Filomat 31:6 (2017), 1619–1625
DOI 10.2298/FIL1706619G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Strong Convergence in Fuzzy Metric Spaces

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Abstract. In this paper we introduce and study the concept of strong convergence in fuzzy metric spaces $(X, M, *)$ in the sense of George and Veeramani. This concept is related with the condition $\bigwedge_{t>0} M(x, y, t) > 0$, which frequently is required or missing in this context. Among other results we characterize the class of s-fuzzy metrics by the strong convergence defined here and we solve partially the question of finding explicitly a *compatible* metric with a given fuzzy metric.

1. Introduction

I. Kramosil and J. Michalek [10] defined the concept of fuzzy metric space which could be considered a reformulation of the concept of Menger space in fuzzy setting. This concept was modified by Grabiec in [2]. Later, George and Veeramani modified this last concept and gave a concept of fuzzy metric space $(X, M, *)$. Many concepts and results can be stated for all the above fuzzy metric spaces mentioned. In particular, if M is any of these fuzzy metrics on X then a topology τ_M deduced from M is defined on X. A sequence $\{x_n\}$ in X is convergent to x_0 if and only if $\lim_n M(x_n, x_0, t) = 1$ for each $t > 0$.

A significant difference between a classical metric and a fuzzy metric is that this last one includes in its definition a parameter t. This fact has been successfully used in engineering applications such as colour image filtering [15–17] and perceptual colour differences [5, 14]. From the mathematical point of view this parameter t allows to define novel well-motivated fuzzy metric concepts which have no sense in the classical case. So, several concepts of Cauchyness and convergence have appeared in the literature (see [2, 3, 6, 12, 18]). Nevertheless, in some cases the natural concepts introduced are non-appropriate. A discussion of this assertion can be found in [4].

From now on by a fuzzy metric space we mean a fuzzy metric space in the sense of George and Veeramani.

Given $x, y \in X$ the real function $M_{xy}(t) :]0, \infty[\to]0, 1]$ defined by $M_{xy}(t) = M(x, y, t)$ is continuous in a fuzzy metric space. Notice that M_{xy} is not defined at $t = 0$. Then, the behaviour of M for values close to 0 turns of interest. For instance, recently, for obtaining fixed point theorems for a self-mapping T on X D. Wardowski [20] and D. Mihet [13] have demanded conditions on M involving T for values of t close to 0. In particular, the Mihet's condition ([13, Theorem 2.4]) can be written as $\bigwedge_{t>0} M(x, T(x), t) > 0$ for some $x \in X$.

2010 *Mathematics Subject Classification*. Primary 54A40; Secondary 54D35, 54E50
Keywords. Fuzzy metric space; s-fuzzy metric space; s-convergence
Received: 12 March 2015; Accepted: 21 May 2015
Communicated by Ljubiša D.R. Kočinac
Valentn Gregori acknowledges the support of the Spanish Ministry of Economy and Competitiveness under Grant MTM2015-64373-P (MINECO/FEDER, UE).
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This condition is related with the condition $\bigwedge_{t>0} M(x,y,t) > 0$ for all $x, y \in X$, which has been studied in [6] and the obtained results are summarized in the next paragraph.

A sequence $\{x_n\}$ is called s-convergent to x_0 if $\lim_n M(x_n, x_0, \frac{1}{n}) = 1$. This is a (strictly) stronger concept than convergence and it is given by a limit, which, as in the classical case, only depends on n. A fuzzy metric space in which every convergent sequence is s-convergent is called s-fuzzy metric space. In a similar way to the class of principal fuzzy metric spaces [3], the class of s-fuzzy metric spaces admits the following characterization by means of a special local base [6]: $(X, M, *)$ is an s-fuzzy metric space if and only if the family $\{\bigcap_{t>0} B(x,r,t) : r \in]0,1[\}$ is a local base at x, for each $x \in X$. On the other hand, if N is a mapping on $X \times X$ given by $N(x,y) = \bigwedge_{t>0} M(x,y,t)$, then $(X, N, *)$ is a stationary fuzzy metric space if and only if $N(x,y) > 0$ for all $x, y \in X$. In a such case, in [6] it is proved that $\tau_N = \tau_M$ if and only if M is an s-fuzzy metric. However, a drawback of the concept of s-convergence, as in the case of standard Cauchy (see [4]), is that it has not a natural Cauchyness compatible pair.

The aim of this paper is to go in depth the understanding of the behaviour of a fuzzy metric M when the parameter t takes values close to 0. Then, motivated by the above works, we study the behaviour of the sequential convergence when simultaneously the parameter t tends to 0. For it, we introduce a stronger concept than convergence called strong convergence, briefly st-convergence. This new concept reminds the classical concept of convergence when it is defined by the role of ϵ and n_0. So, we will say that a sequence $\{x_n\}$ is st-convergence to x_0 if given $\epsilon \in]0,1[$ there exists n_0, depending on ϵ such that $M(x_n, x_0, t) > 1 - \epsilon$ for all $n \geq n_0$ and all $t > 0$. Our first achievement is that $(X, M, *)$ is an s-fuzzy metric space if and only if every convergent sequence is st-convergent. Then, in Remark 3.11 we observe that for a subclass of s-fuzzy metrics M is possible to find a compatible metric deduced explicitly from M. The second achievement is that the natural concept of st-Cauchy sequence (Definition 4.1) deduced from st-convergence is a compatible pair, in the sense of [4] (Definition 4). This new concept fulfils also the following nice properties:

1. st-convergence implies s-convergence, and the converse is false, in general.
2. Every subsequence of a st-convergent sequence is st-convergent.
   A significant difference with respect to s-convergence is:
3. There exist convergent sequences without st-convergent subsequences. Also:
4. In an s-fuzzy metric space Cauchy sequences are not st-Cauchy, in general.

The structure of the paper is as follows. In Section 3, after the preliminary section, we introduce and study the notion of st-convergence. In Section 4 we introduce the corresponding natural concept of st-Cauchyness and we show that it is compatible with st-convergence. At the end, a question related to the obtained results is proposed.

2. Preliminaries

Definition 2.1. (George and Veeramani [1]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x,y,t) > 0$;

(GV2) $M(x,y,t) = 1$ if and only if $x = y$;

(GV3) $M(x,y,t) = M(y,x,t)$;

(GV4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;

(GV5) $M(x,y,\_) :]0,\infty[\to]0,1]$ is continuous.

The continuous t-norms used in this paper are the usual product, denoted by $\cdot$, and the Lukasievicz t-norm, denoted by $\mathfrak{L}$ ($x \mathfrak{L} y = \max\{0, x+y-1\}$), which satisfy that $\cdot \geq \mathfrak{L}$.

Note that if $(X, M, *)$ is a fuzzy metric space and $\diamond$ is a continuous t-norm satisfying $\diamond \leq *$, then $(X, M, \diamond)$ is a fuzzy metric space.

If $(X, M, *)$ is a fuzzy metric space, we will say that $(M, *)$, or simply M, is a *fuzzy metric* on X. This terminology will be also extended along the paper in other concepts, as usual, without explicit mention.

George and Veeramani proved in [1] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$. If confusion is not possible, as usual, we write simply B instead of B_M.

Let (X, d) be a metric space and let M_d a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [1], and M_d is called the *standard fuzzy metric* induced by d. The topology τ_{M_d} coincides with the topology $\tau(d)$ on X deduced from d.

Definition 2.2. (Gregori and Romaguera [9]) A fuzzy metric M on X is said to be *stationary* if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

Proposition 2.3. (George and Veeramani [1]) *Let $(X, M, *)$ a fuzzy metric space. A sequence $\{x_n\}$ in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.*

Definition 2.4. (George and Veeramani [1], Schweizer and Sklar [19]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be *M-Cauchy*, or simply Cauchy, if for each $\epsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. Equivalently, $\{x_n\}$ is M-Cauchy if $\lim_{n,m} M(x_n, x_m, t) = 1$ for all $t > 0$.

As in the classical case convergent sequences are Cauchy.

Definition 2.5. (Gregori and Miñana [4]) Suppose it is given a stronger concept than convergence, say A-convergence. A concept of Cauchyness, say A-Cauchyness, is said to be compatible with A-convergence, and *vice-versa*, if the diagram of implications below is fulfilled

$$\begin{array}{ccc} \text{A}-\textit{convergence} & \rightarrow & \textit{convergence} \\ \downarrow & & \downarrow \\ \text{A}-\textit{Cauchy} & \rightarrow & \textit{Cauchy} \end{array}$$

and there is not any other implication, in general, among these concepts.

From now on $(X, M, *)$, or simply X if confusion is not possible, is a fuzzy metric space.

3. Strong Convergence

The condition of convergence in a fuzzy metric space can be rewritten as follows.

A sequence $\{x_n\}$ converges to x_0 if and only if for all $t > 0$ and for all $\epsilon \in]0, 1[$ there exists $n_{\epsilon,t} \in \mathbb{N}$, depending on ϵ and t, such that

$$M(x_n, x_0, t) > 1 - \epsilon, \text{ for all } n \geq n_{\epsilon,t}.$$

Then we can give a stronger concept than convergence strengthening in a natural way the imposition on t as follows.

Definition 3.1. A sequence $\{x_n\}$ in $(X, M, *)$ is strong convergent, briefly *st-convergent*, to $x_0 \in X$ if given $\epsilon \in]0, 1[$ there exists n_ϵ, depending on ϵ, such that

$$M(x_n, x_0, t) > 1 - \epsilon, \text{ for all } n \geq n_\epsilon \text{ and for all } t > 0.$$

Equivalently, $\{x_n\}$ is *st-convergent* to $x_0 \in X$ if given $\epsilon \in]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$x_n \in B(x_0, \epsilon, t), \text{ for all } n \geq n_\epsilon \text{ and for all } t > 0.$$

Clearly, a *st*-convergent sequence to x_0 is convergent to x_0.
Next, we will give a characterization of *st*-convergent sequences by means of (double) limits.

Proposition 3.2. *A sequence $\{x_n\}$ in $(X, M, *)$ is st-convergent to x_0 if and only if* $\lim_{n,m} M(x_n, x_0, \frac{1}{m}) = 1$

Proof. Suppose $\{x_n\}$ is *st*-convergent to x_0. Let $\epsilon \in]0, 1[$. Then we can find n_ϵ such that $M(x_n, x_0, t) > 1 - \epsilon$ for all $n \geq n_\epsilon$ and for all $t > 0$. In particular $M(x_n, x_0, \frac{1}{m}) > 1 - \epsilon$ for all $n \geq n_\epsilon$ and for all $m \in \mathbb{N}$, i.e., $\lim_{n,m} M(x_n, x_0, \frac{1}{m}) = 1$.

Conversely, suppose $\lim_{n,m} M(x_n, x_0, \frac{1}{m}) = 1$. Let $\epsilon \in]0, 1[$. Then we can find $n_\epsilon \in \mathbb{N}$ such that $M(x_n, x_0, \frac{1}{m}) > 1 - \epsilon$ for all $n, m \geq n_\epsilon$. Take $t > 0$. Then we can find $m_t \geq n_\epsilon$ such that $\frac{1}{m_t} < t$ and so $M(x_n, x_0, t) \geq M(x_n, x_0, \frac{1}{m_t}) > 1 - \epsilon$ for all $n \geq n_\epsilon$, so $\{x_n\}$ is *st*-convergent to x_0. □

The next corollary is immediate.

Corollary 3.3. *Each st-convergent sequence is s-convergent.*

Now we will see that the converse of the last corollary is not true, in general.

Example 3.4. Let $(X, M_d, \cdot)$ be the standard fuzzy metric, where $X = \mathbb{R}$ and d is the usual metric on $\mathbb{R}$.
Consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. The sequence $\{x_n\}$ is *s*-convergent to 0, since

$$\lim_n M_d(x_n, 0, \frac{1}{n}) = \lim_n \frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n^2}} = 1.$$

Now, we will see that $\{x_n\}$ is not *st*-convergent to 0.
Suppose that $\{x_n\}$ is *st*-convergent to 0. Then for each $\epsilon \in]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$ such that $M_d(x_n, 0, t) = \frac{t}{t + \frac{1}{n^2}} > 1 - \epsilon$ for all $t > 0$ and for all $n \geq n_\epsilon$. Therefore, $\frac{1}{n_\epsilon^2} < \frac{t\epsilon}{1-\epsilon}$ for all $t > 0$, a contradiction.

Under the above terminology the following assertions are immediate:

Proposition 3.5.
1. *Constant sequences are st-convergent.*
2. *If M is stationary then convergent sequences are st-convergent.*

Proposition 3.6. *Each subsequence of a st-convergent sequence in X is st-convergent.*

Proof. It is straightforward. □

Remark 3.7. In [6] the authors proved that in a fuzzy metric space each convergent sequence admits an *s*-convergent subsequence. This affirmation is not true for *st*-convergent sequences as we will show in the the next example.

Example 3.8. Consider the standard fuzzy metric space $(X, M_d, \cdot)$ of Example 3.4 and let $\{x_n\}$ be the sequence defined by $x_n = \frac{1}{n}$. Clearly, $\{x_n\}$ converges to 0. Suppose that $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is *st*-convergent to 0. Then for each $\epsilon \in]0, 1[$ there exists $k_\epsilon \in \mathbb{N}$ such that $M_d(x_{n_k}, 0, t) = \frac{t}{t + \frac{1}{n_k}} > 1 - \epsilon$ for all $t > 0$ and for all $k \geq k_\epsilon$. Therefore $\frac{1}{n_{k_\epsilon}} < \frac{t\epsilon}{1-\epsilon}$ for all $t > 0$, a contradiction.

Theorem 3.9. *Every convergent sequence in $(X, M, *)$ is st-convergent if and only if every convergent sequence in X is s-convergent.*

Proof. If every convergent sequence in X is st-convergent then by Corollary 3.3 every convergent sequence in X is s-convergent.

Conversely, suppose that every convergent sequence in X is s-convergent and suppose that there exists a convergent sequence $\{x_n\}$ to x_0 in X which is not st-convergent. Then there exists $\delta \in]0,1[$ such that for each $k \in \mathbb{N}$ there exists $n(k) \geq k$ such that $M(x_{n(k)}, x_0, t(k)) \leq 1 - \delta$, for some $t(k) > 0$.

Next we will construct a convergent sequence $\{y_j\}$ which is not s-convergent.

Take $1 \in \mathbb{N}$, then there exists $n(1) \geq 1$ such that $M(x_{n(1)}, x_0, t(1)) \leq 1 - \delta$. Let $n_1 \in \mathbb{N}$ such that $n_1 \geq \max\{\frac{1}{t(1)}, n(1)\}$ and we define

$$y_1 = y_2 = \cdots = y_{n_1} = x_{n(1)}.$$

Now, for $n_1 \in \mathbb{N}$, there exists $n(n_1) \geq n_1$ such that $M(x_{n(n_1)}, x_0, t(n_1)) \leq 1 - \delta$. Let $n_2 \in \mathbb{N}$ such that $n_2 \geq \max\{\frac{1}{t(n_1)}, n(n_1)\}$. Clearly, $n_2 \geq n_1$. So we define

$$y_{n_1+1} = y_{n_1+2} = \cdots = y_{n_2} = x_{n(n_1)}.$$

By induction on $k \in \mathbb{N}$, for $n_{k-1} \in \mathbb{N}$, there exists $n(n_{k-1}) \geq n_{k-1}$ such that $M(x_{n(n_{k-1})}, x_0, t(n_{k-1})) \leq 1 - \delta$. Let $n_k \in \mathbb{N}$ such that $n_k \geq \max\{\frac{1}{t(n_{k-1})}, n(n_{k-1})\}$. Clearly, $n_k \geq n_{k-1}$. So we define

$$y_{n_{k-1}+1} = y_{n_{k-1}+2} = \cdots = y_{n_k} = x_{n(n_{k-1})}.$$

The constructed sequence $\{y_j\}$ is convergent. Indeed, since $\{x_n\}$ converges to x_0 we have that for each $\epsilon \in]0,1[$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) > 1 - \epsilon$ for all $n \geq n_0$. If we take $k_0 \in \mathbb{N}$ such that $n_{k_0} \geq n_0$ and consider $j_0 = n_{k_0}$, then for each $j \geq j_0$, $y_j = x_{n(n_k)}$, where $n_k \geq n_{k_0}$, and so by construction of $\{y_j\}$ we have that $M(y_j, x_0, t) > 1 - \epsilon$.

Now, we will see that $\{y_j\}$ is not s-convergent to x_0. By construction of $\{y_j\}$ we have that for all $k \in \mathbb{N}$, $M(y_{n_k}, x_0, \frac{1}{n_k}) \leq 1 - \delta$. Therefore there exists $\delta \in]0,1[$ such that for each $j \in \mathbb{N}$ we can find $k(j) \in \mathbb{N}$ such that $n_{k(j)} \geq j$ and so $M(y_{n_{k(j)}}, x_0, \frac{1}{n_{k(j)}}) \leq 1 - \delta$. Thus $\{y_j\}$ is not s-convergent, a contradiction. □

An example of s-fuzzy metric is $(]0,\infty[, M, \cdot)$, where $M(x,y,t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$. On the other hand, the standard fuzzy metric space $(X, M_d, \cdot)$ is s-fuzzy metric if and only if $\tau(d)$ is the discrete topology [6].

The next corollary is obvious taking into account the last theorem and Corollary 3.10 of [6].

Corollary 3.10. *The following are equivalent:*

(i) M is an s-fuzzy metric.
(ii) $\bigcap_{t>0} B(x,r,t)$ is a neighborhood of x for all $x \in X$, and for all $r \in]0,1[$.
(iii) $\{\bigcap_{t>0} B(x,r,t) : r \in]0,1[\}$ is a local base at x, for each $x \in X$.
(iv) Every convergent sequence is st-convergent.

Notice that in an s-fuzzy metric convergence can be defined with a simple limit and that one can find a local base at x for each $x \in X$ depending only on the radius, which reminds the case of classical metrics. This observation is related with the next remark.

Remark 3.11. (Metric deduced explicitly from a fuzzy metric)

We will say that a metric d and a fuzzy metric M, both on X, are compatible if the topologies deduced from d and M coincide, i.e. $\tau(d) = \tau_M$. Recall that a topological space is metrizable if and only if it is fuzzy metrizable [7]. Now, the topological study of a (fuzzy) metrizable space is easier thought a metric or even thought a stationary fuzzy metric because in both cases it does not appear the parameter t.

The reader knows that for a given metric d on X one can find many compatible fuzzy metrics (see [1]) deduced explicitly from d. The converse, up to we know, is an unsolved question. To approach this question, in the next paragraph, we recall some known results.

Given a metric d on X it is easy to find stationary fuzzy metrics compatible with d. For instance, for a fixed $K > 0$, if we define $N_K = \frac{K}{K+d(x,y)}$ for each $x, y \in X$ then $(N_K, \cdot)$ is a stationary fuzzy metric and $\tau(d) = \tau_{N_K}$. Conversely, if $(N, \mathfrak{L})$ is a stationary fuzzy metric on X then $d(x, y) = 1 - N(x, y)$, for each $x, y \in X$, is a metric on X and $\tau(d) = \tau_N$.

Now, let $* \geq \mathfrak{L}$ and suppose that $(M, *)$ is a fuzzy metric on X satisfying $N(x, y) = \bigwedge_{t>0} M(x, y, t) > 0$ for each $x, y \in X$. Then $(N, *)$ is a fuzzy metric on X and $\tau_N = \tau_M$ if and only if M is an s-fuzzy metric (see [6, Theorem 4.2]). Consequently, in this case $d(x, y) = 1 - \bigwedge_{t>0} M(x, y, t)$ is a metric on X with $\tau(d) = \tau_M$ and so d is a compatible metric with M. Clearly, d is deduced explicitly from M.

4. Strong Cauchy Sequences

Next, we will give a concept of strong Cauchy sequence according to Definition 3.1.

Definition 4.1. A sequence $\{x_n\}$ in X is strong Cauchy, briefly *st-Cauchy*, if given $\epsilon \in]0, 1[$ there exists n_ϵ, depending on ϵ, such that

$$M(x_n, x_m, t) > 1 - \epsilon, \text{ for all } n, m \geq n_\epsilon \text{ and for all } t > 0.$$

Clearly, *st*-Cauchy sequences are Cauchy.

In a similar way to the case of *st*-convergence, we give the next characterization of *st*-Cauchyness by means of (triple) limit.

Proposition 4.2. *$\{x_n\}$ is st-Cauchy if and only if* $\lim_{n,m,k} M(x_n, x_m, \frac{1}{k}) = 1$

Proof. The proof is similar to the proof of Proposition 3.2. □

We will see that the concept of *st*-Cauchyness is compatible with the concept of *st*-convergence. First, we will see that the next diagram

$$\begin{array}{ccc} st-convergence & \rightarrow & convergence \\ \downarrow & & \downarrow \\ st-Cauchy & \rightarrow & Cauchy \end{array}$$

is fulfilled. For it, we start showing the next proposition.

Proposition 4.3. *Every st-convergent sequence is st-Cauchy.*

Proof. Let $\{x_n\}$ be a *st*-convergent sequence in a fuzzy metric space $(X, M, *)$. Take $\epsilon \in]0, 1[$. By continuity of $*$, we can find $r \in]0, 1[$ such that $(1 - r) * (1 - r) > 1 - \epsilon$. Since $\{x_n\}$ is *st*-convergent, there exists $x_0 \in X$ and $n_0 \in \mathbb{N}$ such that $M(x_n, x_0, t) > 1 - r$ for all $n \geq n_0$ and all $t > 0$. Therefore, for each $n, m \geq n_0$ and each $t > 0$ we have that

$$M(x_n, x_m, t) \geq M(x_n, x_0, t/2) * M(x_0, x_m, t/2) > (1 - r) * (1 - r) > (1 - \epsilon.)$$

And thus, $\{x_n\}$ is *st*-Cauchy. □

Now, we will see that the implications of the above diagram cannot be reverted in general.

Example 3.4 shows an s-convergent sequence, and so convergent, which is not *st*-convergent. It is easy to verify that it is also an example of convergent (Cauchy) sequence which is not *st*-Cauchy.

The next example shows an *st*-Cauchy sequence, which is not (*st*-)convergent.

Example 4.4. Let $(X, M, *)$ be the stationary fuzzy metric space, where $X =]1, +\infty[$, $M(x, y) = \frac{\min\{x,y\}}{\max\{x,y\}}$ and $*$ is the usual product. It is easy to verify that the sequence $\{x_n\}$, where $x_n = 1 + \frac{1}{n}$ is a *st*-Cauchy sequence in X, which is not (*st*-)convergent.

Therefore, the concepts of *st*-Cauchyness and *st*-convergence are compatible.
Finally, we will see that in an *s*-fuzzy metric space Cauchy sequences are not *st*-Cauchy, in general.

Example 4.5. Consider $(X, M, *)$, where $X =]0, \infty[$, $*$ is the usual product and $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ for each $x, y \in X$ and each $t > 0$. In [6] it is proved that it is an *s*-fuzzy metric space.

Now, if we consider the sequence $\{x_n\}$ in X, where $x_n = \frac{1}{n}$ for each $n \in \mathbb{N}$, it is a Cauchy sequence in X. Indeed,

$$\lim_{n,m} M(x_n, x_m, t) = \lim_{n,m} \frac{\min\{\frac{1}{n}, \frac{1}{m}\} + t}{\max\{\frac{1}{n}, \frac{1}{m}\} + t} = 1.$$

On the other hand, $\{x_n\}$ is not *st*-Cauchy. Indeed, tacking $\epsilon = \frac{1}{2}$, then for each $n \in \mathbb{N}$ we can find $m > n$ and $t > 0$ such that $M(x_n, x_m, t) < \frac{1}{2}$. For instance, given $n \in \mathbb{N}$, if we consider $m = 3n$ and $t \in]0, \frac{1}{3n}[$ we have that

$$M(x_n, x_m, t) = \frac{\frac{1}{3n} + t}{\frac{1}{n} + t} < \frac{\frac{1}{3n} + \frac{1}{3n}}{\frac{1}{n} + \frac{1}{3n}} = \frac{1}{2}.$$

A question concerning our above study is the next.

Problem 4.6. *Characterize those fuzzy metric spaces in which Cauchy sequences are st-Cauchy.*

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