A front-fixing numerical method for a free boundary nonlinear diffusion logistic population model

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Abstract

The spatial-temporal spreading of a new invasive species in a habitat has interest in ecology and is modeled by a moving boundary diffusion logistic partial differential problem, where the moving boundary represents the unknown expanding front of the species. In this paper a front-fixing approach is applied in order to transform the original moving boundary problem into a fixed boundary one. A finite difference method preserving qualitative properties of the theoretical solution is proposed. Results are illustrated with numerical experiments.

Keywords: Diffusive logistic population model, moving boundary, Stefan condition, finite difference, numerical analysis, computing simulation.

1. Introduction

For the sake of clarity in the terminology and as many authors undistinguish the terms free boundary and moving boundary, we recall these concepts following Crank approach [9]. A moving boundary problem is characterized by the fact that the boundary of the domain is not known in advance but it has to be determined as a part of the solution. These problems are often called Stefan problems due to the Stefan condition that links the behavior of the boundary with the unknown solution, see [9, 22]. The term free-boundary problem is

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commonly used when the boundary is independent of the time and typically related to elliptic problems. Moving boundary problems have their origins in physical and engineering problems [9, 14], and more recently in biological and physiological sciences [5], decision and control theory and ecology [13].

Prior to [13] the modelling of biological invasions has been widely studied in [15, 16, 24, 1, 2, 23, 27, 28, 19] under the crucial restriction that in the previous papers the spatial domain is not constrained by the population behavior, that is the essence of the Stefan condition. The first diffusive logistic model related to biological invasions was initiated in 1937, of course without boundary restrictions, independently by Fisher [15] and Kolmogorov-Petrovsky-Piskunov (KPP) [16]. Very recent papers have treated numerically these nonlinear models focusing on the stability and the preservation of the qualitative properties of the theoretical solution [4, 20, 21].

To our knowledge the seminal paper [13] by Du and Lin is the first contribution in the field of spreading of populations where a Stefan condition is used and managing a moving boundary problem of parabolic type. Further developments of this problem have been treated in [11, 12, 26]. The diffusive logistic model of [13] for the density of population of the invasive species $U(t,x)$ depending on time t and spatial variable x states as follows:

$$\frac{\partial U}{\partial t} - D\frac{\partial^2 U}{\partial x^2} = U(a-bU), \;\; t>0, \; 0<x<H(t), \tag{1}$$

together with the boundary conditions

$$\frac{\partial U}{\partial x}(t,0) = 0, \quad U(t,H(t)) = 0, \quad t>0, \tag{2}$$

the Stefan condition

$$H'(t) = -\mu\frac{\partial U}{\partial x}(t,H(t)), \quad t>0, \tag{3}$$

and the initial conditions

$$H(0) = H_0, \quad U(0,x) = U_0(x), \quad 0 \le x \le H_0. \tag{4}$$

The initial function $U_0(x)$ satisfies the following properties:

$$U_0(x) \in C^2([0, H_0]), \quad U_0'(0) = U_0(H_0) = 0, \quad U_0(x) > 0, \;\; 0 \leq x < H_0. \qquad (5)$$

Here $H(t)$ is the unknown moving boundary such that the population is distributed in the interval $[0, H(t)]$, $D > 0$ is the dispersal rate and the positive parameters a and b are the intrinsic growth rate and the intraspecific competition, respectively. The parameter $\mu > 0$ involved in the Stefan condition (3) is the proportionality constant between the population gradient at the front and the speed of the moving boundary. Unlike to the previous models, where only spreading behaviour was admissible, the authors of [13] show by the very first time a dichotomic alternative behavior, vanishing or spreading approach to the habitat carrying capacity a/b, depending on the initial front and population density and the value of the parameter μ appearing in Stefan condition. According to [13] there is a threshold value μ^* whose value is not known in advance splitting the vanishing-spreading behavior.

This paper aims to be a continuation and numerical completion of [13] with the conviction that the best model may be wasted with a careless numerical treatment. Apart from the computation of the population density solution of problem (1)-(4) and the numerical analysis detailed below, this paper has the potential advantage that allows us the computation of the expanding front of the species population as well as the approximation of the crucial parameter μ^* whose existence is guaranteed in [13], but whose value is not known in terms of data problem. A brief numerical treatment of the problem may be found in Section 3.6 of [3].

This paper is organized as follows. In Section 2, and following the trajectory of the authors in the study of finance problems (see [7, 8]), we use the well-known Landau transformation (see [9, 18]), in order to convert the problem (1)-(4) into a fixed spatial domain one, where the moving boundary is included as another variable to solve apart from the population density. We also include in Section 2 the discretization of the transformed problem achieving an explicit finite difference scheme allowing the computation not only of the population but also of the expanding front. Section 3 deals with the study of the consistency of the scheme with the transformed problem. Dealing with population problems it is important to guarantee the positivity of the numerical solution; this qualitative property together with the stability of the numerical solution and the positivity and monotone behavior of the numerical expanding front are studied in Section 4. Section 5 illustrates with numerical examples the dichotomic behavior of the numerical solution of the problem.

2. Transformation and discretization of the continuous problem

Let us begin this section by transforming the moving front problem (1)-(4) into a problem with a fixed domain $[0, 1]$. Let us consider the Landau transformation, [9, 18],

$$z(t,x) = \frac{x}{H(t)}, \quad W(t,z) = U(t,x). \tag{6}$$

Under substitution (6) problem (1)-(4) takes the form:

$$G(t)\frac{\partial W}{\partial t} - G'(t)\frac{z}{2}\frac{\partial W}{\partial z} - D\frac{\partial^2 W}{\partial z^2} = G(t)W(a - bW), \;\; t > 0, \; 0 < z < 1, \tag{7}$$

where:

$$G(t) = H^2(t), \;\; t \geq 0. \tag{8}$$

Boundary conditions (2) and Stefan condition (3) take the form:

$$\frac{\partial W}{\partial z}(t,0) = 0, \quad W(t,1) = 0, \quad t > 0, \tag{9}$$

and

$$G'(t) = -2\mu\frac{\partial W}{\partial z}(t,1), \quad t > 0, \tag{10}$$

respectively, while the initial conditions (4) become:

$$G(0) = H_0^2, \quad W(0,z) = W_0(z) = U_0(zH_0), \quad 0 \leq z \leq 1. \tag{11}$$

Conditions (5) for the initial function $U_0(x)$ are translated to $W_0(z)$ as follows:

$$W_0(z) \in C^2([0,1]), \quad W_0'(0) = W_0(1) = 0, \quad W_0(z) > 0, \;\; 0 \leq z < 1. \tag{12}$$

After the transformation, the new problem lies in solving the nonlinear parabolic partial differential equation (7) in the unbounded fixed domain $(0,\infty)\times(0,1)$ for the variables (t,z). Let us consider the step size discretization $k=\Delta t$, $h=\Delta z=1/M$, and the mesh points (t^n,z_j), with $t^n=kn$, $n\geq 0$, $z_j=jh$, $0\leq j\leq M$ and M positive integer. Let us denote the approximate value of $W(t^n,z_j)$ at the mesh point (t^n,z_j),

$$w_j^n \approx W(t^n,z_j), \tag{13}$$

and let g^n be the approximation of $G(t^n)$. Let us consider the forward approximation of the time derivatives,

$$\frac{w_j^{n+1}-w_j^n}{k} \approx \frac{\partial W}{\partial t}(t^n,z_j), \quad \frac{g^{n+1}-g^n}{k} \approx G'(t^n), \tag{14}$$

and the central approximation of the spatial derivatives,

$$\frac{w_{j+1}^n-w_{j-1}^n}{2h} \approx \frac{\partial W}{\partial z}(t^n,z_j), \quad \frac{w_{j-1}^n-2w_j^n+w_{j+1}^n}{h^2} \approx \frac{\partial^2 W}{\partial z^2}(t^n,z_j). \tag{15}$$

From (14) and (15) the equation (7) is approximated by:

$$g^n\frac{w_j^{n+1}-w_j^n}{k}-\frac{z_j}{2}\frac{w_{j+1}^n-w_{j-1}^n}{2h}\left(\frac{g^{n+1}-g^n}{k}\right)-D\frac{w_{j-1}^n-2w_j^n+w_{j+1}^n}{h^2} \\ = g^n w_j^n(a-bw_j^n), \quad n\geq 0, \quad 0\leq j\leq M-1, \tag{16}$$

that can be written as:

$$w_j^{n+1}=\left[\frac{Dk}{h^2g^n}-\frac{z_j}{4h}\left(\frac{g^{n+1}}{g^n}-1\right)\right]w_{j-1}^n+\left[1+k(a-bw_j^n)-\frac{2Dk}{h^2g^n}\right]w_j^n \\ +\left[\frac{Dk}{h^2g^n}+\frac{z_j}{4h}\left(\frac{g^{n+1}}{g^n}-1\right)\right]w_{j+1}^n, \quad n\geq 0, \quad 0<j\leq M-1. \tag{17}$$

As it is usual in numerics we assume that equation (7) can be also approximated at $z_j=0$. Equation (17) written for $j=0$ involves the fictitious value w_{-1}^n at the point $(t^n,-h)$. This value w_{-1}^n is eliminated from the discretization of the boundary and initial conditions (9) and (11),

$$\frac{w_1^n-w_{-1}^n}{2h}=0, \quad w_M^n=0, \quad n\geq 0. \tag{18}$$

Transformed Stefan condition (10) is discretized using first order forward approximation for $G'(t)$ and three points backward spatial approximation of $\frac{\partial W}{\partial z}(t,1)$:

$$\frac{g^{n+1}-g^n}{k} = -\frac{\mu}{h}(3w_M^n - 4w_{M-1}^n + w_{M-2}^n), \quad n \geq 0, \tag{19}$$

to preserve accuracy of order $\mathcal{O}(k)+\mathcal{O}(h^2)$. From (18) equation (19) can be rewritten as:

$$g^{n+1} = g^n + \frac{k\mu}{h}(4w_{M-1}^n - w_{M-2}^n), \quad n \geq 0. \tag{20}$$

Finally, replacing (20) in the explicit scheme (17), we have:

$$w_j^{n+1} = a_j^n w_{j-1}^n + b_j^n w_j^n + c_j^n w_{j+1}^n, \quad n \geq 0, \quad 0 \leq j \leq M-1, \tag{21}$$

where the coefficients are given by:

$$\begin{aligned} a_j^n &= \frac{k}{h^2}\left(\frac{D}{g^n} - \frac{z_j\mu(4w_{M-1}^n - w_{M-2}^n)}{4g^n}\right), \\ b_j^n &= 1 + k(a - bw_j^n) - \frac{k}{h^2}\frac{2D}{g^n}, \\ c_j^n &= \frac{k}{h^2}\left(\frac{D}{g^n} + \frac{z_j\mu(4w_{M-1}^n - w_{M-2}^n)}{4g^n}\right), \\ & n \geq 0, \quad 0 \leq j \leq M-1. \end{aligned} \tag{22}$$

3. Consistency

Consistency of a numerical scheme with a PDE problem means that the theoretical solution of the problem approximates well the numerical scheme when the step size discretizations tend to zero. So, a numerical scheme can be consistent with an equation and not with another one, see [25], chap. 2. Thus, it is important to address the consistency of a numerical scheme with a problem.

Let us consider the problem (7)-(11), denoted in vector form as $\mathcal{L}(W,G) = (\mathcal{L}_1(W,G), \mathcal{L}_2(W,G), \mathcal{L}_3(W,G))$ where equations (7),(9), (10) are written in the form:

$$\mathcal{L}_1(W,G) = \frac{\partial W}{\partial t} - \frac{G'(t)}{G(t)}\frac{z}{2}\frac{\partial W}{\partial z} - \frac{D}{G(t)}\frac{\partial^2 W}{\partial z^2} - W(a-bW) = 0, \;\; t>0, \; 0<z<1, \tag{23}$$

$$\mathcal{L}_2(W,G) = \frac{\partial W}{\partial z}(t,0) = 0, \;\; t > 0, \tag{24}$$

$$\mathcal{L}_3(W,G) = G'(t) + 2\mu \frac{\partial W}{\partial z}(t,1) = 0, \quad t > 0, \tag{25}$$

and the finite difference scheme (16), (18), (20), written together as $L(w,g) = (L_1(w,g), L_2(w,g), L_3(w,g))$ where:

$$\begin{aligned} L_1(w,g) = \frac{w_j^{n+1} - w_j^n}{k} - \frac{z_j}{2}\frac{w_{j+1}^n - w_{j-1}^n}{2h}\left(\frac{g^{n+1} - g^n}{g^n k}\right) \\ -\frac{D}{g^n}\frac{w_{j-1}^n - 2w_j^n + w_{j+1}^n}{h^2} - w_j^n(a - bw_j^n) = 0, \quad n \geq 0, \quad 0 \leq j \leq M-1, \end{aligned} \tag{26}$$

$$L_2(w,g) = \frac{w_1^n - w_{-1}^n}{2h} = 0, \;\; n \geq 0, \tag{27}$$

$$L_3(w,g) = \frac{g^{n+1} - g^n}{k} - \frac{\mu}{h}(4w_{M-1}^n - w_{M-2}^n) = 0, \quad n \geq 0. \tag{28}$$

In accordance with [25], scheme $L(w,g)$ is said to be consistent with problem $\mathcal{L}(W,G)$ if local truncation error $T_j^n(W,G) = (T(1)_j^n, T(2)_j^n, T(3)_j^n)$,

$$T(1)_j^n(W,G) = L_1(W_j^n, G^n) - \mathcal{L}_1(W_j^n, G^n), \tag{29}$$

$$T(2)_j^n(W,G) = L_2(W_n^j, G^n) - \mathcal{L}_2(W_j^n, G^n), \tag{30}$$

$$T(3)_j^n(W,G) = L_3(W_j^n, G^n) - \mathcal{L}_3(W_j^n, G^n), \tag{31}$$

tend to zero as $k \to 0$, $h \to 0$, where $W_j^n = W(t^n, z_j)$ and $G^n = G(t^n)$ are the values of the exact solution of problem (7)-(11) of both the PDE and the free boundary respectively at the point (t^n, z_j) . Now let us consider the local truncation error $T(1)_j^n$ assuming that the exact solution $W(t,z)$ is continuously partial differentiable four times with respect to z and two times with respect to t. We also assume that $G(t)$ is two times continuously differentiable. By using

Taylor's expansion about (t^n, z_j) one gets:

$$T(1)_j^n(W,G) = E_j^n(1)k - \frac{z_j}{2G^n}G'(t^n)E_j^n(3)h^2 - \frac{z_j}{2G^n}E^n(2)E_j^n(3)kh^2 \\ - \frac{z_j}{2G^n}\frac{\partial W}{\partial z}(t^n, z_j)E^n(2)k - \frac{D}{G^n}E_j^n(4)h^2, \tag{32}$$

where:

$$E_j^n(1) = \frac{1}{2}\frac{\partial^2 W}{\partial t^2}(\tau, z_j), \quad t^n < \tau < t^{n+1}. \tag{33}$$

$$E^n(2) = \frac{1}{2}\frac{d^2 G}{dt^2}(\delta), \quad t^n < \delta < t^{n+1}. \tag{34}$$

$$E_j^n(3) = \frac{1}{6}\frac{\partial^3 W}{\partial z^3}(t^n, \xi_1), \quad z_{j-1} < \xi_1 < z_{j+1}. \tag{35}$$

$$E_j^n(4) = \frac{1}{12}\frac{\partial^4 W}{\partial z^4}(t^n, \xi_2), \quad z_{j-1} < \xi_2 < z_{j+1}. \tag{36}$$

Hence, the local truncation error satisfies:

$$T(1)_j^n(W,G) = \mathcal{O}(k) + \mathcal{O}(h^2). \tag{37}$$

From (24), (27) and (30) one gets that $T(2)_j^n(W,G) = \mathcal{O}(h^2)$ while from (25), (28) and (31) it follows that $T(3)_j^n(W,G) = \mathcal{O}(k) + \mathcal{O}(h^2)$. Summarizing the following result has been established:

Theorem 1. *With previous notation, the scheme $L(w,g)$ is consistent with the problem $\mathcal{L}(W,G)$ and the local truncation error behaves as:*

$$T_j^n(W,G) = \mathcal{O}(k) + \mathcal{O}(h^2). \tag{38}$$

4. Positivity and Stability

4.1. Positivity

Dealing with population models it is necessary to guarantee that the numerical solution is nonnegative. In this section we show that the numerical solution of the scheme (21)-(22) is nonnegative for small enough values of the step size discretization. We also prove that the numerical solution preserves qualitative properties of the exact theoretical solution of the problem obtained by Du and Lin in [13].

We prove the nonnegativity of the solution w_j^n of (21)-(22) as well as the positivity and monotonicity of the free boundary g^n using the induction principle on the index n. For $n = 0$, from the initial conditions (12) $w_j^0 > 0, \; 0 \leq j \leq M-1$ and particularly $w_{M-1}^0 > 0$. From (12) we also have that the left hand side derivative $W_0'(1^-)$ at $z = 1$ and hence the corresponding difference approximation $(3w_M^0 - 4w_{M-1}^0 + w_{M-2}^0)/(2h) = (w_{M-2}^0 - 4w_{M-1}^0)/(2h) < 0$ for small enough values of h. As $g^0 > 0$, from (19) one gets:

$$g^1 > g^0 > 0. \tag{39}$$

Let us suppose that $w_j^l > 0$ and $g^l > g^{l-1} > ... > g^0 > 0, \; 1 \leq l \leq n$. We will prove that $w_j^{n+1} > 0$ and $g^{n+1} > g^n$. By using Taylor's expansion on the left about $z_M = 1$ one gets:

$$w_{M-2}^n = 2w_{M-1}^n + \mathcal{O}(h^2), \;\; n \geq 0. \tag{40}$$

Note that from (20), (40) and using that $w_M^n = 0$ and $w_{M-1}^n = O(h)$, one gets:

$$g^{n+1} = g^n + \frac{k\mu}{h}(4w_{M-1}^n - w_{M-2}^n) = g^n + \frac{k\mu}{h}(2w_{M-1}^n + \mathcal{O}(h^2)) = g^n + \mathcal{O}(k), \tag{41}$$

as it is expected from the differentiability of function $g(t)$, [13].

Coming back to the positivity issues, let us consider the equation (21) for $j = M-1$. From (22) and (40) one gets:

$$w_{M-1}^{n+1} = \left(1 + k(a - bw_{M-1}^n) - \frac{k}{h^2}\frac{z_{M-1}}{g^n}\mu w_{M-1}^n\right)w_{M-1}^n + \mathcal{O}(h^2). \tag{42}$$

As $g^n > g^0$ from the hypothesis and $z_{M-1} < 1$, from (42) we can write:

$$w_{M-1}^{n+1} > \left(1 + k(a - bw_{M-1}^n) - \frac{k}{h^2}\frac{1}{g^0}\mu w_{M-1}^n\right)w_{M-1}^n = \varphi_{M-1}^n w_{M-1}^n, \tag{43}$$

for small enough values of h. As we are interested in showing that $\varphi_{M-1}^n > 0$, let us start bounding w_{M-1}^n. From the expression of (42), for small enough values of h we have that $w_{M-1}^{l+1} < w_{M-1}^l(1 + ka), \; 0 \leq l \leq n-1$. Recursively one gets:

$$w_{M-1}^{l+1} < w_{M-1}^0(1 + ka)^l \leq e^{aT} w_{M-1}^0, \;\; 0 \leq l \leq n \leq N-1; \quad kN = T, \tag{44}$$

for a time reference $T > 0$.

From (44) and the definition of φ^n_{M-1} given in (43) it is easy to show that $w^{n+1}_{M-1} > 0$ under the condition:

$$k < \frac{h^2}{\frac{\mu C}{g^0} + h^2(bC - a)}, \tag{45}$$

where:

$$C = e^{aT} w^0_{M-1}. \tag{46}$$

Once the positivity of w^n_{M-1} is established, it is necessary to show that $w^n_j > 0$ for $0 \leq j \leq M-2$. From (21) and the induction hypothesis, this occurs when coefficients of the scheme are nonnegative. Note that from (22) and (40) every coefficient $c^n_j > 0$ for small enough values of h. From (22), and taking into account that $0 \leq z_j < 1$, coefficient $a^n_j > 0$ if $\mu w^n_{M-1} < 2D$, and thus from (46) one concludes that $a^n_j > 0$ holds true under the condition:

$$w^n_{M-1} < \frac{2De^{-aT}}{\mu}. \tag{47}$$

Let $B(n)$ be defined by $B(n) = \max\{w^n_j;\ ,\ 0 \leq j \leq M\}$. Using that $g^n > g^0 > 0$ by induction hypothesis $b^n_j > 0$ if

$$k < \frac{h^2}{\frac{2D}{g^0} + h^2(bB(n) - a)}. \tag{48}$$

In order to get an explicit expression of $B(n)$ independent of the discretization, note that from positivity of coefficients a^n_j, b^n_j, c^n_j and (21):

$$\begin{aligned} w^{n+1}_j \leq (1 + k(a - bw^n_j))B(n) \leq (1 + ka)B(n) \leq (1 + ka)^2 B(n-1) \\ \leq ... \leq (1 + ka)^n B(0) < e^{aT} B(0), \end{aligned} \tag{49}$$

where $B(0) = \max\{W(0, z)\}$, $0 \leq z \leq 1$. In order to prove the monotonicity of the free boundary g^n, from (40) and (20) one gets that $g^{n+1} > g^n$.

Summarizing, the following results have been established:

Theorem 2. *With previous notation, let k_0 be:*

$$k_0 = \min\left\{ k_1 = \frac{h^2}{\frac{\mu C}{g^0} + h^2(bC - a)}, k_2 = \frac{h^2}{\frac{2D}{g^0} + h^2(be^{aT}B(0) - a)} \right\}. \tag{50}$$

Under condition $k < k_0$ for small enough values of h the solution $\{w_j^n, g^n\}$ of scheme (18), (20) and (21) verifies that g^n is positive monotone increasing and:

$$0 \leq w_j^n \leq B(0)e^{aT}; \ \ 0 \leq j \leq M, \ \ 0 \leq n \leq N, \ \ Nk = T. \tag{51}$$

4.2. Stability

As the concept of stability is somewhat plural in the literature, for the sake of clarity in the presentation we specify the concept of stability we use below (see page 92 of [17], [6]). We recall the definition of the supremum norm of a vector $x = (x_1, x_2, ..., x_n)^T$ in $\mathbb{R}^n$ as $\|x\|_\infty = max(|x_1|, |x_2|, ..., |x_n|)$.

Definition 1. *The numerical scheme (19)-(21) is said to be $\| \cdot \|_\infty$-stable in the domain $[0, T] \times [0, 1]$, if for every partition with $k = \Delta t$, $h = \Delta z$, $Nk = T$ and $Mh = 1$ it holds true that:*

$$\|w^n\|_\infty \leq K\|w^0\|_\infty, \ \ 0 \leq n \leq N, \tag{52}$$

where $w^n = [w_0^n, \ w_1^n, ..., w_M^n]^T$ is the vector solution of the scheme and K is independent of $h, k,$ and n.

From (51), using that $B(n) = \|w^n\|_\infty$ and from Theorem 2 one gets $\|w^n\|_\infty \leq K\|w^0\|_\infty$ with $K = e^{aT}$. Thus the following result has been established:

Theorem 3. *With previous notation, under the condition $k < k_0$ where k_0 is given by (50) and small enough values of h the numerical scheme (19)-(21) is conditionally $\| \cdot \|_\infty$-stable in the domain $[0, T] \times [0, 1]$.*

In the following examples we show that the stability and positivity condition of Theorems 2 and 3 can not be disregarded and that in fact this is a tight condition. In Example 1 the condition is satisfied, however in Example 2 the stability and positivity condition is broken and results become unstable.

Example 1. Consider the logistic diffusion model (1)-(4) with parameters $(D, \mu, a, b, H_0) = (5, 5, 5, 1, 2)$ and $U_0 = cos(\pi x/4)$. For $h = 0.05$ one gets

$k_1 = 0.0076$ and $k_2 = 0.001$. Taking $k = 0.00091$ stability is guaranteed as can be seen in Figure 1.

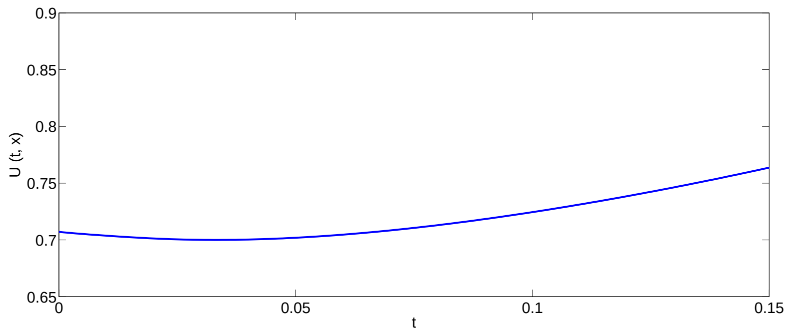

Fig. 1. Numerical solution of Example 1 for $z = 0.5$ under stability condition.

Example 2. With the same parameters and value of $h = 0.05$ as in Example 1, with $k = 0.001179$, the stability condition is broken because $k > k_2$ and Figure 2 shows unstable results.

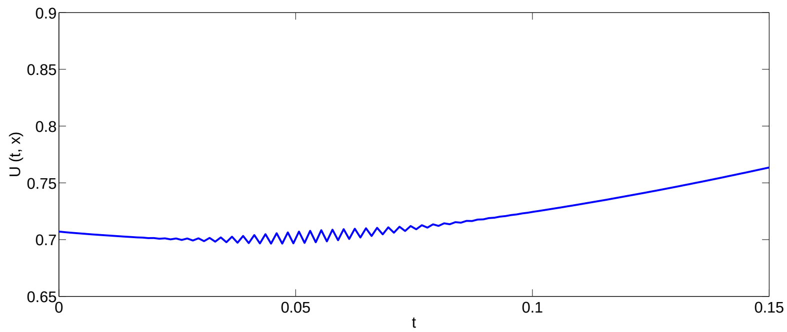

Fig. 2. Numerical solution of Example 2 for $z = 0.5$ when the stability condition is broken.

5. Numerical dichotomy: spreading versus vanishing

Theoretical results in [13] establish that for $H_0 \geq L$ where $L = \frac{\pi}{2}\sqrt{\frac{D}{a}}$ spreading of the species is guaranteed. Even if $H_0 < L$, spreading occurs under condition $\mu > \mu^*$ where μ^* is an unknown threshold depending on U_0, see Theorem 3.9 of [13]. In the spreading case the population density tends to the habitat carrying capacity limit a/b as time tends to infinity, see Lemma 3.2 of [13]. For $H_0 < L$ and $\mu \leq \mu^*$ vanishing happens, satisfying that L is an upper bound of $H(t)$, i.e., $H(t) \leq L, \quad t > 0$. The following example is devoted to spreading case showing that the numerical solution of problem (7)-(11) computed by the proposed scheme (21)-(22) converges to a/b confirming that the numerical spreading occurs.

Example 3. In the logistic diffusion model (1)-(4) with parameters values $(D, \mu, a, b, H_0) = (1, 1, 2, 1, 4)$ and $U_0 = cos(\pi x/8)$, Figure 3 shows the spreading behavior under condition $H_0 = 4.00 > L = 1.11$. Note that as time increases the numerical solution approaches to the habitat carrying capacity a/b.

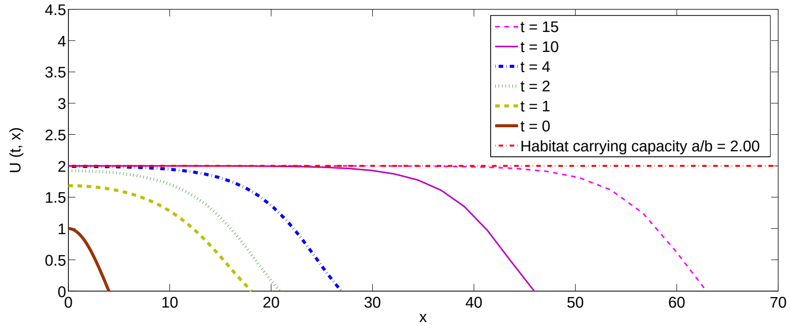

Fig. 3. Numerical solution of Example 3 for several values of time in a spreading case.

The next example illustrates the vanishing behavior of the numerical solution according to the theoretical results of [13].

Example 4. In this example we take $(D, \mu, a, b, H_0) = (0.1, 0.2, 0.04, 0.04, 1)$, with $U_0 = cos(\pi x/2)$. There is vanishing behavior with $H_0 = 1.00 < L = 2.48$ and $\mu = 0.20$. Figure 4 shows that numerical population density tends to zero and the free boundary is always upper bounded by L. Besides, Table 1 exhibits CPU time for several values of the time horizon T considered in the simulation.

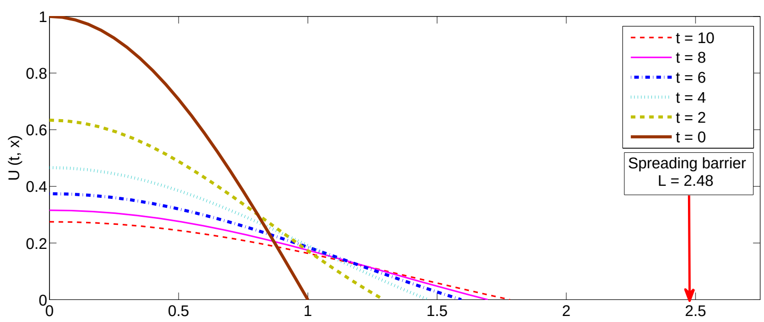

Fig. 4. Numerical solution of Example 4 for several values of time in a vanishing case.

T (years)	CPU (seconds)
2	0.186687
4	0.490169
6	1.097099
8	6.850214
10	31.021795

Table 1: Associated CPU time for several values of T (Example 4).

One of the advantages of the proposed front-fixing numerical approach is to forecast the magnitude of the parameter μ^* whose existence is guaranteed in the theory but whose value is not known. In addition, the numerical solution of the free boundary $H(t^n)$ is obtained explicitly by expression (20), noting that $H(t^n) = \sqrt{g^n}$.

Next examples show the evolution of the free boundary (Example 5) and the speed of spreading behavior (Example 6), taking into account that the forecasted value of the parameter μ^* is the threshold where the solution transits from vanishing to spreading.

Example 5. Choosing the values $(D, \mu, a, b, H_0) = (1, \mu, 1, 1, 1)$ and $U_0 = cos(\pi x/2)$, the evolution of the expanding front $H(t)$ for different values of μ is shown in Figure 5. The parameter μ^* which separates spreading from vanishing behavior is estimated. In the vanishing cases, it can be seen that the "spreading barrier" is an upper bound for the expanding front as the theoretical results predict.

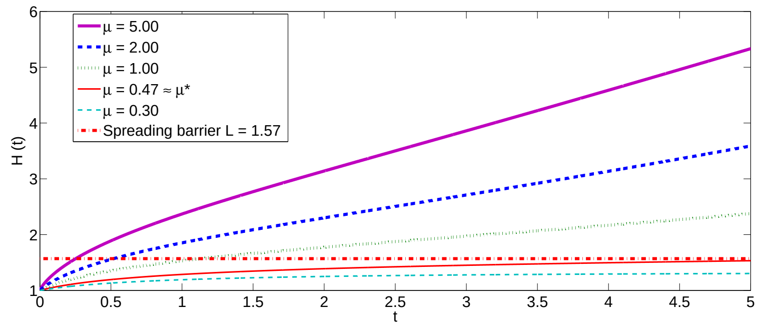

Fig. 5. Expanding front $H(t)$ of Example 5 for several values of μ.

Example 6. With the same values of the previous example, the speed of the front dH/dt is illustrated in Figure 6. In the long term, for the spreading cases, the front speed tends to a nonzero constant value in accordance with [13], Section 4, while in the vanishing case it tends to zero.

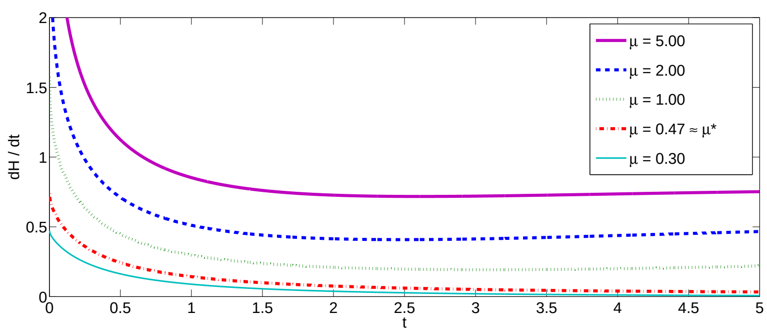

Fig. 6. Expanding front speed dH/dt of Example 6 for several values of μ.

6. Conclusions

In this paper, a front-fixing approach is introduced in such a way that the expanding front becomes a new unknown variable of the transformed problem. It has the advantage of achieving a fixed numerical domain and the availability of computing explicitly the expanding front as well as the approximation of the parameter $\mu*$, whose existence is guaranteed in [13] but its value is not known. We provide a careful numerical analysis of theoretical results given in [13]. Results and techniques are potentially applicable to problems in higher dimensions proposed in [12], or to the presence of two fronts in one dimension [13].

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References

[1] D. G. Aronson, H. F. Weinberger, *Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation*, in Partial Differential Equations and Related Topics, Lectures Notes in Math. 446, Springer, Berlin, 1975, pp. 5-49.

[2] D. G. Aronson, H. F. Weinberger, "Multidimensional nonlinear diffusions arising in population genetics", *Adv. in Math.*, 30 (1978), pp. 33-76.

[3] G. Bunting, Y. Du, K. Krakowski, "Spreading speed revisited: Analysis of a free boundary model", *Netw. Heterog. Media*, 7 (2012), pp. 583-603 (special issue dedicated to H. Matano).

[4] J. R. Branco, J. A. Ferreira, P. de Oliveira, "Numerical methods for the generalized Fisher-Kolmogorov-Petrovskii-Piskunov equation", *Applied Numerical Mathematics*, 57 (2007), 89102.

[5] X. F. Chen, A. Friedman, "A free boundary problem arising in a model of wound healing", *SIAM J. Math. Anal.*, 32 (2000), pp. 778-800.

[6] R. Company, L. Jódar, J.-R. Pintos, "A consistent stable numerical scheme for a nonlinear option pricing model in illiquid markets", *Mathematics and Computers in Simulation*, vol. 82, no. 10, pp. 1972-1985, 2012.

[7] R. Company, V. N. Egorova, L. Jódar, "Solving American Option Pricing Models by the Front Fixing Method: Numerical Analysis and Computing", *Abstract and Applied Analysis*, Volume 2014, Article ID 146745, 9 pages.

[8] R. Company, V. N. Egorova, L. Jódar, "Constructing positive reliable numerical solution for American call options: A new front-fixing approach", *Journal of Computational and Applied Mathematics*, 291 (2016), 422-431.

[9] J. Crank, *Free and Moving Boundary Problems*, Oxford Science Publications, 1984.

[10] J. Crank, "Two methods for the numerical solution of moving-boundary problems in diffusion and heat flow", *Quart. J. Mech. Appl. Math.* 10 (1957), 220-231.

[11] Y. Du, Z.M. Guo, "The Stefan problem for the Fisher-KPP equation", *J. Diff. Eqns.*, 253 (2012), 996-1035.

[12] Y. Du, Z.M. Guo, "Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, II", *J. Diff. Eqns.*, 250 (2011), 4336-4366.

[13] Y. Du, Z. Lin, "Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary", *SIAM J. Math. Anal.*, 42 (2010), no. 1, 377-405.

[14] D. J. Duffy, *Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach*, John Wiley & Sons, 2006.

[15] R. A. Fisher, "The wave of advance of advantageous genes", *Ann. Eugenics*, 7 (1937), pp. 335-369.

[16] A. Kolmogorov, N. Petrovsky, S. Piskunov, "Étude de l'équations de la diffusion avec croissance de la quantité de matière et son application a un

problème biologique", *Bull. Univ. Moskou*, 1 (1937), 125; English translation in *Dynamics of Curved Fronts*, P. Pelcé (ed.), Academic Press, 1988, pp. 105-130.

[17] D. Kröner. *Numerical Schemes for Conservation Laws*, Wiley-Teubner Series, 1997.

[18] H. G. Landau, "Heat condution in a melting solid", *Quarterly of Applied Mathematics*, vol. 8, pp. 81-95, 1950.

[19] X. Liang, X.-Q. Zhao, "Asymptotic speeds of spread and traveling waves for monotone semiflows with applications", *Comm. Pure Appl. Math.*, 60 (2007), pp. 1-40.

[20] J. E. Macías-Díaz, A. Puri, "An explicit positivity-preserving finite-difference scheme for the classical Fisher-Kolmogorov-Petrovsky-Piskunov equation", *Applied Mathematics and Computation*, 218 (2012), 5829-5837.

[21] W. Qin, D. Ding, X. Ding, "Two boundedness and monotonicity preserving methods for a generalized Fisher-KPP equation", *Applied Mathematics and Computation*, 252 (2015), 552-567.

[22] L. I. Rubinstein, "The Stefan Problem", *American Mathematical Society*, Providence, RI, 1971.

[23] N. Shigesada, K. Kawasaki, *Biological Invasions: Theory and Practice*, Oxford Series in Ecology and Evolution, Oxford University Press, Oxford, 1997.

[24] J. G. Skellam, "Random dispersal in theoretical populations", *Biometrika*, 38 (1951), pp. 196-218.

[25] G. D. Smith, *Numerical Solution of Partial Differential Equations: Finite Difference Methods*, 3rd ed., Clarendon Press, Oxford, 1985.

[26] M.X. Wang, "The diffusive logistic equation with a free boundary and sign-changing coefficient", *J. Diff. Eqns.*, (258) 2015, 1252-1266.

[27] H. F. Weinberger, "On spreading speeds and traveling waves for growth and migration models in a periodic habitat", *J. Math. Biol.*, 45 (2002), pp. 511-548.

[28] H. F. Weinberger, M. A. Lewis, B. Li, "Anomalous spreading speeds of cooperative recursion systems", *J. Math. Biol.*, 55 (2007), pp. 207-222.