Valentín Gregori[1], Juan-José Miñana[2], Oscar Valero[2]

[1]*Instituto de Matemática Pura y Aplicada,*
Universitat Politècnica de València, Campus de Gandia
Calle Paranimf 1, 467302 Gandia (SPAIN)
vgregori@mat.upv.es

[2]*Departamento de Ciencias Matemáticas e Informática*
Universidad de las Islas Baleares
Carretera de Valldemossa km. 7.5, 07122 Palma (SPAIN)
jj.minana@uib.es, o.valero@uib.es

Abstract

In this paper we develop a new technique for constructing fuzzy metric spaces, in the sense of George and Veeramani, from metric spaces and by means of the Lukasievicz t-norm. In particular such a technique is based on the use of metric preserving functions in the sense of J. Doboš. Besides, the new generated fuzzy metric spaces are strong and completable, and if we add an extra condition, they are principal. Appropriate examples of such fuzzy metric spaces are given in order to illustrate the exposed technique.

Keywords: Fuzzy metric space, completion, metric preserving function, stationary, strong (non-Archimedean), principal.
2010 MSC: 54A40, 54D35, 54E50

---

1. Introduction

In 1994 George and Veeramani introduced in [3] a concept of fuzzy metric space. Since then, several authors has been studied deeply this concept both from the mathematical point of view (see, for instance, [8, 15, 16, 18]) and from the applied viewpoint (see, for instance, [1, 13, 14, 12]).

Regarding the mathematical point of view, in [8], V. Gregori and S. Romaguera showed that metrizable topological spaces coincide with fuzzy metrizable topological spaces. It follows that topologically, metric and fuzzy metric spaces are identical. However, we can find differences between these two concepts when we focus our attention on the intrinsic (fuzzy) metric properties. An instance of such differences that are worth mentioning is provided in [9] where Gregori and Romaguera proved that there exist fuzzy metric spaces which do not admit completion. An important class of fuzzy metric spaces that are not in general completable are the so-called fuzzy stationary metric spaces (see Definition 2.9). This type of fuzzy metric spaces are the closest to the classical metrics and, in fact, many properties can be extended from metrics to stationary fuzzy metrics in a straightforward manner, specially when the stationary fuzzy metric is defined by means of the Lukasievicz t-norm $\mathfrak{L}$. It is due to the fact that stationary fuzzy metrics enjoy two distinguished properties, they are principal (see Definition 2.12) and strong (non-Archimedean) (see Definition 2.18).

In the matter of applications, a few techniques used in image filtering and in the study of perceptual color difference have been improved when a classical metric has been replaced by a fuzzy metric. Nonetheless, it must be pointed out that the shortage of examples of fuzzy metrics in the literature turns be a drawback when one wants to apply fuzzy metrics to the aforesaid engineering problems.

The aim of this paper is twofold: On the one hand, inspired by the fact that there are fuzzy metric spaces that are not completable, we develop a technique for constructing fuzzy metric spaces from metric spaces and by means of the Lukasievicz t-norm which are completable. In particular such a technique is based on the use of metric preserving functions in the sense of J. Doboš ([2]). Besides, the new generated fuzzy metric spaces are strong and when we add an extra condition they are also principal. Furthermore, we show that some well-known examples can be obtained using our technique. On the other hand, motivated for the aforementioned lack of examples, new examples can be constructed applying our new technique in order to overcome the mentioned drawback.

The paper is organized as follows. Section 2 is devoted to recall the basic notions that will be crucial throughout the paper. In Section 3, we introduce the notion of uniformly continuous mapping between stationary fuzzy metric spaces and metric spaces, and *vice-versa*. Thus we define when they are equivalent. Based on such a notion, we present a technique that allows to construct stationary fuzzy metric spaces from a metric space by means of metric preserving functions with values in $[0,1[$. Moreover, it is showed that the new constructed stationary fuzzy metric spaces are completable provided that the used metric preserving function is a strongly metric preserving function (in the sense of Doboš). In addition, it is proved that the new stationary fuzzy metric spaces are complete if and only if the metric spaces from which are generated are also complete. Section 4 is devoted to generalize the construction presented in Section 3 to the non-stationary case. Thus fuzzy metric spaces are generated from metric spaces by means of a family of metric preserving functions that satisfy a distinguished condition which will be specified later on. These fuzzy metric spaces are always strong and, in addition, they are complete if and only if the metric spaces from which are generated are also complete. Furthermore, they are principal and completable whenever the set of all metric preserving functions belonging to the family under consideration are strongly metric preserving functions. Appropriate examples that illustrate the exposed theory are also yielded.

2. Preliminaries

In the following we will recall the notions that will be crucial in our subsequent work. With this aim, we will divide this section in two parts. In the first part, we will recall those notions related to metric preserving functions. In the second one, we will fix the pertinent notions about fuzzy metric spaces in which our work will be based on.

2.1. Metric Preserving functions

We recall, according to Doboš, the basic and pertinent notions about metric preserving functions (for a detailed treatment we refer the reader to [2]).

Let (X, d) be a metric space. For each $f : [0, \infty[\to [0, \infty[$ denote by d_f the function $d_f : X \times X \to [0, \infty[$ defined as follows

$$d_f(x, y) = f(d(x, y)) \text{ for each } x, y \in X.$$

In the light of the preceding construction we are able to introduce the notion of metric preserving function.

Definition 2.1. A function $f : [0, \infty[\to [0, \infty[$ is a *metric preserving* if for each metric space (X, d) the function d_f is a metric on X.

From now on, we will denote by $\mathcal{M}$ the class of all metric preserving functions. Moreover, we will denote by $\mathcal{O}$ the set of all functions $f : [0, \infty[\to [0, \infty[$ with $f^{-1}(0) = \{0\}$. It is obvious that $\mathcal{M} \subset \mathcal{O}$.

The next result provides a distinguished class of metric preserving functions. In order to state such a result we recall the notion of subadditive function. A function $f : [0, \infty[\to [0, \infty[$ is subadditive provided

$$f(a + b) \leq f(a) + f(b)$$

for each $a, b \in [0, \infty[$.

Theorem 2.2. *If $f \in \mathcal{O}$ and it is non-decreasing and subadditive, then $f \in \mathcal{M}$.*

Following [2], a metric space (X, d) is said to be uniformly discrete whenever there exists $\varepsilon > 0$ such that $d(x, y) > \varepsilon$ for all $x, y \in X$ with $x \neq y$.

The next result characterizes those metric preserving functions that are not continuous on $[0, \infty[$ with the usual topology of the real numbers $\mathbb{R}$.

Proposition 2.3. *Let $f \in \mathcal{M}$. Then f is discontinuous if and only if (X, d_f) is uniformly discrete for every metric space (X, d).*

A special class of metric preserving functions is the so-called strongly metric preserving functions. Let us recall such a notion.

Definition 2.4. A function $f : [0, \infty[\to [0, \infty[$ is *strongly metric preserving* if for each metric space (X, d) the function d_f is a metric on X topologically equivalent to d.

From now on, we will denote by $\mathcal{M}_S$ the class of all strongly metric preserving functions.

The next result characterizes strongly metric preserving functions.

Theorem 2.5. *Let $f \in \mathcal{M}$. Then the following assertions are equivalent:*

1. *f is continuous,*
2. *f is continuous at 0,*
3. *for each $\epsilon > 0$ we can find $x > 0$ such that $f(x) < \epsilon$,*
4. *$f \in \mathcal{M}_S$.*

Following [2], let us recall that two metrics d and ρ on X are called uniformly equivalent if both identity mappings $i : (X, d) \to (X, \rho)$ and $i : (X, \rho) \to (X, d)$ are uniformly continuous.

In the light of Theorem 2.5 the following two results can be obtained.

Theorem 2.6. *Let $f \in \mathcal{M}$. Suppose that (X, d) is a metric space which is not uniformly discrete. Then the metrics d_f and d are uniformly equivalent if and only if $f \in \mathcal{M}_S$.*

Theorem 2.7. *Let $f \in \mathcal{M}$. Suppose that (X, d) is a metric space which is uniformly discrete. Then, the metrics d_f and d are uniformly equivalent.*

2.2. Fuzzy metric spaces

In [3], Goerge and Veeramani introduced the following notion of fuzzy metric space.

Definition 2.8. A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(GV1) $M(x, y, t) > 0$;

(GV2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV3) $M(x, y, t) = M(y, x, t)$;

(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(GV5) $M(x, y, \_) :]0, \infty[\rightarrow]0, 1]$ is a continuous function.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$ is a *fuzzy metric* on X, or simply, M is a fuzzy metric on X.

According to [3], every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d be a fuzzy set on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Following [3], $(X, M_d, \cdot)$ is a fuzzy metric space, where $\cdot$ stands for the product t-norm. The fuzzy metric M_d is called the *standard fuzzy metric* induced by d. Moreover, the topology τ_{M_d} coincides with the topology on X induced by d.

The following notion, which is due to Gregori and Romaguera (see [10]), will play a central role later on.

Definition 2.9. A fuzzy metric space $(X, M, *)$ is said to be *stationary* if M does not depend on t, i.e., if the function $M_{x,y}(t) = M(x, y, t)$ is constant for each $x, y \in X$.

When a fuzzy metric space $(X, M, *)$ is stationary we will say that M is a stationary fuzzy metric and, in addition, we will write $M(x, y)$ instead of $M(x, y, t)$ when no confusion arises.

The next result, given by George and Veeramani in [3], characterizes the notion of convergence in fuzzy metric spaces.

Proposition 2.10. *A sequence $\{x_n\}$ in X converges to x with respect to τ_M if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.*

The following notion was introduced by George and Veeramani in [3] (and previously, by H. Sherwood, in the context of PM-spaces [17]).

Definition 2.11. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is Cauchy if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$, where $\lim_{n,m}$ denotes the double limit as $n \to \infty$, and $m \to \infty$. $(X, M, *)$ (or simply X) is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.

The notion of principal fuzzy metric space, introduced in [4], will be useful in our subsequent study.

Definition 2.12. A fuzzy metric space $(X, M, *)$ is *principal* (or simply, M is principal) if the family $\{B_M(x, r, t) : r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

In the study of completion of fuzzy metric space the notion of isometry introduced by Gregori and Romaguera in [9] plays a central role.

Definition 2.13. Let $(X, M, *_1)$ and $(Y, N, *_2)$ be two fuzzy metric spaces. A mapping f from X to Y is called an *isometry* if for each $x, y \in X$ and each $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$. Moreover if f is a bijective isometry, then $(X, M, *_1)$ and $(Y, N, *_2)$ (or simply X and Y) are called *isometric*.

Thus a *fuzzy metric completion* of $(X, M, *)$ is a complete fuzzy metric space $(X^*, M^*, \diamond)$ such that $(X, M, *)$ is isometric to a dense subspace of X^*. Furthermore, $(X, M, *)$ (or simply X) is called *completable* if it admits a fuzzy metric completion.

In [10] is given the following characterization about completion of a fuzzy metric space which will be useful when we discuss the completion of those fuzzy metric spaces induced by our new technique.

Theorem 2.14. *Let $(X, M, *)$ be a fuzzy metric space, and let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in $(X, M, *)$. Then $(X, M, *)$ is completable if and only if it satisfies the following conditions:*

(C1) The assignment $t \to \lim_n M(a_n, b_n, t)$ is a continuous function from $]0, \infty[$ into $]0, 1]$.

(C2) If $\lim_n M(a_n, b_n, s) = 1$ for some $s > 0$ then $\lim_n M(a_n, b_n, t) = 1$ for all $t > 0$.

Remark 2.15. *Obviously, a stationary fuzzy metric space $(X, M, *)$ is completable if and only if $\lim_n M(a_n, b_n) > 0$ for every two Cauchy sequences $\{a_n\}$ and $\{b_n\}$.*

Taking into account the preceding notions Gregori and Romaguera proved in [9] the following useful result.

Proposition 2.16. *If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.*

Remark 2.17. *Attending to the last proposition, let us recall the construction of the completion of a fuzzy metric space given in [10]. Suppose $(X^*, M^*, \diamond)$ is a fuzzy metric completion of $(X, M, *)$. Then we have that:*

1. *$X \subseteq X^*$, where X^* is the quotient set on the set of M-Cauchy sequences induced by the equivalence relation $\sim$ defined by*

$$\{x_n\} \sim \{y_n\} \Leftrightarrow \lim_n M(x_n, y_n, t) = 1 \text{ for all } t > 0.$$

2. *$\diamond$ can be assumed to be $*$.*
3. *M^* is defined on X^* by*

$$M^*(x^*, y^*, t) = \lim_n M(x_n, y_n, t)$$

for all $x^*, y^* \in X^*$ and for all $t > 0$, where $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$.

A distinguished class of fuzzy metric spaces, that will need to complete our study, is the so-called strong fuzzy metric spaces. According to [7], let us recall such a notion.

Definition 2.18. Let $(X, M, *)$ be a fuzzy metric space. The fuzzy metric space $(X, M, *)$ (or simply the fuzzy metric M) is said to be *strong* (non-Archimedean) if (in addition) it satisfies the following inequality

(GV4') $M(x, z, t) \geq M(x, y, t) * M(y, z, t)$,

for each $x, y, z \in X$ and each $t > 0$.

To finish, we give two observations on the class of strong fuzzy metrics that will be useful in our work.

Remark 2.19. *In [6] the authors showed that the assignment in condition $(C1)$, in the statement of Theorem 2.14, is always a continuous function whenever M is strong. So, as the authors pointed out in Theorem 4.7 of that paper, a strong fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following conditions are fulfilled:*

- *(c1)* $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.
- *(c2)* $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.

Remark 2.20. *Observe that if $(X, M, *)$ is a non-stationary fuzzy metric space, then we can define the family of fuzzy sets $\{M_t : t > 0\}$ where, for each $t > 0$, $M_t : X \times X \times]0, \infty[\to]0, 1]$ is given by $M_t(x, y, s) = M(x, y, t)$ for all $x, y \in X$ and for all $s > 0$. According to [6], $(X, M, *)$ is strong if and only if $(X, M_t, *)$ is a stationary fuzzy metric space for each $t > 0$. In this case the family $\{M_t : t > 0\}$ is called the* family of stationary fuzzy metrics associated to M. *Note that if $(X, M_t, *)$ is a stationary fuzzy metric space, then we can identify the value $M_t(x, y, s)$ with the value $M_t(x, y)$. Moreover, $\tau_M = \bigvee\{\tau_{M_t} : t > 0\}$ provided that $(X, M, *)$ is strong (see [7]). Furthermore, if M is strong and $\tau_M = \tau_{M_t}$ for all $t > 0$, then M is principal.*

3. A technique for inducing stationary fuzzy metric spaces from metric spaces via metric preserving functions

First of all, we introduce two continuity notions that we will need in our subsequent discussion.

Definition 3.1. Let $(X, M, *)$ be a stationary fuzzy metric space and let (Y, ρ) be a metric space. A mapping $f : X \to Y$ is said to be M-ρ uniformly continuous if given $\epsilon > 0$ we can find $\delta \in]0, 1[$ such that $M(x, y) > 1 - \delta$ implies $\rho(f(x), f(y)) < \epsilon$.

Definition 3.2. Let (X, d) be a metric space and let $(Y, N, \diamond)$ be a stationary fuzzy metric space. We will say that the mapping $f : X \to Y$ is d-N uniformly continuous if given $\epsilon \in]0, 1[$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $N(f(x), f(y)) > 1 - \epsilon$.

The following examples illustrate the preceding definitions.

Example 3.3. 1. Let $(X, M, \mathfrak{L})$ be the stationary fuzzy metric space, where $X = [0, 1[$ and

$$M(x, y, t) = 1 - |x - y|$$

for each $x, y \in X$ and $t > 0$. Also, let (Y, ρ) be the metric space where $Y = \mathbb{R}^2$ and

$$\rho((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Define the mapping $f : X \to Y$ by

$$f(x) = (4x, 3x + 1)$$

for all $x \in X$. Next we will see that f is M-ρ uniformly continuous. To this end, fix $\epsilon > 0$ and consider $\delta < \min\{\epsilon/5, 1\}$. Then, for each $x, y \in X$ satisfying $M(x, y) > 1 - \delta$, we have that $\rho(f(x), f(y)) < \epsilon$. Indeed, if $1 - \delta < M(x, y) = 1 - |x - y|$, then $|x - y| < \delta < \epsilon/5$ and so

$$\rho(f(x), f(y)) = \rho((4x, 3x + 1), (4y, 3y + 1)) =$$

$$\sqrt{(4x - 4y)^2 + (3x + 1 - 3y - 1)^2} =$$

$$\sqrt{16(x - y)^2 + 9(x - y)^2} = 5 \cdot |x - y| < 5 \cdot \tfrac{\epsilon}{5} = \epsilon.$$

Therefore, f is M-ρ uniformly continuous.

2. Let (X, d) be the metric space such that $X = [0, 1]$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Also, let $(Y, N, \cdot)$ be the stationary fuzzy metric space with $Y = [1, 2]$ and
$$N(x, y) = \frac{\min\{x, y\}}{\max\{x, y\}}$$
for all $x, y \in Y$. Define the mapping $f : X \to Y$ by $f(x) = x + 1$ for all $x \in X$. Next we will see that f is d-N uniformly continuous.

With this aim, fix $\epsilon > 0$ and consider $\delta < \epsilon$. Then, for each $x, y \in X$ satisfying $d(x, y) < \delta$, we have that $N(f(x), f(y)) > 1 - \epsilon$. Indeed, let $x, y \in [0, 1]$ such that $d(x, y) < \delta$. Since $\max\{x + 1, y + 1\} \geq 1$ we have that
$$\delta > |x - y| > \frac{|x - y|}{\max\{x + 1, y + 1\}}.$$
Furthermore,
$$N(f(x), f(y)) = \frac{\min\{x + 1, y + 1\}}{\max\{x + 1, y + 1\}} = 1 - \frac{|x - y|}{\max\{x + 1, y + 1\}} > 1 - \delta > 1 - \epsilon.$$

Obviously, note that if f is M-ρ (d-N) uniformly continuous then it is continuous. With the above terminology we can prove the next proposition which will be crucial in the development of our new technique.

Proposition 3.4. *Let $(X, M, *)$ be a stationary fuzzy metric space and let (Y, ρ) be metric space. Let $f : X \to Y$ be an M-ρ uniformly continuous mapping. If $\{x_n\}$ is an M-Cauchy sequence then $\{f(x_n)\}$ is a ρ-Cauchy sequence.*

Proof. Let $\epsilon > 0$, and consider a sequence $\{x_n\}$ in X which is M-Cauchy. Since f is M-ρ uniformly continuous we can find $\delta \in]0, 1[$ such that $M(x, y) > 1 - \delta$ implies $\rho(f(x), f(y)) < \epsilon$. Now, since $\{x_n\}$ is M-Cauchy we can find $n_0 \in \mathbb{N}$ such that $M(x_n, x_m) > 1 - \delta$ for all $n, m \geq n_0$, and so $\rho(f(x_n), f(x_m)) < \epsilon$ for all $n, m \geq n_0$. Hence $\{f(x_n)\}$ is ρ-Cauchy. $\square$

Applying a similar reasoning to that given in the proof of Proposition 3.4 we can prove the next one.

Proposition 3.5. *Let (X, d) be a metric space and let $(Y, N, \diamond)$ be a stationary fuzzy metric space. Let $f : X \to Y$ a d-N uniformly continuous mapping. If $\{x_n\}$ is a d-Cauchy sequence then $\{f(x_n)\}$ is an N-Cauchy sequence.*

From now on, if no confusion arises, we will omitt the metric and the fuzzy metric when we refer to a mapping f as uniformly continuous (in the sense of Definitions 3.1 and 3.2).

In the light of the introduced notions, the next result shows that the composition of uniformly continuous mappings among metric spaces and stationary fuzzy metric spaces is uniformly continuous.

Proposition 3.6. *Let $(X, M, *)$ be a stationary fuzzy metric space, and let (Y, ρ) and (Z, d) be two metric spaces. Suppose that $f : X \to Y$ and $g : Y \to Z$ are two uniformly continuous mappings. Then $g \circ f$ is a uniformly continuous mapping.*

Proof. Let $\epsilon > 0$. Since g is uniformly continuous we can find $\delta_1 > 0$ such that $\rho(a, b) < \delta_1$ implies $d(g(a), g(b)) < \epsilon$. Since f is M-ρ uniformly continuous then, given $\delta_1 > 0$, we can find $\delta \in]0, 1[$ such that $M(x, y) > 1 - \delta$ implies $\rho(f(x), f(y)) < \delta_1$. Therefore, for each $x, y \in X$ such that $M(x, y) > 1 - \delta$ it is satisfied that $\rho(f(x), f(y)) < \delta_1$ and so $d(g(f(x), f(y)) < \epsilon$, hence $g \circ f$ is M-d uniformly continuous. □

In an analogous way we can prove the next proposition.

Proposition 3.7. *Let $(X, M, *)$ be a stationary fuzzy metric space, and let (Y, ρ) and (Z, d) be two metric spaces. Suppose that $f : Z \to X$ and $g : Y \to Z$ are two uniformly continuous mappings. Then $f \circ g$ is a uniformly continuous mapping.*

Taking into account Propositions 3.4 and 3.5 we extend the classical concept of uniformly equivalent metric spaces (see Section 2) to our framework as follows.

Definition 3.8. Let d and $(M, *)$ be a metric and a stationary fuzzy metric on X, respectively. Then d and $(M, *)$ are called uniformly equivalent if both identity mappings are M-d and d-M uniformly continuous, respectively.

Observe that if a metric d is uniformly equivalent to a fuzzy metric $(M, *)$ on X, then $\tau_M = \tau_d$, where τ_d denotes the topology induced by the metric d. Moreover, every d-Cauchy sequence is an M-Cauchy sequence and vice-versa.

The next examples provide instances of metric and fuzzy metric spaces that are uniformly equivalent.

Example 3.9. Assume that (X, d) is a bounded metric space, i.e., there exists $K > 0$ such that $d(x, y) \leq K$ for all $x, y \in X$. On account of [7], we have that $(X, M, \mathfrak{L})$ is a stationary fuzzy metric space, where

$$M(x, y) = 1 - \frac{d(x, y)}{1 + K}$$

for all $x, y \in X$. Next we show that (X, d) and $(X, M, \mathfrak{L})$ are uniformly equivalent. Indeed, it is not hard to check that, given $\varepsilon > 0$, then $d(x, y) < \varepsilon$ provided that $M(x, y) > 1 - \delta$ whenever δ is taken as follows:

$$\delta = \begin{cases} \frac{\varepsilon}{1+K} & \text{if } \varepsilon < K \\ \\ \frac{K}{1+K} & \text{if } \varepsilon \geq K \end{cases}$$

Hence the identity mapping is M-d uniformly continuous. Moreover, given $\varepsilon \in]0, 1[$, then $M(x, y) > 1 - \varepsilon$ provided that $d(x, y) < \delta$ whenever δ is taken as $\delta = (1 + K)\varepsilon$. Thus the identity mapping is d-M uniformly continuous. So (X, d) and $(X, M, \mathfrak{L})$ are uniformly equivalent. Of course, it follows that $\tau_M = \tau_d$.

Example 3.10. Assume that $(X, M, \mathfrak{L})$ is a stationary fuzzy metric space. According to [7], the mapping d_M defined on $X \times X$ by

$$d_M(x, y) = 1 - M(x, y)$$

for all $x, y \in X$ is a metric on X. A straightforward computation shows that the identity mapping is M-d_M and d_M-M uniformly continuous. So $(X, M, \mathfrak{L})$ and (X, d_M) are uniformly equivalent. Of course, it follows that $\tau_M = \tau_{d_M}$.

Inspired by the preceding examples we will introduce the promised technique for generating stationary fuzzy metric spaces from metric spaces by means of metric preserving functions which, besides, preserves the spirit of the aforementioned examples. To this end, we will denote by $\mathcal{M}^1$ and by $\mathcal{M}^1_S$ the class of metric preserving functions and strongly metric preserving functions, respectively, satisfying in both cases that $f(x) < 1$ for each $x \in [0, \infty[$.

Proposition 3.11. *Let (X, d) be a metric space and let $f \in \mathcal{M}^1$. Then:*

(i) $(X, M_f, \mathfrak{L})$ *is a stationary fuzzy metric space, where* $M_f(x,y) = 1 - d_f(x,y)$ *for all* $x, y \in X$*. Moreover,* $\tau_{M_f} = \tau_{d_f}$*.*

(ii) If, in addition $f \in \mathcal{M}_S^1$*, then* $(M_f, \mathfrak{L})$ *and* d *are uniformly equivalent and, thus,* $\tau_{M_f} = \tau_d$*.*

Proof. Let (X, d) be a metric space and consider $f \in \mathcal{M}^1$.

(i) It is straightforward.

(ii) Suppose that $f \in \mathcal{M}_S^1$. Then, from Theorems 2.6 and 2.7 we obtain that d and d_f are uniformly equivalent, and so the identity mapping $i_1 : (X, d_f) \to (X, d)$ is uniformly continuous. Next we show that the identity mapping $i_2 : (X, M_f) \to (X, d_f)$ is uniformly continuous. To this end, we can consider $\epsilon \in]0, 1[$, since the metric d_f is bounded with $d_f(x,y) < 1$ for all $x, y \in X$. Then, $M_f(x,y) > 1 - \epsilon$ if and only if $1 - d_f(x,y) > 1 - \epsilon$, or equivalently, if and only if $d_f(x,y) < \epsilon$. Therefore $i_2 : (X, M) \to (X, d_f)$ is uniformly continuous. Thus, by Proposition 3.6, the identity mapping $i : (X, M) \to (X, d)$ is uniformly continuous, since $i = i_1 \circ i_2$. Following similar arguments, but now with the help of Proposition 3.7, we can show that $i : (X, d) \to (X, M)$ is uniformly continuous. Therefore M_f and d are uniformly equivalent and, thus, $\tau_{M_f} = \tau_d$.

□

The next example shows that the condition "$f \in \mathcal{M}_S^1$" cannot be relaxed in the assertion (ii) in the statement of Proposition 3.11.

Example 3.12. Consider $f : [0, \infty[\to [0, \infty[$ given by

$$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{x+1}{x+2}, & \text{if } x > 0. \end{cases}$$

Obviously, f is non-decreasing. We will see that it is also subadditive. Let $a, b \in [0, \infty[$. If one of them is 0 the subadditive condition is obvious. Suppose that $a, b \in]0, \infty[$. Then,

$$f(a+b) = \frac{a+b+1}{a+b+2} < 1 \leq \frac{a+1}{a+2} + \frac{b+1}{b+2} = f(a) + f(b).$$

Therefore, f is a non-decreasing and subadditive function such that $f \in \mathcal{O}$. By Theorem 2.2, $f \in \mathcal{M}$. Moreover, $f(x) < 1$ for each $x \in [0, \infty[$ and it is clearly not continuous at 0. Then, by Theorem 2.5, f does not belong to $\mathcal{M}_S$, i.e., there exists (X, d) which is not topologically equivalent to (X, d_f). Then, by assertion (i) in the statement of Proposition 3.11, $\tau_M = \tau_{d_f}$ but τ_{d_f} is not equivalent to τ_d.

Since every stationary fuzzy metric space is always principal we have the following result as a consequence of Proposition 3.11.

Corollary 3.13. *Let (X, d) be a metric space and let $f \in \mathcal{M}^1$. Then the fuzzy metric space $(X, M_f, \mathfrak{L})$ is principal, where $M_f(x, y) = 1 - d_f(x, y)$ for all $x, y \in X$.*

Attending to Proposition 3.11, we introduce the next definition.

Definition 3.14. Let $f \in \mathcal{M}^1$ and (X, d). The stationary fuzzy metric $(M_f, \mathfrak{L})$ defined on X by $M_f(x, y) = 1 - d_f(x, y)$ for all $x, y \in X$, will be called the stationary fuzzy metric induced by f and (X, d) or, simply, induced by f if no confusion arises.

In the following example we show that some well-known instances of stationary fuzzy metric spaces can be obtained applying the technique introduced in Proposition 3.11. Observe that such a example illustrates Definition 3.14 and, in addition, complementes the examples furnished by Examples 3.9 and 3.10 about fuzzy metric spaces uniformly equivalent to metric spaces.

Example 3.15. Let (X, d) be a metric space and let $K > 0$. Consider the functions f, g and h defined for all $x \in [0, \infty[$ by

1. $f(x) = \min\left\{\frac{x}{1+K}, \frac{K}{1+K}\right\}$,
2. $g(x) = \frac{x}{K+x}$,
3. $h(x) = 1 - \exp^{-\frac{x}{K}}$.

Note that the preceding functions are not decreasing, subadditive and belong to $\mathcal{O}$. So, by Theorem 2.2, they belong to $\mathcal{M}$. Moreover, it is not hard to see that they are continuous and, thus, by Theorem 2.5, we have that they belong to $\mathcal{M}_S$. Since, in addition, they take values into $[0, 1[$ we have that, in fact, they belong to $\mathcal{M}_S^1$.

The corresponding stationary fuzzy metrics $(M_f, \mathfrak{L})$, $(M_g, \mathfrak{L})$ and $(M_h, \mathfrak{L})$ induced by f, g and h and (X, d) are given, respectively, by

1. $M_f(x, y) = 1 - d_f(x, y) = \max\left\{1 - \frac{d(x,y)}{1+K}, \frac{1}{1+K}\right\}$,

2. $M_g(x,y) = 1 - d_g(x,y) = \frac{K}{K+d(x,y)}$,
3. $M_h(x,y) = 1 - d_h(x,y) = \exp^{-\frac{d(x,y)}{K}}$,

for each $x, y \in X$.

Once the technique for generating stationary fuzzy metric spaces have been introduced we are able to discuss their completion.

Theorem 3.16. *Let (X,d) be a metric space and let $f \in \mathcal{M}_S^1$. The following assertions hold:*

(i) *$(X, M_f, \mathfrak{L})$ is complete if and only if (X,d) is complete.*
(ii) *$(X, M_f, \mathfrak{L})$ is completable and the completion of $(X, M_f, \mathfrak{L})$ is $(X^*, M_f^*, \mathfrak{L})$, where M_f^* is the stationary fuzzy metric given by $M_f^*(a^*, b^*) = 1 - d_f^*(a^*, b^*)$ for each $a^*, b^* \in X^*$ and, in addition, (X^*, d^*) is the completion of (X,d).*

Proof. Let (X,d) be a metric space and let $f \in \mathcal{M}_S^1$. Consider the stationary fuzzy metric M_f induced by f and (X,d).

(i) By Proposition 3.11 d and M_f are uniformly equivalent. Hence $\tau_{M_f} = \tau_d$ and, by Propositions 3.4 and 3.5, a sequence in X is M_f-Cauchy if and only if it is d-Cauchy.
(ii) First, we will show that (X, M_f) is completable. With this aim, let $\{a_n\}$ and $\{b_n\}$ be two M_f-Cauchy sequences. By assertion (ii) in the statement of Proposition 3.11 and by Proposition 3.4, they are d-Cauchy. Consider $a^*, b^* \in X^*$ such that $\{a_n\} \in a^*$ and $\{b_n\} \in b^*$. By Theorem 2.5 f is continuous and so we have that

$$\lim_n M_f(a_n, b_n) = \lim_n (1 - d_f(a_n, b_n)) = 1 - \lim_n f(d(a_n, b_n))$$

$$= 1 - f(\lim_n d(a_n, b_n)) = 1 - f(d^*(a^*, b^*)).$$

Since $f \in \mathcal{M}^1$ we have that

$$\lim_n M_f(a_n, b_n) = 1 - f(d^*(a^*, b^*)) > 0.$$

Therefore, by Remark 2.15 we have that (X, M_f) is completable.

Next suppose that $(\tilde{X}, M_f^*, \mathfrak{L})$ is the completion of $(X, M_f, \mathfrak{L})$. We will see that $\tilde{X} = X^*$.

By Proposition 3.11, d and M_f are uniformly equivalent and, thus, $\{x_n\}$ is an M_f-Cauchy sequence in X if and only if $\{x_n\}$ is a d-Cauchy sequence in X. On the other hand, given two d-Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X we have that

$$\lim_n d_f(x_n, y_n) = \lim_n f(d(x_n, y_n) = f(\lim_n d(x_n, y_n)),$$

because, by Theorem 2.5, f is continuous. Hence

$$\lim_n M_f(x_n, y_n) = 1 \Leftrightarrow \lim_n d_f(x_n, y_n) = 0 \Leftrightarrow \lim_n d(x_n, y_n) = 0,$$

since $f \in \mathcal{O}$. Thus, $\tilde{X} = X^*$.

Finally, consider $a^*, b^* \in X^*$ such that $\{a_n\} \in a^*$ and $\{b_n\} \in b^*$. Attending to Remark 2.17, we have that

$$M_f^*(a^*, b^*, t) = \lim_n M_f(a_n, b_n, t),$$

for each $t > 0$. Since $M_f(a_n, b_n, t) = M_f(a_n, b_n)$ for each $t > 0$ and

$$\lim_n M_f(a_n, b_n) = 1 - \lim_n d_f(a_n, b_n) =$$

$$1 - f(\lim_n d(a_n, b_n)) = 1 - f(d^*(a^*, b^*)) = 1 - d_f^*(a^*, b^*),$$

we conclude that

$$M_f^*(a^*, b^*, t) = 1 - d_f^*(a^*, b^*)$$

for all $\{a_n\} \in a^*$ and $\{b_n\} \in b^*$ and $t > 0$.

□

Remark 3.17. *Notice that the notion of uniformly discrete metric space can be adapted to the stationary fuzzy metric context in the following easy way. Indeed, a stationary fuzzy metric space $(X, M, *)$ will be said to be uniformly discrete provided that there exists $\epsilon \in]0, 1[$ such that $M(x, y) < \epsilon$ for all $x, y \in X$ such that $x \neq y$. Of course we have omitted the t in the expression of M because of its stationary nature.*

Taking into account the preceding notion we have the following reasoning regarding the completion of the stationary fuzzy metric $(X, M_f, \mathfrak{L})$.

If the metric preserving function f in the statement of Theorem 3.16 is assumed to be discontinuous, then, by Proposition 2.3, the metric d_f is uniformly discrete and, consequently, the stationary fuzzy metric M_f induced by f and (X,d) is also uniformly discrete. Therefore, the unique M_f-Cauchy sequences on X are the eventually constant sequences and so $(X, M_f, \mathfrak{L})$ is complete.

In the light of Theorem 3.16 and Remark 3.17 we can assert that for each $f \in \mathcal{M}^1$ and each metric space (X,d) the stationary fuzzy metric space $(X, M_f, \mathfrak{L})$ induced by f and (X,d) is always completable.

4. A technique for inducing non-stationary fuzzy metric spaces from metric spaces via metric preserving functions

In Section 3, we have provided a technique which is able to induce stationary fuzzy metric spaces from classical metric spaces by means of metric preserving functions and, in addition, we have studied the completion and completeness of such fuzzy metric spaces. In this section we will extend the aforementioned technique in order to construct non-stationary fuzzy metric spaces induced by classical metric spaces and, now, by a family of metric preserving functions. Moreover we will study the completion and completeness of such fuzzy metric spaces. We begin such a study introducing the next concept.

Definition 4.1. Consider a family $F = \{f_t : t > 0\}$ of real functions defined on $[0, \infty[$. We will say that F is decreasing if $t < s$ implies $f_t(x) \geq f_s(x)$ for each $x \in [0, \infty[$.

Example 4.7 gives a few instances of decreasing families of functions in the sense of Definition 4.1.

In order to introduce the announced technique let us recall a few known facts. According to [7] we have the following:

Proposition 4.2. *Let $\{(X, M_t, \mathfrak{L}) : t > 0\}$ be a family of stationary fuzzy metric spaces associated to a strong fuzzy metric space $(X, M, \mathfrak{L})$. Then the following assertions hold:*

(i) The real function d, defined by $d(x,y) = 1 - \bigwedge_t M_t(x,y)$ for all $x, y \in X$, is a metric on X such that $\tau_d \supseteq \bigvee_t \tau_{M_t} = \tau_M$, where $\bigwedge_t M_t$ is the real function defined on $X \times X$ by $\bigwedge_t M_t(x,y) = \inf\{M_t(x,y) : t > 0\}$ for all $x, y \in X$.

(ii) *The real function d_t, defined by $d_t(x,y) = 1 - M_t(x,y)$ for all $x, y \in X$, is a metric on X for all $t > 0$. Moreover, $d(x,y) = \bigvee_t d_t(x,y)$ for all $x, y \in X$, where $\bigvee_t d_t(x,y) = \sup\{d_t(x,y) : t > 0\}$ for all $x, y \in X$, is a metric on X and $\tau_d \supseteq \tau_{d_t}$ for all $t > 0$.*

Taking into account the preceding proposition we introduce the new technique in the following result.

Theorem 4.3. *Let (X, d) be a metric space and let $F = \{f_t : t > 0\}$ be a decreasing family of functions included in $\mathcal{M}^1$ such that the function f^x is continuous on $]0,\infty[$ for each $x \in [0,\infty[$, where $f^x(t) = f_t(x)$ for all $t > 0$. Then the following assertions hold:*

(i) *$(X, M_F, \mathfrak{L})$ is a fuzzy metric space, where $M_F(x,y,t) = 1 - d_{f_t}(x,y)$ for each $x, y \in X$ and each $t > 0$.*
(ii) *$(X, M_F, \mathfrak{L})$ is strong.*
(iii) *$\tau_{M_F} = \bigvee\{\tau_{M_{F_t}} : t > 0\} = \bigvee\{\tau_{d_{f_t}} : t > 0\}$, where $M_{F_t}(x,y) = M_F(x,y,t)$ for each $x, y \in X$ and $t > 0$.*
(iv) *The function d_F is a metric on X, where d_F is defined by $d_F(x,y) = 1 - \bigwedge_t M_{F_t}(x,y)$ for all $x, y \in X$. Besides, $d_F(x,y) = \bigvee_t d_{f_t}(x,y)$ for all $x, y \in X$ and $\tau_{d_F} \supseteq \tau_{M_F}$.*
(v) *If $F \subseteq \mathcal{M}^1_S$, then $(X, M_F, \mathfrak{L})$ is principal and $\tau_{d_F} \supseteq \tau_{M_F} = \tau_d$.*

Proof. Consider a decreasing family $F = \{f_t : t > 0\}$ of functions in $\mathcal{M}^1$ such that for each $x \in [0,\infty[$ we have that f^x is continuous on $]0,\infty[$. Define $M_F(x,y,t) = 1 - d_{f_t}(x,y)$ for each $x, y \in X$ and each $t > 0$.

(i) Next we will see that $(X, M_F, \mathfrak{L})$ is a fuzzy metric space.
It is obvious that M satisfies axioms $(GV1), (GV2)$ and $(GV3)$. Furthermore, the assumption that the function f^x is continuous on $]0,\infty[$ for all $x \in [0,\infty[$ ensures that $(GV5)$ is fulfilled. We will show that $(GV4)$ is satisfied too.
First, note that for each $x, y \in X$ we have that $M_{F_{x,y}}$ is an increasing function on $]0,\infty[$, where $M_{F_{x,y}}(t) = M_F(x,y,t)$ for each $t > 0$. Indeed, since the family $\{f_t : t > 0\}$ is decreasing, given $0 < t < s$ then

$$M_{F_{x,y}}(s) = M_F(x,y,s) = 1 - d_{f_s}(x,y) \geq$$

$$1 - d_{f_t}(x,y) = M_F(x,y,t) = M_{F_{x,y}}(t).$$

Moreover, on the one hand, $M_F(x,z,t) = 1 - d_{f_t}(x,z) > 0$, since $f_t \in \mathcal{M}_1$. On the other hand, since for each $t > 0$ we have that d_{f_t} is a metric on X, then for each $x, y, z \in X$ and each $t > 0$ we have that $M_F(x,z,t) = 1 - d_{f_t}(x,z) \geq 1 - d_{f_t}(x,y) - d_{f_t}(y,z) = 1 - d_{f_t}(x,y) + 1 - d_{f_t}(y,z) - 1$. Therefore,

$$M_F(x,z,t) \geq M_F(x,y,t)\mathfrak{L}M_F(y,z,t). \tag{1}$$

Finally, given $x, y, z \in X$ and $t, s > 0$, by these two last observations we have

$$M_F(x,z,t+s) \geq M_F(x,z,\max\{t,s\}) \geq$$

$$M_F(x,y,\max\{t,s\})\mathfrak{L}M_F(y,z,\max\{t,s\}) \geq M(x,y,t)\mathfrak{L}M(y,z,s)$$

and so $(GV4)$ is fulfilled. Therefore, $(X, M_F, \mathfrak{L})$ is a fuzzy metric space.

(ii) The inequality (1) shows that the fuzzy metric space $(X, M_F, \mathfrak{L})$ is strong, i.e., that it holds the condition $(GV4')$.

(iii) Since $(X, M_F, \mathfrak{L})$ is strong we deduce, by Remark 2.20, that $\tau_{M_F} = \bigvee\{\tau_{M_{F_t}} : t > 0\}$, where $M_{F_t}(x,y) = M_F(x,y,t)$ for each $x, y \in X$ and $t > 0$. Proposition 3.11 guarantees that $\tau_{M_{F_t}} = \tau_{d_t}$ for each $t > 0$. It follows that $\tau_{M_F} = \bigvee\{\tau_{M_{F_t}} : t > 0\} = \bigvee\{\tau_{d_t} : t > 0\}$.

(iv) By assertion (i) in the statement of Proposition 4.2 we have that the function d_F is a metric on X. By assertion (ii) in the statement of the aforesaid proposition we obtain that $d_F(x,y) = \bigvee_t d_{f_t}(x,y)$ for all $x, y \in X$ is also a metric on X and that $\tau_{d_F} \supseteq \tau_{M_F}$.

(v) Next we see that $(X, M_F, \mathfrak{L})$ is principal provided $f_t \in \mathcal{M}_S^1$ for all $t > 0$. By assertion (ii) in the statement of Proposition 3.11 we have that $\tau_{M_{F_t}} = \tau(d_t) = \tau(d)$ for each $t > 0$. Whence we have that $\tau_{M_F} = \bigvee\{\tau_{d_t} : t > 0\} = \tau(d)$. Thus, $\tau_{M_F} = \tau_{M_{F_t}}$ for each $t > 0$. By Remark 2.20 we conclude that $(X, M_F, \mathfrak{L})$ is principal. Hence $\tau_{d_F} \supseteq \tau_{M_F} = \tau_d$.

□

The next example shows that the condition "f^x is continuous on $]0, \infty[$ for each $x \in [0, \infty[$" cannot be deleted in the statement of Theorem 4.3 in order to guarantee the introduced technique induces a fuzzy metric.

Example 4.4. Consider the family $F = \{f_t : t > 0\}$, where

$$f_t(x) = \begin{cases} \frac{x}{t+x}, & \text{if } 0 < t \leq 1 \text{ and } x \in [0, \infty[, \\ \frac{x}{2t+x}, & \text{if } t > 1 \text{ and } x \in [0, \infty[. \end{cases}$$

It is easy to verify that F is a decreasing family of functions included in $\mathcal{M}^1$. Besides,

$$f^x(t) = \begin{cases} \frac{x}{t+x}, & \text{if } 0 < t \leq 1 \\ \frac{x}{2t+x}, & \text{if } t > 1, \end{cases}$$

for each $x \in X$, which, obviously, is not continuous at $t = 1$.

Let (X, d) be a metric space, if we define the fuzzy set M_F on $X \times X \times]0, \infty[$ as in assertion (i) in the statement of Theorem 4.3, i.e.,

$$M_F(x, y, t) = \begin{cases} \frac{t}{t+d(x,y)}, & \text{if } 0 < t \leq 1 \text{ and } x, y \in X, \\ \\ \frac{2t}{2t+d(x,y)}, & \text{if } t > 1 \text{ and } x, y \in X, \end{cases}$$

it is easy to verify that M_F does not satisfy axiom $(GV5)$ in definition of fuzzy metric space (Definition 2.8).

The next example shows that the assumption "$F \subseteq \mathcal{M}_S^1$" cannot be deleted in the statement of Theorem 4.3 in order to guarantee that the induced fuzzy metric is principal.

Example 4.5. Let (X, d) be a metric space. Consider the family of functions $F = \{f_t : t > 0\}$ defined on $[0, \infty[$ by

$$f_t(x) = \begin{cases} 0, & \text{if } x = 0; \\ 1 - \frac{t^2}{t+x}, & \text{if } x \in]0, \infty[, t \in]0, 1[; \\ \frac{x}{t+x}, & \text{elsewhere.} \end{cases}$$

Note that $f_t \in \mathcal{M}^1$ for each $t \in [1, \infty[$. Now, we will see that $f_t \in \mathcal{M}^1$ for each $t \in]0, 1[$. To this end, note that $f_t \in \mathcal{O}$ and $f_t(x) < 1$ for each $x \in [0, \infty[$ and each $t \in]0, 1[$. Besides, it is easy to see that f_t is non-decreasing for $t \in]0, 1[$.

Next, we will see that f_t is also subadditive for each $t \in]0, 1[$.

With this aim, we fix $t \in]0, 1[$.

If $a = 0$ or $b = 0$, then it is obvious that $f(a + b) \leq f(a) + f(b)$. Now, suppose that $a, b \in]0, \infty[$. Then it is easy to verify that

$$\frac{t}{t+a+b} \geq \frac{t}{t+a} \cdot \frac{t}{t+b}.$$

So

$$\frac{t^2}{t+a+b} \geq \frac{t^2}{t+a} \cdot \frac{t^2}{t+b},$$

since $t \in]0, 1[$. Moreover, $\frac{t^2}{t+a}, \frac{t^2}{t+b} \in [0, 1]$ and taking into account that $x \cdot y \geq x \mathfrak{L} y$ for each $x, y \in [0, 1]$, we have that

$$\frac{t^2}{t+a+b} \geq \frac{t^2}{t+a} \mathfrak{L} \frac{t^2}{t+b} \geq \frac{t^2}{t+a} + \frac{t^2}{t+b} - 1.$$

Whence we deduce that

$$f(a+b) = 1 - \frac{t^2}{t+a+b} \leq 1 - \frac{t^2}{t+a} + 1 - \frac{t^2}{t+b} = f(a) + f(b).$$

Therefore, f_t is subadditve. Theorem 2.2 guarantees that $f_t \in \mathcal{M}^1$.

It is not hard to check that f^x is continuous on $]0, \infty[$ for each $x \in [0, \infty[$, since $f^0(t) = f_t(0) = 0$ for each $t \in]0, \infty[$ and, for each $x \in]0, \infty[$, we have that

$$f^x(t) = \begin{cases} 1 - \frac{t^2}{t+x}, & \text{if } t \in]0, 1[; \\ \\ \frac{x}{t+x}, & \text{if } t \in [1, \infty[. \end{cases}$$

Clearly the family F satisfies all hypothesis in the statement of Theorem 4.3. Thus $(X, M_F, \mathfrak{L})$ is a strong fuzzy metric on X, where M_F is given by

$$M_F(x, y, t) = 1 - f_t(d(x, y)) = \begin{cases} 1, & \text{if } x = y; \\ \\ \frac{t^2}{t+d(x,y)}, & \text{if } x, y \in X \text{ with } x \neq y \text{ and } t \in]0, 1[; \\ \\ \frac{t}{t+d(x,y)}, & \text{if } x, y \in X \text{ with } x \neq y \text{ and } t \in [1, \infty[. \end{cases}$$

According to [5], $(X, M_F, \mathfrak{L})$ is not a principal fuzzy metric space. Besides, notice that f_t is not continuous at 0 for any $t \in]0, 1[$ and, thus, by Theorem 2.5 we have that $f_t \notin \mathcal{M}_S^1$ for any $t \in]0, 1[$.

The following notion have been inspired by Theorem 4.3.

Definition 4.6. Let (X, d) be a metric space and let $F = \{f_t : t > 0\}$ be a decreasing family of functions included in $\mathcal{M}^1$ such that for each $x \in [0, \infty[$ we have that f^x is continuous on $]0, \infty[$, where $f^x(t) = f_t(x)$ for all $t > 0$. Then the fuzzy metric space $(X, M_F, \mathfrak{L})$, where $M_F(x, y, t) = 1 - d_{f_t}(x, y)$ for each $x, y \in X$ and each $t > 0$, will be called the fuzzy metric space induced by the family F and the metric space (X, d). We will also say that $(M_F, \mathfrak{L})$ is the fuzzy metric induced by F and (X, d).

In the following example we show that some well-known instances of strong and principal fuzzy metric spaces can be obtained applying the technique introduced in Theorem 4.3. Such examples illustrate Definition 4.6.

Example 4.7. Let (X, d) be a metric space. Consider the three families of functions $F = \{f_t : t > 0\}$, $G = \{g_t : t > 0\}$ and $H = \{h_t : t > 0\}$ defined on $[0, \infty[$ by:

1. $f_t(x) = \min\{\frac{x}{1+t}, \frac{t}{1+t}\}$,
2. $g_t(x) = \frac{x}{t+x}$,
3. $h_t(x) = 1 - \exp^{\frac{-x}{t}}$.

It is not hard to check that these families of functions fulfil all hypothesis, even that they are included in $\mathcal{M}^1_S$, in the statement of Theorem 4.3. The corresponding strong and principal fuzzy metric spaces induced by F, G and H and the metric space (X, d) are given, respectively, by

1. $M_F(x, y, t) = \max\{1 - \frac{d(x,y)}{1+t}, \frac{1}{t+1}\}$,
2. $M_G(x, y, t) = 1 - \frac{d(x,y)}{t+d(x,y)} = \frac{t}{t+d(x,y)}$,
3. $M_H(x, y, t) = 1 - (1 - \exp^{\frac{-d(x,y)}{t}}) = \exp^{\frac{-d(x,y)}{t}}$,

for each $x, y \in X$ and each $t > 0$. Observe that $(M_G, \mathfrak{L})$ is the standard fuzzy metric induced by the metric d.

After introducing the technique for generating non-stationary fuzzy metric spaces we end the paper focussing our discussion on their completeness and their completion.

Proposition 4.8. *Let (X,d) be a metric space and let $F=\{f_t : t>0\}$ be a decreasing family of functions included in $\mathcal{M}_S^1$ such that f^x is continuous on $]0,\infty[$ for each $x\in[0,\infty[$, where $f^x(t)=f_t(x)$ for all $t>0$. Then the fuzzy metric space $(X,M_F,\mathfrak{L})$ induced by F and (X,d) is complete if and only if (X,d) is complete.*

Proof. Let $(M,\mathfrak{L})$ be the fuzzy metric induced by F and (X,d). We first note that a sequence $\{x_n\}$ in X is M_F-Cauchy if and only if it is d-Cauchy. Indeed, since M_F is strong, then a sequence $\{x_n\}$ is M_F-Cauchy if and only if it is M_{F_t}-Cauchy for all $t>0$. Moreover, d-Cauchy sequences coincide with M_{F_t}-Cauchy sequences for each $t>0$, since d and M_{F_t} are uniformly equivalent by assertion (ii) in the statement of Proposition 3.11. Thus, M_F-Cauchy sequences coincide with d-Cauchy sequences. Furthemore, assertion (v) in the statement of Theorem 4.3 gives that $\tau_M=\tau(d)$.

Therefore, every M_F-Cauchy sequence converges in τ_{M_F} if and only if every d-Cauchy sequence converges in τ_d. Thus $(X,M_F,\mathfrak{L})$ is complete if and only if (X,d) is complete. □

Theorem 4.9. *Let (X,d) be a metric space and let $F=\{f_t : t>0\}$ be a decreasing family of functions included in $\mathcal{M}_S^1$ such that f^x is continuous on $]0,\infty[$ for each $x\in[0,\infty[$, where $f^x(t)=f_t(x)$ for all $t>0$. Then the fuzzy metric space $(X,M_F,\mathfrak{L})$ induced by F and (X,d) is completable and $(X^*,M_F^*,\mathfrak{L})$ is its completion, where $M_F^*(x^*,y^*,t)=1-d_{f_t}^*(x^*,y^*)$ for each $x^*,y^*\in X^*$ and each $t>0$ and, in addition, (X^*,d^*) is the completion of (X,d).*

Proof. Consider the fuzzy metric space $(X,M_F,\mathfrak{L})$ induced by F and (X,d). Let (X^*,d^*) be the completion of (X,d). We begin showing that $(X,M_F,\mathfrak{L})$ is completable. To this end, let $t>0$ and, in addition, let $\{a_n\}$ and $\{b_n\}$ be two Cauchy sequences in X, where $\{a_n\}\in a^*$ and $\{b_n\}\in b^*$. Then taking into account that $F\subseteq\mathcal{M}_S^1$, we have that

$$\lim_n M_F(a_n,b_n,t)=\lim_n(1-d_{f_t}(a_n,b_n))=1-f_t(\lim_n d(a_n,b_n))=1-d_{f_t}^*(a^*,b^*)>0.$$

By Remark 2.19 we deduce that assertion $(C1)$ in the statement of Theorem 2.14 is fulfilled.

Next, suppose that $\lim_n M_F(a_n,b_n,s)=1$ for two M_F-Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X and for some $s>0$. Then, by continuity of f_s, we have that

$$1 = \lim_n M_F(a_n, b_n, s) = \lim_n (1 - d_{f_s}(a_n, b_n)) = 1 - f_s(\lim_n d(a_n, b_n)).$$

So $\lim_n M_F(a_n, b_n, s) = 1$ if and only if $f_s(\lim_n d(a_n, b_n)) = 0$. Since $f_s \in \mathcal{M}_S^1$ we have that $f_s(\lim_n d(a_n, b_n)) = 0$ if and only if $\lim_n d(a_n, b_n) = 0$.

Therefore if $\lim_n M_F(a_n, b_n, s) = 1$ for some $s > 0$, then $f_t(\lim_n d(a_n, b_n) = 0$ for each $t > 0$. Hence $\lim_n M_F(a_n, b_n, t) = 1$ for each $t > 0$. Therefore, assertion $(C2)$ in the statement of Theorem 2.14 is fulfilled. Consequently, Theorem 2.14 yields that $(X, M_F, \mathfrak{L})$ is completable.

Next, we will construct the completion of $(X, M_F, \mathfrak{L})$. Suppose that $(\tilde{X}, M_F^*, \mathfrak{L})$ is the completion of $(X, M_F, \mathfrak{L})$.

First we will see that $\tilde{X} = X^*$. The set of M_F-Cauchy sequences in X coincides with the set of d-Cauchy sequences in X, as we have seen in the proof of Proposition 4.8. Then, given two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X, we have that $\lim_n M_F(x_n, y_n, t) = 1$, for each $t > 0$, if and only if $\lim_n d_{f_t}(x_n, y_n) = 0$, for each $t > 0$. So, $\lim_n d(x_n, y_n) = 0$ if and only if $\lim_n d_{f_t}(x_n, y_n) = 0$, for each $t > 0$, since $f_t \in \mathcal{M}_S^1$ for each $t > 0$. Thus, $\tilde{X} = X^*$.

Finally, given $x^*, y^* \in X^*$ and $t > 0$, on account of Remark 2.17, we have that

$$\begin{aligned} M_F^*(x^*, y^*, t) &= \lim_n M_F(x_n, y_n, t) = 1 - \lim_n d_{f_t}(x_n, y_n) \\ &= 1 - f_t(\lim_n d(x_n, y_n)) = 1 - f_t(d^*(x^*, y^*)) = 1 - d^*_{f_t}(a^*, b^*), \end{aligned}$$

where $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$.

□

Acknowledgements

The authors are very grateful to the referees for their valuable suggestions.

Valentín Gregori acknowledges the support of the Spanish Ministry of Economy and Competitiveness under Grant MTM2015-64373-P (MINECO/FEDER, UE).

Oscar Valero acknowledges the support of the Spanish Ministry of Economy and Competitiveness under Grant TIN2014-56381-REDT (LODISCO).

[1] J. G. Camarena, V. Gregori, S. Morillas, A. Sapena, *Fast detection and removal of impulsive noise using peer groups and fuzzy metrics*, Journal of Visual Communication and Image Representation, **19** (2008) 20-29.

[2] J. Doboš, *Metric preserving functions*, Štroffek, Košice, 1998.

[3] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994) 395-399.

[4] V. Gregori, A. López-Crevillén, S. Morillas, A. Sapena, *On convergence in fuzzy metric spaces*, Topology and its Applications **156** (2009) 3002-3006.

[5] V. Gregori, J-J. Miñana, S. Morillas, *A note on convergence in fuzzy metric spaces*, Iranian Journal of Fuzzy Systems, **11(4)** (2014) 75-85.

[6] V. Gregori, J.J. Miñana, S. Morillas, A. Sapena, *Characterizing a class of completable fuzzy metric spaces*, Topology and its Applications **203** (2016) 3-11.

[7] V. Gregori, S. Morillas, A. Sapena, *On a class of completable fuzzy metric spaces*, Fuzzy Sets and Systems **161** (2010) 2193-2205.

[8] V. Gregori, S. Romaguera, *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems **115** (2000) 485-489.

[9] V. Gregori, S. Romaguera, *On completion of fuzzy metric spaces*, Fuzzy Sets and Systems **130** (2002) 399-404.

[10] V. Gregori, S. Romaguera, *Characterizing completable fuzzy metric spaces*, Fuzzy Sets and Systems **144** (2004) 411-420.

[11] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika **11** (1975), 326-334.

[12] S. Morillas, L. Gomez-Robledo, R. Huertas, M. Melgosa, *Fuzzy analysis for detection of inconsistent data in experimental datasets employed at the development of the CIEDE2000 colour-difference formula*, Journal Of Modern Optics, **56 (13)** (2009) 1447-1456.

[13] S. Morillas, V. Gregori, G. Peris-Fajarnés, *New adaptative vector filter using fuzzy metrics*, Journal of Electronic Imaging, **16 (3)** (2007) 033007:1-15.

[14] S. Morillas, V. Gregori, G. Peris-Fajarnés, A. Sapena *Local self-adaptative fuzzy filter for impulsive noise removal in color image*, Signal Processing **8 (2)** (2008) 390-398.

[15] J. Rodríguez-López, S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems **147** (2004) 273-283.

[16] A. Sapena, *A contribution to the study of fuzzy metric spaces*, Applied General Topology **2** (2001) 63-76.

[17] H. Sherwood, *On the completion of probabilistic metric spaces*, Zeitschrift fr Wahrscheinlichkeitstheorie und Verwandte Gebiete **6 (1)** (1966) 62-64.

[18] P. Veeramani, *Best approximation in fuzzy metric spaces*, Journal of Fuzzy Mathematics **9** (2001) 75-80.