Modelling Acoustics on the Poincaré Half-Plane

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Abstract

Novel advances in the field of metamaterial research have permitted the engineering of devices with extraordinary characteristics. Here, we explore the possibilities in transformation acoustics to implement a model for the simulation of acoustic wave propagation on the Poincaré half-plane—the simplest model possessing hyperbolic geometry and also of considerable historical interest. We start off from a variational principle on the given spacetime manifold to find the design description of the model in the laboratory. After examining some significant geometrical and physical properties of the Poincaré half-plane model, we derive a general formal solution for its acoustic wave propagation. A numerical example for the evolution of the acoustic potential on a rectangular region of the Poincaré half-plane concludes this discussion.

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1. Introduction

In recent years metamaterials have provided researchers and engineers with unprecedented tools for the design and construction of artificial devices with properties exceeding the possibilities found in nature. While optical metamaterials have been the focus of continued interest for the last decade, acoustic metamaterials have only recently drawn the attention of researchers [1]. The central idea

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of metamaterial research is to depart from the traditional assumption of the Euclidean predicate of the underlying physical 3-space geometry, which is the classical view *e.g.* in optics and acoustics. Instead, one exploits the remarkable properties of metamaterials to simulate optical and acoustic phenomena with curved background spacetimes, leading to such impressive effects as metamaterial cloaking and superlenses [2, 3]. In general, the study of physical phenomena with curved spacetimes does not only pose challenges in engineering, but also raises fundamental questions beyond their possible experimental verification, see *e.g.* Ref. [4].

In this work, we demonstrate the use of a novel technique based on a fundamental variational principle in combination with powerful differential-geometric methods to model acoustic wave propagation on a curved spacetime [5, 6, 7]. In particular we show how to implement acoustic wave propagation on the Poincaré half-plane model, $\mathbb{H}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ endowed with the Poincaré metric [8]. It is the simplest and one of the most thoroughly investigated non-Euclidean models of two-dimensional hyperbolic geometry (see *e.g.* [9]), which makes it a suitable spacetime candidate for the implementation and study of an acoustic metamaterial and a particularly fascinating model.

We will comment on the design and implementation of such a spacetime with acoustic metadevices via the corresponding constitutive equations which relate the physical acoustic parameters to the underlying curved spacetime, following our general approach already proposed in 2012, see Refs. [5, 7].

Finally, we outline how to derive within this framework the partial differential equation for the acoustic potential which describes wave propagation on the Poincaré half-plane. Apart from the harmonic time and x-dependence of the potential, it is possible to analytically solve the emerging Sturm-Liouville problem for the y-dependence and formally describe the solutions for the acoustic potential as a superposition of harmonics and modified Bessel functions. We conclude the discussion with a numerical simulation of the acoustic potential on a rectangular region of the Poincaré half-plane.

2. Field formulation of acoustics and variational principle

Variational principles are powerful methods in classical and field mechanics—this includes optics as an electromagnetic field theory—to define in a very concise manner the laws which govern their physical domain. The corresponding equation of motions are extremal solutions of the postulated action integrals and completely determine the physical behaviour of the system. Much of the mathematical charm and sophistication of variational principles lies in its coordinate-frame independent formulation. Moreover, Noether's theorem allows with almost no effort to shed light on the underlying symmetries of the theoretical model. In this formalism, physical laws have their equivalent in equations of motion with self-adjoint differential operators acting on the related field variables [10]. This gives rise to separable partial differential equations that frequently comprise Sturm-Liouville problems for one of the variables, so that its solutions may be obtained in an analytical or at least semi-analytical way.

For acoustics with a smooth background spacetime M (endowed with Lorentzian metric $\mathbf{g}$ and negative signature such that $g = \det \mathbf{g} < 0$), we postulate [5] that the action integral $\mathscr{A}$ is stationary with respect to variations of the acoustic potential $\phi : M \to \mathbb{R}$:

$$\frac{\delta}{\delta\phi}\mathscr{A}[\phi] = \frac{\delta}{\delta\phi}\int\limits_{\Omega} d\mathrm{vol}_g\, \mathscr{L}(x, \phi, \nabla\phi) = 0, \tag{1}$$

where the integration domain $\Omega \subseteq M$ is a bounded, closed set of spacetime and the invariant volume element is denoted by $d\mathrm{vol}_g = \sqrt{-g}\, dx^0 \wedge \ldots \wedge dx^3$ with $x \in M$. If P denotes the ambient space [11] of the acoustic potential, the explicit form of the Lagrangian function $\mathscr{L} : M \times TP \to \mathbb{R}$ is constrained by several symmetry requirements:

(i) locality (only first-order derivatives of the potential);
(ii) free-wave propagation (independence of the potential itself):
(iii) energy-momentum conservation (independence of spacetime position).

Therefore the simplest possible choice is [5]:

$$\mathscr{L}(\nabla\phi) = \tfrac{1}{2}\, \mathbf{g}(\nabla\phi, \nabla\phi). \tag{2}$$

This expression represents a kinetic term in covariant form. In Ref. [5] we have shown that Eq. (2) reduces to the classical acoustic Lagrangian in flat space [12, p. 248], corresponding to an isotropic acoustic wave equation.

In the following, we choose the notation that Latin indices run over the spatial values of tensors alone, whereas Greek indices will be used for the full range of spacetime values. As usual, comma and semicolon denote partial and covariant derivatives, respectively, and the Einstein summation convention is implied for co- and contravariant index pairs. Thus, in local coordinates $x^\mu \in M$, the Lagrangian of Eq. (2) may be rewritten as[1]

$$\mathscr{L}(\phi_{,\mu}) = \tfrac{1}{2}\, g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}. \tag{3}$$

Note that if $\mathbf{v}$ denotes the local fluid velocity, p the acoustic pressure, ϱ_0 the density, and $c > 0$ the time-independent wave speed of the acoustic metamaterial, the gradient appearing in Eq. (3) is

$$\phi_{,\mu} = \begin{pmatrix} p/c\varrho_0 \\ -\mathbf{v} \end{pmatrix}. \tag{4}$$

This expression encapsulates elementary relations of acoustics [13] and holds within a *fixed* laboratory frame.

Finally, substituting Eq. (2) into Eq. (1) yields the Euler-Lagrange equation for the acoustic potential. It is the wave equation which fully determines the dynamics of the acoustic system with underlying spacetime $(M, \mathbf{g})$.

For the actual implementation of such spacetime $(M, \mathbf{g})$ the acoustic engineer requires to fine-tune the mass-density tensor ϱ and bulk modulus κ in the laboratory—also called *physical space*—and relate them to their magnitude in the corresponding space with known acoustic wave propagation—called *virtual space*. Labelling the virtual space by barred quantities, both spaces are connected by the *constitutive relations* [5]:

$$\kappa = \frac{\sqrt{-g}}{\sqrt{-\bar{g}}}\,\bar{\kappa}, \qquad \rho_0\rho^{ij} = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}}\,\bar{g}^{ij}, \tag{5}$$

where without loss of generality $\bar{\rho}/\rho_0 \equiv 1$. For most cases the quantities in virtual space may be conveniently chosen $\bar{\kappa} = 1$ and $\bar{g}^{ij} = \delta^{ij}$.

3. The Poincaré half-plane and acoustic wave propagation

Poincaré's half-plane $\mathbb{H}^2_+ = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ is the upper 2D half-plane endowed with the Poincaré metric, the simplest case of two-dimensional hyperbolic geometry or, alternatively, a surface with a constant negative Gaussian curvature [8]. The line element of the pseudo-Riemannian manifold $M = \mathbb{R} \times \mathbb{H}^2_+$,

[1] Recall that for a scalar quantity the covariant derivative is the same as the partial derivative. Thus, it is irrelevant whether we use comma or semicolon for the derivative of ϕ. Also note that by the index contraction no full Lorentz invariance is implied as acoustics is obviously no relativistic theory.

representing spacetime, is given at point $x \in M$ in terms of the nonholonomic basis 1-forms $\theta^\mu \in T_x^*M$:

$$ds^2 = -(\underbrace{c\,dt}_{\theta^0}) \otimes (c\,dt) + \underbrace{\frac{dx}{y}}_{\theta^1} \otimes \frac{dx}{y} + \underbrace{\frac{dy}{y}}_{\theta^2} \otimes \frac{dy}{y}. \tag{6}$$

Thus, in local coordinates, the components of the metric $\mathbf{g} = g_{\mu\nu} dx^\mu \otimes dx^\nu$ have the following simple diagonal form

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/y^2 & 0 \\ 0 & 0 & 1/y^2 \end{pmatrix}, \quad \text{with } \sqrt{-g} = \sqrt{-\det(g_{\mu\nu})} = 1/y^2. \tag{7}$$

Note that on the Poincaré half-plane for a smooth curve with parametrization $\gamma : I \subset \mathbb{R} \to \mathbb{H}^2_+$, the length between its initial and final point is determined by evaluating

$$\ell = \int_{\gamma(I)} d\ell = \int_I \gamma^* d\ell = \int_I \sqrt{(\gamma^* g_{ij})(\gamma^* dx^i)(\gamma^* dx^j)}, \tag{8}$$

where the pullback is as usual denoted by an upper star. Identifying $x \in I = [a, b]$ as parameter, Eq. (8) readily reduces to

$$\ell = \int_a^b dx \sqrt{g_{ij}(x) \frac{dx^i}{dx} \frac{dx^j}{dx}} = \int_a^b dx \, \frac{\sqrt{1 + y'^2}}{y}, \tag{9}$$

where in the last step we have substituted Eq. (7). The shortest path between the points $\gamma(a), \gamma(b) \in \mathbb{H}^2_+$ is given by the extremal requirement $\delta\ell/\delta x = 0$ for Eq.(9), which by straightforward calculation produces the differential equation

$$yy'' + y'^2 + 1 = 0. \tag{10}$$

It is not difficult to check that the solutions of this equation, being the geodesics on the Poincaré half-plane, are either straight vertical lines ($\theta^1 = 0$), semicircles centered on the x-axis or their partial arcs, which is a well-known result (see *e.g.* Ref. [14]).

The previous observations for the geodesics imply that the spacetime $M = \mathbb{R} \times \mathbb{H}^2_+$ is curved. Cartan's structure equations allow to efficiently compute the curvature 2-form in the nonholonomic frame.

One starts out with Cartan's first structure equation in the local frame $\theta^\mu \in T_x^*M$ to determine the curvature 1-forms $\omega^\mu{}_\nu$:

$$D\theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0, \tag{11}$$

where D denotes the exterior covariant derivative and no torsion is implied. Using Eq. (6) in the condition provided by Eq. (11), one readily obtains

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta^1 \\ 0 & \theta^1 & 0 \end{pmatrix}. \tag{12}$$

Cartan's second structure equation defines the curvature 2-forms $\Omega^\mu{}_\nu$ by applying once more the covariant derivative:

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu. \tag{13}$$

From Eq. (12) it follows in this case that $\omega^\mu{}_\lambda \wedge \omega^\lambda{}_\nu = 0$, which then yields for Eq. (13):

$$\Omega^\mu{}_\nu = d\omega^\mu{}_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -d\theta^1 \\ 0 & d\theta^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta^1 \wedge \theta^2 \\ 0 & \theta^1 \wedge \theta^2 & 0 \end{pmatrix}. \tag{14}$$

As expected $\Omega^1{}_2 = -\Omega^2{}_1$ is the only independent and non-vanishing component in the matrix of Eq. (14) representing all possible curvature 2-forms.

From the relation between curvature form and Riemann curvature tensor, $\Omega^1{}_2 = \hat{R}^1{}_{212}\, \theta^1 \wedge \theta^2 = -d\theta^1 = (-1)\theta^1 \wedge \theta^2$, it also follows that the only independent component of the Riemann tensor is $\hat{R}^1{}_{212} = -1$. All other components vanish. As usual, the Ricci tensor and scalar are obtained by contraction. The components of the Ricci tensor in the nonholonomic frame are $\hat{R}_{00} = 0$ and $\hat{R}_{11} = \hat{R}_{22} = -1$. Thus, the Ricci scalar in both frames, nonholonomic and coordinate frame, is $R = \hat{R} = -2$.

Putting this information together, the associated Einstein tensor, $G_{\mu\nu}$, is also identical in the nonholonomic and coordinate frame with only the following non-zero component

$$G_{00} = R_{00} + \eta_{00} = -1. \tag{15}$$

Matching Eq. (15) with the stress-energy tensor of a perfect fluid,

$$T_{\mu\nu} = \begin{pmatrix} \rho_0\, c^2 & 0 & 0 \\ 0 & p_0 & 0 \\ 0 & 0 & p_0 \end{pmatrix} = -\frac{c^4}{8\pi G} G_{\mu\nu},$$

implies that for an observer falling along a geodesic the spacetime $\mathbb{R} \times \mathbb{H}^2_+$ is pressure-free and it only consists of *exotic matter*. More concretely, the spacetime $\mathbb{R} \times \mathbb{H}^2_+$ has constant negative mass-energy density

$$\rho_0 = -\frac{c^2}{8\pi G}, \tag{16}$$

where c is the speed of light and G is the gravitational constant. Precisely this mass-energy distribution would generate $\mathbb{R} \times \mathbb{H}^2_+$ in physical spacetime.

The analogous acoustic space is implemented by a suitable choice of the physical parameters ϱ and κ. For this we only require the components of metric $\mathbf{g}$, immediately read off from Eq. (6), and the constitutive equations, Eqs. (5). Thus, we obtain the following simple prescription for the acoustic analogue of $\mathbb{R} \times \mathbb{H}^2_+$:

$$\kappa = \left(\frac{y}{y_0}\right)^2 \bar{\kappa}, \qquad \rho_0 \rho^{ij} = \left(\frac{y_0}{y}\right)^2 \delta^{ij}, \tag{17}$$

where $y_0 > 0$ is just a constant to fix the dimension.

The acoustic wave equation on the Poincaré half-plane agrees with the geodesics for field ϕ on a curved spacetime with the underlying metric provided by Eq. (6). Applying the variational principle, Eq. (1), for this spacetime gives the associated Euler-Lagrange equation

$$\Delta_{\mathbb{R}\times\mathbb{H}^2_+} \phi = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + y^2 \, \Delta_{\mathbb{R}^2}\right)\phi = 0, \tag{18}$$

where Δ_M denotes the Laplace-Beltrami operator on manifold M.

Standard techniques [15] yield for the free-wave solution $\phi(t, x, y)$ of Eq. (18) a harmonic dependence on time variable t and for the propagation along the x-axis, so that we may write

$$\phi(t, x, y) = \underbrace{\left[A \cos\left(\sqrt{\lambda} ct\right) + B \sin\left(\sqrt{\lambda} ct\right)\right]}_{\phi_0(t)} \underbrace{\left[\tilde{A} \cos\left(\sqrt{\mu} x\right) + \tilde{B} \sin\left(\sqrt{\mu} x\right)\right]}_{\phi_1(x)} \phi_2(y), \tag{19}$$

where $A, B, \tilde{A}, \tilde{B} \in \mathbb{R}$ are integration constants, and $\lambda, \mu > 0$ are the constants related to the eigensolutions of the harmonic oscillators. As expected, all of the non-trivial behaviour resides in the propagation along the y-axis, which is contained in the contribution $\phi_2(y)$.

In order to determine $\phi_2(y)$, we substitute Eq. (19) into the wave equation, Eq. (18), and obtain following problem of Sturm-Liouville type:

$$\phi_2'' + \left(\frac{\lambda}{y^2} - \mu\right)\phi_2 = 0. \tag{20}$$

The differential equation Eq. (20) has a regular singular point at $y = 0$, and in the limit $y \to \infty$ the metric Eq. (7) degenerates, *i.e.*, $\sqrt{-g} \to 0$. Moreover, in this limit damping occurs for $\mu > 0$. The physical significance of this damping effect is that the free acoustic pressure wave, $p = \rho_0 \partial\phi/\partial t$, will eventually die down when propagating in direction of the y-axis. This effect is due to the character of the underlying hyperbolic space, in which the distances measured by the spatial components of the metric g_{ij} are "squeezed" together with growing height.

In order to solve Eq. (20), we identify $f(y) = \phi_2(y)/\sqrt{y}$ with $y = x/\sqrt{\mu}$, and $\lambda = -\alpha^2 + 1/4$, to obtain upon substitution

$$x^2 f''(x) + x f'(x) - (x^2 + \alpha^2) f(x) = 0, \tag{21}$$

which is just the modified Bessel differential equation.[2] Its two independent solutions are the modified Bessel functions [16] of the first and second kind, $I_\alpha(x)$ and $K_\alpha(x)$, respectively, where the former is an exponentially growing and the latter is an exponentially decaying function.

The general solution of Eq. (20) for the nontrivial y-dependence of the wave equation, Eq. (18), is therefore given by

$$\phi_2(y) = \sqrt{y}\left[C_1 I_{\frac{1}{2}\sqrt{1-4\lambda}}(\sqrt{\mu}y) + C_2 K_{\frac{1}{2}\sqrt{1-4\lambda}}(\sqrt{\mu}y)\right], \tag{22}$$

where C_1 and C_2 are arbitrary constants.

Finally, by the superposition principle and combining Eqs. (19) and (22), the most general formal solution can be given by the expression

$$\begin{aligned}\phi(t,x,y) = \int d\lambda \int d\mu \; & \left[A(\lambda,\mu)\cos\left(\sqrt{\lambda}\,ct\right) + B(\lambda,\mu)\sin\left(\sqrt{\lambda}\,ct\right)\right] \\ & \times \left[\tilde{A}(\lambda,\mu)\cos(\sqrt{\mu}\,x) + \tilde{B}(\lambda,\mu)\sin(\sqrt{\mu}\,x)\right] \\ \times \sqrt{y}\Big[C_1(\lambda,\mu)\, & I_{\frac{1}{2}\sqrt{1-4\lambda}}(\sqrt{\mu}y) + C_2(\lambda,\mu)\, K_{\frac{1}{2}\sqrt{1-4\lambda}}(\sqrt{\mu}y)\Big] \; .\end{aligned} \tag{23}$$

[2] Note that interestingly enough and quite suitably in this context, the modified Bessel differential equation is sometimes also called the *hyperbolic* Bessel differential equation.

The numerical evaluation for given boundary conditions may be worked out by a finite-element analysis of $\phi(t, x, y)$. In case the time and x-dependence of the acoustic potential ϕ can be expressed as a sum of a few lower-frequency harmonics, a semianalytical approach might be carried out by using an expansion of $\phi_2(y)$ in terms of a modified Neumann series [16, 17], while at the same time maintaining the superposition of the explicit harmonic solutions of the form $\phi_0(t)$ and $\phi_1(x)$, *viz.* Eq. (19). Frequently, in these Neumann series only the modified Bessel functions of the second kind $K_\alpha(\sqrt{\mu}y)$ appear, because the first-kind functions produce unphysical results due to the divergent behaviour $I_\alpha(\sqrt{\mu}y) \to \infty$ in the asymptotic limit $y \to \infty$.

We conclude this discussion with the following Cauchy problem for a acoustic wave propagation on the Poincaré half-plane with spacetime $M = \mathbb{R} \times \mathbb{H}^2_+$:

$$\begin{aligned}
\overbrace{-\frac{\partial^2\phi}{\partial t^2} + y^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)}^{\Delta_M\phi} &= 0 \quad \text{on } R = [-10, 10]\times]0, 5] \text{ with } t \in [0, 2\pi] \\
\phi &= e^{-5\left(x-\frac{3}{2}\right)^2+\left(y-\frac{3}{2}\right)^2} \quad \text{for } t = 0,\ (x, y) \in R \\
\frac{\partial\phi}{\partial t} &= 0 \quad \text{for } t = 0 \\
\phi &= 0 \quad \text{for } (x, y) \in \partial R
\end{aligned} \tag{24}$$

At instant $t = 0$, the potential starts with an exponential peak located at position $(1.5, 1.5)$ on the rectangle $R = [-10, 10]\times]0, 5]$ lying within the Poincaré half-plane. Another assumption we make is that initially $\partial\phi/\partial t = 0$, which implies that at $t = 0$ the acoustic pressure vanishes. Moreover, at all times the potential shall be zero on the rectangle border ∂R.

To carry out the numerical simulation of the Cauchy problem, Eq. (24), computing the acoustic potential ϕ, we use the method-of-lines technique implemented with the software package Mathematica 11.1, see [18]. Figure 1 shows several snapshots for successive increments within the time interval $[0, 2\pi]$. The wave fades out with increasing height y, which is not only due to the boundary condition on ∂R for this particular case, but mainly due to the asymptotic behaviour of the metric for $y \to \infty$, as was already observed before.

4. Conclusions

The analogue model of the Poincaré half-plane in transformation acoustics is a particularly fascinating model, since the original approach was among the first with hyperbolic geometry to be throughly studied in history. In oder to create such a spacetime, in Einstein's theory of gravity space would have to be filled with exotic matter, a substance which will perhaps never be attainable. However, we have shown that the implementation within an acoustic metamaterial requires a bulk modulus and isotropic mass-energy density which display a specific dependence on height, *viz.* Eq. (17). This might be a workable alternative for testing in a laboratory in the foreseeable future.

To derive the wave equation for the acoustic potential on the Poincaré half-plane, we have started from a covariant variational principle and arrived at a complete description of acoustic free-wave phenomena with the underlying spacetime $\mathbb{R} \times \mathbb{H}^2_+$. The formal analytical solution for the acoustic wave equation can be expressed by the superposition principle as a combination of harmonic frequency modes and modified Bessel functions.

For a numerical simulation, we have chosen a Cauchy problem with a confined rectangular region of the Poincaré half-plane. As expected, we observe a damping effect of the amplitude with increasing height.

In summary, it is hoped that the variational spacetime approach to transformation acoustics supplies a powerful and tractable tool for the study and design of acoustic metadevices. It may facilitate new research pathways in this field, overcoming pending challenges in the engineering of acoustic phenomena with curved spacetime backgrounds.

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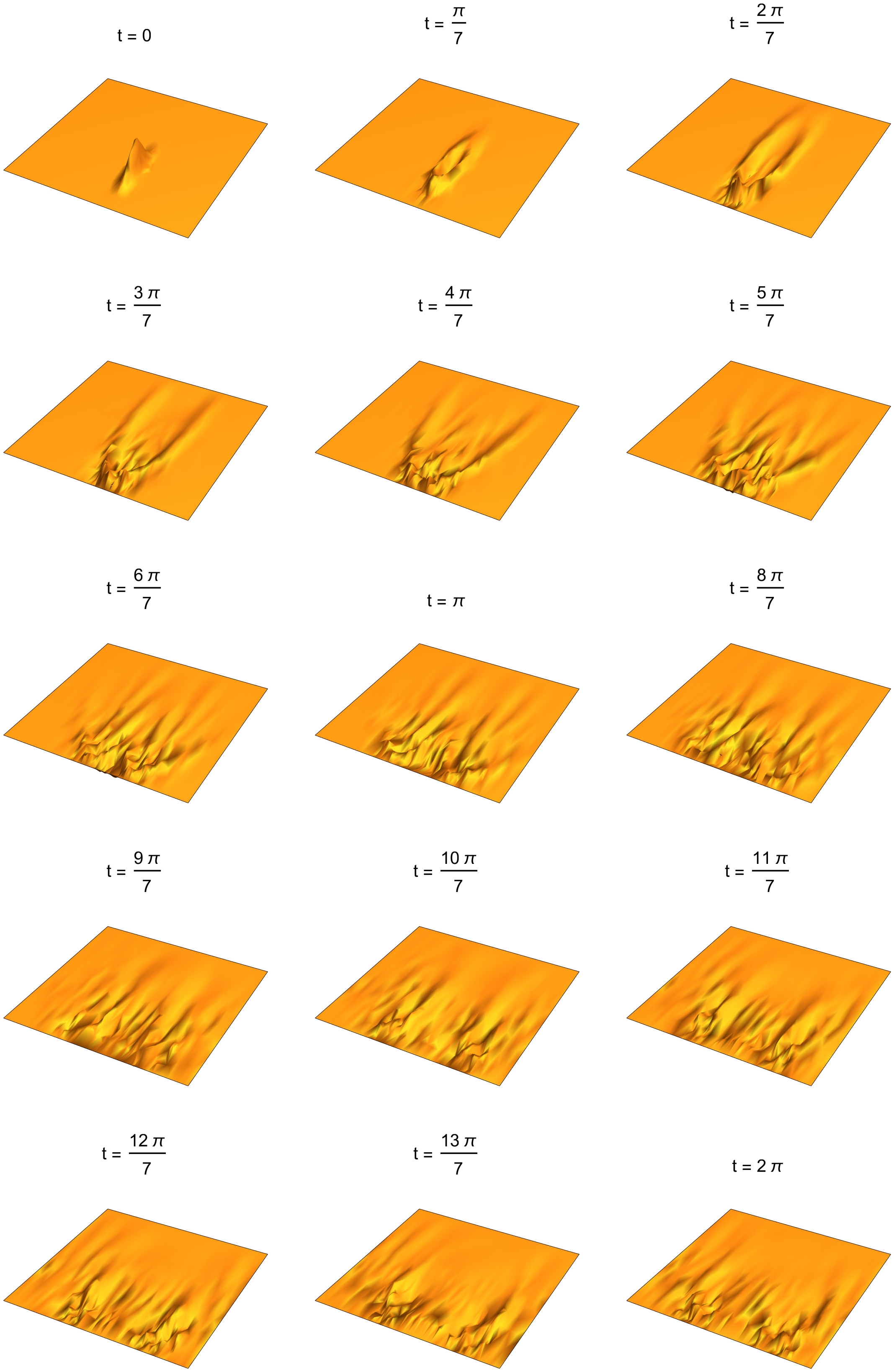

Figure 1: Graphical representation of the solution to the Cauchy problem Eq. (24), describing the evolution of the acoustic potential ϕ on the Poincaré half-plane $\mathbb{H}^2_+$. For the time interval $t \in [0, 2\pi]$, 15 snapshots are taken.