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SOME CHARACTERISATIONS OF GROUPS IN WHICH NORMALITY IS A TRANSITIVE RELATION BY MEANS OF SUBGROUP EMBEDDING PROPERTIES

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Communicated by Patrizia Longobardi

Dedicated to the memory of Mario Curzio

Abstract. In this survey we highlight the relations between some subgroup embedding properties that characterise groups in which normality is a transitive relation in certain universes of groups with some finiteness properties.

1. Introduction

We begin by recalling the definition of the groups in which normality is a transitive relation, or, in short, T-groups.

Definition 1.1. A group G is said to be a *T-group* if $H \trianglelefteq K \trianglelefteq G$ implies $H \trianglelefteq G$.

This is equivalent to stating that all subnormal subgroups are normal. The first explicit mention we have found of T-groups in the literature corresponds to a paper of Best and Taussky [3]. Chapter 2 of [2] summarises some basic results about T-groups in finite groups. The description of T-groups in the infinite case is more complex and can be found in the celebrated paper of D. J. S. Robinson [27].

It is well known that the class of T-groups is not closed under taking subgroups. A typical example of a T-groups with subgroups that are not T-groups is the alternating group A_5 of degree 5, that is

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obviously a T-group since it is simple and its only subnormal subgroups are 1 and A_5, but that has a subgroup isomorphic to A_4 which is not a T-group, since the cyclic subgroups in the Klein 4-group are subnormal, but not normal in A_4. This motivates the following definition.

Definition 1.2. We say that a group G is a *$\bar{T}$-group* if every subgroup of G is a T-group.

In the finite soluble universe, the following classical characterisation of Gaschütz characterises finite soluble T-groups. Recall that a Dedekind group is a group with all subgroups normal and that a power automorphism of a group X is an automorphism of X that stabilises all subgroups of X.

Theorem 1.3. [12, Gaschütz] *A finite soluble group G is a T-group if and only if G has a normal abelian Hall subgroup L of odd order such that G/L is a Dedekind group and L is acted upon by conjugation as a group of power automorphisms by G.*

This result has the virtue of showing that a finite soluble T-group is supersoluble. Moreover, a finite $\bar{\mathrm{T}}$-group must be soluble, since, otherwise, if we have an insoluble $\bar{\mathrm{T}}$-group whose proper subgroups are soluble, then it is a minimal-non-supersoluble group and so it is soluble by a theorem of Doerk [8, Satz A] (see also [13, Kapitel VI, Satz 9.6]). Therefore we have the following result.

Theorem 1.4. *Let G be a finite group.*

(1) *If G is a soluble T-group, then G is a $\bar{T}$-group.*
(2) *If G is a $\bar{T}$-group, then G is soluble.*

Examples of infinite soluble T-groups that are not $\bar{\mathrm{T}}$-groups are constructed in [27] and [19].

The first class of infinite groups we will consider is the class of FC*-groups, that generalise the class of groups of FC-groups or groups in which every conjugacy class is finite. Recall that if H is a subgroup of G, then H^G denotes the normal closure of H in G.

Definition 1.5. We say that a group G is an *FC^0-group* if G is finite. By induction, we say that a group G is an *FC^{n+1}-group* if $G/\mathrm{C}_G(\langle x\rangle^G)$ is an FC^n-group for all $x \in G$. Then G is an FC*-group if G is an FC^n-group for some $n \geq 0$.

Theorem 1.6. [11, Theorem 2.3] *Let G be an FC*-group. Then the following statements are equivalent:*

(1) *G is a soluble T-group.*
(2) *G is a $\bar{T}$-group.*

The following property, introduced by Kaplan in [15], is not exactly a subgroup embedding property *per se*, but describes a class of groups in which the non-normal subgroups are embedded in the group in a particular way.

Definition 1.7. [15] A group G is said to be an *NNM-group* (for "non-normal maximal") if each non-normal subgroup of G is contained in a non-normal maximal subgroup of G.

Theorem 1.8. [15, Theorem 1] *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *All subgroups of G are NNM-groups.*

This result can be also extended to FC*-groups.

Theorem 1.9. [11, Theorem 2.5] *Let G be a FC*-group. Then the following statements are equivalent:*

(1) *G is a soluble T-group.*
(2) *All subgroups of G are NNM-groups.*

We remark that there exist examples of T-groups that are hyperfinite and FC-nilpotent, but that are not NNM-groups [11, Example 2.6].

The other class of groups we are interested in is the class of groups without infinite simple sections. This class contains every FC*-group and it is in fact a subclass of the class of locally graded groups.

Definition 1.10. We say that a group G is *locally graded* if every non-trivial finitely generated subgroup of G has a non-trivial finite homomorphic image.

Theorem 1.11. [6, Theorem 3.6]

(1) *Let G be a group without infinite simple sections. Then:*
  (a) *G is locally graded.*
  (b) *If G is a $\bar{T}$-group, then G is metabelian.*
(2) *Let G be a soluble group. Then G is a $\bar{T}$-group if and only if every ascendant subgroup of G is normal in G.*

The following question is open in the *Kourovka Notebook* (see [22, Question 14.36]).

Question 1.12. Are non-periodic locally graded $\bar{\mathrm{T}}$-groups soluble?

In [9], this question is reduced to the following one.

Question 1.13. Let G be a locally graded $\bar{\mathrm{T}}$-group. If G is torsion-free, can we say that it is abelian?

2. Subgroup embedding properties

In this section we present some subgroup embedding properties that have been used to characterise T-groups.

2.1. Pseudonormal and pronormal subgroups.

The following subgroup embedding property appears in a natural way in the scope of $\bar{\mathrm{T}}$-groups. The first appearance of this property known to us is due to Peng [26].

Definition 2.1. A subgroup X of a group G is said to be *pseudonormal* [7] or *transitively normal* [18] or *to satisfy the subnormaliser condition* [24] if $\mathrm{N}_G(H) \le \mathrm{N}_G(X)$, for each subgroup H of G such that $X \le H \le \mathrm{N}_G(X)$.

This is equivalent to affirming that if $H \le L \le G$ and H is subnormal in L, then $H \trianglelefteq L$ (see [7, Theorem 2.1]).

Pronormality is a well-known subgroup embedding property introduced by Hall in his Cambridge lectures.

Definition 2.2. A subgroup X of a group G is said to be *pronormal* if X and X^g are conjugate in $\langle X, X^g\rangle$, for every element $g \in G$.

For instance, we have the following result for finite groups.

Theorem 2.3. [1, Theorem A] *Let G be a finite group, then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group.*
(2) *Every subgroup of G is pronormal.*
(3) *Every subgroup of G is pseudonormal.*

Theorem 2.3 admits an extension to FC*-groups.

Theorem 2.4. *Let G be an FC*-group, then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group.*
(2) *Every subgroup of G is pronormal.*
(3) *Every subgroup of G is pseudonormal.*

This result follows by [7, Theorem 3.1 and Corollary 3.5] and [5, Theorem 4.6] or [29, Theorem 3.3]. In general, we have the following result.

Theorem 2.5. [7, Theorem 3.1] *A group G is a $\bar{T}$-group if and only if all its subgroups are pseudonormal.*

With respect to pronormality, we have the next result.

Theorem 2.6. [25] *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *G is a $\bar{T}$-group.*
(3) *X is pronormal in G for all $X \le G$.*

The theorem of Peng can be extended to FC*-groups.

Theorem 2.7. [6, Theorem 3.9] *Let G be a soluble FC*-group. Then the following statements are equivalent:*

(1) *G is a T-group.*

(2) *X is pronormal in G for all $X \leq G$.*

Combining the above Theorems 2.6 and 1.11 with [6, Lemma 3.5] we have the following characterisation.

Theorem 2.8. *Let G be a group without infinite simple sections. Then G is a $\bar{T}$-group if and only if every cyclic subgroup is pronormal.*

Kovács, Neumann, and de Vries [17, Theorem 2.1] show the existence of a metabelian $\bar{\mathrm{T}}$-group that contains some non-pronormal Sylow subgroups (see also the comments after [6, Lemma 2.7] for more details). Kuzennyi and Subbotin [19, Example 2] present an example of a group with all primary subgroups pronormal, but with some non-pronormal subgroups.

2.2. **Weakly normal subgroups.** The following concept was introduced by Müller [23].

Definition 2.9. [23] A subgroup X of a group G is said to be *weakly normal* if $X^g \leq \mathrm{N}_G(X)$ implies $g \in \mathrm{N}_G(X)$.

Theorem 2.10. [1, Theorem A] *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *Every subgroup of G is weakly normal.*

Theorem 2.11. [31, Corollary 4], [30, Theorem 2.8] *Let G be a group. Then the following statements are equivalent:*

(1) *G is a $\bar{T}$-group without infinite simple sections.*
(2) *G is a locally graded group whose subgroups are weakly normal.*

2.3. **$\mathcal{H}$-subgroups.** The notion of $\mathcal{H}$-subgroup is due to Bianchi, Gillio Berta Mauri, Herzog, and Verardi [4].

Definition 2.12. [4] A subgroup X of a group G is said to be an *$\mathcal{H}$-subgroup* or that it has the *$\mathcal{H}$-property* in G if $\mathrm{N}_G(X) \cap X^g \leq X$ for all elements g of G.

Theorem 2.13. [4, Theorem 10] *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*
(2) *G is a $\bar{T}$-group.*
(3) *Every subgroup of G is an $\mathcal{H}$-subgroup.*

The previous theorem also holds for groups without infinite simple sections.

Theorem 2.14. [32, Theorem 3.2] *Let G be a group without infinite simple sections. Then the following statements are equivalent:*

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(1) *G is a $\bar{T}$-group.*

(2) *Every subgroup of G has the property $\mathcal{H}$.*

2.4. **NE-subgroups.** The notion of NE-subgroup is due to Li.

Definition 2.15. [20] A subgroup H of a finite group G is called an *NE-subgroup* if it satisfies $\mathrm{N}_G(H) \cap H^G = H$.

Theorem 2.16. [21, Theorem 3.1] *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*

(2) *Every subgroup of G is an NE-subgroup of G.*

Theorem 2.17. [9] *Let G be a group without infinite simple sections. Then the following statements are equivalent:*

(1) *G is a soluble $\bar{T}$-group.*

(2) *Every subgroup of G is an NE-subgroup of G.*

2.5. **φ-subgroups and cr-subgroups.** The following subgroup embedding properties were introduced by Kaplan [14].

Definition 2.18. [14] A subgroup H of a group G is said to be a *φ-subgroup* of G if, for all K, L maximal in H, if it is the case that if K, L are conjugate in G, then K, L are conjugate in H.

Definition 2.19. [14] A subgroup K of a group G is said to be a *cr-subgroup* (for "conjugation restricted") of G if there are no $A < K$, $g \in G$ such that $K = AA^g$.

Theorem 2.20. [14, Theorem 7] *Let G be a finite soluble group. Then the following statements are equivalent:*

(1) *G is a T-group.*

(2) *Every subgroup of G has the property φ.*

(3) *Every subgroup of G is a cr-subgroup.*

This result admits an extension to FC*-groups.

Theorem 2.21. [16, Theorem 5.2] *Let G be a soluble FC*-group. Then the following statements are equivalent:*

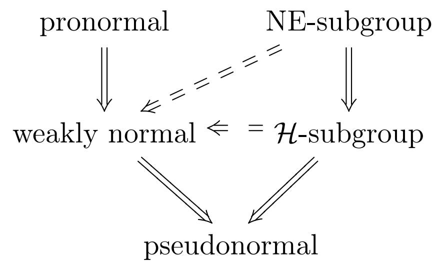
(1) *G is a T-group.*

(2) *Every subgroup of G has the property φ.*

(3) *Every subgroup of G is a cr-subgroup.*

2.6. **Relations between subgroup embedding properties.** Now we will investigate the relations between these subgroup embedding properties. Figure 1 shows the relations between some of the subgroup embedding properties considered in this survey.

Figure 1. Relations between subgroup embedding properties



The broken arrows mean implications that are only known to hold in HNN-free groups (in particular, in finite groups), but whose validity in the general case is not known. We recall that G is HNN-free if $H^g \leq H$ implies that $g \in \mathrm{N}_G(H)$ (see [28]).

The following comments show that no other general implications between the subgroup embedding properties presented in Figure 1 hold, although some partial results have been obtained.

In [1, Remark 1], an example of a weakly normal subgroup that is not pronormal is presented. It is constructed from an irreducible and faithful Σ_3-module V_7 over the field of 7 elements whose restriction to the alternating group A_3 of degree 3 is a direct sum of two irreducible modules $V_7 = W_1 \oplus W_2$ of dimension 1. Let $G = \Sigma_3 \ltimes V_7$. Then $H = A_3W_1$ is weakly normal, but not pronormal in G. The fact that pronormal subgroups are weakly normal has been proved in [1, Proposition 1]. This proof is also valid in the infinite case.

In [1, Lemma 1], it is shown that weakly normal subgroups satisfy the subnormaliser condition. This proof is also valid for infinite groups. An example of Mysovskikh [24] (see also [2, Example 1.5.16]) shows that the converse is false. It consists of a semidirect product $G = A_4 \ltimes W$ of A_4 by an irreducible and faithful module of dimension 3 over the field of 3 elements obtained by considering the A_4-invariant subgroup $W = \langle w_4w_1^{-1}, w_4w_2^{-1}, w_4w_3^{-1}\rangle$ of the base subgroup of the natural wreath product $C_3 \wr A_4$. Then $D = \langle (1,2)(3,4)\rangle W$ is pseudonormal, but not weakly normal in G.

Suppose that $H^G \cap \mathrm{N}_G(H) = H$, then $H^g \cap \mathrm{N}_G(H) \leq H^G \cap \mathrm{N}_G(H) = H$ and so all NE-subgroups are $\mathcal{H}$-subgroups. The converse is false, because in $\mathrm{SL}_2(3)$, a Sylow 3-subgroup H is an $\mathcal{H}$-subgroup that is not an NE-subgroup.

In [4, Lemma 5], it is shown that $\mathcal{H}$-subgroups are pseudonormal in finite groups. For infinite groups, the result also holds, we have to modify slightly the argument: if $H \leq K \leq \mathrm{N}_G(H)$ and $g \in \mathrm{N}_G(K)$, then $K^g = K^{g^{-1}} = K$ and so H^g, $H^{g^{-1}} \leq K \leq \mathrm{N}_G(H)$. Therefore $H^g = H^g \cap \mathrm{N}_G(H) \leq H$ and $H^{g^{-1}} = H^{g^{-1}} \cap \mathrm{N}_G(H) \leq H$, that is, $H \leq H^g$. Consequently $H^g = H$ and $g \in \mathrm{N}_G(H)$. The previous example of Mysovskikh gives a pseudonormal subgroup that is not an $\mathcal{H}$-subgroup.

The fact that $\mathcal{H}$-subgroups are weakly normal in finite groups was indicated in [1]: if $H^g \leq \mathrm{N}_G(H)$ and H is an $\mathcal{H}$-subgroup, then $H^g \leq H^g \cap \mathrm{N}_G(H) = H$. This gives that $g \in \mathrm{N}_G(H)$ if G is HNN-free. A similar argument shows that for HNN-free groups, NE-subgroups are weakly normal. However, we do not know whether these implications hold in the general case. This is left open in [32, Question 2]. Conversely, the subgroup $\langle(1,2,3,4)\rangle$ of the symmetric group Σ_4 of degree 4 is an example of a weakly normal subgroup that is not an $\mathcal{H}$-subgroup (see [1, Example 1]) and, hence, not an NE-subgroup. In [1, Theorems 4 and 5], some sufficient conditions for a weakly normal subgroup H of a supersoluble group to be an $\mathcal{H}$-subgroup are considered, namely H being a p-group or H having all its subgroups weakly normal.

The above presented group of [1, Remark 1] is also an example of an $\mathcal{H}$-subgroup that is not pronormal.

The subgroup $\langle(1,2,3,4)\rangle$ of Σ_4 of [1, Example 1] is also a pronormal subgroup, but not an $\mathcal{H}$-subgroup and, consequently, not an NE-subgroup.

Finally, in [9], we prove that the $\bar{\mathrm{T}}$-groups with no infinite simple sections coincide with the locally graded groups whose subgroups are NE-subgroups. Based on this result, we see that the constructions of Kovács, Neumann, and de Vries and Kuzennyi and Subbotin that we have mentioned after Theorem 2.8 give examples of NE-subgroups that are not pronormal.

On the other hand, the properties of being φ-subgroups and cr-subgroups seem to be essentially different from the other properties we have considered before. We know by [14, Theorem 4.1] that φ-subgroups that are normal are cr-subgroups and that soluble cr-subgroups are φ-subgroup, with some counterexamples given when the hypothesis of normality or solubility of the corresponding subgroup is removed. It is clear that all subgroups of prime order are both φ-subgroups and cr-subgroups, but $K = \langle(1,2)(3,4)\rangle$ is not pseudonormal in the alternating group A_4. Moreover, the fact that $V = \langle(1,2)(3,4),(1,3)(2,4)\rangle$ has two subgroups $\langle(1,2)(3,4)\rangle$, $\langle(1,3)(2,4)\rangle$ that are not conjugate in G, shows that normal subgroups are not necessarily φ-subgroups nor cr-subgroups.

3. Systems of subgroups satisfying embedding properties and $\bar{\mathrm{T}}$-groups

We can summarise the previous characterisations of $\bar{\mathrm{T}}$-groups by means of subgroup embedding properties, as well as other characterisations presented in [7, 9, 32], in the following results.

Theorem 3.1. *Let G be a periodic, locally graded group. The following statements are pairwise equivalent.*

(1) *G is a $\bar{T}$-group.*
(2) *G is locally finite and all cyclic subgroups of G are pronormal.*
(3) *All subgroups of G are $\mathcal{H}$-subgroups.*
(4) *G is locally finite and all cyclic subgroups of G are $\mathcal{H}$-subgroups (see* [32, Theorem 3.1]*).*
(5) *All subgroups of G are weakly normal.*
(6) *G is locally finite and all cyclic subgroups of G are weakly normal (see* [7, Lemma 3.2]*).*
(7) *All subgroups of G are NE-subgroups.*
(8) *G is locally finite and all cyclic subgroups of G are NE-subgroups (see* [9]*).*
(9) *All subgroups of G are pseudonormal.*

(10) *G is locally finite and all cyclic subgroups of G are pseudonormal (see* [7, Lemma 3.2]*).*

Moreover, if one of the above conditions hold, then G is metabelian.

Theorem 3.2. *Let G be a non-periodic group without infinite simple sections. The following statements are pairwise equivalent.*

(1) *All subgroups of G are pronormal.*
(2) *All cyclic subgroups of G are pronormal.*
(3) *All subgroups of G are NE-subgroups.*
(4) *All subgroups of G are weakly normal.*
(5) *All subgroups of G are $\mathcal{H}$-subgroups.*
(6) *G is abelian.*

In the comment after [32, Theorem 3.1] it is shown that the dihedral infinite group D_∞ is an example of a group with all cyclic subgroups $\mathcal{H}$-subgroups, but with a non-$\mathcal{H}$-subgroup. In [9] we also show that this group has all cyclic subgroups NE-subgroups, but it contains a subgroup which is not an NE-subgroup.

We do not know whether a non-periodic soluble group with all cyclic subgroups weakly normal is abelian.

We conclude by summarising the characterisations for $\bar{\mathrm{T}}$-groups that hold in the universe of all FC*-groups.

Theorem 3.3. *Let G be a soluble FC*-group. The following statements are pairwise equivalent.*

(1) *G is a T-group.*
(2) *G is a $\bar{T}$-group.*
(3) *All subgroups of G are NNM-groups.*
(4) *All subgroups of G are cr-subgroups.*
(5) *All subgroups of G are φ-subgroups.*

A scheme of the characterisations contained in this section can be found in [10].

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