On the domain of formal balls of the Sorgenfrey quasi-metric space

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Abstract

We show that the poset of formal balls of the Sorgenfrey quasi-metric space is an ω-continuous domain, and deduce that it is also a computational model, in the sense of R.C. Flagg and R. Kopperman, for the Sorgenfrey line. Furthermore, we study its structure of quantitative domain in the sense of P. Waszkiewicz.

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1 Introduction and preliminaries

In this paper, the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ will denote the set of all real numbers, the set of all non-negative real numbers and the set of all positive integer numbers, respectively.

Ou basic references for quasi-metric spaces are [6, 10].

By a quasi-metric on a set X we mean a non-negative real-valued function d on $X \times X$ such that, for each $x, y, z \in X$: (i) $x = y \iff d(x, y) = d(y, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is a pair (X, d) where X is a set and d is a quasi-metric on X.

According to [14] a quasi-metric d on X is said to be weightable if there is a non-negative real-valued function w on X such that $d(x, y) + w(x) = d(y, x) + w(y)$ for all $x, y \in X$. In this case we say that w is a weight function for d.

Note that if d is a quasi-metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, for all $x, y \in X$, is also a quasi-metric on X, and the function d^s defined on $X \times X$ by $d^s(x, y) = d(x, y) \vee d^{-1}(x, y)$, for all $x, y \in X$, is a metric on X.

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Each quasi-metric d on X induced a T_0 topology τ_d on X for which the family of open balls $\{B_d(x,r) : x \in X, r > 0\}$, is a base, where $B_d(x,r) = \{y \in X : d(x,y) < r\}$.

A topological space (X,τ) is said to be quasi-metrizable if there is a quasi-metric d on X such that the topologies τ and τ_d agree on X. In this case, we say that d is compatible with τ.

The Sorgenfrey line $(\mathbb{R}, \tau_S)$ is a distinguished example of a quasi-metrizable topological space. In fact, it is well known, and easy to check, that the quasi-metric d_S on $\mathbb{R}$ given by $d_S(x,y) = y - x$ if $x \leq y$, and $d_S(x,y) = 1$ otherwise, is compatible with the Sorgenfrey topology τ_S.

In the sequel, we will refer to $(\mathbb{R}, d_S)$ as the Sorgenfrey quasi-metric space.

Next we recall several necessary concepts and properties of domain theory, which may be found in [7].

A partially ordered set, or poset for short, is a (non-empty) set X equipped with a (partial) order $\sqsubseteq$. It will be denoted by $(X, \sqsubseteq)$ or simply by X if no confusion arises.

An element x of X is said to be maximal if the condition $x \sqsubseteq y$ implies $x = y$. The set of all maximal points of X will be denoted by $\mathrm{Max}((X, \sqsubseteq))$ or simply $\mathrm{Max}(X)$ if no confusion arises.

A subset D of a poset X is directed provided that it is non-empty and every finite subset of D has upper bound in D.

A poset X is said to be directed complete, and is called a dcpo, if every directed subset of X has a least upper bound. The least upper bound of a subset D of X is denoted by $\sqcup D$ if it exists.

Let X be a poset and $x, y \in X$; we say that x is way below y, in symbols $x \ll y$, if for each directed subset D of X having least upper bound $\sqcup D$, the relation $y \sqsubseteq \sqcup D$ implies the existence of some $u \in D$ with $x \sqsubseteq u$.

A poset X is continuous if it has a basis B, where B is said to be a basis for X if for all $x \in X$, the set $\Downarrow x := \{b \in B : b \ll x\}$ is directed with least upper bound x.

The Scott topology $\sigma((X, \sqsubseteq))$ of a continuous poset $(X, \sqsubseteq)$ is the topology on X that has as a base the collection of all sets of the form $\Uparrow x$, $x \in X$, where $\Uparrow x := \{y \in X : x \ll y\}$.

The lower (or weak) topology $\omega((X, \sqsubseteq))$ of a continuous poset $(X, \sqsubseteq)$ is the topology on X that has as a base the collection of all sets of the form $X \setminus \uparrow x$, $x \in X$, where $\uparrow x := \{y \in X : x \sqsubseteq y\}$.

The Lawson topology $\lambda((X, \sqsubseteq))$ of a continuous poset $(X, \sqsubseteq)$ is the supremum topology of $\sigma((X, \sqsubseteq))$ and $\omega((X, \sqsubseteq))$.

A continuous poset which is also a dcpo is called a continuous domain or, simply, a domain.

A domain having a countable basis is said to be an ω-continuous domain or, simply, an ω-domain.

Recall [22] that a constructive maximal point of a domain $(X, \sqsubseteq)$ is an element x of X such that every $\lambda((X, \sqsubseteq))$-neighborhood of x contains a $\sigma((X, \sqsubseteq))$-neighborhood of x. The set of all constructive maximal points is denoted by $\mathrm{CMax}(X)$.

Since the theory of metric spaces and domain theory provide suitable mathematical structures in theoretical computer science, several authors have studied the question of obtaining connections between them. In this direction, Lawson characterized in [11] separable completely metrizable spaces in terms of ω-domains. This important result suggested the following concepts introduced by Martin [12] (see also [9, 19]), and Flagg and Kopperman [5], respectively.

Definition 1 ([12])**.** A model for a topological space (X, τ) is a pair (Y, ϕ) where Y is a domain and ϕ is a homeomorphism between (X, τ) and $\mathrm{Max}(Y)$, when $\mathrm{Max}(Y)$ is endowed with the restriction of the Scott topology of Y.

Definition 2 ([5])**.** A computational model (maximal point model in [2]) for a topological space (X, τ) is a model (Y, ϕ) for (X, τ) such that Y is an ω-domain and the restrictions to $\mathrm{Max}(Y)$ of the Scott topology and of the Lawson topology, agree on $\mathrm{Max}(Y)$.

On the other hand, Edalat and Heckmann [3] obtained, in an elegant and natural way, theoretic domain characterizations of complete and of separable complete metric spaces with the help of the simple notion of a formal ball. More precisely, they proved, among other results, that for a metric space (X, d) the following hold: (A) its poset of formal balls is continuous; (B) (X, d) is complete if and only if its poset of formal balls is a domain; (C) (X, d) is separable and complete if and only if its poset of formal balls is an ω-domain.

It then follows from [3, Theorem 13] that the domain of formal balls of a complete metric space (X, d) is a model for the topological space (X, τ_d), whereas the ω-domain of formal balls of a separable complete metric space (X, d) is a computational model for (X, τ_d).

Extensions and generalizations of Edalat and Heckmann's results to ultrametric spaces, Banach spaces, hyperspaces, uniform spaces, partial metric spaces, quasi-metric spaces and fuzzy metric spaces, may be found, for instance, in [1, 4, 5, 8, 9, 15, 16, 17, 18, 19, 20, 21].

In particular, it was observed in [1, 19] that, as in the metric case, if (X, d) is a quasi-metric space, the relation $\sqsubseteq_d$ defined on the set $\mathbf{B}X := X \times [0, \infty)$ of formal balls of X, as

$$(x, r) \sqsubseteq_d (y, s) \Longleftrightarrow d(x, y) \leq r - s,$$

for all $(x, r), (y, s) \in \mathbf{B}X$, is a partial order on $\mathbf{B}X$, so $(\mathbf{B}X, \sqsubseteq_d)$ is a poset, called the poset of formal balls of (X, d).

In [20] it was proved that the poset $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ of formal balls of the Sorgenfrey quais-metric space $(\mathbb{R}, d_S)$ is a domain. In this paper we shall show that $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is actually an ω-domain and hence it has the structure of a quantitative domain in the sense of Waszkiewicz (see Definition 4 in Section 3). In fact, we shall construct a relatively simple and somewhat surprising weightable quasi-metric q on $\mathbf{B}\mathbb{R}$ whose induced topology is weaker than the Scott topology of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ and such that the topology induced by the metric q^s agrees with the Lawson topology of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$. We also prove that the ω-domain $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is a computational model for the Sorgenfrey line.

2 The ω-domain of formal balls of the Sorgenfrey quasi-metric space

We start this section with three useful lemmas which provide a complete description of the way-below relation for $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$. (In the sequel, given a quasi-metric space (X, d), the way below relation associated to $\sqsubseteq_d$ will be denoted by $\ll_d$.)

Lemma 1 ([1]). *For any quasi-metric space (X,d) the following holds:*

$$(x,r) \ll_d (y,s) \Longrightarrow d(x,y) < r - s.$$

Lemma 2 ([20]). *For every $(x,r),(y,s) \in \mathbf{B}\mathbb{R}$ with $x \neq y$, the following holds:*

$$d_S(x,y) < r - s \Longrightarrow (x,r) \ll_{d_S} (y,s).$$

Lemma 3. *Let $x \in \mathbb{R}$ and $r,s \in \mathbb{R}^+$. Then*

$$(x,r) \ll_{d_S} (x,s) \Longleftrightarrow s+1 < r.$$

Proof. Suppose that $s+1 < r$. Let D be a directed subset of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ with least upper bound (z,t) such that $(x,s) \sqsubseteq_{d_S} (z,t)$.

Choose $\varepsilon \in]0,1[$ such that $s+1+\varepsilon < r$. Then, there is $(a,u) \in D$ such that $u < t+\varepsilon$ and $d_S(a,z) < \varepsilon$. Hence $0 \leq z - a < \varepsilon$. If $z = a$, we deduce that $d_S(x,a) \leq s - t < r - 1 - t < r - u$. If $z > a$, we deduce that

$$d_S(x,a) \leq d_S(x,z) + d_s(z,a) \leq s - t + 1 < r - \varepsilon - u + \varepsilon = r - u.$$

We have proved that $(x,r) \sqsubseteq_{d_s} (a,u)$, and thus $(x,r) \ll_{d_S} (x,s)$.

Conversely, construct the directed subset D of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ given by

$$D = \{(x - \frac{1}{n+1}, s + \frac{1}{n}) : n \in \mathbb{N}\}.$$

Clearly $\sqcup D = (x,s)$. Since for each $n \in \mathbb{N}$, $d_S(x, x-1/(n+1)) = 1 > (s+1)-(s+1/n)$, we deduce that $(x, s+1)$ is not way-below (x,s), so (x,r) is not way-below (x,s) whenever $s+1 \geq r$. ■

Combining the above lemmas we have the following consequence which will be used in Section 3.

Corollary 1. *For every $(x,r),(y,s) \in \mathbf{B}\mathbb{R}$ the following hold:*

(1) *If $x < y$, then*

$$(x,r) \ll_{d_S} (y,s) \Longleftrightarrow y - x < r - s.$$

(2) *If $x \geq y$, then*

$$(x,r) \ll_{d_S} (y,s) \Longleftrightarrow s+1 < r.$$

In [20] it was proved that $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is a domain. Next we show that actually it is an ω-domain.

Theorem 1. $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ *is an ω-domain.*

Proof. We will show that the domain $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ has a countable basis. Indeed, denote by $\mathbb{Q}$ and $\mathbb{Q}^+$ the set of all rational real numbers and the set of all non-negative rational numbers, respectively. Put $B = \mathbb{Q} \times \mathbb{Q}^+$. Then B is a countable subset of $\mathbf{B}\mathbb{R}$.

Choose an arbitrary $(x, r) \in \mathbf{B}\mathbb{R}$. Let $(r_n)_n$ be a strictly decreasing sequence in $\mathbb{Q}^+$ such that $\lim_n r_n = r$. Construct a strictly increasing sequence $(q_n)_n$ in $\mathbb{Q}$ satisfying $q_n < x < q_n + r_n - r$ and $q_{n+1} < q_n + r_n - r_{n+1}$ for all $n \in \mathbb{N}$. Clearly $(q_n, r_n)_n$ is an ascending sequence in B with least upper bound (x, r). Moreover $(q_n, r_n) \ll_{d_S} (x, r)$ for all $n \in \mathbb{N}$, by Lemma 2. So, in particular, $\Downarrow (x, r)_B \neq \emptyset$.

Now let $(y, s), (z, t) \in \Downarrow (x, r)_B$. Then, there is $n \in \mathbb{N}$ such that $(y, s) \sqsubseteq_{d_S} (q_n, r_n)$ and $(z, t) \sqsubseteq_{d_S} (q_n, r_n)$, and hence $\Downarrow (x, r)_B$ is directed.

Finally, let $(z, t) \in \mathbf{B}\mathbb{R}$ be an upper bound of $\Downarrow (x, r)_B$. Then $d_S(q_n, z) \leq r_n - t$ for all $n \in \mathbb{N}$.

If $x \leq z$ we have $q_n < z$, so $d_S(q_n, z) = z - q_n \leq r_n - t$ for all $n \in \mathbb{N}$, and thus $d_S(x, z) = z - x < z - q_n \leq r_n - t$ for all $n \in \mathbb{N}$. Since $\lim_n r_n = r$ it follows that $d_S(x, z) \leq r - t$.

If $x > z$, we have $q_n > z$ eventually, so $1 = d_S(q_n, z) \leq r_n - t$ eventually, and thus $1 \leq r - t$. Hence $d_S(x, z) = 1 \leq r - t$.

We have shown that $\sqcup(\Downarrow (x, r)_B) = (x, r)$. This concludes the proof. ■

Improving results of [1] we now state the following.

Theorem 2. *$(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is a computational model for $(\mathbb{R}, \tau_S)$ via the embedding mapping $i : \mathbb{R} \to \mathbf{B}\mathbb{R}$ given by $i(x) = (x, 0)$* for all $x \in \mathbb{R}$.

Proof. We first observe that $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}) = \{(x, 0) : x \in \mathbb{R}\}$. Thus i is clearly a bijection between $\mathbb{R}$ and $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$.

Now let $(y, s) \in \mathbf{B}\mathbb{R}$. If $0 < s \leq 1$, we obtain

$$i^{-1}(\Uparrow (y, s) \cap \mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) = \{x \in \mathbb{R} : (y, s) \ll_{d_S} (x, 0)\} = [y, y + s[,$$

and if $s > 1$, we obtain

$$i^{-1}(\Uparrow (y, s) \cap \mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) =]-\infty, y + s[.$$

Therefore $i^{-1}(\Uparrow (y, s) \cap \mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$ is open for τ_{d_S}.

Moreover, for each $x \in \mathbb{R}$ and each $\varepsilon \in]0, 2/3[$ it follows that

$$i(B_{d_S}(x, \varepsilon)) = \{(y, 0) : d_S(x, y) < \varepsilon\} = (\Uparrow (x - \frac{\varepsilon}{2}, \frac{3\varepsilon}{2})) \cap \mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}),$$

and thus $i(B_{d_S}(x, \varepsilon))$ is open for the restriction of the Scott topology to $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$.

Hence i is an homeomorphism between (X, τ_{d_S}) and $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$, when $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is endowed with the restriction of the Scott topology of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_s})$.

Finally, we show that the restriction of the Scott topology and the restriction of the Lawson topology agree on $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$. To this end, let $(x, 0) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y, s)$. Then $d_S(y, x) > s$, so $x \neq y$.

If $x > y$, we have $x - y > s$. Choose $n \in \mathbb{N}$ such that $x - y > s + 1/n$. Put $V = \Uparrow (]x - 1/(n+1), 1/n)$. Then V is an open set for the Scott topology with $(x, 0) \in V$. Let $(z, t) \in V \cap \mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$. Then $t = 0$ and thus $d_s(x - 1/(n+1), z) < 1/n$, so, in particular, $0 \leq z - (x - 1/(n+1))$. Therefore

$$z - y \geq x - y - 1/(n+1) > x - y - 1/n > s,$$

which implies that $d_S(y,z) = z - y > s$, so $(z,t) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$.

If $x < y$, we first note that $d_S(y,x) = 1 > s$. Now choose $n \in \mathbb{N}$ such that $y - x > 1/n(n+1)$. As in the above case, put $V = \Uparrow (x - 1/(n+1), 1/n)$ and let $(z,t) \in V \cap \mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$. Then $t = 0$ and thus $d_S(x - 1/(n+1), z) < 1/n$, so, $z - (x - 1/(n+1)) < 1/n$, and hence $z < x + 1/n(n+1) < y$. Therefore $d_S(y,z) = 1 > s$, so $(z,t) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$. ■

3 The ω-domain $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ as a quantitative domain

Improving some results of [3] cited above, Heckmann proved in [8] that for any metric space (X,d), the function $d^H : \mathbf{B}X \times \mathbf{B}X \to \mathbb{R}^+$ given by

$$d^H((x,r),(y,s)) = (d(x,y) \vee |r-s|) + s - r,$$

for all $(x,y),(r,s) \in \mathbf{B}X$, is a weightable quasi-metric on $\mathbf{B}X$, with weight function w given by $w((x,r)) = 2r$ for all $(x,r) \in \mathbf{B}X$, that extends the metric d to $\mathbf{B}X$ and such that its induced topology agrees with the Scott topology of the continuous poset $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ (actually, Heckmann did his construction in the realm of partial metrics, a framework equivalent to the one given by weightable quasi-metrics [14, Theorems 4.1 and 4.2]).

According to [20, p. 466], in the sequel we shall refer to the quasi-metric d^H as the Heckmann quasi-metric of (X,d).

It is interesting to point out that Heckmann's construction provides a model for metric spaces of a quantitative nature. This fact motivated, in part, the following notions due to Schelleknes [21] and Waszkiewicz [23], and Waszkiewicz [24], respectively.

Definition 3 ([21, 23])**.** A domain X is called a quantifiable domain if there is a weightable quasi-metric q on X whose induced topology agrees with the Scott topology of X.

Definition 4 ([24])**.** A quantitative domain is a domain X such that there is a weightable quasi-metric q on X, with weight function w, satisfying the following conditions:

(a) the topology induced by q is weaker than the Scott topology of X;
(b) the weight function w is a measurement on X in the sense of [13];
(c) $\ker w = \mathrm{CMax}(X)$, where $\ker w = \{x \in X : w(x) = 0\}$;
(d) The metric q^s induces the Lawson topology of X.

Remark 1. Schellekens [21] and Waszkiewicz [23] independently proved that for each ω-domain its Scott topology is induced by a weightable quasi-metric, so, by Theorem 1, $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is a quantifiable domain. For instance, the following is an adaptation of Schellekens' construction to $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$.

Put $\mathbb{Q} \times \mathbb{Q}^+ = \{(q_n, u_n) : n \in \mathbb{N}\}$ and denote by d_S^w the function on $\mathbf{B}\mathbb{R} \times \mathbf{B}\mathbb{R}$ defined by

$$d_S^w((x,r),(y,s)) = \sum \left\{2^{-n} : (q_n,u_n) \in \Downarrow (x,r) \setminus \Downarrow (y,s)\right\}.$$

Then [21], d_S^w is a weightable quasi-metric on $\mathbf{B}\mathbb{R}$ that induces the Scott topology of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ and such that $(d_S^w)^{-1}$ is also weightable.

Next we show that, nevertheless, there is no quasi-metric q on $\mathbf{B}\mathbb{R}$ inducing the Scott topology of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ and such that the embedding i of Theorem 2 provides an isometry between $(\mathbb{R}, d_S)$ and $\mathrm{Max}(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ endowed with the restriction of the quasi-metric q :

Let q be a quasi-metric on $\mathbf{B}\mathbb{R}$ such that $\tau_q = \sigma(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$. Fix $x \in \mathbb{R}$. Since the sequence $((x - 1/(n+1), 1/n))_n$ converges to $(x, 0)$ for the Scott topology (compare the proof of Lemma 3), there is $k \in \mathbb{N}$ such that $q((x, 0), (x - 1/(k+1), 1/k)) < 1$. Choose $y \in]x-1/(k+1), x[$. Then $(x-1/(k+1), 1/k) \sqsubseteq_{d_S} (y, 0)$, so $q((x-1/(k+1), 1/k), (y, 0)) = 0$. Then, it follows from the triangle inequality that $q((x, 0), (y, 0)) < 1$. However $d_S(x, y) = 1$ because $x > y$.

On the other hand, Waszkiewicz proved in [24] that each ω-domain is also a quantitative domain, so, again by Theorem 1, $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ is a quantitative domain.

We conclude the paper by showing the somewhat surprising fact that the Heckmann quasi-metric of the ω-domain of formal balls of the metric space $(\mathbb{R}, e_1)$, where e_1 is the metric on $\mathbb{R}$ given by $e_1(x, y) = |x - y| \wedge 1$ for all $x, y \in \mathbb{R}$, verifies conditions (a), (c) and (d) of Definition 4, as well as a weak form of (b).

To this end, we first recall that $(\mathbb{R}, e_1)$ is a separable complete metric space, and hence $(\mathbf{B}\mathbb{R}, \sqsubseteq_{e_1})$ is an ω-domain whose Scott topology is induced by the Heckmann quasi-metric $(e_1)^H$ of $(\mathbb{R}, e_1)$ (recall that $(e_1)^H$ is weightable with weight function w given by $w((x, r)) = 2r$).

Now we prove the following results.

Proposition 1. *$\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{e_1}))$ is strictly weaker than $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.*

Proof. Let $(x, r) \in \mathbf{B}\mathbb{R}$ and let $0 < \varepsilon < 1$. We show that $\Uparrow (x - \varepsilon/2, r + \varepsilon) \subset B_{(e_1)^H}((x, r), 2\varepsilon)$.

Indeed, let $(y, s) \in \Uparrow (x - \varepsilon/2, r + \varepsilon)$. Then $d_S(x - \varepsilon/2, y) < r + \varepsilon - s$. We have three cases:

Case 1. $y < x - \varepsilon/2$. Then $1 < r + \varepsilon - s$, so $e_1(x, y) < r + \varepsilon - s$. Hence

$$(e_1)^H((x, r), (y, s)) < ((r + \varepsilon - s) \vee |r - s|) + s - r = \varepsilon.$$

Case 2. $x - \varepsilon/2 \leq y < x$. Then $e_1(x, y) < \varepsilon/2$, and $0 \leq y - (x - \varepsilon/2) < r + \varepsilon - s$, so, in particular, $s < r + \varepsilon$. Hence $(e_1)^H((x, r), (y, s)) < \varepsilon/2$ whenever $r \geq s$, and $(e_1)^H((x, r), (y, s)) < 2\varepsilon$ whenever $r < s$.

Case 3. $x \leq y$. Then $0 \leq y - (x - \varepsilon/2) < r + \varepsilon - s$, so $0 \leq y - x < r - s + \varepsilon/2$, and, in particular, $s < r + \varepsilon/2$. Hence

$$(e_1)^H((x, r), (y, s)) < ((r - s + \varepsilon/2) \vee |r - s|) + s - r < \varepsilon.$$

We conclude that $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{e_1}))$ is weaker than $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Finally, let $\varepsilon \in]0, 1/3[$ and consider the $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$-open neighborhood of $(0, 1)$, $\Uparrow (-\varepsilon, 1 + 2\varepsilon)$. Then, we have $(e_1)^H((0, 1), (-\varepsilon, 2\varepsilon)) = 0$, but by Lemma 3, $(-\varepsilon, 2\varepsilon) \notin \Uparrow (-\varepsilon, 1 + 2\varepsilon)$.

This completes the proof. $\blacksquare$

Remark 2. Since $\tau_{(e_1)^H} = \sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{e_1})$, it follows from Proposition 1 that the weightable quasi-metric $(e_1)^H$ satisfies condition (a) of Definition 4.

Lemma 4. *Let* $(x, r) \in \mathbf{B}\mathbb{R}$. *Then, the countable family*

$$\left\{ \Uparrow (x - 2^{-(n+1)}, r + 2^{-n}) : n \in \mathbb{N} \right\},$$

is a base of neighborhoods of (x, r) *for* $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Proof. First note that, by Lemma 2, $(x, r) \in \Uparrow (x - 2^{-(n+1)}, r + 2^{-n})$ for all $n \in \mathbb{N}$ because $d_S(x - 2^{-(n+1)}, x) = 2^{-(n+1)} < (r + 2^{-n}) - r$.

Now suppose that there exist $k \in \mathbb{N}$ and a sequence $(z_n, t_n)_n$ in $\mathbf{B}\mathbb{R}$ such that

$$(z_n, t_n) \in \Uparrow (x - 2^{-(n+1)}, r + 2^{-n}) \backslash B_{d_S^w}((x, r), 2^{-k}),$$

for all $n \in \mathbb{N}$, where, we recall, d_S^w is the weigthable quasi-metric of Remark 1.

Then, since $d_S^w((x, r), (z_n, t_n)) \geq 2^{-k}$, we can assume, without loss of generality, that there is $m < k + 2$ such that

$$(q_m, u_m) \in \Downarrow (x, r) \backslash \Downarrow (z_n, t_n),$$

for all $n \in \mathbb{N}$.

Without loss of generality we distinguish the following two cases:

Case 1. $z_n = q_m$ for all $n \in \mathbb{N}$.

Since $(q_m, u_m) \notin \Downarrow (q_m, t_n)$, it follows, from Lemma 3, that $1 + t_n \geq u_m$ for all $n \in \mathbb{N}$, and thus $1 + \inf_n t_n \geq u_m$.

Moreover

$$d_S(x - 2^{-(n+1)}, q_m) < r + 2^{-n} - t_n,$$

for all $n \in \mathbb{N}$ except, possibly, for a unique n_0 (in case that $x - 2^{-(n_0+1)} = q_m$), and thus $\inf_n t_n \leq r$.

If $q_m < x$, it follows that $q_m < x - 2^{-(n+1)}$ eventually, so $1 < r + 2^{-n} - t_n$ eventually, and thus $1 + \inf_n t_n \leq r$. Hence $u_m \leq r$, which contradicts the fact that $(q_m, u_m) \in \Downarrow (x, r)$ (see Lemma 1).

If $x \leq q_m$, it follows that $r + 1 < u_m$ by Corollary 1. Hence $r < u_m - 1 \leq \inf_n t_n \leq r$, a contradiction.

Case 2. $z_n \neq q_m$ for all $n \in \mathbb{N}$.

Since $(q_m, u_m) \notin \Downarrow (z_n, t_n)$, it follows from Lemma 2 that $d_S(q_m, z_n) \geq u_m - t_n$ for all $n \in \mathbb{N}$.

Moreover

$$d_S(x - 2^{-(n+1)}, z_n) < r + 2^{-n} - t_n,$$

for all $n \in \mathbb{N}$.

If $q_m < x$, it follows that $q_m < x - 2^{-(n+1)}$ eventually, and, by Lemma 2, $x - q_m < u_m - r - \varepsilon$ for some $\varepsilon \in]0, 1[$.

Hence

$$\begin{aligned}
u_m - t_n &\leq d_S(q_m, z_n) \leq d_S(q_m, x - 2^{-(n+1)}) + d_S(x - 2^{-(n+1)}, z_n) \\
&< x - q_m - 2^{-(n+1)} + r + 2^{-n} - t_n \\
&< u_m - r - \varepsilon + 2^{-(n+1)} + r - t_n,
\end{aligned}$$

eventually. Therefore $\varepsilon < 2^{-(n+1)}$, eventually, a contradiction.

If $x \le q_m$, it follows that $r+1 < u_m$ by Corollary1. Hence

$$\begin{aligned} u_m - t_n &\le d_S(q_m, z_n) \le d_S(q_m, x - 2^{-(n+1)}) + d_S(x - 2^{-(n+1)}, z_n) \\ &< 1 + r + 2^{-n} - t_n, \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore $u_m \le 1 + r$, which contradicts the fact that $r + 1 < u_m$. This concludes the proof. ■

Let us recall that a measurement on a domain $(X, \sqsubseteq)$ is a function $\mu : X \to \mathbb{R}^+$ satisfying the following conditions:

(i) μ is Scott continuous from $(X, \sqsubseteq)$ into $(\mathbb{R}^+, \le_{-1})$, where $r \le_{-1} s$ if and only if $s \le r$.

(ii) for each $V \in \sigma((X, \sqsubseteq))$ and each $y \in V$ there is $\varepsilon > 0$ such that

$$\{z \in X : z \sqsubseteq y \text{ and } \mu(z) < \mu(y) + \varepsilon\} \subseteq V.$$

The following example shows that the weight function w of $(e_1)^H$ is not a measurement on $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$:

Choose any $x \in \mathbb{R}$ and $k \in \mathbb{N}$. Put $V = \Uparrow (x - 2^{-(k+1)}, 2^{-k})$. By Lemma 4, $V \in \sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$. We have $(x - 2^{-(k+1)}, 0) \in V$. However, for each $\varepsilon > 0$, the formal ball $(z, \varepsilon/4)$, where $z = x - 2^{-(k+1)} - \varepsilon/4$, satisfies $(z, \varepsilon/4) \sqsubseteq_{d_S} (x - 2^{-(k+1)}, 0)$ and $w((z, \varepsilon/4)) < w((x - 2^{-(k+1)}, 0)) + \varepsilon$, but $(z, \varepsilon/4) \notin V$.

Nevertheless, we can prove the following.

Proposition 2. *The weight function w for $(e_1)^H$, given by $w((x,r)) = 2r$, is a Scott continuous function on $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$ such that for each $(x,r) \in \mathbf{B}\mathbb{R}$ and each $k \in \mathbb{N}$ the following holds:*

Whenever $(y,s) \in \Uparrow (x - 2^{-(k+1)}, r + 2^{-k})$, with $y \ne x - 2^{-(k+1)}$, there is $\varepsilon > 0$ such that $\{(z,t) \in \mathbf{B}\mathbb{R} : (z,t) \sqsubseteq_{d_S} (y,s) \text{ and } w((z,t)) < w((y,s)) + \varepsilon\} \subseteq \Uparrow (x - 2^{-(k+1)}, r + 2^{-k})$.

Proof. Scott continuity of w is an immediate consequence of the fact that if D is a directed subset of $(\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})$, then $\sqcup D := (x_0, r_0)$ satisfies $r_0 = \inf_{(x,r) \in D} r$.

Now let $(x,r) \in \mathbf{B}\mathbb{R}$, $k \in \mathbb{N}$ and $(y,s) \in \Uparrow (x - 2^{-(k+1)}, r + 2^{-k})$, with $y \ne x - 2^{-(k+1)}$. We distinguish two cases:

Case 1. $y < x - 2^{-(k+1)}$. Then $d_S(x - 2^{-(k+1)}, y) = 1 < r + 2^{-k} - s$. Choose $\varepsilon \in]0,1[$ such that $1 + \varepsilon/2 < r + 2^{-k} - s$. Then, for $(z,t) \sqsubseteq_{d_S} (y,s)$ with $2t < 2s + \varepsilon$, we have $d_S(z,y) \le t - s < \varepsilon/2 < 1$, so $z \le y$. Therefore

$$d_S(x - 2^{-(k+1)}, z) = 1 < r + 2^{-k} - s - \varepsilon/2 < r + 2^{-k} - t,$$

i.e., $(z,t) \in \Uparrow (x - 2^{-(k+1)}, r + 2^{-k})$.

Case 2. $x - 2^{-(k+1)} < y$. Then $d_S(x - 2^{-(k+1)}, y) = y - (x - 2^{-(k+1)}) < r + 2^{-k} - s$. Choose $\varepsilon \in]0,1[$ such that $\varepsilon < y - (x - 2^{-(k+1)})$ and $\varepsilon + y - (x - 2^{-(k+1)}) < r + 2^{-k} - s$. Then, for $(z,t) \sqsubseteq_{d_S} (y,s)$ with $2t < 2s + \varepsilon$, we have as in Case 1, $d_S(z,y) < \varepsilon/2$, so $0 \le y - z < \varepsilon/2$. Consequently $z - (x - 2^{-(k+1)}) > y - \varepsilon/2 - (x - 2^{-(k+1)}) > 0$, and hence

$$\begin{aligned} d_S(x - 2^{-(k+1)}, z) &= z - (x - 2^{-(k+1)}) \le y - (x - 2^{-(k+1)}) \\ &< r + 2^{-k} - s - \varepsilon/2 < r - 2^{-k} - t, \end{aligned}$$

i.e., $(z,t) \in \Uparrow (x - 2^{-(k+1)}, r + 2^{-k})$. $\blacksquare$

Proposition 3. *The weight function* w *for* $(e_1)^H$, *given by* $w((x,r)) = 2r$, *satisfies* $\ker w = \mathrm{CMax}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Proof. We prove that $\mathrm{Max}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) = \mathrm{CMax}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Let $(x,0) \in \mathrm{Max}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$. In order to show that $(x,0) \in \mathrm{CMax}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$, it suffices to prove that if $(x,0) \notin \uparrow (y,s)$, then there exists a $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$-neighborhood V of $(x,0)$ satisfying $V \subseteq \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$. Indeed, suppose $(x,0) \notin \uparrow (y,s)$. Then $d_S(y,x) > s$, and we distinguish two cases.

Case 1. $y > x$. Then $1 > s$. Choose $k \in \mathbb{N}$ such that $y > x + 2^{-k}$. Define $V = \Uparrow (x - 2^{-(k+1)}, 2^{-k})$ and let $(z,t) \in V$. Then $d_S(x - 2^{-(k+1)}, z) < 2^{-k} - t < 1$, so, $0 \le z - (x - 2^{-(k+1)}) < 2^{-k} - t < y - x$, and hence $z < y$. Therefore $d_S(y,z) = 1 > s \ge s - t$, i.e., $(z,t) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$.

Case 2. $y < x$. Then $x - y > s$. Choose $k \in \mathbb{N}$ such that $x - y > s + 2^{-k}$. Define $V = \Uparrow (x - 2^{-(k+1)}, 2^{-k})$ and let $(z,t) \in V$. As in Case 1, $0 \le z - (x - 2^{-(k+1)})$. Hence $y + s < x - 2^{-k} < z$, and consequently $d_S(y,z) = z - y > s > s - t$, i.e., $(z,t) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$.

Now let $(x,r) \in \mathrm{CMax}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$ and suppose that $r > 0$. Choose $\varepsilon \in]0, r[$. Since $(x,r) \in \mathrm{CMax}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$ there exists $k \in \mathbb{N}$ such that $\Uparrow (x - 2^{-(k+1)}, r + 2^{-k}) \subseteq \mathbf{B}\mathbb{R} \setminus \uparrow (x, r - \varepsilon)$. However $(x, r - \varepsilon) \in \Uparrow (x - 2^{-(k+1)}, r + 2^{-k}) \cap \uparrow (x, r - \varepsilon)$, which provides a contradiction. Therefore $r = 0$, so $(x,r) \in \mathrm{Max}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

The fact that $\ker w = \mathrm{Max}((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$ concludes the proof. $\blacksquare$

Proposition 4. $\tau_{((e_1)^H)^s} = \lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Proof. We first prove that $\tau_{((e_1)^H)^s} \subseteq \lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$. To this end, note that from Proposition 1 (or Remark 2), $\tau_{(e_1)^H} \subseteq \lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Now we prove that $\tau_{((e_1)^H)^{-1}} \subseteq \lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

Let $(x,r) \in \mathbf{B}\mathbb{R}$ and $\varepsilon \in]0,1[$. Take $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$ and construct the $\lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$-neighborhood V of (x,r) defined as

$$V := \Uparrow (x - 2^{-(k+1)}, r + 2^{-k}) \cap (\mathbf{B}\mathbb{R} \setminus \uparrow (x + \varepsilon, r + \varepsilon)) \cap (\mathbf{B}\mathbb{R} \setminus \uparrow (x - \varepsilon, r + \frac{\varepsilon}{2})).$$

We are going to show that $V \subseteq B_{((e_1)^H)^{-1}}((x,r), 3\varepsilon/2)$. Indeed, choose any $(y,s) \in V$. Then

(I_1) $d_S(x - 2^{-(k+1)}, y) < r + 2^{-k} - s$,

(I_2) $d_S(x + \varepsilon, y) > r + \varepsilon - s$,

(I_3) $d_S(x - \varepsilon, y) > r + \varepsilon/2 - s$,

Since $2^{-k} < \varepsilon$, it follows from inequality (I_1) that $s - r < \varepsilon$, and by inequalities (I_1) and (I_2) that $x - 2^{-(k+1)} \le y$, and we shall distinguish two cases.

Case 1. $y < x$. Then $x - y \le 2^{-(k+1)} < \varepsilon/2$, and by ($\mathrm{I}_3$), $y - (x - \varepsilon) > r + \varepsilon/2 - s$, which implies that $r - s < \varepsilon/2$. Therefore

$$((e_1)^H)^{-1}((x,r),(y,s)) = (e_1(x,y) \vee |r - s|) + r - s < 3\varepsilon/2.$$

Case 2. $x \le y$. Then, by (I_1) and (I_2), $y < x+\varepsilon$. Since by (I_3), $y-(x-\varepsilon) > r+\varepsilon/2-s$, we deduce that $r - s < \varepsilon/2$. Hence, as in Case 1, $((e_1)^H)^{-1}((x,r),(y,s)) < 3\varepsilon/2$.

Therefore $\tau_{((e_1)^H)^{-1}} \subseteq \lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$. We conclude that $\tau_{((e_1)^H)^s} \subseteq \lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$.

It remains to prove that $\lambda((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) \subseteq \tau_{((e_1)^H)^s}$. To this end, we first show that $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) \subseteq \tau_{((e_1)^H)^s}$.

Let $(x,r) \in \mathbf{B}\mathbb{R}$ and $V := \Uparrow (x-2^{-(k+1)}, r+2^{-k})$ be a basic $\sigma((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S}))$-neighborhood of (x,r). We are going to check that $B_{((e_1)^H)^s}((x,r), 2^{-(k+2)}) \subseteq V$. Indeed, let $(y,s) \in B_{((e_1)^H)^s}((x,r), 2^{-(k+2)})$. Then $e_1(x,y) < 2^{-(k+2)}$ and $|r-s| < 2^{-(k+2)}$. Since $|x-y| < 2^{-(k+2)}$ we deduce that $x - 2^{-(k+1)} < y$. Hence

$$\begin{aligned} d_S(x - 2^{-(k+1)}, y) &= y - (x - 2^{-(k+1)}) < 2^{-(k+2)} + 2^{-(k+1)} \\ &= 2^{-k} - 2^{-(k+2)} < 2^{-k} + s - r, \end{aligned}$$

and thus $(y,s) \in V$.

Finally we prove that $\omega((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) \subseteq \tau_{((e_1)^H)^s}$. Let $(x,r) \notin \uparrow (y,s)$. We are going to show that there is $\varepsilon \in]0,1[$ such that $B_{((e_1)^H)^s}((x,r), \varepsilon) \subseteq \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$.

If $x < y$, we have $d_S(y,x) = 1 > s - r$. Choose $\varepsilon \in]0,1[$ such that $\varepsilon < (y-x) \wedge (1 + r - s)$. Then, for any $(z,t) \in B_{((e_1)^H)^s}((x,r), \varepsilon)$, we deduce that $|x - z| < \varepsilon$ and $|r - t| < \varepsilon$. It immediately follows that $z < y$ and $1 > s - t$, so $d_S(y,z) > s - t$, i.e., $(z,t) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$.

If $y \le x$, we have $d_S(y,x) = x - y > s - r$. Choose $\varepsilon \in]0,1[$ such that $x - y > 2\varepsilon \vee (s - r + 2\varepsilon)$. Then, for any $(z,t) \in B_{((e_1)^H)^s}((x,r), \varepsilon)$ we obtain $z - y > x - \varepsilon - y > s - r + \varepsilon > s - t$, and consequently $(z,t) \in \mathbf{B}\mathbb{R} \setminus \uparrow (y,s)$.

Therefore $\omega((\mathbf{B}\mathbb{R}, \sqsubseteq_{d_S})) \subseteq \tau_{((e_1)^H)^s}$. This finishes the proof. ■

Remark 3. It follows from Propositions 3 and 4 that the weightable quasi-metric $(e_1)^H$ satisfies conditions (c) and (d) of Definition 4.

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