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Research Article

On Some Growth Properties of Entire Functions Using Their Maximum Moduli Focusing (p, q)th Relative Order

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We discuss some growth rates of composite entire functions on the basis of the definition of relative (p, q)th order (relative (p, q)th lower order) with respect to another entire function which improve some earlier results of Roy (2010) where p and q are any two positive integers.

1. Introduction, Definitions, and Notations

Let f be an entire function defined in the open complex plane and let

$$M_f(r) = \max\{|f(z)| : |z| = r\} \quad (1)$$

be its maximum modulus function. If f is nonconstant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1}(r) : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that

$$\lim_{s\to\infty} M_f^{-1}(s) = \infty. \quad (2)$$

We use the standard notations and definitions in the theory of entire functions which are available in [1]. In the sequel we use the following notation:

$$\begin{aligned} &\log^{[0]} x = x, \qquad \log^{[k]} x = \log\left(\log^{[k-1]} x\right) \quad \text{for } k = 1, 2, 3, \ldots, \\ &\exp^{[0]} x = x, \qquad \exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \quad \text{for } k = 1, 2, 3, \ldots. \end{aligned} \quad (3)$$

The following definitions are well known.

Definition 1. The order ρ_f and the lower order λ_f of an entire function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}, \qquad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}. \quad (4)$$

Juneja et al. [2] defined the (p, q)th order and (p, q)th lower order of an entire function f, respectively, as follows:

$$\rho_f(p, q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}, \qquad \lambda_f(p, q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}, \quad (5)$$

where p, q are any two positive integers with $p \geq q$.

If $p = l$ and $q = 1$ then we write $\rho_f(l, 1) = \rho_f^{[l]}$ and $\lambda_f(l, 1) = \lambda_f^{[l]}$.

Also for $p = 2$ and $q = 1$ we, respectively, denote $\rho_f(2, 1)$ and $\lambda_f(2, 1)$ by ρ_f and λ_f.

In this connection we just recall the following definition.

Definition 2 (see [2]). An entire function f is said to have index-pair (p,q), $p \geq q \geq 1$, if $b < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1)$ is not a nonzero finite number, where $b=1$ if $p=q$ and $b=0$ if $p>q$. Moreover if $0<\rho_f(p,q)<\infty$, then

$$\begin{aligned} \rho_f(p-n,q) &= \infty \quad \text{for } n<p, \\ \rho_f(p,q-n) &= 0 \quad \text{for } n<q, \\ \rho_f(p+n,q+n) &= 1 \quad \text{for } n=1,2,\ldots. \end{aligned} \tag{6}$$

Similarly for $0<\lambda_f(p,q)<\infty$, one can easily verify that

$$\begin{aligned} \lambda_f(p-n,q) &= \infty \quad \text{for } n<p, \\ \lambda_f(p,q-n) &= 0 \quad \text{for } n<q, \\ \lambda_f(p+n,q+n) &= 1 \quad \text{for } n=1,2,\ldots. \end{aligned} \tag{7}$$

An entire function for which (p,q)th order and (p,q)th lower order are the same is said to be of regular (p,q)-growth. Functions which are not of regular (p,q)-growth are said to be of irregular (p,q)-growth.

Bernal [3] introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\left\{\mu>0 : M_f(r) < M_g(r\mu)\ \ \forall r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned} \tag{8}$$

The definition coincides with the classical one [4] if $g=\exp$.

Similarly one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as follows:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \tag{9}$$

In the case of relative order, it therefore seems reasonable to define suitably the relative (p,q)th order of entire functions. Lahiri and Banerjee [5] also introduced such definition in the following manner.

Definition 3 (see [5]). Let p and q be any two positive integers with $p>q$. The relative (p,q)th order of f with respect to g is defined by

$$\begin{aligned} \rho_g^{(p,q)}(f) &= \inf\left\{\mu>0 : M_f(r) < M_g\left(\exp^{[p-1]}\left(\mu \log^{[q]} r\right)\right)\right. \\ &\qquad \left.\forall r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\log^{[q]} r}. \end{aligned} \tag{10}$$

If $q=1$, $k\geq 1$, and $p=k+1$ then $\rho_g^{(p,q)}(f)=\rho_g^k(f)$. If $g=\exp z$ then $\rho_g^{(p,q)}(f)=\rho_f(p,q)$.

Sánchez Ruiz et al. [6] gave a more natural definition of relative (p,q)th order of an entire function in light of index-pair which is as follows.

Definition 4. Let f and g be any two entire functions with index-pairs (m_1,q) and (m_2,p), respectively, where $m_1 = m_2 = m$ and p,q, and m are all positive integers such that $m\geq p$ and $m\geq q$. Then the relative (p,q)th order of f with respect to g is defined as

$$\begin{aligned} \rho_g^{(p,q)}(f) &= \inf\left\{\mu>0 : M_f(r) < M_g\right. \\ &\qquad \times\left[\exp^{[p]}\left\{\log^{[m_2]}\exp^{[m_1]}\left(\mu\log^{[q]} r\right)\right\}\right] \\ &\qquad \left.\forall r > r_0(\mu) > 0\right\} \\ &= \inf\left\{\mu>0 : M_f(r) < M_g\left(\exp^{[p]}\left(\mu\log^{[q]} r\right)\right)\right. \\ &\qquad \left.\forall r > r_0(\mu) > 0\right\} \\ &= \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}. \end{aligned} \tag{11}$$

Similarly one can define the relative (p,q)th lower order of an entire function f with respect to another entire function g denoted by $\lambda_g^{(p,q)}(f)$ where p and q are any two positive integers in the following way:

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}. \tag{12}$$

In fact Definition 4 improves Definition 3 ignoring the restriction $p\geq q$.

In this paper we wish to prove some results related to the growth rates of entire functions on the basis of relative (p,q)th order and relative (p,q)th lower order with respect to another entire function extending some earlier results for any two positive integers p and q.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 (see [7]). *If f and g are any two entire functions with $g(0)=0$. then*

$$M_{f\circ g}(r) \geq M_g\left(\frac{r}{2}\right) \tag{13}$$

for all sufficiently large values of $r \geqslant r_0$.

Lemma 2 (see [7]). *Let f be entire and let g be a transcendental entire function of finite lower order. Then, for any $\delta > 0$,*

$$M_{f\circ g}\left(r^{1+\delta}\right) \geq M_f\left(M_g\left(r\right)\right) \quad \text{for all sufficiently large values of } r \geqslant r_0. \tag{14}$$

Lemma 3 (see [8]). *If f and g are any two entire functions with $g(0) = 0$. then, for any $0 < c < 1$,*

$$M_{f\circ g}\left(r\right) \geq M_f\left(cM_g\left(\frac{r}{2}\right)\right) \quad \text{for all sufficiently large values of } r \geqslant r_0, \tag{15}$$

Lemma 4 (see [9]). *If f and g are any two entire functions then for all sufficiently large values of $r \geqslant r_0$*

$$M_{f\circ g}\left(r\right) \leq M_f\left(M_g\left(r\right)\right). \tag{16}$$

3. Theorems

In this section we present the main results of the paper.

Theorem 5. *Let f be an entire function and let g be any polynomial such that $f \circ g$ has got finite relative (p, q)th order with respect to h where h is a transcendental entire function and p, q are any two positive integers. Then $\rho_h^{(p,q)}(f) < \infty$.*

Proof. Given that $f \circ g$ is of finite relative (p, q)th order with respect to h, we have from Definition 4, for a suitable finite number $\mu > 0$ and for all sufficiently large values of r, that

$$M_{f\circ g}\left(r\right) < M_h\left(\exp^{[p]}\left(\mu\log^{[q]}r\right)\right). \tag{17}$$

Now let m be the order of the polynomial g so that

$$g\left(z\right) = c_1 z + c_2 z^2 + \cdots + c_m z^m, \quad c_m \neq 0. \tag{18}$$

Then by Cauchy's inequality we get from (18) that

$$\left|c_m\right| r^m \leq M_g\left(r\right), \qquad |z| = r. \tag{19}$$

Now given $0 < c < 1$, in view of Lemma 3 and from (17) it follows for all sufficiently large values of r that

$$M_f\left(c\left|c_m\right|\left(\frac{r}{2}\right)^m\right) \leq M_{f\circ g}\left(r\right) \leq M_h\left(\exp^{[p]}\left(\mu\log^{[q]}r\right)\right). \tag{20}$$

We rewrite the above to the equivalent for all sufficiently large values of r that

$$M_f\left(r\right) \leq M_h\left(\exp^{[p]}\left(\mu\log^{[q]}\left(\left(c\left|c_m\right|\right)^{-1}2^m r^{1/m}\right)\right)\right). \tag{21}$$

Therefore from (21) we get for all sufficiently large values of r that

$$\begin{gathered} M_h^{-1}M_f\left(r\right) \leq \exp^{[p]}\left(\mu\log^{[q]}\left(\left(c\left|c_m\right|\right)^{-1}2^m r^{1/m}\right)\right), \\ \text{i.e., } \log^{[p]}M_h^{-1}M_f\left(r\right) \leq \mu\log^{[q]}\left(\left(c\left|c_m\right|\right)^{-1}2^m r^{1/m}\right). \end{gathered} \tag{22}$$

Case I. Assume $q = 1$. Then we have from (22) for all sufficiently large values of r that

$$\frac{\log^{[p]}M_h^{-1}M_f\left(r\right)}{\log r} \leq \frac{\mu}{m}\frac{\log r + \left|O\left(1\right)\right|}{\log r}, \tag{23}$$

where $O(1)$ stands for the constant expression, $m\log((c|c_m|)^{-1}2^m)$. Then

$$\begin{gathered} \limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_f\left(r\right)}{\log r} \leq \frac{\mu}{m}\limsup_{r\to\infty}\frac{\log r + \left|O\left(1\right)\right|}{\log r}, \\ \text{i.e., } \rho_h^p\left(f\right) \leq \frac{\mu}{m} < \infty. \end{gathered} \tag{24}$$

Case II. Let us now assume $q > 1$. Then we obtain from (22) for all sufficiently large values of r that

$$\frac{\log^{[p]}M_h^{-1}M_f\left(r\right)}{\log^{[q]}r} \leq \mu\frac{\log^{[q]}r + \left|O\left(1\right)\right|}{\log^{[q]}r}, \tag{25}$$

where $O(1)$ stands for a bounded quantity. Then

$$\begin{gathered} \limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_f\left(r\right)}{\log^{[q]}r} \leq \mu\limsup_{r\to\infty}\frac{\log^{[q]}r + \left|O\left(1\right)\right|}{\log^{[q]}r}, \\ \text{i.e., } \rho_h^{(p,q)}\left(f\right) \leq \mu < \infty. \end{gathered} \tag{26}$$

Thus the theorem follows from (24) and (26). □

In the forthcoming proofs we will assume the natural number q to be $q > 1$, the reasonings being easily adapted for $q = 1$.

Theorem 6. *Let f, g, and h be any three transcendental entire functions and let p and q be two positive integers. If, for any α, β with $0 < \alpha < 1$, $\beta > 0$, and $\alpha(\beta + 1) > 1$, it holds that the two limits $A, B \in \mathbb{R}^+$ of some of either*

(i) $\limsup_{r\to\infty}(\log^{[p]}M_h^{-1}(M_g(r))/(\log^{[q]}r)^\alpha) = A$, $\liminf_{r\to\infty}(\log^{[p]}M_h^{-1}(M_f(r))/(\log^{[p]}M_h^{-1}(r))^{\beta+1}) = B$,

(ii) $\liminf_{r\to\infty}(\log^{[p]}M_h^{-1}(M_g(r))/(\log^{[q]}r)^\alpha) = A$, $\limsup_{r\to\infty}(\log^{[p]}M_h^{-1}(M_f(r))/(\log^{[p]}M_h^{-1}(r))^{\beta+1}) = B$, *or*

(iii) $\liminf_{r\to\infty}(\log^{[p]}M_h^{-1}(M_g(r))/(\log^{[q]}r)^\alpha) = A$, $\liminf_{r\to\infty}(\log^{[p]}M_h^{-1}(M_f(r))/(\log^{[p]}M_h^{-1}(r))^{\beta+1}) = B$

exist, then $\rho_h^{(p,q)}(f \circ g) = \infty$.

Proof. (i) The existence of A and B implies that given any $\varepsilon > 0$, for sufficiently large values of r,

$$\log^{[p]} M_h^{-1}\left(M_g(r)\right) \geq (A-\varepsilon)\left(\log^{[q]} r\right)^{\alpha},$$
$$\log^{[p]} M_h^{-1}\left(M_f(r)\right) \geq (B-\varepsilon)\left(\log^{[p]} M_h^{-1}(r)\right)^{\beta+1}. \tag{27}$$

Since $M_g(r)$ is a continuous, increasing, and unbounded function of r, we get from above for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right) \geq (B-\varepsilon)\left(\log^{[p]} M_h^{-1}\left(M_g(r)\right)\right)^{\beta+1}. \tag{28}$$

Also $M_h^{-1}(r)$ is an increasing function of r; it follows from Lemma 2, (27), and (28) that given $\delta > 0$, for a sequence of values of r tending to infinity, the following holds:

$$\begin{aligned}
&\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right) \\
&\quad \geq \log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right) \\
&\quad \geq (B-\varepsilon)\left(\log^{[p]} M_h^{-1}\left(M_g(r)\right)\right)^{\beta+1} \\
&\quad \geq (B-\varepsilon)\left[(A-\varepsilon)\left(\log^{[q]} r\right)\alpha\right]^{\beta+1},
\end{aligned} \tag{29}$$

$$\text{i.e.,}\quad \frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]}\left(r^{1+\delta}\right)} \geq \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log^{[q]} r\right)^{\alpha(\beta+1)}}{\log^{[q]}\left(r^{1+\delta}\right)}.$$

Hence

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]}\left(r^{1+\delta}\right)} \geq \liminf_{r\to\infty} \frac{(B-\varepsilon)(A-\varepsilon)^{\beta+1}\left(\log^{[q]} r\right)^{\alpha(\beta+1)}}{\log^{[q]} r + |O(1)|} \tag{30}$$

for all sufficiently large values of r. Since $\varepsilon > 0$ is arbitrary and $\alpha(\beta+1) > 1$ it follows that

$$\rho_h^{(p,q)}(f\circ g) = \infty. \tag{31}$$

Under (ii) or (iii) a similar argument applies. □

Theorem 7. *Let f, g, and h be any three transcendental entire functions and let p and q be two positive integers. If, for any α, β with $\alpha > 1$, $0 < \beta < 1$, and $\alpha\beta > 1$, it holds that the two limits $A, B \in \mathbb{R}^+$ of either*

(i) $\limsup_{r\to\infty}(\log^{[p]} M_h^{-1}(M_g(r))/(\log^{[q+1]} r)^{\alpha}) = A$, $\liminf_{r\to\infty}(\log[\log^{[p]} M_h^{-1}(M_f(r))/\log^{[p]} M_h^{-1}(r)]/[\log^{[p]} M_h^{-1}(r)]^{\beta}) = B$,

(ii) $\liminf_{r\to\infty}(\log^{[p]} M_h^{-1}(M_g(r))/(\log^{[q+1]} r)^{\alpha}) = A$, $\limsup_{r\to\infty}(\log[\log^{[p]} M_h^{-1}(M_f(r))/\log^{[p]} M_h^{-1}(r)]/[\log^{[p]} M_h^{-1}(r)]^{\beta}) = B$, *or*

(iii) $\liminf_{r\to\infty}(\log^{[p]} M_h^{-1}(M_g(r))/(\log^{[q+1]} r)^{\alpha}) = A$, $\liminf_{r\to\infty}(\log[\log^{[p]} M_h^{-1}(M_f(r))/\log^{[p]} M_h^{-1}(r)]/[\log^{[p]} M_h^{-1}(r)]^{\beta}) = B$

exist, then $\rho_h^{(p,q)}(f\circ g) = \infty$.

Proof. (i) Given any $\varepsilon > 0$, for a sequence of values of r tending to infinity, we get that

$$\log^{[p]} M_h^{-1}\left(M_g(r)\right) \geq (A-\varepsilon)\left(\log^{[q+1]} r\right)^{\alpha} \tag{32}$$

and for all sufficiently large values of r that

$$\log\left[\frac{\log^{[p]} M_h^{-1}\left(M_f(r)\right)}{\log^{[p]} M_h^{-1}(r)}\right] \geq (B-\varepsilon)\left[\log^{[p]} M_h^{-1}(r)\right]^{\beta},$$
$$\text{i.e.,}\quad \frac{\log^{[p]} M_h^{-1}\left(M_f(r)\right)}{\log^{[p]} M_h^{-1}(r)} \geq \exp\left[(B-\varepsilon)\left[\log^{[p]} M_h^{-1}(r)\right]^{\beta}\right]. \tag{33}$$

Since $M_g(r)$ is a continuous, increasing, and unbounded function of r, we get from above for all sufficiently large values of r that

$$\frac{\log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right)}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \geq \exp\left[(B-\varepsilon)\left[\log^{[p]} M_h^{-1}\left(M_g(r)\right)\right]^{\beta}\right]. \tag{34}$$

Also $M_h^{-1}(r)$ is an increasing function of r; thus from Lemma 2, (32), and (34) it follows that, given that $\delta > 0$, for a sequence of values of r tending to infinity,

$$\begin{aligned}
&\frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]}\left(r^{1+\delta}\right)} \\
&\quad \geq \frac{\log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right)}{\log^{[q]}\left(r^{1+\delta}\right)} \\
&\quad \geq \frac{\log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right)}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \cdot \frac{\log^{[p]} M_h^{-1}\left(M_g(r)\right)}{\log^{[q]} r + |O(1)|}.
\end{aligned} \tag{35}$$

Therefore

$$\begin{aligned}
&\frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]}\left(r^{1+\delta}\right)} \\
&\quad \geq \exp\left[(B-\varepsilon)\left[\log^{[p]} M_h^{-1}\left(M_g(r)\right)\right]^{\beta}\right] \\
&\qquad \cdot \frac{(A-\varepsilon)\left(\log^{[q+1]} r\right)^{\alpha}}{\log^{[q]} r + |O(1)|} \\
&\quad \geq \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[q+1]} r\right)^{\alpha\beta}\right] \\
&\qquad \cdot \frac{(A-\varepsilon)\left(\log^{[q+1]} r\right)^{\alpha}}{\log^{[q]} r + |O(1)|} \\
&\quad = \exp\left[(B-\varepsilon)(A-\varepsilon)^{\beta}\left(\log^{[q+1]} r\right)^{\alpha\beta-1}\log^{[q+1]} r\right] \\
&\qquad \cdot \frac{(A-\varepsilon)\left(\log^{[q+1]} r\right)^{\alpha}}{\log^{[q]} r + |O(1)|} \\
&\quad \geq \left(\log^{[q]} r\right)^{(B-\varepsilon)(A-\varepsilon)\beta(\log^{[q+1]} r)^{\alpha\beta-1}} \\
&\qquad \cdot \frac{(A-\varepsilon)\left(\log^{[q+1]} r\right)^{\alpha}}{\log^{[q]} r + |O(1)|}.
\end{aligned} \tag{36}$$

Hence

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]}\left(r^{1+\delta}\right)} \\
&\quad \geq \liminf_{r\to\infty} \left(\log^{[q]} r\right)^{(B-\varepsilon)(A-\varepsilon)\beta(\log^{[q+1]} r)^{\alpha\beta-1}} \\
&\qquad \cdot \frac{(A-\varepsilon)\left(\log^{[q+1]} r\right)^{\alpha}}{\log^{[q]} r + |O(1)|}.
\end{aligned} \tag{37}$$

Since $\varepsilon > 0$ is arbitrary and $\alpha > 1$, $\alpha\beta > 1$, it follows that

$$\rho_h^{(p,q)}(f\circ g) = \infty. \tag{38}$$

Under (ii) or (iii) a similar argument may be used. □

Theorem 8. *Let f, g, and h be any three transcendental entire functions such that $0 < \lambda_h^{(p,q)}(g) \leq \rho_h^{(p,q)}(g) < \infty$ where p and q are any two positive integers. If the limit $A \in \mathbb{R}$ exists in either*

(i) $\limsup_{r\to\infty}(\log^{[p]} M_h^{-1}(M_f(r))/\log^{[p]} M_h^{-1}(r)) = A$ *or*

(ii) $\liminf_{r\to\infty}(\log^{[p]} M_h^{-1}(M_f(r))/\log^{[p]} M_h^{-1}(r)) = A$,

then

$$\lambda_h^{(p,q)}(f\circ g) \leq A\lambda_h^{(p,q)}(g) \leq \rho_h^{(p,q)}(f\circ g) \leq A\rho_h^{(p,q)}(g). \tag{39}$$

Proof. (i) Since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemmas 2 and 4, given $\delta > 0$, for all sufficiently large values of r, that

$$M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right) \geq M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}, \tag{40}$$

$$M_h^{-1} M_{f\circ g}(r) \leq M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}, \tag{41}$$

respectively.

Therefore from (40) we get for all sufficiently large values of r that

$$\begin{aligned}
\frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]} r^{1+\delta}} &\geq \frac{\log^{[p]} M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}}{\log^{[q]} r^{1+\delta}} \\
&\geq \frac{\log^{[p]} M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \\
&\quad \cdot \frac{\log^{[p]} M_h^{-1}\left(M_g(r)\right)}{\log^{[q]} r + |O(1)|}.
\end{aligned} \tag{42}$$

From here it follows that

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[q]} r} \\
&\quad \geq \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \\
&\qquad \times \liminf_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_g(r)\right)}{\log^{[q]} r + |O(1)|} \\
&\text{i.e., } \rho_h^{(p,q)}(f\circ g) \geq A\lambda_h^{(p,q)}(g).
\end{aligned} \tag{43}$$

Similarly from (41) it follows for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r) \leq \log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right). \tag{44}$$

Therefore

$$\begin{aligned}
&\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[q]} r} \\
&\quad \leq \frac{\log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right)}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \cdot \frac{\log^{[p]} M_h^{-1}\left(M_g(r)\right)}{\log^{[q]} r}.
\end{aligned} \tag{45}$$

Hence

$$\begin{aligned}
\liminf_{r\to\infty} & \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[q]} r} \\
& \leq \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \\
& \quad \times \liminf_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_g(r)\right)}{\log^{[q]} r}, \\
\text{i.e., } & \lambda_h^{(p,q)}(f\circ g) \leq A\lambda_h^{(p,q)}(g).
\end{aligned} \tag{46}$$

Also from (45) we obtain for all sufficiently large values of r that

$$\begin{aligned}
\limsup_{r\to\infty} & \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[q]} r} \\
& \leq \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left\{M_f\left(M_g(r)\right)\right\}}{\log^{[p]} M_h^{-1}\left(M_g(r)\right)} \\
& \quad \times \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_g(r)\right)}{\log^{[q]} r}, \\
\text{i.e., } & \rho_h^{(p,q)}(f\circ g) \leq A\cdot \rho_h^{(p,q)}(g).
\end{aligned} \tag{47}$$

Then the thesis follows from (43), (46), and (47).

(ii) follows with a similar argument. □

Theorem 9. *Let f, g, and h be any three transcendental entire functions with $g(0) = 0$. If p, q, and m are any three positive integers with $m > q$, then $\rho_h^{(p,q)}(f\circ g) = \infty$ under any of the following conditions:*

(i) $\rho_h^{(p,q)}(g) = \infty$;

(ii) $\min(\rho_h^{(p,q)}(f), \lambda_g(m,q)) > 0$;

(iii) $\min(\rho_g(m,q), \lambda_h^{(p,q)}(f)) > 0$.

Proof. (i) If $\rho_h^{(p,q)}(g) = \infty$, since $M_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, for all sufficiently large values of r, that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}(r)\right) \geq \log^{[p]} M_h^{-1}\left(M_g\left(\frac{r}{2}\right)\right). \tag{48}$$

Therefore

$$\begin{aligned}
\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}(r)\right)}{\log^{[q]} r} & \geq \frac{\log^{[p]} M_h^{-1}\left(M_g(r/2)\right)}{\log^{[q]} r} \\
& \geq \frac{\log^{[p]} M_h^{-1}\left(M_g(r/2)\right)}{\log^{[q]}(r/2) + |O(1)|}.
\end{aligned} \tag{49}$$

Then

$$\begin{aligned}
\limsup_{r\to\infty} & \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}(r)\right)}{\log^{[q]}} \\
& \geq \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_g(r/2)\right)}{\log^{[q]}(r/2) + |O(1)|}, \\
\text{i.e., } & \rho_h^{(p,q)}(f\circ g) \geq \rho_h^{(p,q)}(g) = \infty.
\end{aligned} \tag{50}$$

(ii) Suppose $\rho_h^{(p,q)}(f) > 0$ and $\lambda_g(m,q) > 0$.

As $M_h^{-1}(r)$ is an increasing function of r, we get from Lemma 2 that given $\delta > 0$ and any $\varepsilon > 0$, for all sufficiently large values of r,

$$\begin{aligned}
\log^{[p]} & M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right) \\
& \geq \log^{[p]} M_h^{-1}\left(M_f\left(M_g(r)\right)\right) \\
& \geq \left(\rho_h^{(p,q)}(f) - \varepsilon\right) \log^{[q]} M_g(r) \\
& \geq \left(\rho_h^{(p,q)}(f) - \varepsilon\right) \exp^{[m-q-1]}\left(\log^{[q-1]} r\right)^{(\lambda_g(m,q)-\varepsilon)}.
\end{aligned} \tag{51}$$

Thus

$$\begin{aligned}
& \frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]} r^{1+\delta}} \\
& \quad \geq \frac{\left(\rho_h^{(p,q)}(f) - \varepsilon\right) \exp^{[m-q-1]}\left(\log^{[q-1]} r\right)^{(\lambda_g(m,q)-\varepsilon)}}{\log^{[q]} r^{1+\delta}}.
\end{aligned} \tag{52}$$

Hence

$$\begin{aligned}
\limsup_{r\to\infty} & \frac{\log^{[p]} M_h^{-1} M_{f\circ g}\left(r^{1+\delta}\right)}{\log^{[q]} r^{1+\delta}} \\
\geq \liminf_{r\to\infty} & \frac{\left(\rho_h^{(p,q)}(f) - \varepsilon\right) \exp^{[m-q-1]}\left(\log^{[q-1]} r\right)^{(\lambda_g(m,q)-\varepsilon)}}{\log^{[q]} r + |O(1)|}, \\
\text{i.e., } & \rho_h^{(p,q)}(f\circ g) = \infty.
\end{aligned} \tag{53}$$

Under (iii) a similar argument to (i) applies. □

In the line of Theorem 9 one can easily prove the following result.

Theorem 10. *Let f, g, and h be any three transcendental entire functions with $g(0) = 0$. If p,q, and m are any three positive integers with $m > q$, then $\lambda_h^{(p,q)}(f\circ g) = \infty$ if any of the following facts happens:*

(i) $\lambda_h^{(p,q)}(g) = \infty$;

(ii) $\min(\lambda_h^{(p,q)}(f), \lambda_g(m,q)) > 0$.

Theorem 11. *Let f, g, and h be any three transcendental entire functions such that $g(0) = 0$. If p, q, and m are any three positive integers with $m > q$ and any of the following two facts happens*

(i) $\min(\rho_h^{(p,q)}(f), \lambda_g(m,q)) > 0$ *or*

(ii) $\min(\lambda_h^{(p,q)}(f), \lambda_g(m,q)) > 0$,

then

$$\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}(r)\right)}{\log^{[p]} M_h^{-1}\left(M_f(r)\right)} = \infty. \tag{54}$$

Proof. (i) Since

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}(r)\right)}{\log^{[p]} M_h^{-1}\left(M_f(r)\right)} \\
&\quad \geq \limsup_{r\to\infty} \frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}(r)\right)}{\log^{[q]} r} \\
&\quad \times \liminf_{r\to\infty} \frac{\log^{[q]} r}{\log^{[p]} M_h^{-1}\left(M_f(r)\right)} \\
&\quad = \rho_h^{(p,q)}(f\circ g) \frac{1}{\rho_h^{(p,q)}(f)}
\end{aligned} \tag{55}$$

the result follows from Theorem 9.

(ii) The proof can be carried out in the line of (i) and Theorem 10. □

4. Conclusion

After modifying the notion of relative order of higher dimensions in case of entire functions in [6], where a number of examples of relative order between functions were provided, in this paper we have obtained some growth properties of composite entire functions on the basis of relative (p,q)th order and relative (p,q)th lower order. In this process, Theorem 5 and the first part of Theorem 6 and Theorems 7 and 8 can be regarded as extensions of some results of [10].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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