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# CONJUGACY CLASSES CONTAINED IN NORMAL SUBGROUPS: AN OVERVIEW

ANTONIO BELTRÁN, MARA JOSÉ FELIPE AND CARMEN MELCHOR*

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Abstract. We survey known results concerning how the conjugacy classes contained in a normal subgroup and their sizes exert an influence on the normal structure of a finite group. The approach is mainly presented in the framework of graphs associated to the conjugacy classes, which have been introduced and developed in the past few years. We will see how the properties of these graphs, along with some extensions of the classic Landau's Theorem on conjugacy classes for normal subgroups, have been used in order to classify groups and normal subgroups satisfying certain conjugacy class numerical conditions.

## 1. **Introduction**

We will assume that every group is finite. The study of the structural properties of a group when taking into account the information related to its conjugacy classes is a classical field in Finite Group Theory, which has been widely developed in the last two decades. We will pay attention to the conjugacy classes contained in normal subgroups, so this will be an expository paper in which we outline the major results related to the influence of these classes on the structure of the normal subgroup. We provide references to the literature for their proofs.

Let $N$ be a normal subgroup of a group $G$. For each element $x \in N$, the $G$-conjugacy class of $x$, denoted by $x^G$, is the set of elements of $N$ which are conjugate to $x$ in $G$. We will denote by

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*Corresponding author.

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$\mathrm{Con}_G(N) = \{x^G, x \in N\}$ the set of $G$-classes of elements of $N$, and by $\mathrm{cs}_G(N) = \{|x^G|, x \in N\}$ the set of their sizes. Each element in $\mathrm{Con}_G(N)$ is the join of conjugacy classes of $N$, and it easily follows that its size is a multiple of a single $N$-class size. Recent results have put forward that $\mathrm{cs}_G(N)$ continues to maintain a strong influence on the structure of $N$, in spite of the fact that there may exist primes dividing the $G$-class sizes which, however, do not divide $|N|$. As a starting point, the problem concerning those groups having a normal subgroup $N$ such that $|\mathrm{cs}_G(N)| = 2$ has been deeply studied. In [22], the nilpotency of $N$ was shown under the additional hypothesis that $N$ contains some Sylow subgroup of $G$. This result was completely extended in [3] by eliminating the latter hypothesis. Precisely, it was proved that if $\mathrm{cs}_G(N) = \{1, m\}$, for some integer $m$, then $N$ is either abelian or the direct product of a $p$-group by a central subgroup of $G$.

Regarding groups having normal subgroups with three class sizes, the normal structure under certain arithmetical conditions on these sizes was described in [1]. Concretely, it was demonstrated that if $\mathrm{cs}_G(N) = \{1, m, n\}$ with $(m, n) = 1$, then $N$ is either a quasi-Frobenius group with abelian kernel and complement or, up to central factors in $G$, is a $p$-group for some prime $p$. The solvability of $N$ without assuming additional hypotheses was finally reached in [2], by employing new techniques based on the study of class sizes in normal sections. Nevertheless, a complete classification of the structure of $N$ in the case in which $\mathrm{cs}_G(N) = \{1, m, n\}$ where $m$ divides $n$ still remains open. We refer the reader to the surveys [4] and [5] for more specific details on this study. There is much more research work about the influence of the class sizes on the structure of the group (see for instance [10] or [17]) but our goal is not to give here an exhaustive list of them.

This paper is divided into three sections. In section 2, we look at several graphs constructed from the sets of conjugacy classes. We highlight that two interesting surveys dealing with conjugacy class graphs have already been published (See [13, 26, Section 5]). However, the aim of the different approach that we present here is to compare the results in regard to ordinary conjugacy classes with those on conjugacy classes contained in a normal subgroup. Essentially, we analyze the common divisor graph associated to ordinary conjugacy classes of a group $G$ and the corresponding graph associated to $G$-classes contained in a normal subgroup. As we will see, the properties firstly obtained for the ordinary graph are not inherited by the graph of $G$-classes, which is certainly a subgraph. Likewise, we examine the diameter and the connectivity of both graphs as well as the relation to the structure of the groups and normal subgroups. In section 3, we present several results relating the structure of groups and normal subgroups to the simplest structures that these graphs may have: when the graphs consists of exactly one, two ore three vertices. These properties are used to obtain the structure of the normal subgroup when the graph of $G$-classes has no triangles. Finally, in section 4, we show an application of the graph of $G$-classes and two extensions of the well-known Landau's Theorem [25] for $G$-classes and for $G$-classes of elements of prime power order contained in a normal subgroup.

## 2. Graphs associated to conjugacy classes of normal subgroups in finite groups

In 1990, E. A. Bertram, M. Herzog and A. Mann defined in [12] the graph $\Gamma(G)$ associated to the sizes of the ordinary conjugacy classes of $G$ as follows: the vertices of $\Gamma(G)$ are represented by the non-central conjugacy classes of $G$ and two vertices $C$ and $D$ are connected by an edge if $|C|$ and $|D|$ have a common prime divisor. Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and let $d(\Gamma(G))$ be its diameter. They proved that $n(\Gamma(G))$ is at most 2 and that when the graph is connected, then $d(\Gamma(G)) \leq 4$. Furthermore, they characterize the disconnected case by proving that quasi-Frobenius groups with abelian kernel and complements are the only groups whose graphs have two connected components. Let us recall that a group $G$ is said to be quasi-Frobenius if $G/\mathbf{Z}(G)$ is a Frobenius group. In this case, the inverse image in $G$ of the kernel and complement of $G/\mathbf{Z}(G)$ are called the kernel and complement of $G$, respectively. Later, in [18], C. Chillag, Herzog and Mann obtained the best bound of the diameter of this graph, which is 3.

If we look at the conjugacy classes contained in a normal subgroup $N$ of $G$, then the graph $\Gamma_G(N)$, which is a subgraph of $\Gamma(G)$, appears.

**Definition 2.1.** *Let $G$ be a finite group and let $N$ be a normal subgroup in $G$. We define the graph $\Gamma_G(N)$ in the following way: the set of vertices is the set of non-central elements of $\mathrm{Con}_G(N)$, and two vertices $x^G$ and $y^G$ are joined by an edge if and only if $|x^G|$ and $|y^G|$ have a common prime divisor.*

As we pointed out in the introduccion, the fact that the number of connected components and the diameter of $\Gamma(G)$ are bounded does not directly imply that the corresponding parameters for $\Gamma_G(N)$ must be bounded too. There is no relation between $\Gamma(N)$ and $\Gamma_G(N)$ either. For instance, $\Gamma(N)$ can be disconnected while $\Gamma_G(N)$ is not. An example (based on the semilinear affine group $\Gamma(p^n)$ for appropiate $p$ and $n$) can be found in [6]. Likewise, the diameters of $\Gamma_G(N)$ and $\Gamma(N)$ are not related either. For instance, let $P$ be an extraspecial group of order $p^3$ with $p \neq 2$. If $G = P \times S_3$ and $N = P \times A_3$, we have that $\Gamma(N)$ is a complete graph (all non-trivial $N$-classes have size $p$) while $\Gamma_G(N)$ has diameter 2, since $\mathrm{cs}_G(N) = \{2, p, 2p\}$. In spite of these facts, $n(\Gamma_G(N))$ and $d(\Gamma_G(N))$ are actually bounded. The authors determine the best bounds for both parameters and describe the structure of $N$ when $\Gamma_G(N)$ is disconnected.

**Theorem 2.2.** [6, Theorem A] *Let $G$ be a finite group and let $N$ be a normal subgroup of $G$. Then $n(\Gamma_G(N)) \leq 2$.*

**Theorem 2.3.** [6, Theorem B] *Let $G$ be a finite group and let $N$ be a normal subgroup of $G$.*

(1) *If $n(\Gamma_G(N)) = 1$, then $d(\Gamma_G(N)) \leq 3$.*
(2) *If $n(\Gamma_G(N)) = 2$, then each connected component is a complete graph.*

**Theorem 2.4.** [6, Theorem E] *Let $G$ be a finite group and $N \trianglelefteq G$. If $\Gamma_G(N)$ has two connected components then, either $N$ is quasi-Frobenius with abelian kernel and complement or $N = P \times A$ where $P$ is a $p$-group and $A \leqslant \mathbf{Z}(G)$.*

As it has been said at the begining of this section, a characterization of the structure of $G$ for the disconnected case of $\Gamma(G)$ was obtained in [12]. Nonetheless, the converse of Theorem 2.4 is false. It is known that the special linear group $H = SL(2,5)$ acts Frobeniusly on $K \cong \mathbb{Z}_{11} \times \mathbb{Z}_{11}$. We consider $P \in \mathrm{Syl}_5(H)$ and $\mathbf{N}_H(P)$ acting Frobeniusly on $K$. We define the semidirect product $N := KP$, which is trivially a normal subgroup of $G := K\mathbf{N}_H(P)$. Thus, $N$ is a Frobenius group with abelian kernel and complement and $N$ decomposes into the following disjoint union

$$N = \{1\} \cup (K \setminus \{1\}) \cup (\bigcup_{k \in K} P^k \setminus \{1\}),$$

and $K \setminus \{1\}$ is partitioned into $N$-classes of cardinality 5, whereas the elements of $\bigcup_{k\in K}(P^k \setminus \{1\})$ are grouped into $N$-classes of cardinality 121. Therefore, the set of $N$ classes is $\{1, 5, 121\}$. As $G$ is a Frobenius group with kernel $K$ and complement $\mathbf{N}_H(P)$, it follows that $K$ is decomposed exactly into the trivial class and $G$-classes of size $|\mathbf{N}_H(P)| = 20$. That is to say, the $N$-classes contained in $K \setminus \{1\}$ are grouped 4 by 4 to form $G$-classes. On the other hand, the four $N$-conjugacy classes contained in $\bigcup_{k\in K} P^k \setminus \{1\}$ of size 121, are grouped in pairs and become two $G$-classes of size $121 \times 2$. Thus, $\mathrm{cs}_G(N) = \{1, 20, 242\}$ and so $\Gamma_G(N)$ is a connected graph.

We also remark that the case in which $N$ is a $p$-group in Theorem 2.4 actually occurs. For instance, let $G$ be the group of the SmallGroups library ([11]) of GAP ([21]) with number 324#8 (the $m$-th group of order $n$ in the SmallGroups library is identified by $n\#m$). One can check that $G$ has an abelian normal subgroup $N \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with $\mathrm{cs}_G(N) = \{1, 2, 3\}$, so $\Gamma_G(N)$ is disconnected.

Concerning ordinary classes, L. S. Kazarin characterizes in [23] the structure of a group $G$ having two "isolated classes". We recall that a group $G$ is said to have isolated classes if there exist elements $x, y \in G$ such that every element of $G$ has a conjugacy class size coprime to either $|x^G|$ or $|y^G|$. Particularly, Kazarin determined the structure of those groups $G$ with $d(\Gamma(G)) = 3$. It should be noted that similar results have also been studied for other graphs. In [19], S. Dolfi defines the dual graph of $\Gamma(G)$ whose vertices are the primes which occur as divisors of the class sizes of $G$, and two vertices $p$ and $q$ are joined by an edge if there exists a conjugacy class in $G$ whose size is a multiple of $pq$. In [16], Dolfi and C. Casolo describe all finite groups $G$ for which the dual graph of $\Gamma(G)$ is connected and has diameter three. The corresponding dual graph of $\Gamma_G(N)$ was defined in [6], but the problem of determining the normal structure when its diameter is exactly three is still open.

Regarding $\Gamma_G(N)$, the structure of the normal subgroup $N$ is determined in [8] when its diameter is as large as possible, that is, it is equal to 3. From now on, if $G$ is a finite group, $\pi(G)$ denotes the set of primes dividing $|G|$, and analogously, if $X$ is a set, then $\pi(X)$ denotes the set of primes dividing $|X|$. The following is the above mentioned result.

**Theorem 2.5.** [8, Theorem 1.1] *Let $G$ be a finite group and $N \trianglelefteq G$. Suppose that $x^G$ and $y^G$ are two non-central $G$-conjugacy classes of $N$ such that any $G$-conjugacy class of $N$ has size coprime with $|x^G|$ or $|y^G|$. Let $\pi_x = \pi(x^G)$, $\pi_y = \pi(y^G)$ and $\pi = \pi_x \cup \pi_y$. Then $N = \mathbf{O}_{\pi'}(N) \times \mathbf{O}_{\pi}(N)$ with $x, y \in \mathbf{O}_{\pi}(N)$, which is either a quasi-Frobenius group with abelian kernel and complement or $\mathbf{O}_{\pi}(N) = P \times A$ with $A \leq \mathbf{Z}(N)$ and $P$ is a $p$-group for a prime $p$.*

Notice that under the hypotheses of Theorem 2.5, there exist only two possibilities: $d(\Gamma_G(N)) \leq 2$ or $d(\Gamma_G(N)) = 3$. In the former case, $\Gamma_G(N)$ is disconnected, and the structure of $N$ is already determined in Theorem 2.4. Nonetheless, this result is shightly improved in [8, Corollary 1.2]. In the second case, $\Gamma_G(N)$ is connected, since otherwise, each connected component is a complete graph (Theorem 2.3).

## 3. Normal subgroups with few conjugacy classes

The simplest cases for the graph $\Gamma_G(N)$ occur when it consists of one, two or three vertices. We point out that the structure of those normal subgroups which are union of exactly three or four $G$-conjugacy classes had already appeared in [30] and [29], respectively.

**Theorem 3.1.** [30, Main theorem] *Let $G$ be a finite group and let $H$ be a normal subgroup of $G$ which is a union of three conjugacy classes of $G$. Then $H$ is an elementary abelian $p$-group for some odd prime $p$, a metabelian $p$-group, or an extension of an elementary abelian group with a certain cyclic group.*

The following discusses the case which $N$ is union of exactly four $G$-classes.

**Theorem 3.2.** [29, Main theorem] *Let $G$ be a finite group and let $H$ be a normal subgroup of $G$ which is a union of four conjugacy classes of $G$. Then either $H$ is the alternating group $A_5$ or a metabelian $p$-group or a group of order $p^a q^b$ which is abelian or Frobenius.*

From now on, we will only consider non-central conjugacy classes and Theorem 3.1 and Theorem 3.2 will be extended. For any group $G$, it is elementary that the case in which $\Gamma(G)$ has exactly one vertex cannot happen. However, when considering $G$-classes inside a normal subgroup, this situation changes.

**Theorem 3.3.** [9, Theorem 3.1] *If $G$ is a group and $N$ is a normal subgroup of $G$ such that $\Gamma_G(N)$ has only one vertex, then $N$ is a $p$-group for some prime $p$ and $N/(N \cap \mathbf{Z}(G))$ is an elementary abelian $p$-group.*

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The structure of $G$ when $\Gamma(G)$ has vertices but no edges was obtained in [12]. In this case, by taking into account that $n(\Gamma(G)) \leq 2$, this graph consists of exactly two vertices and it was proved that $G$ is isomorphic to $S_3$. This situation does not happen when dealing with $\Gamma_G(N)$. Just take for instance $G = S_3$ and $N = A_3$. Actually, a more general structure for $N$ arises.

**Theorem 3.4.** [9, Theorem 4.1.2] *Let $N$ be a normal subgroup of a group $G$ such that $\Gamma_G(N)$ has two vertices and no edge. Then $N$ is a 2-group or a Frobenius group with $p$-elementary abelian kernel $K$, and complement $H$, which is cyclic of order $q$, for two different primes $p$ and $q$. In particular, $|N| = p^n q$ with $n \geq 1$.*

The case where the graph consists of exacty two vertices and one edge does not occur for the ordinary graph. In fact, in [20] it was given a complete list of all groups $G$ such that $\Gamma(G)$ has no triangles, and none of such graphs has exactly two vertices and one edge. Later on, we will present this classification. However, for normal subgroups and $G$-classes the following properties were obtained.

**Theorem 3.5.** [9, Lemma 4.2.2] *Let $N$ be a normal subgroup of a group $G$ such that $\Gamma_G(N)$ has exactly two vertices and one edge. Then one of the following possibilities holds:*

(1) *$N$ is a $p$-group for a prime $p$.*
(2) *$N = P \times Q$ with $P/(\mathbf{Z}(G) \cap P)$ an elementary abelian $p$-group with $p$ an odd prime, and $Q \subseteq \mathbf{Z}(G) \cap N$ and $Q \cong \mathbb{Z}_2$.*
(3) *$N$ is a Frobenius group with $p$-elementary abelian kernel $K$ and complement $H \cong \mathbb{Z}_q$ for some distinct primes $p$ and $q$. In particular, $|N| = p^a q$ for some $a \geq 1$ and the $G$-classes of $N$ have cardinality $1$, $(p^a - 1)$ and $p^a(q-1)$.*

On the other hand, it is proved in [20] that $\Gamma(G)$ is disconnected with three vertices if and only if $G \cong D_{10}$ or $G \cong A_4$ (Theorem 3.9). Such a situation for $\Gamma_G(N)$ is contemplated in the following.

**Theorem 3.6.** [9, Theorem 5.1.1] *Let $N$ be a normal subgroup of a group $G$. If $\Gamma_G(N)$ has three vertices and one edge, then $N$ is a $\{p, q\}$-group for two primes $p$ and $q$. Furthermore, either*

1) *$N$ is a $p$-group, or*
2) *$N$ is a quasi-Frobenius group with abelian kernel and complement. In this case, $|N \cap \mathbf{Z}(G)| = 1$ or $2$.*

The case of exactly three vertices connected in a line do not occur for the ordinary graph. This follows again by using the classification of [20]. For $\Gamma_G(N)$, however, the following result is proved.

**Theorem 3.7.** [9, Theorem 5.3.1] *Let $N$ be a normal subgroup of a group $G$. If $\Gamma_G(N)$ has three vertices in a line, then $\mathbf{Z}(G) \cap N = 1$ and one of the following cases is satisfied:*

(1) *$N$ is a 2-group of exponent at most 4.*

(2) *$N = P \times Q$, where $P$ and $Q$ are elementary abelian $p$ and $q$-groups.*

(3) *$N$ is a Frobenius group with complement isomorphic to $\mathbb{Z}_q$, $\mathbb{Z}_{q^2}$ or $Q_8$. In the former case, the kernel of $N$ is a $p$-group with exponent $\leq p^2$ and in the last two cases, the kernel of $N$ is $p$-elementary abelian.*

*In all cases, $|N|$ is divisible by at most two primes.*

The structure of $G$ when $\Gamma(G)$ consists of exactly one triangle was obtained in [9]. It was proved that in this case, $G \cong Q_8$ or $G \cong D_8$. In addition, the authors gave an extension for $\Gamma_G(N)$.

**Theorem 3.8.** [9, Theorem C] *Let $N$ be a normal subgroup of a finite group $G$. If $\Gamma_G(N)$ has exactly one triangle, then one of the following possibilities holds:*

(1) *$N$ is a $p$-group for some prime $p$.*

(2) *$N = P \times Q$, with $P$ $p$-elementary abelian and $Q$ $q$-elementary abelian for some primes $p$ and $q$, and $\mathbf{Z}(G) \cap N = 1$.*

(3) *$N = P \times Q$, with $P$ a $p$-group for a prime $p \neq 3$, and $Q \subseteq \mathbf{Z}(G) \cap N$, $Q \cong \mathbb{Z}_3$ and $P/(\mathbf{Z}(G) \cap P)$ has exponent $p$.*

(4) *$N = PQ$, where $P$ is a Sylow $p$-subgroup, $p \neq 2$ and $Q$ is a Sylow 2-subgroup of $N$. In addition, $P$ has exponent $p$, $|\mathbf{Z}(G) \cap N| = 2$ and $Q/(\mathbf{Z}(G) \cap N)$ is 2-elementary abelian.*

(5) *Either $N$ is a Frobenius group with complement $\mathbb{Z}_q$, $\mathbb{Z}_{q^2}$ or $Q_8$ for a prime $q$, or there are two primes $p$ and $q$ such that $N/\mathbf{O}_p(N)$ is a Frobenius group of order $pq$ and $\mathbf{O}_p(N)$ has exponent $p$. In this case, $\mathbf{Z}(G) \cap N = 1$.*

(6) *$N \cong A_5$ and $G = (N \times K)\langle x\rangle$ for some $K \leq G$ and $x \in G$, with $x^2 \in N \times K$ and $G/K \cong N\langle x\rangle \cong S_5$.*

As an application of the above results, a theorem for graphs without triangles was achieved. As we mentioned above, in [20] it was given a complete list of those groups $G$ whose graph $\Gamma(G)$ has no triangles, which is the following.

**Theorem 3.9.** [20, Main theorem] *Let $G$ be a non-abelian finite group. Then $\Gamma(G)$ is a graph without triangles if and only if $G$ is isomorphic to one of the following solvable groups:*

- *the symmetric group $S_3$;*
- *the dihedral groups $D_{10}$ and $D_{12}$;*
- *the alternating group $A_4$;*
- *the group $T_{12}$ of order 12 given by $T_{12} = \langle a, b : a^6 = 1, b^2 = a^3, ba = a^{-1}b\rangle$;*
- *the group $T$ of order 21 given by $T = \langle a, b : a^3 = b^7 = 1, ba = ab^2\rangle$.*

*It turns out that the property that $\Gamma(G)$ has no triangles is equivalent to the one that $\Gamma(G)$ is a disjoint union of two connected trees.*

The structure of $N$ when $\Gamma_G(N)$ has no triangles is determined in [9] by appealing to certain classification theorems concerning CP-groups, that is, groups having all elements of prime-power order. The authors first needed to prove the solvability of $N$.

**Theorem 3.10.** [9, Theorem 6.1] *Let $N$ be a normal subgroup of a group $G$ such that $\Gamma_G(N)$ has no triangles. Then $N$ is solvable.*

**Theorem 3.11.** [9, Theorem A] *Let $N$ be a non-central normal subgroup of a finite group $G$ such that $\Gamma_G(N)$ has no triangles. Then $N$ is a $\{p, q\}$-group and satisfies one of these properties*

(1) *$N$ is a $p$-group.*
(2) *$N = P \times Q$ with $P$ a $p$-group and $Q \subseteq \mathbf{Z}(G) \cap N$, $Q \cong \mathbb{Z}_2$.*
(3) *$N = P \times Q$ with $P$ a $p$-group and $Q$ a $q$-group both elementary abelian with $p$ and $q$ odd primes. In this case $\mathbf{Z}(G) \cap N = 1$.*
(4) *$N$ is a quasi-Frobenius group with abelian kernel and complement and $\mathbf{Z}(G) \cap N \cong \mathbb{Z}_2$.*
(5) *$N$ is a Frobenius group with complement isomorphic to $\mathbb{Z}_q$, $\mathbb{Z}_{q^2}$ or $Q_8$. In the first case, the kernel of $N$ is a $p$-group with exponent less or equal than $p^2$ and in the last two cases, the kernel of $N$ is $p$-elementary abelian.*

Examples for each one of the cases of this section can be found in [9].

## 4. Landau's theorem on conjugacy classes for normal subgroups

Landau's theorem on conjugacy classes asserts that there are only finitely many finite groups, up to isomorphism, with exactly $k$ conjugacy classes for any positive integer $k$. No upper bound in terms of $k$ for the order of such groups was provided in the original theorem. M. Newman [28] demonstrated that $|G| \leq k^{2^{k-1}}$, or equivalently, that $k$ goes to infinity with the order of $G$ according to the inequality

$$k \geq \frac{\log_2(\log_2|G|)}{2}.$$

Since then many results have appeared in the literature regarding logarithmic bounds for certain classes of groups, for instance, nilpotent or "almost nilpotent" groups [15], or solvable groups [14], as well as extensions of Landau's result, such as [24] and [27], where only conjugacy classes of prime-power order elements and of $p$-regular elements are taken into account.

When dealing with classes contained in a normal subgroup $N$, a natural question is whether there exist finitely many groups having a normal subgroup which is the union of a fixed number of $G$-classes. The answer is negative if the index $|G : N|$ is not fixed. Indeed, if we take $N$ to be a $p$-elementary abelian group of order $p^s$ and $G$ is the holomorph group of $N$, then since $\mathrm{Aut}(N)$ acts transitively on $N \setminus \{1\}$, it follows that $N$ consists only of two $G$-classes, $\{1\}$ and $N \setminus \{1\}$. Nevertheless, $\mathrm{Aut}(N) \cong \mathrm{GL}(s, p)$ and so $|G : N| = |\mathrm{GL}(s, p)|$ may increase as much as we wish. On the contrary, if the index $|G : N|$ is fixed, then the answer to the question is affirmative and upper bounds for $|G|$ and $|N|$ depending on the number of non-central $G$-classes lying in $N$ (instead of all $G$-classes) can be provided. Precisely, the authors prove in [7] the following extension of Landau's theorem.

**Theorem 4.1.** [7, Theorem A] *Let $s, n \in \mathbb{N}$ such that $s, n \geq 1$. There exists at most a finite number of isomorphism classes of finite groups $G$ which contain a normal subgroup $N$ such that $|G : N| = n$ and $N$ has exactly $s$ non-central $G$-classes. Moreover,*

$$|G| < n^{2^s+1}(s+1)\prod_{i=0}^{s-1}(s+1-i)^{2^{s-1-i}}$$

*and*

$$|N| < n^{2^s}(s+1)\prod_{i=0}^{s-1}(s+1-i)^{2^{s-1-i}}.$$

When $n = 1$, the previous formula is an improvement of Newman's bound in terms of the number of non-central classes in a group. As an application of Theorem 4.1, the authors classified groups and normal subgroups with certain number of non-central $G$-classes by using GAP. In order to expedite this classification, they employed properties of the graph $\Gamma_G(N)$. They explicitly classified those groups $G$ having a normal subgroup $N$ with one or two non-central $G$-conjugacy classes of coprime sizes for some concrete indices. The reason why they did not deal with the case of two non-central $G$-classes with non-coprime size is because it cannot be ensured that $\mathbf{Z}(G) \cap N = 1$. Thus, the bound of Theorem 4.1 cannot be improved and this is too large to be used in an efficient algorithm. However, when the sizes are coprime this equality always holds, and the bound of $|N|$ can be improved. The considered indices allow to classify groups by means of the library SmallGroups of GAP.

To improve the algorithm efficiency in order to find normal subgroups with only one non-central $G$-class, the bound of Theorem 4.1 was improved and those normal subgroups that do not satisfy the conditions of Theorem 3.3 were discarded.

**Theorem 4.2.** [7, Theorem 3.1] *Let $N$ be a normal subgroup of a group $G$ with $|G : N| = n$. If $N$ has exactly one non-central $G$-conjugacy class, then $|G| < n(n+1)^2$.*

In Table 1, we indicate the index, the bound for $|G|$ of Theorem 4.2 and the number of groups with a normal subgroup containing a single non-central conjugacy class of $G$.

In Table 2, we show the complete classification for the indices that appear in Table 1. It is remarkable that for index 5 there is no group satisfying the desired conditions.

In order to get normal subgroups with two non-central $G$-classes of coprime sizes the authors used [9, Lemma 4.1.1], which asserts that in this clase $\mathbf{Z}(G) \cap N = 1$. Also, Theorem 3.4 is used to improve the efficiency of the algorithm. Moreover, they gave the following improvement of the bounds of Theorem 4.1.

TABLE 1. Number of groups with normal subgroups having one non-central $G$-class.

| $\|G:N\| = n$ | $\|G\| \leq n(n+1)^2$ | Number of groups |
|---|---|---|
| 2 | 18 | 3 |
| 3 | 48 | 2 |
| 4 | 100 | 21 |
| 5 | 180 | 0 |
| 6 | 294 | 16 |
| 7 | 448 | 1 |

TABLE 2. Groups having normal subgroups with one non-central $G$-class for certain indices.

| $\|G:N\| = n$ | $G$ | | | | | |
|---|---|---|---|---|---|---|
| 2 | 6#1 | 8#3 | 8#4 | | | |
| 3 | 12#3 | 24#3 | | | | |
| 4 | 12#1 | 12#4 | 16#3 | 16#4 | 16#6 | 16#7 |
| | 16#8 | 16#9 | 16#11 | 16#12 | 16#13 | 20#3 |
| | 32#27 | 32#28 | 32#29 | 32#30 | 32#31 | 32#32 |
| | 32#33 | 32#34 | 32#35 | | | |
| 6 | 18#1 | 18#3 | 18#4 | 24#12 | 24#13 | 24#4 |
| | 24#6 | 24#8 | 24#10 | 24#11 | 42#1 | 48#28 |
| | 48#29 | 48#32 | 48#33 | 54#8 | | |
| 7 | 56#11 | | | | | |

**Theorem 4.3.** [7, Theorem 3.2] *Let $N$ be a normal subgroup of a group $G$ with $|G:N| = n$. Suppose that $G$ has exactly two non-central conjugacy classes $x_1^G$ and $x_2^G$ in $N$ and these two classes have coprime sizes. Let $n_1 = |\mathbf{C}_G(x_1)|$ and $n_2 = |\mathbf{C}_G(x_2)|$ such that $n_1 < n_2$. Then*

(i) $n + 1 \leq n_1 \leq 3n - 1$.
(ii) $E[\frac{nn_1}{n_1-n}] + 1 \leq n_2 \leq E[\frac{2n_1 n}{n_1-n}]$, *where $E[x]$ denotes the integer part of $x$.*
(iii) $|G| \leq n(n+1)(n^2+n+1)$.

In Table 3 we indicate the index, the improved bound for $|G|$ appearing in Theorem 4.3, and the number of groups with a normal subgroup containing two non-central classes of $G$ of coprime sizes. The complete classification can be found in [7].

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Table 3. Groups with normal subgroups having two non-central $G$-classes with coprime sizes.

| $\lvert G:N\rvert = n$ | $\lvert G\rvert \leq n(n+1)(n^2+n+1)$ | Number of groups |
|---|---|---|
| 2 | 42 | 3 |
| 3 | 156 | 2 |
| 4 | 420 | 7 |
| 5 | 930 | 2 |
| 6 | 1806 | 8 |
| 7 | 3192 | 1 |
| 8 | 5256 | 22 |
| 9 | 8190 | 5 |
| 10 | 12210 | 7 |
| 11 | 17556 | 2 |
| 13 | 33306 | 2 |
| 17 | 93942 | 1 |

As we pointed out before, L. Héthelyi and B. Külshammer [24] proved an extension of Landau's theorem for prime-power order elements. They did not give, however, any numerical expression for the upper bound of the group order. Here is their result.

**Theorem 4.4.** [24, Theorem 1.1] *For any positive integer $k$, there are only finitely many finite groups, up to isomorphism, with exactly $k$ conjugacy classes of elements of prime power order.*

In [7], we provide an explicit function of the bound for solvable groups. If $G$ is a finite group and $N \trianglelefteq G$, then $kpp(G)$ denotes the number of conjugacy classes of prime-power order elements of $G$ and $kpp_G(N)$ the number of $G$-classes of prime-power order elements of $N$.

**Theorem 4.5.** [7, Theorem 4.7] *If $G$ is a finite solvable group such that $kpp(G) = k$, then $|G| \leqslant \gamma(k)$ where $\gamma$ is defined as follows: $\gamma(1) = 1$ and $\gamma(k) = k\gamma(k-1)^2$ for every $k \geqslant 2$. Consequently*

$$|G| \leqslant \prod_{i=0}^{k-1} (k-i)^{2^i}.$$

An extenssion of Landau's result for prime-power order elements lying in a normal subgroup was also given in [7]. However, one cannot restrict to just non-central $G$-classes of prime-power order elements contained in a normal subgroup $N$ of $G$. This happens because one can easily see that $|N|$ and $|G|$ cannot be bounded in terms of the number of such classes although the index $|G : N|$ is fixed. For instance, suppose that $N \trianglelefteq G$ with $|G : N| = n$ and that $N$ has just one non-central $G$-class (necessarily of prime-power order elements). Then $N$ is a $p$-group for some prime $p$ and we can take an arbitrary abelian finite $p'$-group $H$ and construct $N_0 = N \times H$ and $G_0 = G \times H$. It follows that

$N_0 \trianglelefteq G_0$ has index $n$ too, and $N_0$ contains exactly one non-central $G_0$-class of prime-power order elements. Nevertheless, $|N_0|$ and $|G_0|$ need not be bounded. Therefore, all $G$-classes, central and non-central, must be considered. The bounds for $|G|$ and $|N|$ in terms of $kpp_G(N)$ are given in the following.

**Corollary 4.6.** [7, Theorem B] *Let $G$ be a finite solvable group and let $N \trianglelefteq G$ such that $|G : N| = n$ and $kpp_G(N) = k$. Then $|N| \leqslant \prod_{i=0}^{nk-1}(nk-i)^{2^i}$ and $|G| \leqslant n\prod_{i=0}^{nk-1}(nk-i)^{2^i}$.*

These bounds allow to obtain, with the help of GAP, a classification of all groups such that $kpp(G)$ is 2, 3, 4 or 5. They are listed in Table 4.

TABLE 4. Classification of solvable groups with small $kpp(G)$.

| $kpp(G)$ | $G$ | | | |
|---|---|---|---|---|
| 2 | 2#1 | | | |
| 3 | 3#1 | 6#1 | | |
| 4 | 4#1 | 4#2 | 6#2 | 10#1 |
| | 12#3 | | | |
| 5 | 5#1 | 8#3 | 8#4 | 12#1 |
| | 12#4 | 14#1 | 20#3 | 21#1 |
| | 24#3 | 24#12 | 30#3 | 42#1 |

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**Antonio Beltrán**

Departamento de Matemáticas, Universidad Jaume I, 12071, Castellón, Spain

Email: `abeltran@mat.uji.es`

**María José Felipe**

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, 46022, Valencia, Spain

Email: `mfelipe@mat.upv.es`

**Carmen Melchor**

Departamento de Matemáticas, Universidad Jaume I, 12071, Castellón, Spain

Email: `cmelchor@uji.es`