse some streets in a specified way, most applications of ARPs must be modeled with a mixed graph. Although several exact algorithms have been proposed, no polyhedral investigations have been done for ARPs on a mixed graph. In this paper we deal with the Mixed General Routing Problem which consists of finding a minimum cost traversal of a given link subset and a given vertex subset of a mixed graph. A formulation is given that uses only one variable for each link (edge or arc) of the graph. Some properties of the associated polyhedron and some large families of facet-inducing inequalities are described. A preliminary cutting-plane algorithm has produced very good lower bounds over a set of 100 randomly generated instances of the Mixed Rural Postman Problem. Finally, applications of this study to other known routing problems are described.

Key Words: Polyhedral Combinatorics, Facets, Routing, Arc Routing, Rural Postman Problem, General Routing Problem, Mixed Chinese Postman Problem.

1   Introduction

Arc Routing Problems, ARPs, have their origin in the celebrated Königsberg Bridge Problem solved by Euler and consist, basically, of finding a set of routes over the arcs (or edges) of a graph satisfying certain restrictions. There are many real situations in which such a set of routes is required: collection or delivery of goods, mail distribution, network maintenance (electrical lines or gas mains inspection), school

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bus transportation, garbage collection, etc. For this reason, a big number of Arc Routing Problems have been studied in the last 30 years. Papers by Eiselt, Gendreau & Laporte (1995a,1995b), by Assad & Golden (1995) and the recent book edited by Dror (2000) survey the state of the art and real-life applications of the ARPs. In most applications, there are some restrictions for the vehicle to traverse some streets in a specified way, and the problem needs to be modeled with a mixed graph. As a rule, one-way streets are represented by arcs and two-way streets by edges. We will use the word link to refer to both an edge or an arc.

In this paper we study the General Routing Problem defined on a mixed graph, the Mixed General Routing Problem (MGRP), that we define next:

Given a strongly connected mixed graph $G = (V, E, A)$ with vertex set V, edge set E, arc set A, a cost c_e for each $e \in E \cup A$, a set $E_R \subseteq E$ of *required edges*, a set $A_R \subseteq A$ of *required arcs* and a set $V_R \subseteq V$ of *required vertices*, the *Mixed General Routing Problem* (MGRP) is the problem of finding a minimum cost vehicle route passing through each $e \in E_R \cup A_R$ and through each $i \in V_R$ at least once.

The MGRP has several well known routing problems as special cases. Given an undirected and connected graph G=(V,E) with nonnegative costs associated with its edges, the Chinese Postman Problem (CPP) consists of finding a minimum cost tour (a closed walk) traversing, at least once, all the edges in E. This well known problem is solvable in polynomial time (Edmonds & Johnson, 1973). However, if the tour has to traverse only a subset, E_R, of the set of edges E and this subset induces in G a disconnected graph, the problem, called the Rural Postman Problem (RPP), turns out to be NP-hard (Lenstra & Rinnooy-Kan, 1976). Furthermore, if the condition for the tour is to traverse both a given subset of edges $E_R \subset E$ and a given subset of vertices $V_R \subset V$, the problem is then called the General Routing Problem (GRP) and it is also NP-hard. On the other hand, if the CPP is defined over a mixed graph, the problem of traversing all the arcs and edges in G (the Mixed Chinese Postman Problem, MCPP) is again NP-hard (Papadimitriou, 1976).

Although the MGRP is not strictly an ARP, it is a generalization of both the Mixed RPP (MRPP) and the Graphical Asymmetric TSP (GATSP). The MRPP contains the (undirected) Rural Postman Problem and the Mixed Chinese Postman Problem as particular cases and, thus, is a NP-hard problem. The Graphical Asymmetric TSP consists of finding a minimum cost tour on a strongly connected directed graph $G = (V, A)$ traversing each vertex of G at least once. This problem, recently studied by Chopra and Rinaldi (1996), is clearly a special case of MGRP when $E = \emptyset$, all the arcs are non-required and all the vertices are required.

The existing optimal approaches for ARP's on mixed graphs are related to the MCPP. Christofides et al. (1984) present a formulation for the MCPP in which a variable is associated with each arc, two variables with each edge (representing the number of times the edge is traversed in each direction) and another variable with each

vertex. Then, a branch and bound algorithm was implemented that solved 34 small size instances. Grötschel & Win (1992) propose a cutting-plane algorithm to solve the Windy Postman Problem (WPP), the problem of finding a minimum cost tour traversing all edges of an undirected graph with two costs associated with each edge (the cost of traversing an edge depends on the direction of travel). The WPP, that contains the MCPP as a special case, is also formulated using two variables associated with each edge. Nine medium size MCPP instances were solved to optimality with this cutting-plane algorithm. The formulations in both papers are based on transforming the original graph into a directed graph that is eulerian if symmetry conditions on the vertices of the graph are satisfied. A somewhat different approach that uses only one variable associated with each edge was proposed by Nobert & Picard (1996). It is based on the characterization of an eulerian mixed graph given by Ford & Fulkerson (1962): *a strongly connected mixed graph is eulerian if and only if the degree of each node is even (*evenness condition*) and for every proper subset of vertices S, the number of arcs entering S minus the number of arcs leaving S is less than or equal to the number of edges between S and V \ S (*balanced sets condition*).* Nobert & Picard present a cutting-plane algorithm that was able to solve 148 instances out of 180 randomly generated instances with sizes up to 169 vertices, 2876 arcs and 1849 edges.

Since there is reasonably strong evidence that the last approach is computationally superior to the previous one, we chose to formulate the MGRP in this way. In Section 2 we present the formulation of the problem and in Section 3 we study its associated polyhedron. Some basic inequalities that are valid for the polyhedron and that, under certain conditions, define facets of it, are presented in Section 4: trivial, connectivity, balanced-set and R-odd cut constraints. In Sections 5 and 6, two big families of inequalities, Path Bridge and Path Bridge$_{02}$ inequalities, are presented and shown to be facet inducing of the MGRP polyhedron. Some computational results are presented in Section 7, where a preliminary cutting-plane algorithm, based on a part of the given description of the MGRP polyhedron, has produced very good lower bounds over a set of 100 randomly generated instances of the MRPP. Finally, in Section 8 we apply the results from this study to other known Routing Problems.

2   Problem definition and notation

Note that if $i \in V$ is a vertex incident to any required link $e \in E_R \cup A_R$, the condition on the tour passing through edge e contains the condition of visiting vertex i. Therefore, in the following, we will assume that V_R contains the set of vertices incident to the required edges.

As it is usual when the subgraph induced by the required vertices and links is not connected, we first transform the original graph in order to simplify both the problem structure and the formulation. This transformation is done in a similar way to that of Christofides et al. (1981) for the undirected RPP (see Eiselt, Gendreau & Laporte, 1995b, or Christofides et al., 1986, for an illustration of the procedure):

1. Add to $G_R = (V_R, E_R \cup A_R)$ an arc between every pair of vertices in V_R having a cost equal to the shortest path length on G if such a path does not uses required links.

2. Delete: a) one of every two arcs in parallel if they have the same cost, b) every arc parallel to an edge with the same cost, and c) all arcs $e = (i,j) \notin E_R$ such that $c_{ij} = c_{ik} + c_{kj}$ for some vertex k.

This transformation can occasionally increase the number of edges in G, but it frequently decreases it. Notice that all vertices in the simplified graph are either required or incident to a required link and that $E \setminus E_R = \emptyset$. Hence, we can assume that we work with a (simplified) strongly connected graph $G = (V, E, A) := (V_R, E_R, A_R \cup A_{NR})$.

Let $G^R = G(V, E, A_R)$ be the graph obtained by deleting in G all the non required arcs. In general, graph G^R is not connected. Let p be the number of connected components of G^R and let $V_1, V_2, \ldots, V_p$ be the vertex sets corresponding to the p connected components of G^R, which will be called R-sets, with $V_1 \cup \ldots \cup V_p = V$. We will represent by $C_i = G(V_i)$, $i = 1, \ldots, p$, the subgraphs of G induced by the R-sets and they will be referred to as R-connected components. Notice that every isolated required vertex is a R-connected component of G. Given two disjoint sets of vertices S_1, $S_2 \subset V$ and a set $S \subset V$, we will use the following notation:

$$(S_1 : S_2) = \{(i,j) \in E \cup A : i \in S_1, j \in S_2 \quad or \quad i \in S_2, j \in S_1\}$$
$$A(S_1 : S_2) = \{(i,j) \in A : i \in S_1, j \in S_2\}$$
$$E(S_1 : S_2) = (S_1 : S_2) \cap E$$
$$\delta(S) = (S : V \setminus S) \text{ (called link cut-set of } G \text{ defined by } S)$$
$$\gamma(S) = \{(i,j) \in E \cup A : i, j \in S\}$$
$$A^+(S) = A(S : V \setminus S)$$
$$A^-(S) = A(V \setminus S : S)$$
$$A(S) = A^+(S) \cup A^-(S)$$
$$E(S) = E(S : V \setminus S)$$

The above sets are defined in a similar way referring to required links only and to non-required links only: $(S_1 : S_2)_R$, $A_{NR}(S_1 : S_2)$, $A_R^+(S)$, $A_{NR}^-(S)$, $\delta_R(S)$, etc. Given $x \in \mathbb{R}^{E \cup A}$ and given $T \subset E \cup A$, $x(T)$ denotes $\sum_{e \in T} x_e$.

A *tour for the MGRP* is a family $\mathcal{F}$ of links of G such that the graph $(V, E^{\mathcal{F}} \cup A^{\mathcal{F}})$ is unicursal (also called eulerian) and contains all the required links, where $E^{\mathcal{F}} \cup A^{\mathcal{F}}$ is obtained by considering each copy of a link in $\mathcal{F}$ as a different element. Applying sufficient conditions for a connected mixed graph to be unicursal (Ford and Fulkerson, 1962), we can state that $\mathcal{F}$ is a tour for the MGRP when the following conditions are satisfied:

- $\mathcal{F}$ contains all the required links.

- Graph $(V, E^{\mathcal{F}} \cup A^{\mathcal{F}})$ is even, i.e., every vertex is incident to an even number of links.

- Graph $(V, E^{\mathcal{F}} \cup A^{\mathcal{F}})$ is connected.

- Graph $(V, E^{\mathcal{F}} \cup A^{\mathcal{F}})$ is balanced, i.e., for every $S \subset V$, the difference between the number of arcs in $A^+(S)$ (leaving S) and the number of arcs in $A^-(S)$ (entering S) is less than or equal to the number of edges in $E(S)$ (*balanced-set condition* for set S).

We formulate the MGRP with respect to semitours. We call the family of links obtained from any tour for the MGRP by deleting one copy of every link in $E \cup A_R$ a *semitour for the MGRP*. We associate with each tour (semitour) an integer *incidence vector* $x = (x_e : \quad e \in E \cup A) \in I\!R^{E \cup A}$, where x_e denotes the number of times a link $e \in E \cup A$ appears in the tour (semitour). For notational convenience, we will use also the word tour (semitour) to denote this vector. Let x^R denotes the incidence vector of required links. Then, x is a semitour for the MGRP on G if, and only if, $x + x^R$ is a tour. Vector $x + x^R$ will be referred to as the corresponding tour of a given semitour x. A vertex $v \in V$ will be called *R-odd* if it is incident to an odd number of required links, otherwise it will be called *R-even*. Note that every isolated required vertex is R-even. Given $S \subset V$, let $u_S = |A_R^+(S)| - |A_R^-(S)| - |E(S)|$. It is easy to see that the set of semitours for the MGRP is the set of vectors $x \in I\!R^{E \cup A}$ satisfying:

$$
\begin{align}
x_e \geq 0 \ \ and \ \ integer, &\quad \forall e \in E \cup A \tag{1} \\
x(\delta(\{i\})) \equiv 0 \bmod 2, &\quad \forall i \in V \ : \ v \ is \ R-even \tag{2} \\
x(\delta(\{i\})) \equiv 1 \bmod 2, &\quad \forall i \in V \ : \ v \ is \ R-odd \tag{3} \\
x(A^+(S)) \geq 1, &\quad \forall S = \cup_{k \in Q} V_k, \ \ Q \subset \{1, \ldots, p\} \tag{4} \\
-x(A^+(S)) + x(A^-(S)) + x(E(S)) \geq u_S, &\quad \forall S \subset V \tag{5}
\end{align}
$$

where conditions (2) and (3), (4) and (5) force the graph represented by the tour to be, respectively, even, connected and balanced.

Let us describe briefly constraints (5) (see figure 1, where the required links are represented in solid lines and the non required in dash lines). Given that $u_S = 1$, constraint (5) can be expressed as $x(A^-(S)) + x(E(S)) \geq 1 + x(A^+(S))$. It could be understood that we must add, at least, $u_S = 1$ link in the set $A^-(S) \cup E(S)$ in order to obtain a balanced graph. If we consider now $\bar{S} = V \setminus S$, then $u_{\bar{S}} = |A_R^-(S)| - |A_R^+(S)| - |E(S)| = -3$ and the corresponding constraint (5) can be expressed as $x(A^-(S)) \leq 3 + x(A^+(S)) + x(E(S))$. It means that we can not add more than 3 links in $A^-(S)$, if we do not add also links in $A^+(S) \cup E(S)$.

Consider now a set $S \subset V$ such that $E(S) = \emptyset$. Then $u_S = -u_{\bar{S}}$ and constraints (5) corresponding to S and $\bar{S}$ imply the so called *symmetry equation* $x(A^+(S)) + u_S = x(A^-(S))$. Hence, system (1) to (5) includes an equation associated with each set $S \subset V$ with $E(S) = \emptyset$, most of them linearly dependent. In order to discuss this, let us represent by $K_1, K_2, \ldots, K_q$ the sets of vertices of the connected components of the graph (V, E). Some sets K_i could consist in a single vertex. Each set K_i is contained
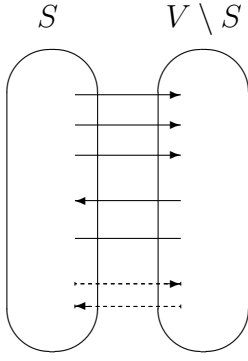
$$u_S=1: \quad x(A^-(S)) + x(E(S)) \geq 1 + x(A^+(S))$$

$$u_{\bar{S}}=-3: \quad x(A^-(S)) \leq 3 + x(A^+(S)) + x(E(S))$$

Figure 1: Some examples of balanced set constraints.

in a set V_j. We will call the q subgraphs of G induced by sets K_i *edge-connected components* of G. The q equations

$$x(A^+(K_i)) + u_{K_i} = x(A^-(K_i)), \quad i = 1, 2, \ldots, q \tag{6}$$

will be referred as *the system equations*. It can be shown that any $q - 1$ of them are linearly independent.

3 Basic Properties

Let MGRP(G) be the convex hull in $\mathbb{R}^{E \cup A}$ of all the semitours for the MGRP on G, i.e., of all vectors $x \in \mathbb{R}^{E \cup A}$ satisfying (1) to (5). It is not hard to see that MGRP(G) is an unbounded polyhedron. In order to study the dimension of MGRP(G), we will use the following result, due to Chopra & Rinaldi (1996). Let GATSP(G) be the convex hull of the tours for the GATSP on a directed graph G. Then,

Lemma 1 (Chopra & Rinaldi, 1996) *Let $G = (V, A)$ be a directed graph. If G is strongly connected, dim(GATSP(G)) = $|A| - |V| + 1$*

Theorem 1 *dim(MGRP(G)) = $|E \cup A| - q + 1$, where q is the number of edge-connected components of G.*

Proof: As every semitour satisfies equations (6) and $q - 1$ of them are linearly independent, dim(MGRP(G)) $\leq |E \cup A| - q + 1$ holds. In order to prove the equality, we will find $|E \cup A| - q + 2$ affinely independent semitours for the MGRP on G.

Let $F = A \setminus (\gamma(K_1) \cup \ldots \cup \gamma(K_q))$ be the set of arcs joining vertices from two different sets K_i, K_j. Let G_s be the graph obtained by shrinking node sets K_i, $i = 1, \ldots, q$ into a single vertex each. Graph G_s is a strongly connected directed graph with q vertices and with an arc corresponding to each arc in F. Note that G_s can have parallel arcs. From lemma 1, dim(GATSP(G_s))= $|F| - q + 1 := m$ and hence we can find $m + 1$ affinely independent tours for the GATSP on G_s. If we add to these tours the incidence

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vector of a cycle on G_s traversing all the arcs corresponding to the required arcs in F, we obtain $m + 1$ new tours for the GATSP on G_s that are also affinely independent. Now, it is easy to transform these tours by adding edges and arcs in $\gamma(K_1) \cup \ldots \cup \gamma(K_q)$ in order to obtain $m + 1$ tours for the MGRP on G. Let $x^0, x^1, \ldots, x^m$ denote their corresponding $m + 1$ semitours.

For each edge $e \in \gamma(K_1) \cup \ldots \cup \gamma(K_q)$, consider now the semitour obtained by adding to x^0 (for instance) two copies of e. For each arc $a \in \gamma(K_i)$, $i = 1, \ldots, q$, consider the semitour obtained by adding to x^0 a copy of a and a copy of each edge in a path in $G(K_i)$ using only edges (such a path exists because of the definition of K_i). Then we have $|\gamma(K_1) \cup \ldots \cup \gamma(K_q)|$ more semitours for the MGRP on G.

Consider the matrix obtained by expressing the semitours as rows and the links as columns. Subtracting the row corresponding to x^0 from the remaining rows, we obtain the matrix in figure 2, where I is the identity matrix and B is the matrix, with rank m, corresponding to the tours for the GATSP. Then, the rank of the full matrix is $| \cup \gamma(K_i)| + m = | \cup \gamma(K_i)| + |F| - q + 1 = |E \cup A| - q + 1$ and we have found $|E \cup A| - q + 2$ affinely independent semitours for the MGRP on G. ♦

$(\cup \gamma(K_i)) \cap A$	$(\cup \gamma(K_i)) \cap E$	F
I	*	*
0	2I	*
0	0	B

Figure 2: Matrix appearing in the proof of theorem 1

Given that MGRP(G) is not full-dimensional, different inequalities can induce the same facet. Such inequalities are called *equivalent*. We next study conditions for inequalities (1) without the integrality condition, which will be called *trivial inequalities*, to be facet-inducing. A link $e \in E \cup A$ will be called a cut-link of G if, after removing e from G, the resulting graph is not strongly connected.

Theorem 2 (Trivial inequalities) : *Inequalities $x_e \geq 0$, $\forall e \in E \cup A$, are facet-inducing for MGRP(G) if e is not a cut-link of G (the condition is also necessary for links $e \in E \cup A_{NR}$).*

Proof: The proof is simple and is omitted here for the sake of brevity. ♦

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Before studying the remaining inequalities in the formulation, we describe some general results that will be useful in what follows.

Let $G = (V, E, A)$ be a mixed graph. A *configuration* $\mathcal{C}$ on G is a triplet $(\mathcal{B}, c^E, c^A)$, where $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$ is a partition of V, c^E and c^A are real functions defined on $\mathcal{B} \times \mathcal{B}$ and the following conditions are satisfied:
($\mathcal{C}$-a) Every subgraph $G(B_i)$ is strongly connected.
($\mathcal{C}$-b) $c^E(B_i, B_j) = c^E(B_j, B_i) > 0$ for every pair $B_i \neq B_j$ such that $E(B_i : B_j) \neq \emptyset$
($\mathcal{C}$-c) There is no closed cycle $B_i, B_j, \ldots, B_m, B_i$ with total c-cost negative.

Associated with a configuration $\mathcal{C}$ on G we have a *configuration graph* $G_\mathcal{C}$ having node set $\mathcal{B}$, a required edge (B_i, B_j) for each required edge (u, v) of G with $u \in B_i$, $v \in B_j$, a required arc (B_i, B_j) for each required arc (u, v) of G with $u \in B_i$, $v \in B_j$ and a non required arc (B_i, B_j) for each pair B_i, B_j such that $A_{NR}(B_i : B_j) \neq \emptyset$. In other words, $G_\mathcal{C}$ is the graph resulting from shrinking node sets B_i, $i = 1, \ldots, r$ into a single vertex each and, after that, shrinking each set of non required parallel arcs into one single arc, but keeping all the required edges and arcs. A configuration $\mathcal{C}$ defines a *configuration inequality*, $\sum_{e \in E \cup A} c_e x_e \geq c_0$, where:

(a) $c_e = 0$ for every $e \in \gamma(B_i)$, $i = 1, \ldots, r$
(b) $c_e = c^E(B_i, B_j)$ for every edge $e \in E(B_i : B_j)$
(c) $c_a = c^A(B_i, B_j)$ for every arc $a \in A(B_i : B_j)$
(d) c_0 is the c-length of the shortest semitour for the MGRP on $G_\mathcal{C}$ (and on G).

Notice that it is possible to have $c^A(B_i, B_j) \neq c^A(B_j, B_i) \neq c^E(B_i, B_j)$ and also $c^A(B_i, B_j) < 0$ (see connectivity and balanced-set inequalities (4) and (5)).

Theorem 3 *All facet-inducing inequalities for MGRP(G), except those equivalent to trivial ones, are configuration inequalities.*

Proof: The proof is similar to that in Naddef and Rinaldi (1991) for the Graphical TSP, and is omitted here for the sake of brevity. ♦

The following results are two 'lifting' theorems that state conditions for a given configuration inequality which is facet-inducing for MGRP($G_\mathcal{C}$) to be also facet-inducing for MGRP(G). Given a mixed graph $G = (V, E, A)$, $q(G)$ will denote the number of edge-connected components of G, i.e., the number of connected components of the graph (V, E).

Theorem 4 *Let G be a mixed graph and let $\mathcal{C}$ be a configuration on G. The associated configuration inequality is facet-inducing for MGRP(G) if the configuration inequality associated with $\mathcal{C}$ on graph $G_\mathcal{C}$ is facet-inducing for MGRP($G_\mathcal{C}$) and the following condition is satisfied:*

$$q(G) = q(G_\mathcal{C}) + \sum_{i=1}^{r} (q(G(B_i)) - 1) \qquad (L1)$$

Proof: By induction, it suffices to prove the result for a graph G obtained from $G_{\mathcal{C}}$ by replacing node B_1 (for example) by a strongly connected graph $G(B_1)$. For simplicity, $F(x) \geq c_0$ will denote both the inequality on G and the inequality on $G_{\mathcal{C}}$. We first show validity. Consider any semitour x for the MGRP on G. Shrinking $G(B_1)$ into a single vertex we obtain a semitour x' for the MGRP on $G_{\mathcal{C}}$ satisfying $F(x) = F(x') \geq c_0$.

Let $K = \dim(\text{MGRP}(G_{\mathcal{C}}))$ and let us suppose that $c_0 \neq 0$. Then, there exists K linearly independent semitours for the MGRP on $G_{\mathcal{C}}$ satisfying $F(x) = c_0$, say $x^1, x^2, \ldots, x^K$. If $c_0 = 0$ there would be only $K - 1$ semitours, but the proof would be similar. After replacing B_1 by the graph $G(B_1)$, some non required arcs may appear between the vertices of $G(B_1)$ and nodes B_i, $i \neq 1$. Let us call $G'_{\mathcal{C}}$ the graph obtained from $G_{\mathcal{C}}$ by adding Q non required arcs $e'_1, e'_2, \ldots, e'_Q$, where each e'_i is parallel to a non required arc (not necessarily distinct) e_i, $i = 1, 2, \ldots, Q$, of $G_{\mathcal{C}}$. For each e'_i, let x^j be the semitour for the MGRP on $G_{\mathcal{C}}$ traversing e_i and consider the semitour $x^j \setminus \{e_i\} \cup \{e'_i\}$. Then, we have $K + Q$ linearly independent semitours for the MGRP on $G'_{\mathcal{C}}$ satisfying $F(x) = c_0$. Given that $G(B_1)$ is strongly connected, each of them can be completed with links of $G(B_1)$ in order to obtain semitours for the MGRP on G also satisfying $F(x) = c_0$. Let us call them $x^1, x^2, \ldots, x^{K+Q}$.

Consider the MGRP defined on graph $G(B_1)$, let $M = \dim(\text{MGRP}(G(B_1)))$ and let $y^1, y^2, \ldots, y^M$ be M linearly independent semitours for the MGRP on $G(B_1)$. Then, $x^1 + y^1, x^1 + y^2, \ldots, x^1 + y^M$ are also semitours for the MGRP on G satisfying $F(x) = c_0$. Expressing the $K + Q + M$ semitours as rows of a matrix and subtracting the first row x^1 from the last M rows, we obtain a full rank matrix. Hence, we have found $K + Q + M$ linearly independent semitours for the MGRP on G satisfying $F(x) = c_0$. The proof will be completed when $\dim(\text{MGRP}(G)) = K + Q + M$ is showed.

Given that graphs $G_{\mathcal{C}}$ and $G'_{\mathcal{C}}$ have the same set of edges, it holds that $q(G_{\mathcal{C}}) = q(G'_{\mathcal{C}})$ and condition (L1) can be expressed as

$$-q(G) = -q(G'_{\mathcal{C}}) - q(G(B_1)) + 1 \tag{7}$$

On the other hand, if $nlink(G)$ represents the number of links on graph G, we have $nlink(G) = nlink(G'_{\mathcal{C}}) + nlink(G(B_1))$ and, from equation (7), $nlink(G)\text{-}q(G)\text{+}1 = nlink(G'_{\mathcal{C}})\text{-}q(G'_{\mathcal{C}}) + 1 + nlink(G(B_1)) \text{ - } q(G(B_1)) + 1$, i.e.,
$\dim(\text{MGRP}(G)) = \dim(\text{MGRP}(G'_{\mathcal{C}}) + \dim(\text{MGRP}(G(B_1)) = K + Q + M$. $\blacklozenge$

Note: Condition (L1) is always satisfied when $G_{\mathcal{C}}$ is a directed graph: after replacing a node B_i by a graph $G(B_i)$, each edge-connected component in $G(B_i)$ is also an edge-connected component in the resulting graph G, and $q(G) = q(G_{\mathcal{C}}) + q(G(B_i)) - 1$ holds. For example, graphs $G_{\mathcal{C}}$ and G shown in figure 3 (the edge-conected components of each $G(B_i)$ are represented by black circles), satisfy $q(G_{\mathcal{C}}) = 3$ and $q(G) = 6 = 3 + (2 - 1) + (3 - 1)$.

Nevertheless, when $G_{\mathcal{C}}$ is a mixed graph, after replacing a node B_i by a graph $G(B_i)$, different edge-connected components in $G(B_i)$ can be connected by edges in the resulting graph G and $q(G) < q(G_{\mathcal{C}}) + q(G(B_i)) - 1$ would hold. In such a case, condition (L1) would not be satisfied. For example, graphs $G_{\mathcal{C}}$ and G in figure 4 satisfy $q(G_{\mathcal{C}}) = 1$ and $q(G) = 3 \neq 1 + (2 - 1) + (3 - 1)$, while those in figure 5 satisfy condition (L1) since $q(G_{\mathcal{C}}) = 1$ and $q(G) = 4 = 1 + (2 - 1) + (3 - 1)$. However, note that if $G_{\mathcal{C}}$ has

edges only between two given nodes, say B_i and B_j, condition (L1) is satisfied if all the edges link the same edge-connected component of $G(B_i)$ to the same edge-connected component of $G(B_j)$.
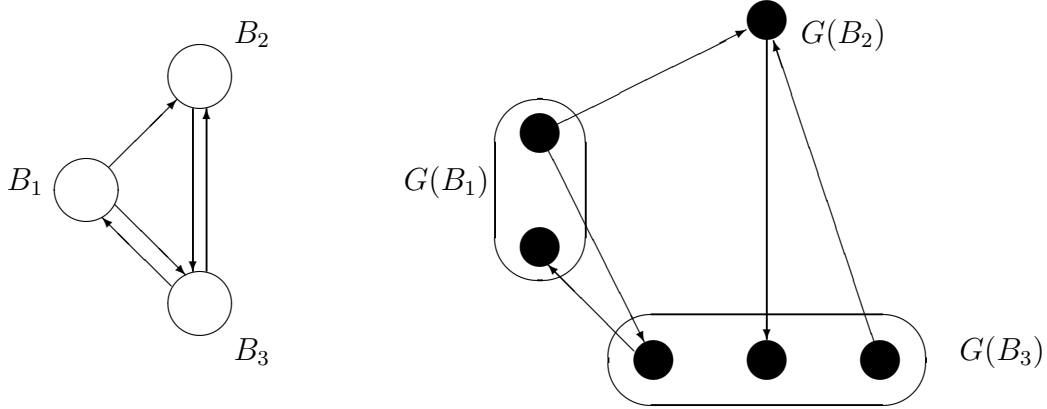

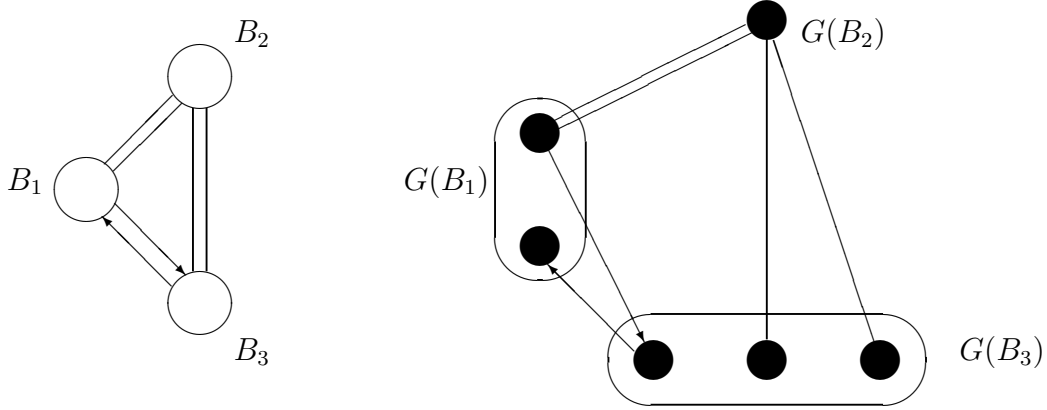Figure 3: (L1) is satisfied when $G_{\mathcal{C}}$ is a directed graph.

Figure 4: $G_{\mathcal{C}}$ is a mixed graph and (L1) is not satisfied.

When condition (L1) is not satisfied, the following second lifting theorem will be tried.

Theorem 5 *Let G be a mixed graph and let $\mathcal{C}$ be a configuration on G. The associated configuration inequality is facet-inducing for MGRP(G) if the configuration inequality $F(x) \geq c_0$ associated with $\mathcal{C}$ on graph $G_{\mathcal{C}}$ is facet-inducing for MGRP($G_{\mathcal{C}}$) and the following conditions are satisfied:*

L2(a) *Given any proper subset of edges in $G_{\mathcal{C}}$, it is possible to replace each edge by an arc in such a way that inequality $F(x) \geq c_0$ is also facet-inducing for the MGRP polyhedron associated with the resulting graph, $G'_{\mathcal{C}}$.*

L2(b) *Given any of the previous graphs $G'_{\mathcal{C}}$ and given any arc $\mathbf{a}$ replacing an edge in $G_{\mathcal{C}}$, if we change arc $\mathbf{a}$ to an arc with the opposite direction, there exists at least one semitour x^* for the MGRP on the resulting graph satisfying $F(x^*) = c_0$*
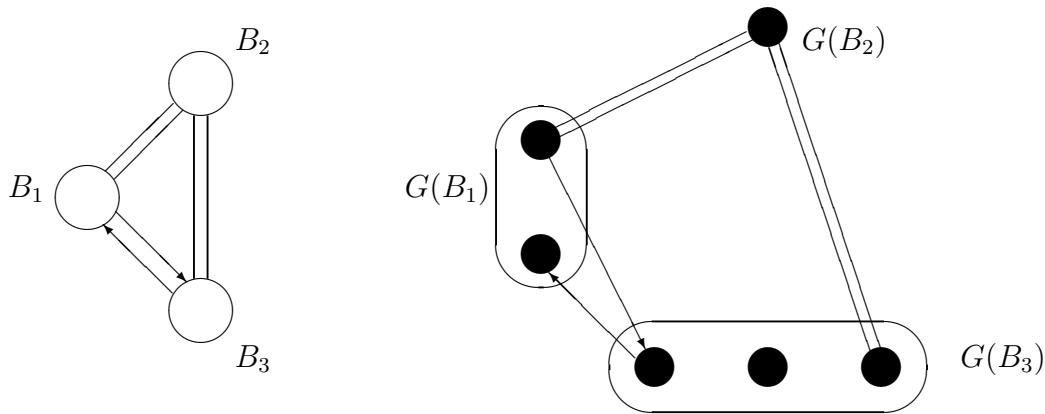
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Figure 5: $G_\mathcal{C}$ is a mixed graph and (L1) is satisfied.

L2(c) *Graph $G_\mathcal{C}$ has edges only between two given nodes, say B_1 and B_2.*

Proof: Again, $F(x) \geq c_0$ will denote both the inequality on G and on $G_\mathcal{C}$. Let us call $\omega = q(G_\mathcal{C}) + \sum_{i=1}^{r} \big(q(G(B_i))-1\big)$ - $q(G) \geq 0$, where $q(G)$ is defined as before. We will proceed by induction on the value of ω. If $\omega = 0$ then condition (L1) is satisfied and, from theorem 4, $F(x) \geq c_0$ is facet inducing of MGRP(G). Assume that the result is true for every pair of graphs $G, G_\mathcal{C}$ with $\omega \leq n - 1$ and let $G, G_\mathcal{C}$ be graphs with $\omega = n \geq 1$.

As, from L2(c), the only edges of graph $G_\mathcal{C}$ link B_1 and B_2, if $\omega \geq 1$ not all the edges in G are linking the same edge-connected component of $G(B_1)$ to the same edge-component of $G(B_2)$, i.e., if we consider the bipartite graph in figure 6 having node sets formed by the edge-connected components of $G(B_1)$ and $G(B_2)$, say $\{S_1, S_2, \ldots, S_m\}$ and $\{T_1, T_2, \ldots, T_n\}$, and having an edge (S_i, T_j) for each $E(S_i : T_j) \neq \emptyset$, then it has, at least, two edges. Consider a node that it is not a cut-vertex of this graph, say S_1 in $G(B_1)$. By construction, there exists another node, say S_2, also in $G(B_1)$, which is incident to some edges. If we delete from G and from $G_\mathcal{C}$ the edges incident to S_1
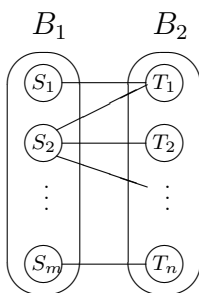
Figure 6: Bipartite graph used in the proof of theorem 5

(transforming them into arcs, for example), the resulting graphs G' and $G'_\mathcal{C}$ satisfy, respectively, $q(G') = q(G)+1$ (because S_1 is not a cut-vertex) and $q(G'_\mathcal{C}) = q(G_\mathcal{C})$ (because $E(S_2 : B_2) \neq \emptyset$ and $G_\mathcal{C}$ has edges only between B_1 and B_2). Hence, we have two graphs G' and $G'_\mathcal{C}$ with $\omega = n - 1$. From L2(a), we can assign a direction to

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each edge in $E(S_1)$ in such a way that inequality $F(x) \geq c_0$ is facet-inducing for the MGRP polyhedron associated with the resulting graph, $G'_\mathcal{C}$. From the hypothesis of induction, $F(x) \geq c_0$ is also facet-inducing for the MGRP polyhedron associated with G'. Let us assume that $c_0 \neq 0$ (otherwise, the proof would be similar). Therefore, there exists $nlink(G')$ - $q(G')$ + 1 linearly independent semitours for the MGRP on G' satisfying $F(x) = c_0$. Given that these semitours are also semitours for the MGRP on G and given that $nlink(G') = nlink(G)$ and $q(G')=q(G)+1$, we have $nlink(G)$ - $q(G)$ semitours and only one more tour is needed for $F(x) \geq c_0$ to be facet-inducing for MGRP(G).

Let e be one of the edges of $G_\mathcal{C}$ in $E(S_1)$ that had been changed into an arc in $G'_\mathcal{C}$. Let $G"_\mathcal{C}$ be the graph obtained by changing the arc e into an arc with the opposite direction. From L2(b), there exists a semitour x^* for the MGRP on $G"_\mathcal{C}$ satisfying $F(x^*) = c_0$. Given that $G(B_i)$ are strongly connected graphs, we can complete x^* with links of $\cup \gamma(B_i)$ to obtain a semitour for the MGRP on G, also called x^*, satisfying $F(x^*) = c_0$. Finally, in order to show that x^* is not a linear combination of the previous $nlink(G)$ - $q(G)$ semitours, we only have to notice that these last semitours satisfy the equation $x(A^+(S_1)) + |A^+(S_1)| = x(A^-(S_1)) + |A^-(S_1)|$ considered in graph G', whilst x^* satisfies equation $x(A^+(S_1) \setminus \{e\}) + |A^+(S_1)| - 1 = x(A^-(S_1)) + x_e + |A^-(S_1)| + 1$ also considered in graph G'. ♦

4 Basic inequalities

Let us begin studying inequalities (4) and (5), which are valid for MGRP(G) and will be referred to as connectivity and balanced-set inequalities, respectively.

4.1 Connectivity inequalities

Theorem 6 : *Inequalities (4), $x(A^+(S)) \geq 1$, $\forall S = \cup_{k \in Q} V_k$, $Q \subset \{1, \ldots, p\}$, are facet-inducing for MGRP(G) if graphs $G(S)$ and $G(V \setminus S)$ are strongly connected.*

Proof: The configuration graph $G_\mathcal{C}$ has only two nodes, say 1 and 2, corresponding to S and to $V \setminus S$, respectively. As $E(S) = A_R(S) = \emptyset$, vertices 1 and 2 are joined by a pair of opposite non required arcs. Therefore, dim(MGRP$(G_\mathcal{C})$)$= 2 - 2 + 1 = 1$ and, since the semitour $x_{1,2} = x_{2,1} = 1$ satisfies $x^+(S) = x_{1,2} = 1$, the inequality is facet-inducing for MGRP$(G_\mathcal{C})$. Given that $(S : V \setminus S)$ does not contain edges, condition $(L1)$ of the first lifting theorem 4 is satisfied and the inequality is also facet-inducing for MGRP(G). ♦

Even if $G(S)$ is not strongly connected, there may exist a subset $S_0 \subset S$ such that $G(S_0)$ is strongly connected, $S_0 = \cup_{k \in Q'} V_k$, $Q' \subset Q$ and $A^+(S) = A^+(S_0)$. In this case, inequality (4) is facet inducing. Otherwise, it can be shown that inequality (4) is not facet inducing because it is dominated by a balanced-set inequality or by another connectivity inequality.

4.2 Balanced-set inequalities

Theorem 7 : *Let $S \subset V$ be such that $G(S)$ and $G(V \setminus S)$ are strongly connected. The balanced-set inequality (5), $-x(A^+(S)) + x(A^-(S)) + x(E(S)) \geq u_S$, where $u_S = |A_R^+(S)| - |A_R^-(S)| - |E(S)|$, is facet-inducing for MGRP(G) if and only if the following conditions are satisfied:*

(i) $A^+(S) \neq \emptyset$
(ii) $E(S) \neq \emptyset$
(iii) $q(G) = q(G(S)) + q(G(V \setminus S))$ - 1

Proof: *The conditions are sufficient:* Consider the corresponding graph $G_{\mathcal{C}}$ having only two vertices, say 1 and 2, a number $|A_R^+(S)|$ of required (parallel) arcs from 1 to 2, a number $|A_R^-(S)|$ of required arcs from 2 to 1, a number $|E(S)|$ of edges and, possibly, a pair of opposite non required arcs. Condition (ii) implies that $q(G_{\mathcal{C}})=1$ and then (iii) implies condition (L1) of the first lifting theorem. Hence, it suffices to show the result on graph $G_{\mathcal{C}}$, by finding an appropriate number of affinely independent semitours satisfying (5) to equality. This is equivalent to find $|E \cup A|$ affinely independent tours satisfying

$$x(A^+(1)) = x(A^-(1)) + x(E(1))$$

Notice that, in such tours, edges have to be traversed from vertex 2 to 1. Hence, in what follows, we will not differentiate between edges (in $E(1)$) and arcs in $A^-(1)$. Let us call $K = |A^+(1)|$ and $M = |E(1) \cup A^-(1)|$. We study two cases:

(a) $K = M$. Consider the tour x^* defined as $x_e^* = 1$ for all $e \in A^+(1) \cup A^-(1) \cup E(1)$. Let now e_1 be a given link in $A^-(1) \cup E(1)$. For each arc $e \in A^+(1)$, let x^e be defined as $x_e^e = x_{e_1}^e = 2$ and $x_a^e = 1$ for all $a \neq e, e_1$. Let now a_1 be a given arc in $A^+(1)$. For each link $e \in E(1) \cup A^-(1)$, let x^e be defined as $x_e^e = x_{a_1}^e = 2$ and $x_a^e = 1$ for all $a \neq e, a_1$. We have defined $|E \cup A| + 1$ different tours satisfying $x(A^+(1)) = x(A^-(1)) + x(E(1))$. Again, by arranging them as the rows of a matrix (figure 7(a)) and subtracting the row 1 (associated with x^*) from all the remaining rows, we obtain a matrix with rank $|E \cup A| - 1$. Thus, $|E \cup A|$ of these tours are affinely independent.

(b) $K \neq M$. Let us assume, for example, that $K < M$ and denote $\sigma = M - K + 1 \neq 1$. For each arc $e \in A^+(1)$, let x^e be defined as $x_e^e = \sigma$ and $x_a^e = 1$ for all $a \neq e$. Let a_1 be a given arc in $A^+(1)$. For each link $e \in E(1) \cup A^-(1)$, let x^e be defined as $x_e^e = 2$, $x_{a_1}^e = \sigma + 1$ and $x_a^e = 1$ for all $a \neq e, a_1$. These $|E \cup A|$ tours satisfy $x(A^+(1)) = x(A^-(1)) + x(E(1))$. Again, if we arrange them as the rows of a matrix (figure 7(b)) and we subtract the first row from all the remaining rows, we obtain a matrix with rank $|E \cup A| - 1$.

The conditions are necessary:
(i) If $A^+(S) = \emptyset$, inequality (5) can be written as $x(E(S)) + x(A^-(S)) \geq -|A_R^-(S)| - |E(S)|$, that it is obviously dominated by the sum of several trivial inequalities.
(ii) If $E(S) = \emptyset$, inequality (5) becomes $x(A^+(S)) + u_S \leq x(A^-(S))$ which is satisfied to equality (symmetry condition) by all the semitours for the MGRP on G.

	$A^+(1)$				$A^-(1) \cup E(1)$			
x^*	1	1	…	1	1	1	…	1
	2	1	…	1	2	1	…	1
	1	2	…	1	2	1	…	1
			⋱				…	1
	1	1	…	2	2	1	…	1
	2	1	…	1	2	1	…	1
	2	1	…	1	1	2	…	1
			…				⋱	
	2	1	…	1	1	1	…	2

(a)

	$A^+(1)$			$A^-(1) \cup E(1)$			
σ	1	…	1	1	1	…	1
1	σ	…	1	1	1	…	1
	⋱					…	1
1	1	…	σ	1	1	…	1
$\sigma+1$	1	…	1	2	1	…	1
$\sigma+1$	1	…	1	1	2	…	1
	…				⋱		
$\sigma+1$	1	…	1	1	1	…	2

(b)

Figure 7: Matrices appearing in the proof of theorem 7

(iii) If $q(G) < q(G(S)) + q(G(V \setminus S))$ - 1, there exists a cut-set $(S_0 : S_1)$ in $G(S)$ (or in $G(V \setminus S)$) such that $E(S_0 : S_1) = \emptyset$ and $E(S_0 : V \setminus S) \neq \emptyset \neq E(S_1 : V \setminus S)$. It can be shown that the balanced-set inequality corresponding to set S is obtained as the sum of the balanced-set inequalities corresponding to sets S_0 and S_1. ♦

Note: Let $S \subset V$ be such that there exists a cut-set $(S_0 : S_1)$ in $G(S)$ with $E(S_0 : S_1) = \emptyset$, $E(S_0 : V \setminus S) \neq \emptyset$ and $E(S_1 : V \setminus S) = \emptyset$. As before, it can be shown that the balanced-set inequality corresponding to S is the sum of the balanced-set inequality corresponding to S_0 and the equation $x(A^+(S_1)) + u_{S_1} = x(A^-(S_1))$. Hence, balanced-set inequalities associated with S and S_0 are equivalent and, in practice, we could only consider balanced-set inequalities corresponding to sets S such that $S \subset K_i$ for any edge-connected component K_i of G. Notice also that, unlike the connectivity inequalities, the balanced-set inequalities corresponding to S and $V \setminus S$ are not equivalent.

4.3 R-odd cut inequalities

A cut-set $\delta(S)$ will be called R-odd if it contains an odd number of required links or, equivalently, if both S and $V \setminus S$ contain an odd number of R-odd vertices. Any tour must cross any given link cut-set an even number of times. This means that if $\delta(S)$ is R-odd, then the tour must traverse all their required links and, at least, one more link. Hence, any semitour must use, at least, one link in each R-odd cut-set and the following inequalities, which will be called R-odd cut inequalities, are valid for MGRP(G):

$$x(\delta(S)) \geq 1, \ \forall \delta(S) \ \ R-odd \ \ cut-set \tag{8}$$

Theorem 8 : *Let $\delta(S)$ be an R-odd cut-set of G such that $G(S)$ and $G(V \setminus S)$ are strongly connected. The R-odd cut inequality (8), $x(\delta(S)) \geq 1$, is facet-inducing for MGRP(G) if and only if $|\alpha - \beta| < \epsilon$, where $\alpha = |A_R^+(S)|$, $\beta = |A_R^-(S)|$ and $\epsilon = |E(S)|$*

Proof: *The condition is necessary. If $|\alpha - \beta| = \epsilon$, then $\alpha + \beta + \epsilon = |\delta_R(S)|$ is not odd. If $|\alpha - \beta| > \epsilon$, the R-odd cut inequality is dominated by a balanced-set inequality. To show*

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this, let us assume that $\alpha \geq \beta$. Then, $\alpha - \beta > \epsilon$ and therefore $\alpha - \beta - \epsilon = u_S \geq 1$. From the balanced-set inequality associated with S, $x(A^-(S)) + x(E(S)) \geq x(A^+(S)) + u_S \geq 1$ holds and dominates (8).
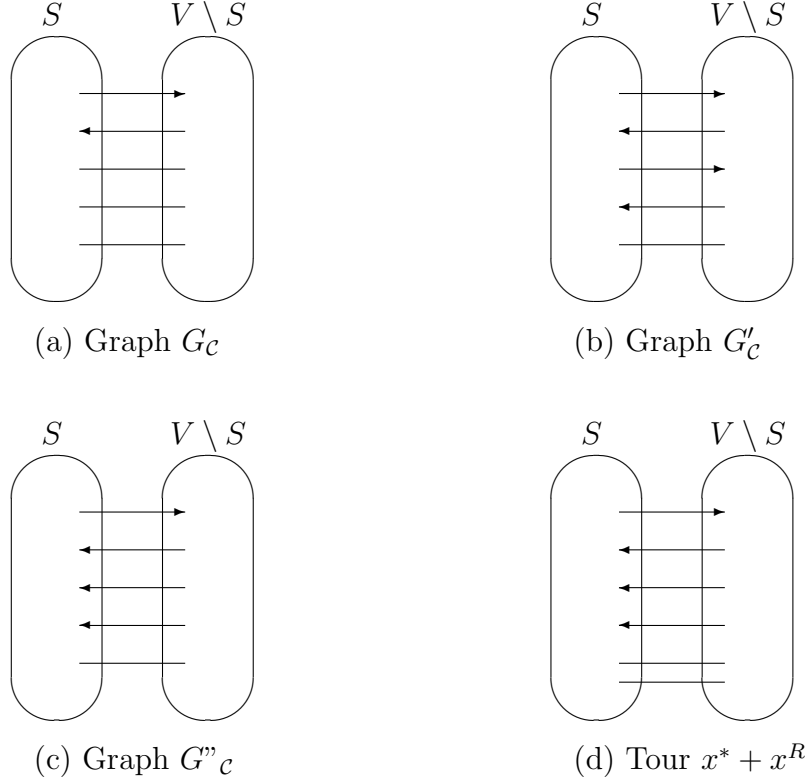

(a) Graph $G_{\mathcal{C}}$

(b) Graph $G'_{\mathcal{C}}$

(c) Graph $G"_{\mathcal{C}}$

(d) Tour $x^* + x^R$

Figure 8: Graphs $G_{\mathcal{C}}$, $G'_{\mathcal{C}}$ and $G"_{\mathcal{C}}$ and tour $x^* + x^R$ used in the proof of theorem 8.

The condition is sufficient. We first show the result for the graph $G_{\mathcal{C}}$. This graph has two vertices, say 1 and 2, a number α of required (parallel) arcs from 1 to 2, a number β of required arcs from 2 to 1, a positive number ϵ of edges and, possibly, a pair of opposite non required arcs. Hence, $\dim(\text{MGRP}(G_{\mathcal{C}})) = nlink(G_{\mathcal{C}})$. Condition $\epsilon \geq |\alpha - \beta| + 1$ implies that there are enough edges in $G_{\mathcal{C}}$ to balance a copy of every required arc and even one more. Hence, the following vectors are semitours for the MGRP on $G_{\mathcal{C}}$:

- For each link e (required or not), let x^e be defined as $x^e_e = 1$ and $x^e_a = 0$ for each link $a \neq e$.

We have defined a number $nlink(G_{\mathcal{C}})$ of linearly independent semitours satisfying $x(\delta(S)) = 1$. Hence, the R-odd cut inequality is facet-inducing for $\text{MGRP}(G_{\mathcal{C}})$, provided that $\epsilon \geq |\alpha - \beta| + 1$. In order to show that it is also facet-inducing for $\text{MGRP}(G)$, we will prove that condition (L2) of the second lifting theorem 5 is satisfied:
- L2(a) Given that $\epsilon \geq |\alpha - \beta| + 1$ holds in $G_{\mathcal{C}}$, for any proper subset of edges in $G_{\mathcal{C}}$ it will always be possible to assign a direction to each edge in such a way that $\epsilon' \geq |\alpha' - \beta'| + 1$ is also satisfied in the resulting graph $G'_{\mathcal{C}}$ (see figure 8). Hence, as it has been proved before, the R-odd cut inequality is facet-inducing for $\text{MGRP}(G'_{\mathcal{C}})$.

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- L2(b) Let us now assume that in one of the previous graphs $G'_\mathcal{C}$ with $\epsilon' = |\alpha' - \beta'| + 1$ we change the direction of one arc obtaining a graph $G''_\mathcal{C}$ where $\epsilon'' = |\alpha'' - \beta''| - 1$ in the worst case. Consider the semitour x^* on $G''_\mathcal{C}$ defined as $x_e^* = 1$, where e is an arbitrary edge and $x_a^* = 0$ for the remaining links $a \neq e$ (see figure 8). This semitour satisfies $x(\delta(S)) = 1$ and $x^* + x^R$ satisfies the balanced-set condition, since it has $\epsilon'' + 1$ edges and a difference in arcs of $|\alpha'' - \beta''|$, with $\epsilon'' + 1 \geq |\alpha'' - \beta''|$.
- L2(c) Obvious. ♦

If $G(S)$ (or $G(V \setminus S)$) is not connected, the R-odd cut inequality $x(\delta(S)) \geq 1$ is not facet-inducing: let $G(S_0)$ be a connected component of $G(S)$ such that $\delta(S_0)$ is also R-odd; then, the R-odd cut inequality corresponding to S is dominated by the R-odd cut inequality corresponding to S_0. If $G(S)$ is connected but it is not strongly connected, R-odd cut inequalities may be or not facet-inducing.

5 Path-Bridge inequalities

Path-Bridge inequalities were introduced by Letchford (1997) for the undirected GRP. In this section we will generalize these inequalities to the MGRP.

A *Path-Bridge configuration* (figure 9) will be defined by two integers P and B with $P \geq 1$, $B \geq 0$, $P + B \geq 3$ and odd, by $n_i \geq 2$ integers, $i = 1, 2, \ldots, P$, by a partition of V into subsets $\{A, Z, M_j^i : i = 1, 2, \ldots P, \ j = 1, 2, \ldots n_i\}$ and by two cost functions. The partition must satisfy that each R-set V_i is contained in exactly one of the node sets $A \cup Z$, M_j^i, $i = 1, 2, \ldots P$, $j = 1, 2, \ldots n_i$ (i.e., each required link either lies in some $\gamma(M_j^i)$ or crosses from A to Z), the induced subgraphs $G(M_j^i)$, $i = 0, 1, \ldots, P$, $j = 1, 2, \ldots n_i + 1$, are strongly connected (where, for convenience, for all i we identify M_0^i with A and $M_{n_i+1}^i$ with Z) and $(A : Z)$ contains a number B of required links. Cost functions are defined as:

$$c^A(A, Z) = c^A(Z, A) = c^E(A, Z) = 1$$

$$c^A(M_j^i, M_q^i) = \frac{|j - q|}{n_i - 1}, \quad \forall j, q \in \{0, 1, 2, \ldots, n_i + 1\}, \ \ 0 < |j - q| < n_i + 1$$

$$c^A(M_j^i, M_q^r) = \frac{1}{n_i - 1} + \frac{1}{n_r - 1} + \left| \frac{j - 1}{n_i - 1} - \frac{q - 1}{n_r - 1} \right|,$$

$$\forall i, r \in \{1, 2, \ldots, P\}, \ i \neq r, \ j \in \{1, 2, \ldots, n_i\}, \ q \in \{1, 2, \ldots, n_r\}$$

Furthermore, we will assume that:

- Sets $A(M_j^i : M_{j+1}^i)$ and $A(M_{j+1}^i : M_j^i)$, $i = 1, 2, \ldots, P$, $j = 0, 1, \ldots, n_i$, are nonempty. We will denote the set of such arcs by $\mathcal{E}x$ (external arcs).
- Sets $A(M_j^i : M_q^i)$, $i = 1, 2, \ldots, P$, $|j - q| > 1$, $\{j, q\} \neq \{0, n_i + 1\}$, could be empty or not. We will denote the set of such arcs by $\mathcal{I}n$ (internal arcs).
- Sets $A(M_j^i : M_q^r)$, $i, r \in \{1, 2, \ldots, P\}$, $i \neq r$, $j \in \{1, 2, \ldots, n_i\}$, $q \in \{1, 2, \ldots, n_r\}$, could be empty or not. We will denote the set of such arcs by $\mathcal{C}r$ (crossing arcs).

The partition $\mathcal{B} = \{A, Z, M_j^i : i = 1, 2, \ldots P, \; j = 1, 2, \ldots n_i\}$ and the functions c^A, c^E define the configuration graph $G_{\mathcal{C}}$ whose skeleton is showed in figure 9. It has P paths from A to Z, each of them with $n_i + 2$ nodes and $n_i + 1$ pairs of opposite external arcs. Internal arcs (M_j^i, M_q^i), not represented in figure 9, have a cost equal to the length of the shortest path from M_j^i to M_q^i using external arcs.
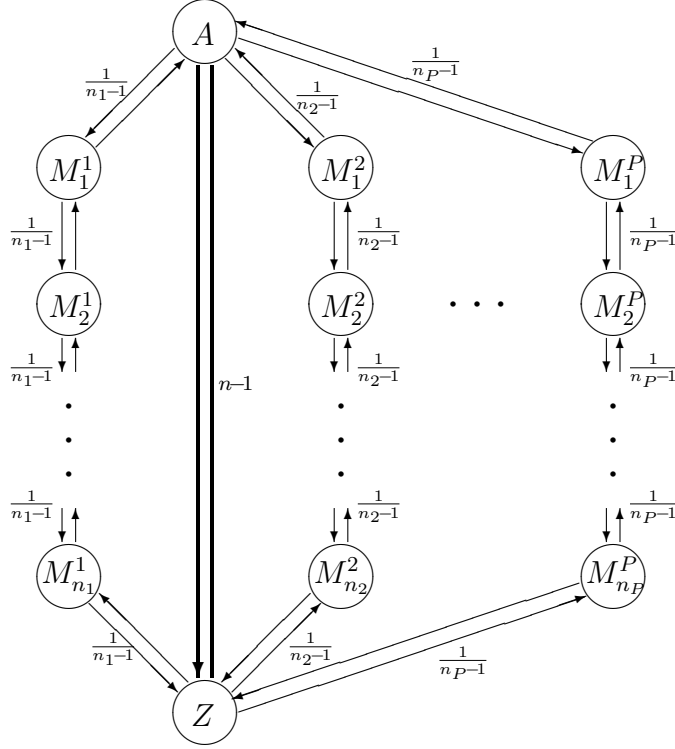
Figure 9: Path-Bridge configuration.

The *Path-Bridge inequality* corresponding to this configuration is defined by:

$$\sum_{e \in E \cup A} c_e x_e \geq 1 + \sum_{i=1}^{P} \frac{n_i + 1}{n_i - 1} \tag{9}$$

We can define the *n-regular Path-Bridge inequality* in which n_i for each path is the same number n. In this case, by multiplying by $n - 1$, the inequality (9) can be rewritten in the following way:

$$\sum_{e \in E \cup A} c_e x_e \geq Pn + P + n - 1 \tag{10}$$

where

$$c^A(A, Z) = c^A(Z, A) = c^E(A, Z) = n - 1$$

$$c^A(M_j^i, M_q^i) = |j - q|, \qquad \forall j, q \in \{0, 1, 2, \ldots, n + 1\}, \;\; 0 < |j - q| < n + 1$$

$$c^A(M_j^i, M_q^r) = |j - q| + 2 \quad \forall i, r \in \{1, 2, \ldots, P\}, \; i \neq r, \;\; j, q \in \{1, 2, \ldots, n\}$$

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Again, internal arcs (M^i_j, M^i_q) have a cost equal to the length of the shortest path from M^i_j to M^i_q using external arcs. Crossing arcs (M^i_j, M^r_q), $i \neq r$ $j, q \in \{1, 2, \ldots, n\}$ have now a cost equal to their 'difference in level', $|i - j|$, plus two.

These inequalities will be also called *Path-Bridge (PB)* because they can be considered as the mixed version of the *Path-Bridge* inequalities for the undirected GRP (Letchford, 1997). These last inequalities include the *Path* inequalities for the (undirected) Graphical TSP, GTSP, proposed by Cornuèjols, Fonlupt & Naddef (1985). It is worthwhile to point out that all these inequalities have the same coefficients and RHS.

PB inequalities reduce to the *s-Path inequalities* for the GATSP, introduced by Chopra & Rinaldi (1996), when there are no required links crossing from A to Z ($B = 0$ and, hence, $P \geq 3$ and odd). In such a case, sets A and Z are allowed to be empty and the inequalities are called *s-Wheelbarrow* and *s-Bicycle* inequalities. However, in general, the set $(A : Z)$ can contain one or more required links ($B > 0$).

Given that Path-Bridge inequalities are proved to be valid for the undirected GRP, they are also valid for MGRP(G). In order to show that, under certain conditions, they are facet-inducing of MGRP(G), we have to consider several cases: $P = 1$, $P \geq 3$ and odd, $P \geq 2$ and even.

When $P = 1$, the PB configuration becomes the so called *K-C configuration* (figure 10). It is defined by an integer $K \geq 3$, a partition of V into $K + 1$ subsets $\{M_0, M_1, M_2, \ldots, M_{K-1}, M_K\}$ such that each R-set V_i, $1 \leq i \leq p$, is contained in exactly one of the node sets $M_0 \cup M_K, M_1, M_2, \ldots, M_{K-1}$, the induced subgraphs $G(M_i)$, $i = 0, 1, 2, \ldots, K$, are strongly connected and $(M_0 : M_K)$ contains a positive and even number of required links, and by the cost functions defined as:

$$c^A(M_0, M_K) = c^A(M_K, M_0) = c^E(M_0, M_K) = K - 2$$
$$c^A(M_i, M_j) = |i - j|, \quad \forall i, j \; : \; \{i, j\} \neq \{0, K\}$$
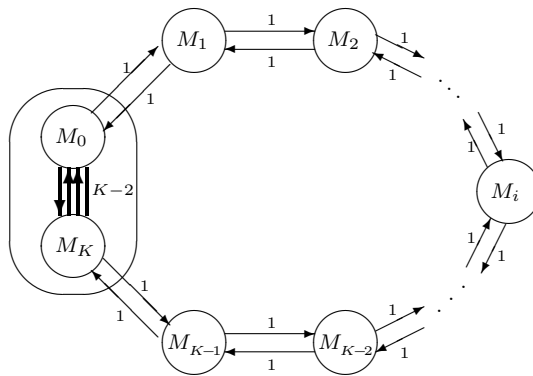
Figure 10: K-C configuration.

Note that M_0 and M_K correspond to A and Z in the general PB configuration and $K - 1$ is exactly n_1.

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The partition $\mathcal{B} = \{M_0, M_1, M_2, \ldots, M_{K-1}, M_K\}$ and the functions c^A, c^E define the configuration graph $G_\mathcal{C}$ whose skeleton is showed in figure 10. Internal arcs (M_i, M_j) not represented in figure 10 have a cost equal to the length of the shortest path from M_i to M_j using external arcs. The associated *K-C inequality* is:

$$(K-2)\, x\Big((M_0 : M_K)\Big) + \sum_{\substack{0 \le i < j \le K \\ (i,j) \ne (0,K)}} |i-j|\, x\Big((M_i : M_j)\Big) \ge 2(K-1) \qquad (11)$$

Constraints (11) are called K-C inequalities because they are identical to the K-C inequalities for the Undirected GRP (Corberán and Sanchis, 1994,1998). The effect of K-C inequalities is to separate 'solution' vectors x^* as shown in figure 11, where $x_e^* = 1$ for non required arcs (in thin lines) and $x_e^* = 0$ for the required links (in bold lines). This vector satisfies all the facet-inducing inequalities described in previous sections but violates the K-C inequality $F(x) \ge 6$ since $F(x^*) = 4$.
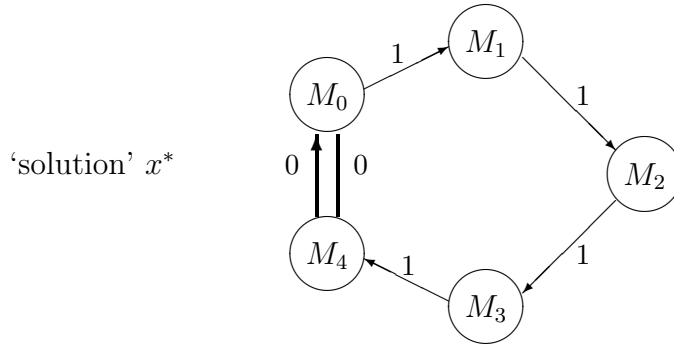
Figure 11: Vector violating a K-C inequality.

Theorem 9 *K-C inequalities (PB inequalities with $P = 1$) are facet-inducing for MGRP(G) if $\epsilon \ge |\alpha - \beta|$, where $\alpha = |A_R(M_K : M_0)|$, $\beta = |A_R(M_0 : M_K)|$ and $\epsilon = |E(M_0 : M_K)|$.*

Proof: Let $F(x) \ge 2(K-1)$ denote the K-C inequality. We will consider two cases: $\epsilon = 0$ and $\epsilon \ge 1$. Let us assume first that $\epsilon = 0$ and, hence, $\alpha = \beta$. We will show first that $F(x) \ge 2(K-1)$ is facet-inducing for MGRP($G_\mathcal{C}$). Given that $\epsilon = 0$, graph $G_\mathcal{C} = (\mathcal{B}, A_\mathcal{C})$ is a directed graph with $K + 1$ edge-connected components. Then, $\dim(\text{MGRP}(G_\mathcal{C})) = |A_\mathcal{C}| - K$. The following vectors are semitours for the MGRP on $G_\mathcal{C}$ and satisfy $F(x) = 2(K-1)$:
(a) Semitours formed by all the pairs of external opposite arcs linking M_0 and M_1, M_1 and M_2, ..., M_{K-1} and M_K except one of them (figure 12(a)).
(b) For each arc $a \in A(M_K : M_0)$, required or not, consider the semitour x consisting of arc a and the arcs in the path $M_0, M_1, \ldots, M_K$ (figure 12(b)). In the same way, for each arc $a \in A(M_0 : M_K)$ consider the semitour obtained by adding to a the arcs in the path $M_K, M_{K-1}, \ldots, M_1, M_0$.
(c) For each arc $a = (M_i, M_j) \in \mathcal{I}n$ consider the semitour obtained by adding to a the arcs in the path from M_j to M_i using external arcs and the pairs of arcs (M_t, M_{t+1}) and (M_{t+1}, M_t), $t \in \{0, 1, \ldots, i-1, j, j+1, \ldots, K-1\}$ except one of them (figure 12(c)).
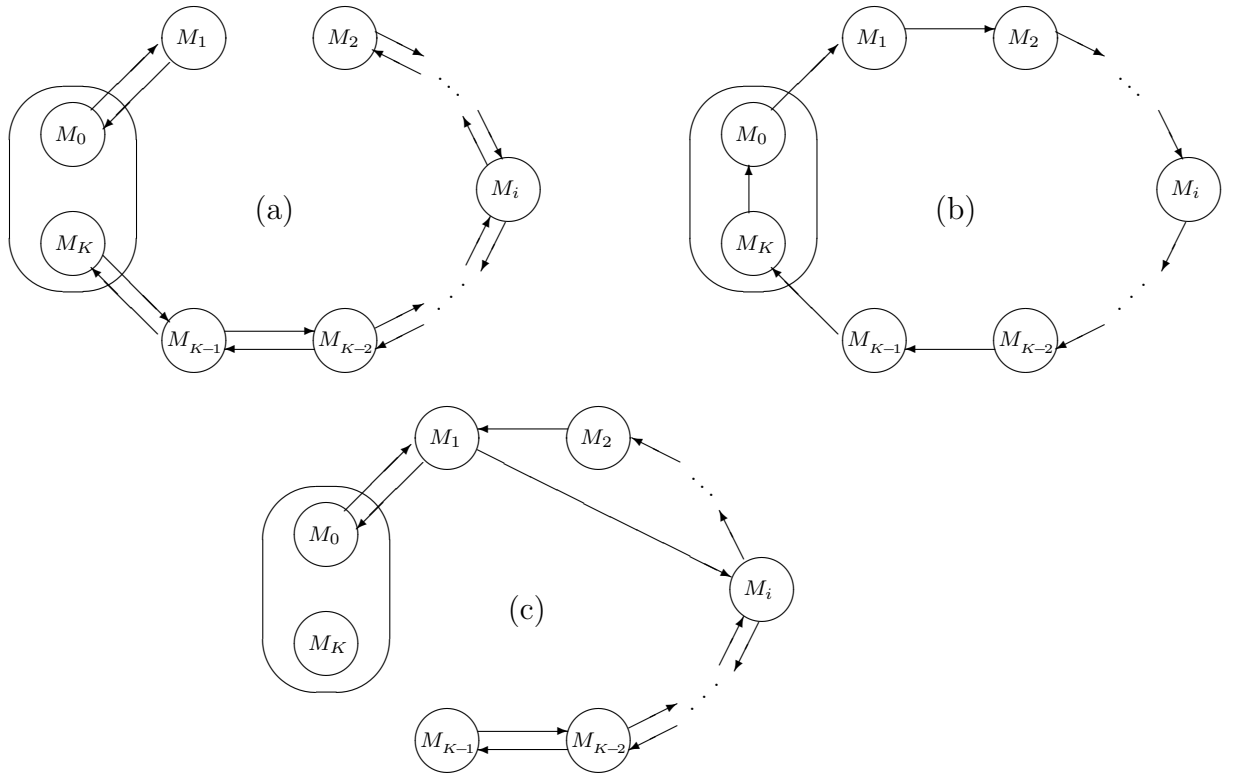
Figure 12: Semitours defined in the proof of theorem 9.

We have defined $K + |(M_0 : M_K)| + |\mathcal{I}n| = |\mathcal{E}x| - K + |(M_0 : M_K)| + |\mathcal{I}n| = |A_\mathcal{C}| - K$ semitours all satisfying $F(x) = 2(K - 1)$. If we express these semitours as rows of a matrix (sorted as (c),(b) and (a) before) and the arcs as columns (sorted as $\mathcal{I}n$, $(M_0 : M_K)$ and $\mathcal{E}x$), we obtain the matrix showed in figure 13. The block B, corresponding to rows (a) and columns (M_0, M_1), (M_1, M_2), ..., (M_{K-1}, M_K), has full rank. Hence, the $|A_\mathcal{C}| - K$ semitours are affinely independent and $F(x) \geq 2(K - 1)$ is facet-inducing for MGRP$(G_\mathcal{C})$. Furthermore, as the configuration graph $G_\mathcal{C}$ has no edges, condition (L1) of lifting theorem 4 is satisfied and, therefore, $F(x) \geq 2(K - 1)$ is also facet-inducing for MGRP(G).

Let us assume now that $\epsilon \geq 1$ and, hence, $\dim(\text{MGRP}(G_\mathcal{C}) = |E_\mathcal{C} \cup A_\mathcal{C}| - K + 1$. Similarly to the case $\epsilon = 0$, we can define such a number of affinely independent semitours for the MGRP on $G_\mathcal{C}$ satisfying $F(x) = 2(K - 1)$. Therefore, the inequality is facet-inducing for MGRP$(G_\mathcal{C})$. In order to show that it is also facet-inducing for MGRP(G), it can be proved, as in theorem 8, that condition (L2) of lifting theorem 5 is satisfied. $\blacklozenge$

Theorem 10 *PB inequalities with $P \geq 3$ and odd are facet-inducing for MGRP(G) if $\epsilon \geq |\alpha - \beta|$, where $\alpha = |A_R(A : Z)|$, $\beta = |A_R(Z : A)|$ and $\epsilon = |E(A : Z)|$.*

Proof: Let $F(x) \geq c_0$ denote the PB inequality. We will consider two cases: $\epsilon = 0$ and $\epsilon \geq 1$. Let us assume first that $\epsilon = 0$ and, hence, $\alpha = \beta$. We will show first that

$\mathcal{In}$	$(M_o : M_K)$	$\mathcal{E}x$	
I	0	*	
0	I	*	
0	0	B	B

$$B = \begin{pmatrix} 0 & 1 & & 1 & 1 \\ 1 & 0 & & 1 & 1 \\ & & \ddots & & \\ 1 & 1 & & 0 & 1 \\ 1 & 1 & & 1 & 0 \end{pmatrix}$$

Figure 13: Matrices appearing in the proof of theorem 9

$F(x) \geq c_0$ is facet-inducing for $\mathrm{MGRP}(G_{\mathcal{C}})$. Given that $\epsilon = 0$, graph $G_{\mathcal{C}} = (\mathcal{B}, A_{\mathcal{C}})$ is a directed graph. Consider the GATSP instance obtained from graph $G_{\mathcal{C}}$ by making all arcs non-required. Then, the corresponding s-Path inequality defines a facet of $\mathrm{GATSP}(G_{\mathcal{C}})$ (Chopra & Rinaldi, 1996). As $\alpha = \beta$, every GATSP tour satisfying the s-Path inequality as an equality is also a semitour for the MGRP on $G_{\mathcal{C}}$. Given that $\dim(\mathrm{MGRP}(G_{\mathcal{C}})) = |A_{\mathcal{C}}| - |\mathcal{B}| + 1 = \dim(\mathrm{GATSP}(G_{\mathcal{C}}))$, the s-Path inequality is facet-inducing for $\mathrm{MGRP}(G_{\mathcal{C}})$. Furthermore, as the configuration graph $G_{\mathcal{C}}$ has no edges, condition (L1) of lifting theorem 4 is satisfied and, therefore, $F(x) \geq c_0$ is also facet-inducing for $\mathrm{MGRP}(G)$ in this case.

Let us assume now that $\epsilon \geq 1$. Given that B is even and $\epsilon \geq |\alpha - \beta|$, the ϵ edges in $G_{\mathcal{C}}$ can be oriented as arcs to obtain a new directed graph $G_{\mathcal{C}}'$ satisfying $\alpha' = \beta'$ (and $\epsilon' = 0$). Then, we can find $\dim(\mathrm{MGRP}(G_{\mathcal{C}}'))$ semitours for the MGRP on $G_{\mathcal{C}}'$ (and on $G_{\mathcal{C}}$) satisfying $F(x) = c_0$. Given that $\epsilon \geq 1$,

$$\dim(\mathrm{MGRP}(G_{\mathcal{C}})) = |A_{\mathcal{C}}| - (|\mathcal{B}| - 1) + 1 = \dim(\mathrm{MGRP}(G_{\mathcal{C}}')) + 1,$$

and we need an extra semitour for the MGRP on $G_{\mathcal{C}}$. Let $e \in (A : Z)$ and consider the GATSP instance $G"_{\mathcal{C}}$ obtained from graph $G_{\mathcal{C}}$ by deleting all the links in $(A : Z)$ except the edge e, which is transformed into an arc a with the opposite direction to that associated to e in $G_{\mathcal{C}}'$. It is known that there exists a GATSP tour x^* on $G"_{\mathcal{C}}$ using only external arcs and exactly one copy of a and satisfying the s-Path inequality as an equality. By adding to x^* the original required links in $(A : Z)$, as $\epsilon \geq |\alpha - \beta|$, we obtain a (balanced) tour for the MGRP on $G_{\mathcal{C}}$. Then, x^* is a semitour for the MGRP on $G_{\mathcal{C}}$. Furthermore, x^* is linearly independent with the previously mentioned semitours due to the different orientation of edge e. Therefore, the inequality $F(x) \geq c_0$ is facet-inducing for $\mathrm{MGRP}(G_{\mathcal{C}})$. In order to show that it is also facet-inducing for $\mathrm{MGRP}(G)$, it is not difficult to see that, as in theorem 8, condition (L2) of lifting theorem 5 is satisfied. $\blacklozenge$

The effect of PB inequalities for P even is to separate 'solutions' as that shown in figure 14. It can be seen that vector x^*, where $x_e^* = 1$ for the non required arcs (in thin lines) and $x_e^* = 0$ for the required link (in bold line), satisfies all the facet-

inducing inequalities described in previous sections, but violates the regular PB inequality $F(x) \geq 7$ since $F(x^*) = 6$.
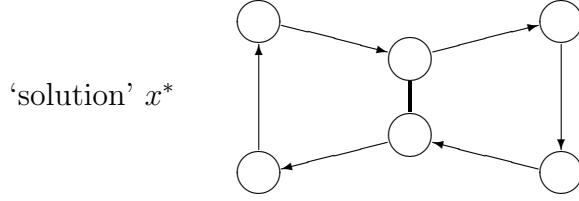

'solution' x^*

Figure 14: Vector violating a 2-PB inequality.

Theorem 11 *PB inequalities with P even are facet-inducing for MGRP(G) if $\epsilon \geq |\alpha - \beta| + 1$, where $\alpha = |A_R(A : Z)|$, $\beta = |A_R(Z : A)|$ and $\epsilon = |E(A : Z)|$.*

Proof: Let $F(x) \geq c_0$ denote the PB inequality. Again, we will show first that $F(x) \geq c_0$ is facet-inducing for MGRP($G_{\mathcal{C}}$). Given that $\epsilon \geq 1$,

$$\dim(\text{MGRP}(G_{\mathcal{C}}) = |(A : Z) \cup \mathcal{I}n \cup \mathcal{C}r| + 2\sum(n_i + 1) - (\sum n_i + 1) + 1 =$$
$$= |(A : Z) \cup \mathcal{I}n \cup \mathcal{C}r| + \sum(n_i + 2)$$

and this is the number of linearly independent semitours satisfying $F(x) = c_0$ we have to find.

As for the Path inequalities for the GTSP (see Cornuèjols et al., 1985), it can be seen that for each link $e \in (A : Z) \cup \mathcal{I}n \cup \mathcal{C}r$, there always exists a semitour for the MGRP in $G_{\mathcal{C}}$ satisfying $F(x) = c_0$ and using e exactly once and external arcs. Hence, only $\sum(n_i + 2)$ more linearly independent semitours, using only external arcs, are needed.

(a) For each path $i^* \in \{1, 2, \ldots, P\}$ and for each $j^* \in \{0, 1, 2, \ldots, n_{i^*}\}$, we consider the semitour that uses the two opposite arcs in each $(B_j^{i^*} : B_{j+1}^{i^*})$, except for $j = j^*$, and one single arc in each $(B_j^i : B_{j+1}^i)$, $i \neq i^*$, in such a way that $\frac{p}{2}$ paths are oriented from A to Z and the remaining $\frac{p}{2} - 1$ paths are oriented from Z to A. We will assume that these $n_{i^*} + 1$ semitours use the same orientation for the other $P - 1$ paths. In other words, $P - 1$ of the paths joining A and Z are used once, while the last path is broken into two parts and each part is used twice. These sets of arcs are semitours for the MGRP on $G_{\mathcal{C}}$ because, given that $\epsilon \geq |\alpha - \beta| + 1$, by adding to them the required links we obtain an eulerian graph. All these semitours satisfy $F(x) = c_0$.

(b) For each path i, it is possible to build $n_i + 1$ different semitours using the opposite orientations to those used in (a) for the other $P - 1$ paths. Among them we select just the semitour corresponding to the first of those described in (a) for each i.

Therefore, we have $n_i + 2$ semitours for the MGRP, satisfying $F(x) = c_0$, associated to each path i. In order to show that these $\sum(n_i + 2)$ semitours are linearly independent, consider their incidence matrix relative to the external arcs. In this matrix, the columns correspond to the external arcs and are grouped by paths. Columns associated to arcs

in each path are sorted in the following way: the first half correspond to the arcs oriented from A to Z and the second one to those oriented from Z to A.

By expressing the semitours as rows of this matrix (sorted as (a) and (b) before), we obtain the matrix showed in figure 15. Submatrix B $((n_i+1) \times (n_i+1))$ is shown in figure 13, $[k]$ represents a $(n_i+1) \times (n_j+1)$ matrix with all its entries equal to k, (k) represents a vector with n_i+1 components equal to k and B_1 is the first row of matrix B.

(a)	B	B	[1]	[0]	[0]	[1]	$\cdots$	[1]	[0]	[0]	[1]	[1]	[0]
	[1]	[0]	B	B	[0]	[1]	$\cdots$	[1]	[0]	[0]	[1]	[1]	[0]
	[1]	[0]	[0]	[1]	B	B	$\cdots$	[1]	[0]	[0]	[1]	[1]	[0]
							$\ddots$						
	[1]	[0]	[0]	[1]	[1]	[0]	$\cdots$	B	B	[0]	[1]	[1]	[0]
	[1]	[0]	[0]	[1]	[1]	[0]	$\cdots$	[0]	[1]	B	B	[1]	[0]
	[1]	[0]	[0]	[1]	[1]	[0]	$\cdots$	[0]	[1]	[1]	[0]	B	B
(b)	B_1	B_1	(0)	(1)	(1)	(0)	$\cdots$	(0)	(1)	(1)	(0)	(0)	(1)
	(0)	(1)	B_1	B_1	(1)	(0)	$\cdots$	(0)	(1)	(1)	(0)	(0)	(1)
	(0)	(1)	(1)	(0)	B_1	B_1	$\cdots$	(0)	(1)	(1)	(0)	(0)	(1)
							$\ddots$						
	(0)	(1)	(1)	(0)	(0)	(1)	$\cdots$	B_1	B_1	(1)	(0)	(0)	(1)
	(0)	(1)	(1)	(0)	(0)	(1)	$\cdots$	(1)	(0)	B_1	B_1	(0)	(1)
	(0)	(1)	(1)	(0)	(0)	(1)	$\cdots$	(1)	(0)	(0)	(1)	B_1	B_1

Figure 15: Matrix appearing in the proof of theorem 11

By substracting to each row in (b) the corresponding row in (a) and by resorting then the columns, the matrix in figure 16 is obtained. This matrix has complete range if P is even and $n_i \geq 2$ for all i. Therefore, the inequality $F(x) \geq c_0$ is facet-inducing for MGRP$(G_\mathcal{C})$. Again, it is not difficult to see that, as in theorem 8, condition (L2) of lifting theorem 5 is satisfied and, hence, the inequality is also facet-inducing for MGRP(G). ♦

6 Other Path-Bridge inequalities

Consider the vectors x^*, x^{**} described in figure 17 with $x_e^* = x_e^{**} = 1$ for the non required arcs (expressed in thin lines) and $x_e^* = x_e^{**} = 0$ for the required links (bold

B	[1]	[1]	⋯	[1]	[1]	B	[0]	[0]	⋯	[0]	[0]
[0]	B	[1]	⋯	[1]	[1]	[1]	B	[0]	⋯	[0]	[0]
[0]	[0]	B	⋯	[1]	[1]	[1]	[1]	B	⋯	[0]	[0]
			⋱						⋱		
[0]	[0]	[0]	⋯	B	[1]	[1]	[1]	[1]	⋯	B	[0]
[0]	[0]	[0]	⋯	[0]	B	[1]	[1]	[1]	⋯	[1]	B
(0)	(-1)	(-1)	⋯	(-1)	(-1)	(0)	(1)	(1)	⋯	(1)	(1)
(1)	(0)	(-1)	⋯	(-1)	(-1)	(-1)	(0)	(1)	⋯	(1)	(1)
(1)	(1)	(0)	⋯	(-1)	(-1)	(-1)	(-1)	(0)	⋯	(1)	(1)
			⋱						⋱		
(1)	(1)	(1)	⋯	(0)	(-1)	(-1)	(-1)	(-1)	⋯	(0)	(1)
(1)	(1)	(1)	⋯	(1)	(0)	(-1)	(-1)	(-1)	⋯	(-1)	(0)

Figure 16: Matrix appearing in the proof of theorem 11

lines). Vectors $x^* + x^R$ and $x^{**} + x^R$ are not tours for the MGRP although vectors x^* and x^{**} can be obtained as vertices of the polyhedron defined by all previously described inequalities.
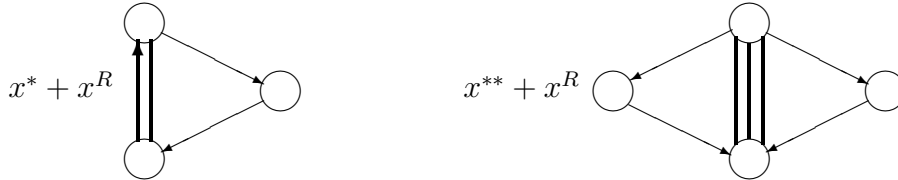
Figure 17: Two vectors satisfying all PB inequalities.

We next describe some new inequalities that 'cut-off' such 'solutions', which will be called PB$_{02}$ inequalities.

Consider a PB configuration as in the previous section except that now n_i can be equal to 1 in every path $i = 1, 2, \ldots, P$. Let us divide the set of paths into two types: paths of type $\mathcal{AZ}$ and paths of type $\mathcal{ZA}$, and let us redefine the cost functions c^E and c^A as follows:

- For each path i of type $\mathcal{AZ}$, we define: $c^A(M_0^i, M_1^i) = 0$ and $c^A(M_1^i, M_0^i) = \frac{2}{n_i}$. The remaining arcs in the path have coefficient $\frac{1}{n_i}$ and, for the internal arcs, $c^A(M_j^i, M_q^i)$ is the cost of the shortest path from M_j^i to M_q^i using only external arcs.

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- For each path i of type $\mathcal{ZA}$, we define $c^A(M_0^i, M_1^i) = \frac{2}{n_i}$ and $c^A(M_1^i, M_0^i) = 0$. The remaining arcs in the path have coefficient $\frac{1}{n_i}$. For each internal arc, $c^A(M_j^i, M_q^i)$ is the cost of the shortest path from M_j^i to M_q^i using only external arcs.

- Order the crossing arcs (if any) in an arbitrary way $a_1, a_2, \ldots, a_h \in \mathcal{C}r$. For $i = 1$ to h let c_{a_i} be the maximum value such that a_i belongs to a tour of c-length $P + 1$ using only links from $(E_{\mathcal{C}} \cup A_{\mathcal{C}}) \setminus \mathcal{C}r$ and $\{a_1, \ldots, a_i\}$ (sequential lifting).

In other words,

$$c^A(A, Z) = c^A(Z, A) = c^E(A, Z) = 1$$

$$c^A(M_j^i, M_q^i) = \frac{|j - q|}{n_i}, \qquad\qquad \forall j, q \in \{1, 2, \ldots, n_i + 1\}$$

$$c^A(M_0^i, M_q^i) = \frac{q - 1}{n_i} \text{ and } c^A(M_q^i, M_0^i) = \frac{q + 1}{n_i}$$
$$\text{if path } i \text{ is of type } \mathcal{AZ}, \quad \forall q \in \{1, 2, \ldots, n_i\}$$

$$c^A(M_0^i, M_q^i) = \frac{q + 1}{n_i} \text{ and } c^A(M_q^i, M_0^i) = \frac{q - 1}{n_i}$$
$$\text{if path } i \text{ is of type } \mathcal{ZA}, \quad \forall q \in \{1, 2, \ldots, n_i\}$$

Such a configuration will be called PB_{02} *configuration* (see figure 18).
Note that each path has a pair of opposite external arcs with coefficients 0 and $\frac{2}{n_i}$, while the other external edges have coefficient $\frac{1}{n_i}$. Furthermore, the direction of the arc with coefficient 0 determines whether the corresponding path is of type $\mathcal{AZ}$ or type $\mathcal{ZA}$.

The PB_{02} *inequality* corresponding to this new configuration is defined by:

$$\sum_{e \in E \cup A} c_e x_e \geq P + 1 \tag{12}$$

Again, we can define the special *n-regular PB_{02} inequality* in which $n_i = n$ for each path i. In this case, by multiplying by n, the inequality (12) can be rewritten in the following way:

$$\sum_{e \in E \cup A} c_e x_e \geq Pn + n \tag{13}$$

where

$$c^A(A, Z) = c^A(Z, A) = c^E(A, Z) = n$$

$$c^A(M_j^i, M_q^i) = |j - q|, \qquad\qquad \forall j, q \in \{1, 2, \ldots, n_i + 1\}$$

$$c^A(M_0^i, M_q^i) = q - 1 \text{ and } c^A(M_q^i, M_0^i) = q + 1$$
$$\text{if path } i \text{ is of type } \mathcal{AZ}, \quad \forall q \in \{1, 2, \ldots, n_i\}$$

$$c^A(M_0^i, M_q^i) = q + 1 \text{ and } c^A(M_q^i, M_0^i) = q - 1$$
$$\text{if path } i \text{ is of type } \mathcal{ZA}, \quad \forall q \in \{1, 2, \ldots, n_i\}$$

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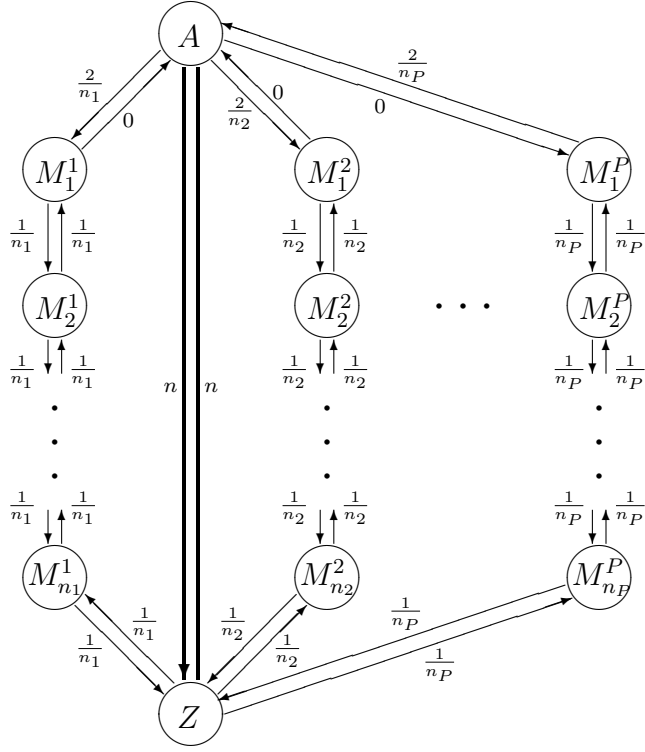
Figure 18: PB$_{02}$ configuration.

Constraints (12) and (13) are called PB$_{02}$ inequalities because they are similar to the standard PB inequalities, being the most relevant difference that a pair of external opposite arcs in each path have changed their coefficients to 0 and $\frac{2}{n_i}$.

Theorem 12 *PB$_{02}$ inequalities are valid for MGRP(G).*

Proof: It suffices to prove validity in the configuration graph $G_{\mathcal{C}}$. Let $F(x) \geq P + 1$ be a PB$_{02}$ inequality. By the definition of the coefficients for internal and crossing arcs, it suffices to prove validity for each semitour x for the MGRP on $G_{\mathcal{C}}$ using only links in $(A : Z) \cup \mathcal{E}x$.

For each path i, if there exists a $j^* \in \{0, 1, 2, \ldots, n_i\}$ such that $x(M_{j^*}^i : M_{j^*+1}^i) = x(M_{j^*+1}^i : M_{j^*}^i) = 0$, then connectivity and evenness conditions for the semitour x in this path implies that $x(M_j^i : M_{j+1}^i) \geq 2$ for all $j = 0, 1, \ldots, j^*-1, j^*+1, \ldots, n_i$. The associated c-cost is, at least, 2. Otherwise, the semitour uses either at least one copy of each arc in the path $A, M_1^i, M_2^i, \ldots, M_{n_i}^i, Z$ or one copy of each arc in the path $Z, M_{n_i}^i, \ldots, M_1^i, A$. In this case, the associated c-cost is, at least, 1.

Given that in a PB$_{02}$ configuration $P + B$ is odd, if a semitour x traverses every path exactly once, it must traverse also a link in $(A : Z)$ (with coeficient 1) and, therefore, $F(x) \geq P + 1$. Otherwise, it traverses, at least, a path twice or a path with a j^* as defined above. In any case, $F(x) \geq 2 + (P - 1) = P + 1$. ♦

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Theorem 13 *PB$_{02}$ inequalities are facet-inducing for MGRP(G) if*
$\epsilon \geq |\alpha + AZ - \beta - ZA| + 1$ *where* $\epsilon = |E(A : Z)|$, $\alpha = |A_R(A : Z)|$, *AZ is the number of paths of type* $\mathcal{AZ}$, $\beta = |A_R(Z : A)|$ *and ZA is the number of paths of type* $\mathcal{ZA}$.

Proof: Let $F(x) \geq P+1$ denote the PB$_{02}$ inequality. We will show first that $F(x) \geq c_0$ is facet-inducing for MGRP$(G_\mathcal{C})$. Given that $\epsilon \geq 1$,

$$\dim(\text{MGRP}(G_\mathcal{C}) = |(A : Z) \cup \mathcal{I}n \cup \mathcal{C}r| + 2\sum(n_i + 1) - (\sum n_i + 1) + 1 =$$
$$= |(A : Z) \cup \mathcal{I}n \cup \mathcal{C}r| + \sum(n_i + 2)$$

and this is the number of linearly independent semitours satisfying $F(x) = P + 1$ we have to find.

As for the standard PB inequalities, for each link $e \in (A : Z) \cup \mathcal{I}n \cup \mathcal{C}r$, there always exists a semitour for the MGRP in $G_\mathcal{C}$ satisfying $F(x) = P + 1$ and using e exactly once and external arcs. Hence, only $\sum(n_i + 2)$ more linearly independent semitours, using only external arcs, are needed.

(a) For each path $i^* \in \{1, 2, \ldots, P\}$ and for each $j^* \in \{0, 1, 2, \ldots, n_{i^*}\}$, we consider the semitour that uses the two opposite arcs in each $(B_j^{i^*} : B_{j+1}^{i^*})$, except for $j = j^*$, and one single arc in each $(B_j^i : B_{j+1}^i)$, $\forall \ i \neq i^*$ (if any), in such a way that paths of type $\mathcal{AZ}$ are oriented from A to Z and paths of type $\mathcal{ZA}$ are oriented from Z to A. In other words, $P - 1$ of the paths joining A and Z are used once, in the direction given by the arc with zero coefficient, while the last path is broken into two parts and each part is used twice. These sets of arcs are semitours for the MGRP on $G_\mathcal{C}$ because, given that $\epsilon \geq |\alpha + AZ - \beta - ZA| + 1$, by adding to them the required links we obtain an eulerian graph. Furthermore, all these semitours satisfy $F(x) = P + 1$, since the $P - 1$ paths used once have a c-cost of 1 and the "broken" path used twice has a c-cost of 2.

(b) For each path i, $i = 1, 2, \ldots, P$, we build a "special" semitour considering two copies of each arc in the path i, in the direction given by the arc with zero coefficient, and one copy of each arc in the remaining paths, always in the direction given by the arc with zero coefficient. Note that for this semitour $F(x) = P + 1$, since arcs in the path i have a total c-cost of 2, while we obtain a total c-cost of $P - 1$ for the remaining paths.

Therefore, we have built $n_i + 2$ semitours for the MGRP, satisfying $F(x) = P + 1$, associated to each path i. In order to show that these $\sum(n_i + 2)$ semitours are linearly independent, consider their incidence matrix relative to the external arcs. In this matrix, the columns correspond to the external arcs and are grouped by paths. Columns associated to arcs in each path are sorted in the following way. For the paths of type $\mathcal{AZ}$, the first half correspond to the arcs oriented from A to Z and the second one to those oriented from Z to A, while for the paths of type $\mathcal{ZA}$ the ordering is the opposite.

By expressing the semitours as rows of this matrix (sorted as (a) and (b) before), we obtain the matrix showed in figure 19. We have used the same notation as in the proof of theorem 11.

By resorting the columns, the matrix in figure 20 is obtained. This matrix has complete range for $P \geq 1$ and $n_i \geq 1$ for all i. Therefore, the inequality $F(x) \geq P + 1$

is facet-inducing for MGRP($G_\mathcal{C}$). Again, as in theorem 8, condition (L2) of lifting theorem 5 is satisfied and therefore the inequality is also facet-inducing for MGRP(G).

♦

	B	B	[1]	[0]	[1]	[0]	⋯	[1]	[0]
	[1]	[0]	B	B	[1]	[0]	⋯	[1]	[0]
	[1]	[0]	[1]	[0]	B	B	⋯	[1]	[0]
(a)						⋱			
	[1]	[0]	[1]	[0]	[1]	[0]	⋯	B	B
	(2)	(0)	(1)	(0)	(1)	(0)	⋯	(1)	(0)
	(1)	(0)	(2)	(0)	(1)	(0)	⋯	(1)	(0)
	(1)	(0)	(1)	(0)	(2)	(0)	⋯	(1)	(0)
(b)						⋱			
	(1)	(0)	(1)	(0)	(1)	(0)	⋯	(2)	(0)

Figure 19: Matrix appearing in the proof of theorem 13

B	[0]	[0]	⋯	[0]	[0]	B	[1]	[1]	⋯	[1]	[1]
[0]	B	[0]	⋯	[0]	[0]	[1]	B	[1]	⋯	[1]	[1]
[0]	[0]	B	⋯	[0]	[0]	[1]	[1]	B	⋯	[1]	[1]
		⋱						⋱			
[0]	[0]	[0]	⋯	B	[0]	[1]	[1]	[1]	⋯	B	[1]
[0]	[0]	[0]	⋯	[0]	B	[1]	[1]	[1]	⋯	[1]	B
(0)	(0)	(0)	⋯	(0)	(0)	(2)	(1)	(1)	⋯	(1)	(1)
(0)	(0)	(0)	⋯	(0)	(0)	(1)	(2)	(1)	⋯	(1)	(1)
(0)	(0)	(0)	⋯	(0)	(0)	(1)	(1)	(2)	⋯	(1)	(1)
		⋱						⋱			
(0)	(0)	(0)	⋯	(0)	(0)	(1)	(1)	(1)	⋯	(2)	(1)
(0)	(0)	(0)	⋯	(0)	(0)	(1)	(1)	(1)	⋯	(1)	(2)

Figure 20: Matrix appearing in the proof of theorem 13

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In the PB$_{02}$ configuration, coefficients $(0, 2)$ can be associated with any pair of opposite external arcs $(M_j^i : M_{j+1}^i)$ in any path i, $i \in \{0, 1, 2, \ldots, P\}$. Nevertheless, all these inequalities induce the same facet of MGRP(G) (given the same ordering for the crossing edges in the sequential lifting process).

Standard PB and PB$_{02}$ inequalities induce different facets of MGRP(G). Note that, unlike standard PB inequalities, PB$_{02}$ inequalities also induce facets of MGRP(G) when $n_i = 1$ for some path i.

For $P = 1$, we obtain the K-C$_{02}$ configuration whose skeleton is showed in figure 21. Note that internal arcs (M_i, M_j) (not represented in figure 21) have a cost equal to the length of the shortest path from M_i to M_j using external arcs. The corresponding K-C$_{02}$ inequality is:

$$\sum_{e \in E \cup A} c_e x_e \geq 2(K-1) \tag{14}$$

These inequalities cut-off the 'solution' x^* presented in figure 17, since the K-C$_{02}$ inequality is $F(x) \geq 2$ but $F(x^*) = 1$.
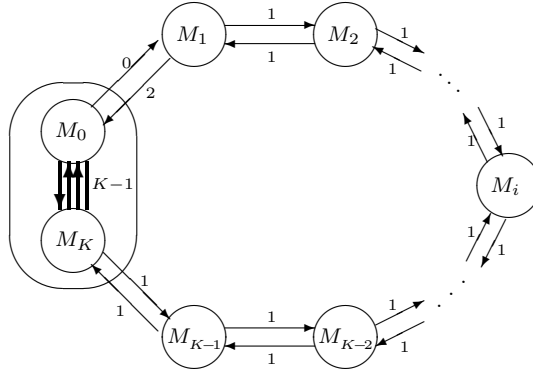
Figure 21: K-C$_{02}$ configuration.

In the definition of the K-C$_{02}$ configuration we have required the existence of two opposite arcs in each $(M_i : M_{i+1})$, for $i = 1, 2, \ldots, K - 1$. Nevertheless, in some cases this condition can be relaxed:
(a) For $K = 2$, the three K-C$_{02}$ inequalities corresponding to the configurations of figure 22 are facet-inducing if $\epsilon \geq \max\{\alpha - \beta \,,\, \beta - \alpha + 2\}$.
(b) For $K \geq 3$, if the subgraph induced by the links in $\mathcal{E}x \cup \mathcal{I}n$ is not strongly connected, i.e., there are no arcs from nodes $\{M_{j+1}, M_{j+2} \ldots, M_K\}$ to nodes $\{M_0, M_1, \ldots, M_j\}$ for some j, then the corresponding K-C$_{02}$ inequality can be shown to be dominated by the 2-C$_{02}$ inequality corresponding to the configuration with node sets: $M'_0 = \{M_0, M_1, \ldots, M_{j-1}\}$, $M'_1 = M_j$ and $M'_2 = \{M_{j+1}, M_{j+2} \ldots, M_K\}$.
(c) For $K \geq 3$, if the subgraph induced by the links in $\mathcal{E}x \cup \mathcal{I}n$ is strongly connected, the corresponding K-C$_{02}$ inequality can be, or not, facet-inducing.
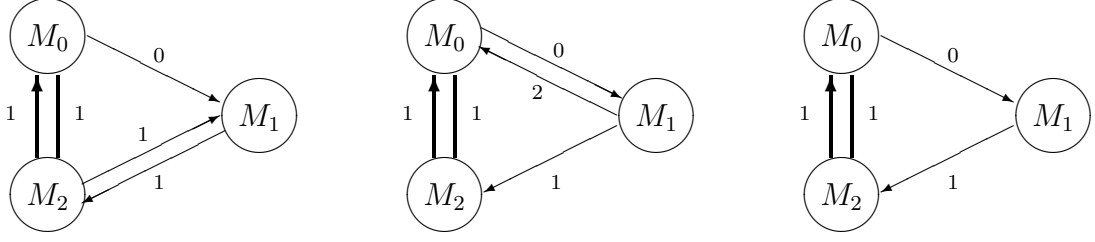
Figure 22: Other 2-C_{02} facet-inducing inequalities.

Finally, when $\epsilon = 0$, although $\alpha = \beta$, the K-C_{02} inequality can be shown to be dominated by the standard K-C inequality and, therefore, it is not facet-inducing.

If $P = 2$ and $n_i = n$ for all i, we obtain the regular 2-PB_{02} configuration. Figure 23 shows two skeletons (in the regular case) out of the 3 possible configurations that result from the assignment of the 2 paths to the types $\mathcal{AZ}$ and $\mathcal{ZA}$. Again, internal arcs (not represented in the figure) have a cost equal to the length of the shortest path using external arcs. The corresponding *regular 2-PB$_{02}$ inequality* is:

$$\sum_{e \in E \cup A} c_e x_e \geq 3K \tag{15}$$
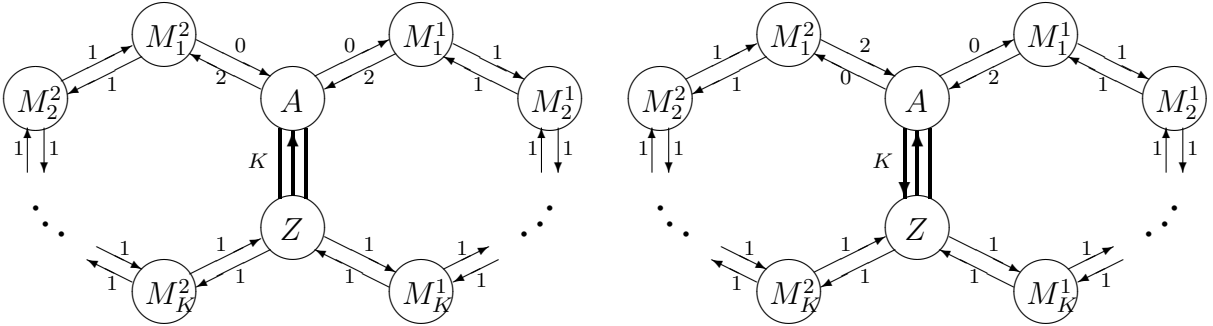
Figure 23: Two Regular 2-PB_{02} configurations.

A 2-PB_{02} inequality cut-off the vector x^{**} in figure 17, since the inequality is $F(x) \geq 3$ while $F(x^{**}) = 2$.

7 Computational Results.

We have devised a cutting-plane algorithm with separation procedures for the connectivity, R-odd cut and balanced-set inequalities. The initial LP relaxation includes the system equations (6), one connectivity inequality (4) for each R-set, one balanced-set inequality (5) for each 'unbalanced' vertex (i.e., the balanced-set inequality corresponding to each vertex $i \in V$ such that $|E(i)| < |A_R^+(i)| - |A_R^-(i)|$

or the balanced-set inequality corresponding to $V \setminus \{i\}$ for each vertex $i \in V$ such that $|E(i)| < |A_R^-(i)| - |A_R^+(i)|$), and one R-odd cut inequality (8) for each R-odd 'balanced' vertex.

The separation problems associated with connectivity and R-odd cut inequalities are solvable in polynomial time by means of max-flow calculations and the Padberg & Rao (1982) procedure to find minimum odd cut-sets. Standard heuristics for both problems were also implemented. Nobert & Picard (1996) showed that identifying violated balanced-set inequalities for the MCPP can also be solved in polynomial time (see Benavent, Corberán and Sanchis, 2000, for a short and direct proof of this important result). This result is easily extended to the MGRP. In the course of the algorithm, additional inequalities of the above mentioned classes are generated as they are found to be violated. When this is no longer possible and the LP solution is still not integral, we invoke branch and bound. If the resulting integer solution is feasible for the MGRP, it is optimal. Otherwise, the procedure terminates with a tight lower bound, but no feasible MGRP solution.

This algorithm has been coded in C and run on a Personal Computer with a 400 MHz Pentium II processor. It is a rough algorithm implemented to provide lower bounds to a set of 100 MRPP instances randomly generated for testing the constructive and the tabu search algorithms described in Corberán, Martí & Romero (2000).

The graph generator works as follows: a given number of vertices, $|V|$, is equally distributed among the desired number p of R-sets. Initially, each R-set has $\lfloor \frac{|V|}{p} \rfloor$ vertices, except possibly the last one which also has the difference up to $|V|$. Arcs and edges are generated, up to a prefixed number, by randomly selecting two vertices and adding the corresponding link. To generate required links, if the two vertices do not belong to the same R-set they are rejected and a new selection is made. Note that, as the required links may not connect all the vertices in a R-set, the final number of R-sets may be diferent to the one initially desired. Furthermore, the final number of non-required arcs might be larger, in some instances, than the prefixed number $|A \setminus A_R|$, in order to obtain a strongly connected graph. This generator allows up to four links between two vertices: one for each kind of link (required and non-required arc, required and non-required edge). We have randomly generated 100 instances with the following ranges: $3 \leq p \leq 12$, $20 \leq |V| \leq 100$, $15 \leq |E| \leq 220$, $5 \leq |E_R| \leq 150$, $50 \leq |A| \leq 350$, $5 \leq |A_R| \leq 200$. The cost of each link has been randomly generated in the range $1-20$.

Tables 1, 2 and 3 present the characteristics of the instances, the lower bound obtained with our cutting-plane procedure (column LB), the lower bound obtained with the branch and bound (column B&B) and the number of nodes of the B&B tree (column Nd), the best upper bound obtained with the mentioned heuristics (column UB) and the CPU time in seconds (column Time). An entry marked with an asterisk means that an optimal solution is reached. Note that the cutting-plane procedure is capable of finding an optimal solution in 71 out of the 100 MRPP instances and

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that this number increases up to 93 after invoking branch and bound. Although general MGRP instances will be harder to solve, we consider that these are encouraging results in order to implement a better cutting-plane or branch and cut algorithm.

8   Relationship with other Routing Problems.

In previous sections we have studied the polyhedron associated with the MGRP. We now present the application of our results to other known routing problems: the Mixed Chinese Postman Problem, the Undirected, Directed and Mixed Rural Postman Problems and the Graphical Asymmetric Traveling Salesman Problem.

The conclusions are conditioned by the fact that the original graph is transformed into a new simplified graph without non-required edges (they have been replaced by a pair of opposite non-required arcs). Hence, the results obtained so far for the MGRP can only be applied to instances and Problems without non-required edges.

Mixed CPP:
This problem is a particular case of the MGRP and arises when all the links of the graph are required to be traversed. The approach described in Nobert & Picard (1996) uses trivial inequalities (1), balanced-set inequalities (5) and R-odd cut inequalities (8) to design an algorithm which has proven to be highly effective. However, the issue of whether these inequalities induce or not facets of the associated polyhedron has not been studied. Given an instance $G = (V, E \cup A)$ of MCPP, our results for MGRP imply that MCPP(G) is a polyhedron of dimension $|E \cup A| - q + 1$, where again q is the number of connected components of graph (V, E), and the above inequalities (1), (5) and (8) are facet-inducing under the conditions described in their respective sections.

Undirected RPP:
Results obtained for the Mixed GRP cannot be directly applied to the Undirected RPP due to the existence of non-required edges. To study the MGRP without replacing the non required edges by two arcs may be more attractive from a theoretical point of view, but it is also quite more difficult. Consider, for example, a K-C configuration in which there may exist, or not, edges between any pair of nodes M_i, M_j. In this case, the dimension of MGRP$(G_\mathcal{C})$ depends on the number and the disposal of such edges.

Directed RPP:
Given that this problem is defined on directed graphs, all the results obtained for the MGRP not implied by the existence of required edges are also valid for the DRPP. Indeed, given a (simplified) instance $G = (V_R, A_R \cup A_{NR})$ of DRPP, we have that $DRPP(G)$ is a polyhedron of dimension $|A_R \cup A_{NR}| - |V| + 1$, and the following inequalities are facet-inducing:

- Trivial inequalities: $x_a \geq 0$ if $a \in A_R \cup A_{NR}$ is not a cut-arc.

- Connectivity inequalities: $x(A^+(S)) \geq 1$, where $S \subset V$ is the union of R-sets, if $G(S)$ and $G(V \setminus S)$ are strongly connected.
- Standard K-C inequalities if $\alpha = \beta$ (see section 5).

K-C$_{02}$ inequalities described in section 5 and PB inequalities described in section 5 are also valid for DRPP(G), but it is not possible to directly conclude if they are or not facet-inducing.

Asymmetric GTSP:
As it was mentioned before, GATSP is a special case of MGRP when $E = A_R = \emptyset$. Therefore, the results obtained in previous sections not involving required links also apply to GATSP. Hence, for example, GATSP(G) is an unbounded and non full-dimensional polyhedron, trivial inequalities are facet-inducing, all the remaining facet-inducing inequalities are configuration inequalities, connectivity inequalities and standard PB inequalities with $(A : Z)_R = \emptyset$ are facet inducing. These last inequalities correspond to the known s-path, s-wheelbarrow and s-bicycle inequalities. All these results were previously obtained by Chopra and Rinaldi (1996).

Conversely, it is worthwhile to point out that every non trivial valid (or facet-inducing) inequality for the GATSP polyhedron provides a valid (or facet-inducing) inequality for MGRP(G). Given an instance $G = (V, E \cup A_R \cup A_{NR})$ of MGRP, if we shrink the R-sets into a single node each and we remove the loops, we obtain an instance for the GATSP. This is also true if we shrink, into a single node, every vertex set B_j in any partition $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$ of V satisfying that each R-set V^i is contained into one set B_j. Let us call *GATSP configuration* to any configuration $\mathcal{C} = (\mathcal{B}, c^E, c^A)$ on G where $\mathcal{B} = \{B_1, B_2, \ldots, B_r\}$ satisfies the previous condition.

Theorem 14 *Let $\mathcal{C}$ be a GATSP configuration on G and let $G_\mathcal{C}$ be its corresponding configuration graph. If the configuration inequality associated with $G_\mathcal{C}$ is facet-inducing for GATSP($G_\mathcal{C}$), then the configuration inequality associated with G is facet-inducing for MGRP(G).*

Proof: Given that the graph $G_\mathcal{C}$ has no required links, GATSP$(G_\mathcal{C})$=MGRP$(G_\mathcal{C})$ and, hence, the inequality associated with $G_\mathcal{C}$ is facet-inducing for MGRP$(G_\mathcal{C})$. Furthemore, given that $G_\mathcal{C}$ is a directed graph, condition $(L1)$ in the first lifting theorem 4 is obviously satisfied. Therefore, the configuration inequality associated with G is facet-inducing for MGRP(G). ♦

9 Conclusions

In this paper we have studied the polyhedron associated with the General Routing Problem on a mixed graph. To the best of our knowledge, this is the first polyhedral study of a routing problem defined on a mixed graph. MGRP generalizes several important and difficult Arc and Node Routing Problems as CPP, RPP and GTSP

either defined on an undirected, directed or mixed graph.

We have defined several families of inequalities and shown that they define facets for the MGRP polyhedron. Some of these families are derived from facet-inducing inequalities for other routing problems polyhedra: connectivity, R-odd cut, standard K-C and standard PB inequalities for the undirected RPP and GRP (Corberán & Sanchis 1994, 1998, Letchford 1997, 1999), balanced-set inequalities for the MCPP (Nobert & Picard 1996) and all the known families for the GATSP (Chopra & Rinaldi, 1996).

Some computational results obtained with a preliminary cutting-plane algorithm based on some of these inequalities are also given. We consider that these results show that the polyhedral description obtained in this paper can be useful when solving MGRP instances.

Acknowledgements

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	p	$|V|$	$|E|$	$|E_R|$	$|A|$	$|A_R|$	LB	B&B	Nd	UB	Time
g3c1	3	20	50	10	80	30	464*				0,98
g3c2	3	30	140	40	200	50	946*				2,58
g3c3	3	50	90	60	250	150	2375*				20,43
g3c4	3	50	140	70	190	100	1697*				4,73
g3c5	3	60	140	90	240	80	1791*				5,71
g3c6	3	70	180	100	120	50	1663*				4,23
g3c7	3	80	160	100	300	100	2320	2321*	13		22,97
g3c8	5	90	130	40	200	50	1244*				6,48
g3c9	3	100	220	150	300	200	4092*				33,23
g3c10	3	100	200	130	250	150	3287	3288*	85		49,57
g4c1	4	20	105	25	80	50	793	793*	2		4,12
g4c2	3	40	140	20	150	100	1480*				3,13
g4c3	4	50	60	40	250	150	2310*				9,72
g4c4	4	50	110	80	250	100	2079*				9,99
g4c5	4	60	100	50	220	120	1852*				3.57
g4c6	5	70	100	30	200	50	1010*				3,25
g4c7	4	80	190	90	350	50	1581*				4,66
g4c8	3	90	150	100	320	200	3395*				28,62
g4c9	15	100	150	50	340	40	1186*				5,33
g4c10	2	100	150	100	220	130	2919*				28,62
g5c1	5	20	50	20	80	10	369*				2,25
g5c2	5	40	90	50	150	60	1194*				3,95
g5c3	5	50	180	30	250	100	1388*				3,79
g5c4	4	50	140	40	210	60	1070*				5,28
g5c5	6	60	110	30	250	50	1104*				3,89
g5c6	5	70	160	100	270	200	3416	3416*	1		17,53
g5c7	5	80	170	90	190	100	2337,75	2338*	1		11,92
g5c8	5	90	180	80	300	150	2708	2708*	1		20,21
g5c9	4	100	130	60	350	200	3118*				10,27
g5c10	4	100	110	60	250	150	2796*				8,79
g6c1	6	20	60	10	105	15	324*				2,96
g6c2	6	40	100	30	180	80	1203,5	1204*	1		4,44
g6c3	5	50	110	20	270	70	1106*				3,96
g6c4	6	50	160	40	170	100	1667*				4,12
g6c5	6	60	110	80	270	150	2545*				5,33
g6c6	6	70	130	30	270	70	1259*				2,85
g6c7	6	80	160	90	170	50	1679*				5,49
g6c8	5	90	150	100	170	70	1986*				13,02
g6c9	2	100	140	50	230	150	2750*				10,71
g6c10	6	100	180	120	220	130	2952*				28,90

Table 1: Computational Results 1.

	p	$|V|$	$|E|$	$|E_R|$	$|A|$	$|A_R|$	LB	B&B	Nd	UB	Time
g7c1	6	20	55	5	130	10	235,5	236*	9		1,87
g7c2	6	40	140	20	100	50	816*				8,46
g7c3	8	50	140	50	280	30	847*				5,93
g7c4	7	50	150	90	170	100	2043	2044*	3		7,69
g7c5	7	60	170	40	240	150	2198*				6,37
g7c6	7	70	150	50	300	200	2867,5	2868*	25		12,04
g7c7	7	80	170	100	210	130	2642,5	2643	1	2650	9,39
g7c8	6	90	170	80	270	150	2630	2632	26	2634	8,91
g7c9	8	100	150	100	260	60	1902*				19,44
g7c10	6	100	200	120	220	150	3024*				16,48
g8c1	8	30	70	20	120	30	650,5	651*	1		4,18
g8c2	8	40	90	20	190	90	1192*				4,45
g8c3	7	50	50	30	360	60	1035*				5,44
g8c4	8	50	120	70	220	120	1969*				3,74
g8c5	8	60	120	40	180	100	1815*				7,58
g8c6	8	70	150	90	230	70	1678,67	1679*	12		9,84
g8c7	6	80	160	50	240	150	2620*				23,07
g8c8	8	90	220	100	160	90	2213*				14,67
g8c9	7	100	150	60	230	100	2365*				23,67
g8c10	7	100	190	120	300	150	2974*				14,11
g9c1	8	30	45	15	95	15	372*				3,63
g9c2	9	40	80	30	210	60	1025*				4,67
g9c3	9	50	90	60	270	70	1537*				6,98
g9c4	9	50	80	40	150	50	966	967*	10		4,56
g9c5	9	60	150	50	270	150	2444*				6,04
g9c6	8	70	150	100	250	150	2915*				5,39
g9c7	8	80	210	120	200	50	1994	1994*	1		7,69
g9c8	7	90	150	50	200	120	2136*				11,09
g9c9	7	100	130	70	290	200	3388*				15,44
g9c10	7	100	150	100	240	70	1939	1940*	2		16,59
g10c1	10	30	20	10	180	30	447	449	1	452	5,11
g10c2	10	40	40	20	170	80	1153*				3,08
g10c3	10	50	80	50	290	40	950,5	951*	12		3,69
g10c4	10	50	80	40	150	50	1157*				5,16
g10c5	10	60	160	40	280	200	2633*				7,30
g10c6	10	70	160	80	300	130	2162	2163*	24		6,28
g10c7	10	80	190	130	270	150	3055,17	3058	19	3072	13,19
g10c8	9	90	150	100	250	160	2596*				6,81
g10c9	8	100	160	70	210	100	2119*				6,04
g10c10	10	100	210	150	200	70	2504,33	2505*	39		8,19

Table 2: Computational Results 2.

	p	$|V|$	$|E|$	$|E_R|$	$|A|$	$|A_R|$	LB	B&B	Nd	UB	Time
g11c1	3	30	15	5	105	5	187*				1,38
g11c2	9	40	35	15	220	20	505*				4,06
g11c3	12	50	210	10	80	30	501*				2,59
g11c4	10	50	80	40	150	50	1146	1146*	1		4,89
g11c5	11	60	150	50	210	120	2029*				5,05
g11c6	11	70	170	90	240	150	2625	2626	1	2642	5,60
g11c7	11	80	170	100	250	150	2926*				5,38
g11c8	10	90	170	50	240	150	2395*				4,29
g11c9	10	100	180	120	190	70	2166	2166*	1		5,11
g11c10	10	100	170	70	220	100	2128*				9,28
g12c1	7	30	105	5	55	5	131*				1,92
g12c2	11	40	60	10	160	10	308*				2,03
g12c3	15	50	25	15	320	20	480	480*	5		2,69
g12c4	12	50	160	70	190	110	2053,87	2055*	22		7,59
g12c5	12	60	160	40	170	70	1309*				3,24
g12c6	12	70	180	80	210	120	2231	2232	42	2248	7,10
g12c7	12	80	180	110	220	90	2252*				6,81
g12c8	11	90	170	60	250	150	2498*				11,04
g12c9	11	100	190	120	170	100	2341,5	2342	3	2355	6,98
g12c10	10	100	190	70	270	150	2614*				11,81

Table 3: Computational Results 3.