# On Valdivia strong version of Nikodym boundedness property[☆]

Dedicated to the memory of Professor Manuel Valdivia (1928-2014)

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**Abstract**

Following Schachermayer, a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of $\Omega$ is said to have the *N-property* if a $\mathcal{B}$-pointwise bounded subset $M$ of $ba(\mathcal{A})$ is uniformly bounded on $\mathcal{A}$, where $ba(\mathcal{A})$ is the Banach space of the real (or complex) finitely additive measures of bounded variation defined on $\mathcal{A}$. Moreover $\mathcal{B}$ is said to have the *strong N-property* if for each increasing countable covering $(\mathcal{B}_m)_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has the $N$-property. The classical Nikodym-Grothendieck's theorem says that each $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ has the *N-property.* The Valdivia's theorem stating that each $\sigma$-algebra $\mathcal{S}$ has the strong $N$-property motivated the main measure-theoretic result of this paper: We show that if $(\mathcal{B}_{m_1})_{m_1}$ is an increasing countable covering of a $\sigma$-algebra $\mathcal{S}$ and if $(\mathcal{B}_{m_1,m_2,\ldots,m_p,m_{p+1}})_{m_{p+1}}$ is an increasing countable covering of $\mathcal{B}_{m_1,m_2,\ldots,m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p$, then there exists a sequence $(n_i)_i$ such that each $\mathcal{B}_{n_1,n_2,\ldots,n_r}$, $r \in \mathbb{N}$, has the strong $N$-property. In particular, for each increasing countable covering $(\mathcal{B}_m)_m$ of a $\sigma$-algebra $\mathcal{S}$ there exists $\mathcal{B}_n$ which has the strong $N$-property, improving mentioned Valdivia's theorem. Some applications to localization of bounded additive vector measures are provided.

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## 1. Introduction

Let $\mathcal{B}$ be a subset of an algebra $\mathcal{A}$ of subsets of a set $\Omega$ (in brief, set-algebra $\mathcal{A}$). The normed space $L(\mathcal{B})$ is the $span\{\chi_C : C \in \mathcal{B}\}$ of the characteristic functions of each set $C \in \mathcal{B}$ with the supremum norm $\|\cdot\|$ and $ba(\mathcal{A})$ is the Banach space of finitely additive measures on $\mathcal{A}$ with bounded variation endowed with the variation norm, i.e., $|\cdot| := |\cdot|(\Omega)$. If $\{C_i : 1 \leqslant i \leqslant n\}$ is a measurable partition of $C \in \mathcal{A}$ and $\mu \in ba(\mathcal{A})$ then $|\mu|(C) = \Sigma_i |\mu|(C_i)$ and, as usual, we represent also by $\mu$ the linear form in $L(\mathcal{A})$ determined by $\mu(\chi_C) := \mu(C)$, for each $C \in \mathcal{A}$. By this identification we get that the dual of $L(\mathcal{A})$ with the dual norm is isometric to $ba(\mathcal{A})$ (see e.g., [2, Theorem 1.13]).

Polar sets are considered in the dual pair $< L(\mathcal{A}), ba(\mathcal{A}) >$, $M^\circ$ means the polar of a set $M$ and if $\mathcal{B} \subset \mathcal{A}$ the topology in $ba(\mathcal{A})$ of pointwise convergence in $\mathcal{B}$ is denoted by $\tau_s(\mathcal{B})$. $(E', \tau_s(E))$ is the vector space of all continuous linear forms defined on a locally convex space $E$ endowed with the topology $\tau_s(E)$ of the pointwise convergence in $E$. In particular, the topology $\tau_s(L(\mathcal{A}))$ in $ba(\mathcal{A})$ is $\tau_s(\mathcal{A})$.

The convex (absolutely convex) hull of a subset $M$ of a topological vector space is denoted by $co(M)$ ($absco(M)$) and $absco(M) = co(\cup\{rM : |r| = 1\})$. An equivalent norm to the supremum norm in $L(\mathcal{A})$

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is the Minkowski functional of $absco(\{\chi_C : C \in \mathcal{A}\})$ ([14, Propositions 1 and 2]) and its dual norm is the $\mathcal{A}$-supremum norm, i.e., $\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$. The closure of a set is marked by an overline, hence if $P \subset L(\mathcal{A})$ then $\overline{span(P)}$ is the closure in $L(\mathcal{A})$ of the linear hull of $P$. $\mathbb{N}$ is the set $\{1, 2, \ldots\}$ of positive integers.

Recall the classical Nikodym-Dieudonné-Grothendieck theorem (see [1, page 80, named as Nikodym-Grothendieck boundedness theorem]): *If $\mathcal{S}$ is a $\sigma$-algebra of subsets of a set $\Omega$ and $M$ is a $\mathcal{S}$-pointwise bounded subset of $ba(\mathcal{S})$ then $M$ is a bounded subset of $ba(\mathcal{S})$* (*i.e.,* $\sup\{|\mu(C)| : \mu \in M, C \in \mathcal{S}\} < \infty$, or, equivalently, $\sup\{|\mu|(\Omega) : \mu \in M\} < \infty$). This theorem was firstly obtained by Nikodym in [11] for a subset $M$ of countably additive complex measures defined on $\mathcal{S}$ and later on by Dieudonné for a subset $M$ of $ba(2^\Omega)$, where $2^\Omega$ is the $\sigma$-algebra of all subsets of $\Omega$, see [3].

It is said that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of a set $\Omega$ has the *Nikodym property*, $N$-property in brief, if the Nikodym-Dieudonné-Grothendieck theorem holds for $\mathcal{B}$, i.e., *if each $\mathcal{B}$-pointwise bounded subset $M$ of $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$* (see [12, Definition 2.4] or [15, Definition 1]). Let us note that in this definition we may suppose that $M$ is $\tau_s(\mathcal{A})$-closed and absolutely convex. If $\mathcal{B}$ has $N$-property then the polar set $\{\chi_C : C \in \mathcal{B}\}^\circ$ is bounded in $ba(\mathcal{A})$, hence $\{\chi_C : C \in \mathcal{B}\}^{\circ\circ} = \overline{absco\{\chi_C : C \in \mathcal{B}\}}$ is a neighborhood of zero in $L(\mathcal{A})$, whence $L(\mathcal{B})$ is dense in $L(\mathcal{A})$.

It is well known that *the algebra of finite and co-finite subsets of* $\mathbb{N}$ *fails $N$-property* [2, Example 5 in page 18] and that Schachermayer proved that *the algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has $N$-property* (see [12, Corollary 3.5] and a generalization in [4, Corollary]). A recent improvement of this result for the algebra $\mathcal{J}(K)$ of Jordan measurable subsets of a compact $k$-dimensional interval $K := \Pi\{[a_i, b_i] : 1 \leqslant i \leqslant k\}$ in $\mathbb{R}^k$ has been provided in [15, Theorem 2], where Valdivia proved that *if $\mathcal{J}(K)$ is the increasing countable union $\cup_m \mathcal{B}_m$ there exists a positive integer $n$ such that $\mathcal{B}_n$ has $N$-property* (see [8, Theorem 1] for a strong result in $\mathcal{J}(K)$). This fact motivated to say that a subset $\mathcal{B}$ of a set-algebra $\mathcal{A}$ has the *strong Nikodym property*, $sN$-property in brief, if for each increasing covering $\cup_m \mathcal{B}_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has $N$-property. As far as we know this result suggested the following very interesting Valdivia's open question (2013):

**Problem 1** ([15, Problem 1])**.** *Let $\mathcal{A}$ be an algebra of subsets of $\Omega$. Is it true that $N$-property of $\mathcal{A}$ implies $sN$-property?*

Note that the Nikodym-Dieudonné-Grothendieck stating that every $\sigma$-algebra $\mathcal{S}$ of subsets of a set $\Omega$ has property $N$ is a particular case of the following Valdivia's theorem.

**Theorem 1** ([14, Theorem 2])**.** *Each $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ has $sN$-property.*

Following [7, Chapter 7, 35.1] a family $\{B_{m_1,m_2,\ldots,m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ of subsets of $A$ is an *increasing web in $A$* if $(B_{m_1})_{m_1}$ is an increasing covering of $A$ and $(B_{m_1,m_2,\ldots,m_p,m_{p+1}})_{m_{p+1}}$ is an increasing covering of $B_{m_1,m_2,\ldots,m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p$. We will say that *a set-algebra $\mathcal{A}$ of subsets of $\Omega$ has the web strong $N$-property* (*web-sN-property*, in brief) *if for each increasing web $\{\mathcal{B}_{m_1,m_2,\ldots,m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ in $\mathcal{A}$ there exists a sequence $(n_i)_i$ in $\mathbb{N}$ such that each $\mathcal{B}_{n_1,n_2,\ldots,n_i}$ has $sN$-property, for each $i \in \mathbb{N}$.*

The main measure-theoretic result of this paper is the following theorem, motivated by Theorem 1 and covering all mentioned results for $\sigma$-algebras.

**Theorem 2.** *Each $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ has web-sN-property.*

In particular, if $\mathcal{B}_{m_1,m_2,\ldots,m_p} = \mathcal{B}_{m_1}$ for each $p \in N$, we have the following improvement of Theorem 1: If $(\mathcal{B}_m)_m$ *is an increasing covering of a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ there exists an index $n$ so that $\mathcal{B}_n$ has $sN$-property.*

Next section provides properties concerning $N$-property of subsets of a set-algebra $\mathcal{A}$ and unbounded subsets of $ba(\mathcal{A})$. These results will be used in Section 3 to provide necessary facts to complete the proof of our main result (Theorem 2).

Last section deals with applications of Theorem 2 to localizations of bounded finite additive vector measures.

A characterization of $sN$-property of a set-algebra $\mathcal{A}$ by a locally convex property of $L(\mathcal{A})$ was obtained in [15, Theorem 3]. Analogously a characterization of *web-*$sN$-property of a set-algebra $\mathcal{A}$ by a locally convex property of $L(\mathcal{A})$ may be found easily following [5] and [10].

## 2. Nikodym property and deep unbounded sets

To keep the paper self-contained we provided a short proof of the next (well known) proposition.

**Proposition 3.** *Let $\mathcal{A}$ be an algebra of subsets of $\Omega$ and let $M$ be an absolutely convex $\tau_s(\mathcal{A})$-closed subset of ba$(\mathcal{A})$. The following properties are equivalent:*

1. *For each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$ the set $M \cap \mathcal{Q}^\circ$ is an unbounded subset of ba$(\mathcal{A})$.*
2. *For each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$ such that span$\{M^\circ\} \cap$ span$\{\mathcal{Q}\} = \{0\}$ the set $M \cap \mathcal{Q}^\circ$ is unbounded in ba$(\mathcal{A})$.*
3. *$M^\circ$ is not a neighborhood of zero in span$\{M^\circ\}$ or the codimension of span$\{M^\circ\}$ in $L(\mathcal{A})$ is infinite.*

*If $M$ is unbounded and $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$ then $M$ verifies the previous properties.*

*Proof.* To prove these equivalences recall that if $M$ is a $\tau_s(\mathcal{A})$-closed and absolutely convex subset of ba$(\mathcal{A})$ then $M^{\circ\circ} = M$ [7, Chapter 4 20.8.5].

(1) $\iff$ (2). Let $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$. First we prove that if there exists $m_1 \in M^\circ$ such that $\chi_{Q_1} = h_1 m_1 + \Sigma_{2\leqslant i\leqslant r} h_i \chi_{Q_i}$ and if $h := 2 + \Sigma_{1\leqslant i\leqslant r} |h_i|$ then

$$\text{absco}(M^\circ \cup \mathcal{Q}) \subset h\,\text{absco}(M^\circ \cup \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\}). \tag{1}$$

In fact, if $x \in \text{absco}(M^\circ \cup \mathcal{Q})$ then $x = \lambda_0 m_0 + \Sigma_{1\leqslant i\leqslant r}\lambda_i\chi_{Q_i}$, with $m_0 \in M^\circ$ and $\Sigma_{0\leqslant i\leqslant r}|\lambda_i| \leqslant 1$, whence $x = \lambda_0 m_0 + \lambda_1 h_1 m_1 + \Sigma_{2\leqslant i\leqslant r}(\lambda_1 h_i + \lambda_i)\chi_{Q_i}$. From $m_2 := (1 + |\lambda_0| + |\lambda_1 h_1|)^{-1}(\lambda_0 m_0 + \lambda_1 h_1 m_1) \in M^\circ$ we get the representation $x = (1 + |\lambda_0| + |\lambda_1 h_1|)m_2 + \Sigma_{2\leqslant i\leqslant r}(\lambda_1 h_i + \lambda_i)\chi_{Q_i}$ which verifies the inequality $1 + |\lambda_0| + |\lambda_1 h_1| + \Sigma_{2\leqslant i\leqslant r}|\lambda_1 h_i + \lambda_i| \leqslant h$, whence $x \in h\,\text{absco}(M^\circ \cup \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\})$. Taking polar sets in (1) we obtain that

$$M \cap \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\}^\circ \subset h(M \cap \mathcal{Q}^\circ),$$

hence if $M \cap \{\mathcal{Q}\backslash\{\chi_{Q_1}\}\}^\circ$ is unbounded one gets that $M \cap \mathcal{Q}^\circ$ is also unbounded. The rest of this equivalence is obvious.

(2) $\iff$ (3). If $M^\circ$ is a neighborhood of zero in span$\{M^\circ\}$ and if $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ is a cobase of span$\{M^\circ\}$ in $L(\mathcal{A})$ then absco$(M^\circ \cup \mathcal{Q})$ is a neighborhood of zero in $L(\mathcal{A})$, hence

$$(\text{absco}(M^\circ \cup \mathcal{Q}))^\circ = M \cap \mathcal{Q}^\circ$$

is a bounded subset of ba$(\mathcal{A})$.

If $M^\circ$ is not a neighborhood of zero in span$\{M^\circ\}$ or if the codimension of span$\{M^\circ\}$ in $L(\mathcal{A})$ is infinite, then for each finite set $\mathcal{Q} := \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ such that span$\{M^\circ\} \cap$ span$\{\mathcal{Q}\} = \{0\}$ the set absco$(M^\circ \cup \mathcal{Q})$ is not a neighborhood of zero in $L(\mathcal{A})$, whence the set $(\text{absco}(M^\circ \cup \mathcal{Q}))^\circ = M \cap \mathcal{Q}^\circ$ is unbounded in ba$(\mathcal{A})$.

If $M$ is an unbounded subset of ba$(\mathcal{A})$ then $M^\circ$ is not a neighborhood of zero in $L(\mathcal{A})$. If, additionally, $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$ we have, by denseness, that $M^\circ$ is not a neighborhood of zero in span$\{M^\circ\}$ and we obtain that $M$ verifies (3). □

The fact that if a subset $M$ of *ba*$(\mathcal{A})$ verifies (1) in Proposition 3 then its subsets $M \cap \mathcal{Q}^\circ$ are unbounded, for each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$, motivates the following definition.

**Definition 1.** Let $B$ be an element of the algebra $\mathcal{A}$ of subsets of $\Omega$. A subset $M$ of ba$(\mathcal{A})$ is deep $B$-unbounded if each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$ verifies that

$$\sup\{|\mu(C)| : \mu \in M \cap \mathcal{Q}^\circ,\, C \in \mathcal{A},\, C \subset B\} = \infty. \tag{2}$$

or, equivalently, $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} = \infty$.

In particular, a subset $M$ of $ba(\mathcal{A})$ is deep $\Omega$-unbounded if $M \cap \mathcal{Q}^\circ$ is an unbounded subset of $ba(\mathcal{A})$, for each finite subset $\mathcal{Q}$ of $\{\chi_A : A \in \mathcal{A}\}$. Therefore an absolutely convex $\tau_s(\mathcal{A})$-closed subset $M$ of $ba(\mathcal{A})$ is deep $\Omega$-unbounded if and only if $M$ verifies condition (2) or (3) in Proposition 3. If, additionally, $\overline{span\{M^\circ\}} = L(\mathcal{A})$ then $M$ is deep $\Omega$-unbounded if and only if it is unbounded.

Next proposition furnishes sequences of deep $\Omega$-unbounded subsets of $ba(\mathcal{A})$. The particular case $\cup_m \mathcal{B}_m = \mathcal{A}$ is Theorem 1 in [15].

**Proposition 4.** *Let $\mathcal{A}$ be an algebra of subsets of $\Omega$ and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of $\mathcal{A}$ such that each $\mathcal{B}_m$ does not have $N$-property and $\overline{span\{\chi_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$. There exists $n_0 \in \mathbb{N}$ such that for each $m \geqslant n_0$ there exists a deep $\Omega$-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_m$ of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{B}_m$, i.e.,* $\sup\{|\mu(C)| : \mu \in M_m\} < \infty$ *for each $C \in \mathcal{B}_m$. In particular this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has $N$-property.*

*Proof.* If for each $m \in \mathbb{N}$ the subspace $H_m := \overline{\text{span}\{\chi_C : C \in \mathcal{B}_m\}}$ has infinite codimension in $L(\mathcal{A})$ then, by (3) in Proposition 3, the polar set of $P_m := \overline{\text{absco}\{\chi_C : C \in \mathcal{B}_m\}}$ is the deep $\Omega$-unbounded set $M_m := P_m^\circ$. The definition of polar set implies that $\sup\{|\mu(C)| : \mu \in M_m\} \leqslant 1$, for each $C \in \mathcal{B}_m$. Whence we get the proposition with $n_0 = 1$.

If there exists $p$ such that the codimension of $F := \overline{\text{span}\{\chi_C : C \in \mathcal{B}_p\}}$ in $L(\mathcal{A}) = \overline{\text{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}}$ is the finite positive number $q$ then $\{\chi_C : C \in \cup_m \mathcal{B}_m\} \not\subset F$, whence there exists $m_1 \in \mathbb{N}$ and $D \in \mathcal{B}_{p+m_1}$ such that $\chi_D \notin F$ and then the codimension of $\overline{\text{span}\{\chi_C : C \in \mathcal{B}_{p+m_1}\}}$ in $L(\mathcal{A})$ is less or equal than $q-1$. Therefore there exists $n_0$ such that $\overline{\text{span}\{\chi_C : C \in \mathcal{B}_m\}} = L(\mathcal{A})$, for each $m \geqslant n_0$. As for each $m \geqslant n_0$ the set $\mathcal{B}_m$ does not have $N$-property there exists an absolutely convex $\tau_s(\mathcal{A})$-closed unbounded subset $M_m$ of ba$(\mathcal{A})$ such that $\sup\{|\mu(C)| : \mu \in M_m\} < k_C < \infty$, for each $C \in \mathcal{B}_m$, and then it follows that $\{k_C^{-1}\chi_C : C \in \mathcal{B}_m\} \subset M_m^\circ$. This inclusion implies that $\text{span}\{\chi_C : C \in \mathcal{B}_m\} \subset \text{span}\{M_m^\circ\}$, whence $\overline{\text{span}\{M_m^\circ\}} = L(\mathcal{A})$, because $\overline{\text{span}\{\chi_C : C \in \mathcal{B}_m\}} = L(\mathcal{A})$. Then, by Proposition 3, the unbounded set $M_m$ is deep $\Omega$-unbounded for each $m \geqslant n_0$.

If $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has $N$-property then $\overline{\text{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$ and this proposition holds. □

Next Proposition 5 it follows from [15, Proposition 1]. We give a simplified proof according to our current notation.

**Proposition 5.** *Let $B$ be an element of an algebra $\mathcal{A}$ and $\{C_1, C_2, \ldots, C_q\}$ a finite partition of $B$ by elements of $\mathcal{A}$. If $M$ is a deep $B$-unbounded subset of $ba(\mathcal{A})$ there exists $C_i$, $1 \leqslant i \leqslant q$, such that $M$ is deep $C_i$-unbounded.*

*Proof.* If for each $i$, $1 \leqslant i \leqslant q$, there exists a finite set $\mathcal{Q}^i$ of characteristic functions of elements of $\mathcal{A}$ such that $\sup\{|\mu|(C_i) : \mu \in M \cap (\mathcal{Q}^i)^\circ\} < H_i$, $i \in \{1, 2, \ldots, q\}$, then we get the contradiction that the set $\mathcal{Q} = \cup_{1 \leqslant i \leqslant q} \mathcal{Q}^i$ verifies that $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} < \Sigma_{1 \leqslant i \leqslant q} H_i$. □

If $t = (t_1, t_2, \ldots, t_p)$, $s = (s_1, s_2, \ldots, s_q)$, $T$ and $U$ are two elements and two subsets of $\cup_s \mathbb{N}^s$ we define $t(i) := (t_1, t_2, \ldots, t_i)$ if $1 \leqslant i \leqslant p$, $t(i) := \emptyset$ if $i > p$, $T(m) := \{t(m) : t \in T\}$, for each $m \in \mathbb{N}$, $t \times s := (t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_{p+q})$, with $t_{p+j} := s_j$, for $1 \leqslant j \leqslant q$, and $T \times U := \{t \times u : t \in T,\ u \in U\}$. We simplify $(t_1)$, $(n)$ and $T \times \{(n)\}$ by $t_1$, $n$ and $T \times n$. The *length of* $t = (t_1, t_2, \ldots, t_p)$ *is* $p$ and the cardinal of a set $C$ is denoted by $|C|$.

If $v \in \cup_s \mathbb{N}^s$ and $t \times v \in U$ then $t \times v$ *is an extension of* $t$ *in* $U$. A sequence $(t^n)_n$ of elements $t^n = (t_1^n, t_2^n, \ldots, t_n^n, \ldots) \in \cup_s \mathbb{N}^s$ is an *infinite chain* if for each $n \in \mathbb{N}$ the element $t^{n+1}$ is an extension of the section $t^n(n)$, i.e., $\emptyset \neq t^n(n) = t^{n+1}(n)$.

A subset $U$ of $\cup_n \mathbb{N}^n$ is *increasing* at $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ if $U$ contains $p$ elements $t^1 = (t_1^1, t_2^1, \ldots)$ and $t^i = (t_1, t_2, \ldots, t_{i-1}, t_i^i, t_{i+1}^i, \ldots)$, $1 < i \leqslant p$, such that $t_i < t_i^i$, for each $1 \leqslant i \leqslant p$. *A non-void subset $U$ of $\cup_s \mathbb{N}^s$ is increasing (increasing respect to a subset $V$ of $\cup_s \mathbb{N}^s$) if $U$ is increasing at each $t \in U$ (at each $t \in V$),* hence $U$ is increasing if $|U(1)| = \infty$ and $|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty$, for each $t = (t_1, t_2, \ldots, t_p) \in U$ and $1 \leqslant i < p$.

If $\{B_u : u \in \cup_s \mathbb{N}^s\}$ is an increasing web in $A$ and $U$ is an increasing subset of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ then $\mathcal{B} := \{B_{u(i)} : u \in U,\ 1 \leqslant i \leqslant \text{length } u\}$ verifies that $(B_{u(1)})_{u\in U}$ is an increasing covering of $A$ and for each $u = (u_1, u_2, \dots, u_p) \in U$ and each $i < p$ the sequence $(B_{u(i)\times n})_{u(i)\times n\in U(i+1)}$ is an increasing covering of $B_{u(i)}$. If, additionally, each element $u \in U$ has an extension in $U$ then renumbering the indexes in the elements of $\mathcal{B}$ we get an increasing web.

The Definition 2 deals with increasing subsets of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ and it is motivated by the technical Example 1 which will be used onwards to complete the proof of Theorem 2. A particular class of increasing trees, named $NV$-trees -surely reminding Nikodym and Valdivia-, is considered in [9, Definition 1].

**Definition 2.** An increasing tree $T$ is an increasing subset of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ without infinite chains.

An increasing tree $T$ is *trivial* if $T = T(1)$; then $T$ is an infinite subset of $\mathbb{N}$. The sets $\mathbb{N}^i$, $i \in \mathbb{N}\backslash\{1\}$, and the set $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial increasing trees.

An increasing subset $S$ of an increasing tree $T$ is an increasing tree. From this observation it follows the Claim 6.

**Claim 6.** *If $(S_n)_n$ is a sequence of non-void subsets of an increasing tree $T$ such that for each $n \in \mathbb{N}$ the set $S_{n+1}$ is increasing respect to $S_n$, then $S := \cup_n S_n$ is an increasing tree.*

*Proof.* It is enough to notice that $S$ is an increasing subset of $T$. □

**Example 1.** *Let $\mathcal{B} := \{\mathcal{B}_{m_1,m_2,\dots,m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ be an increasing web in an algebra $\mathcal{A}$ of subsets of $\Omega$ with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $q \in \mathbb{N}$ such that $\mathcal{B}_{m_1,m_2,\dots,m_q}$ does not have $sN$-property. Then there exists an increasing web $\mathcal{C} := \{\mathcal{C}_{m_1,m_2,\dots,m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ in $\mathcal{A}$ and an increasing tree $T$ such that for each $(t_1, t_2, \dots, t_p) \in T$ there exists a deep $\Omega$-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{t_1,t_2,\dots,t_p}$ of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{C}_{t_1,t_2,\dots,t_p}$, i.e.,*

$$\sup\{|\mu(C)| : \mu \in M_{t_1,t_2,\dots,t_p}\} < \infty, \tag{3}$$

*for each $C \in \mathcal{C}_{t_1,t_2,\dots,t_p}$.*

*Proof.* If each $\mathcal{B}_{m_1}$, $m_1 \in \mathbb{N}$, does not have $N$-property then the example is given by $\mathcal{C} := \mathcal{B}$ and $T := \mathbb{N}\backslash\{1, 2, \dots, n_0 - 1\}$, where $n_0$ is the natural number obtained in Proposition 4 applied to the increasing covering $(\mathcal{B}_{m_1})_{m_1}$ of $\mathcal{A}$. Hence we may suppose that there exists $m_1 \in \mathbb{N}$ such that $\mathcal{B}_{t_1}$ has $N$-property for each $t_1 \geqslant m_1$ and then:

$(i_1)$ Either $\mathcal{B}_{t_1}$ does not have $sN$-property for each $t_1 \in \mathbb{N}$ and the inductive process finish defining $T_0 := \{t_1 \in \mathbb{N} : t_1 \geqslant m_1\}$.

$(ii_1)$ Or there exists $m'_1 \in \mathbb{N}$ such that $\mathcal{B}_{t_1}$ has $sN$-property for each $t_1 \geqslant m'_1$. Then we write $Q_1 := \emptyset$ and $Q'_1 := \{t_1 \in \mathbb{N} : t_1 \geqslant m'_1\}$.

Let us assume that for each $j$, with $2 \leqslant j \leqslant i$, we have obtained by induction two disjoint subsets $Q_j$ and $Q'_j$ of $\mathbb{N}^j$ such that each $t = (t_1, t_2, \dots, t_j) \in Q_j \cup Q'_j$ verifies:

1. $t(j-1) = (t_1, t_2, \dots, t_{j-1}) \in Q'_{j-1}$.
2. If $t \in Q_j$ the set $\mathcal{B}_t$ has $N$-property but it does not have $sN$-property and $S_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is a cofinite subset of $\mathbb{N}$ such that $t(j-1) \times S_{t(j-1)} \subset Q_j$.
3. If $t \in Q'_j$ the set $\mathcal{B}_t$ has $sN$-property and $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is a cofinite subset of $\mathbb{N}$ such that $t(j-1) \times S'_{t(j-1)} \subset Q'_j$.

If $t := (t_1, t_2, \dots, t_i) \in Q'_i$ then $\mathcal{B}_{t_1,t_2,\dots,t_i}$ has $sN$-property and $(\mathcal{B}_{t_1,t_2,\dots,t_i,n})_n$ is an increasing covering of $\mathcal{B}_{t_1,t_2,\dots,t_i}$, hence there exists $m_{i+1}$ such that $\mathcal{B}_{t_1,t_2,\dots,t_i,n}$ has $N$-property for each $n \geqslant m_{i+1}$. Then we may have two possible cases:

$(i_{i+1})$ Either $\mathcal{B}_{t_1,t_2,\ldots,t_i,n}$ does not have $sN$-property for each $n \in \mathbb{N}$ and we define $S_{t_1,t_2,\ldots,t_i} := \{n \in \mathbb{N} : m_{i+1} \leqslant n\}$ and $S'_{t_1,t_2,\ldots,t_i} := \emptyset$,

$(ii_{i+1})$ or there exists $m'_{i+1} \in \mathbb{N}$ such that $\mathcal{B}_{t_1,t_2,\ldots,t_i,n}$ has $sN$-property for each $n \geqslant m'_{i+1}$. In this case let $S_{t_1,t_2,\ldots,t_i} := \emptyset$ and $S'_{t_1,t_2,\ldots,t_i} := \{n \in \mathbb{N} : m'_{i+1} \leqslant n\}$.

We finish this induction procedure by setting $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$. By construction $Q_{i+1}$ and $Q'_{i+1}$ verify the properties 1., 2. and 3. with $j = i + 1$.

The fact that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $j \in \mathbb{N}$ such that $\mathcal{B}_{m_1,m_2,\ldots,m_j}$ does not have $sN$-property imply that $T_0 := \cup\{Q_i : i \in \mathbb{N}\}$ does not contain infinite chains, because if $(m_1, m_2, \ldots, m_p) \in Q_p$ then $\mathcal{B}_{m_1,m_2,\ldots,m_{p-1}}$ has $sN$-property, whence for each $(t_1, t_2, \ldots, t_k) \in Q'_k$ there exists $q \in \mathbb{N}$ and $(t_{k+1}, \ldots, t_{k+q}) \in \mathbb{N}^q$ such that $(t_1, t_2, \ldots, t_k, t_{k+1}, \ldots, t_{k+q}) \in Q_{k+q}$ and then $T_0(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$. These equalities imply that $T_0$ is increasing, because $|T_0(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \ldots, t_p) \in T_0$ the the sets $S'_{t(i-1)}$, $1 < i < p$, and $S_{t(p-1)}$ are cofinite subsets of $\mathbb{N}$.

This increasing tree $T_0$ as well as the trivial increasing tree obtained in $(i_1)$, also named $T_0$, verify that for each $t = (t_1, t_2, \ldots, t_p) \in T_0$ the family $\mathcal{B}_{t_1,t_2,\ldots,t_p}$ has $N$-property and it does not have $sN$-property, whence $\mathcal{B}_{t_1,t_2,\ldots,t_p}$ has an increasing covering $(\mathcal{B}'_{t_1,t_2,\ldots,t_p,n})_n$ such that each $\mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$ does not have $N$-property. By Proposition 4 there exist $n_0 \in \mathbb{N}$ such that for each $n \geqslant n_0$ there exists a deep $\Omega$-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{t_1,t_2,\ldots,t_p,n}$ of $\mathrm{ba}(\mathcal{A})$ which is $\mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$ pointwise bounded, i.e., $\sup\{|\mu(C)| : \mu \in M_{t_1,t_2,\ldots,t_p,n}\} < \infty$, for each $C \in \mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$. We assume $n_0 = 1$, removing $\mathcal{B}'_{t_1,t_2,\ldots,t_p,n}$ when $n < n_0$ and changing $n$ by $n - n_0 + 1$.

Then we get the example with the increasing tree $T := T_0 \times \mathbb{N}$ and with the increasing web $\mathcal{C} := \{\mathcal{C}_t : t \in \cup_s \mathbb{N}^s\}$ in the algebra $\mathcal{A}$ such that for each $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ either $\mathcal{C}_t := \mathcal{B}'_{t(i)}$ if $i \leqslant p$ and $t(i) \in T$ or $\mathcal{C}_t := \mathcal{B}_t$ if $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \emptyset$. $\square$

Let $U$ be a subset of $\cup_s \mathbb{N}^s$. *An element* $t \in \cup_s \mathbb{N}^s$ *admits increasing extension in* $U$ *if the set of* $\{v \in \cup_s \mathbb{N}^s : t \times v \in U\}$ *contains an increasing subset.* We need the following obvious properties $(a)$, $(b_1)$ and $(b_2)$ to prove Proposition 7, stating that if a subset $U$ of an increasing tree $T$ does not contain an increasing tree then $T \backslash U$ contains an increasing tree.

$(a)$ If $U$ is a subset of $\cup_s \mathbb{N}^s$ and $U$ does not contain an increasing tree then there exists $m_1 \in \mathbb{N}$ such that each $n \in \mathbb{N} \backslash \{1, 2, \ldots, m_1\}$ does not admit increasing extension in $U$.

$(b)$ Let $t \in \cup_s \mathbb{N}^s$ and let $U$ be a subset of the increasing tree $T$. Suppose that $t$ does not admit increasing extension in $U$ and that $T_t := \{v \in \cup_s \mathbb{N}^s : t \times v \in T\} \neq \emptyset$. Then

$(b_1)$ if the increasing tree $T_t$ is trivial there exists $m_{i+1} \in \mathbb{N}$ such that the set

$$(t \times \{\mathbb{N} \backslash \{1, 2, \ldots, m_{i+1}\}) \cap T$$

is an infinite subset of $T \backslash U$,

$(b_2)$ if $T_t$ is non-trivial there exists $m'_{i+1} \in \mathbb{N}$ such that each element of

$$\big(t \times \{\mathbb{N} \backslash \{1, 2, \ldots, m'_{i+1}\}\big) \cap T(i + 1)$$

does not admit increasing extension in $U$.

**Proposition 7.** *Let $U$ be a subset of an increasing tree $T$. If $U$ does not contain an increasing tree then $T \backslash U$ contains an increasing tree.*

*Proof.* It is enough to prove that $T \backslash U$ contains an increasing subset $W$. Now we follow the scheme of the proof in Example 1. In fact, if $T$ is a trivial increasing tree the proposition is obvious. Hence we may suppose that $T$ is a non-trivial increasing tree. Then we define $Q_1 := \emptyset$ and by $(a)$ there exists $m'_1 \in \mathbb{N}$ such that each element of the set $Q'_1 := \{n \in T(1) : m'_1 \leqslant n\}$ does not admit increasing extension in $U$. Notice that $Q'_1 \subset T(1) \backslash T$.

Let us suppose that we have obtained for each $j$, with $2 \leqslant j \leqslant i$, two disjoint subsets $Q_j$ and $Q'_j$ such that $Q_j \subset T(j) \cap (T \backslash U)$, $Q'_j \subset T(j) \backslash T$ and each $t \in Q_j \cup Q'_j$ verifies the following properties:

1. $t(j-1) \in Q'_{j-1}$.
2. If $t \in Q_j$ then the cardinal of $S_{t(j-1)} := \{n \in \mathbb{N} : t(j-1)\times n \in Q_j \cup Q'_j\}$ is infinite and $t(j-1)\times S_{t(j-1)} \subset Q_j$.
3. If $t \in Q'_j$ then $t$ does not admit increasing extension in $U$, the cardinal of $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is infinite and $t(j-1) \times S'_{t(j-1)} \subset Q'_j$.

If $t \in Q'_i$ then $t \in T(i)\backslash T$ and it does not admit increasing extension in $U$. If $T_t = \{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$ then, by $(b_1)$ and $(b_2)$, it follows that the following two cases may happen:

*i.* If $T_t$ is trivial then there exists $m_{i+1} \in \mathbb{N}$ such that the infinite set $S_t := \{n \in \mathbb{N} : m_{i+1} \leqslant n,\, t \times n \in T(i+1)\}$ verifies that $t \times S_t \subset T\backslash U$ and we define $S'_t := \emptyset$.

*ii.* If $T_t$ is non-trivial then there exists $m'_{i+1} \in \mathbb{N}$ such that the infinite set $S'_t := \{n \in \mathbb{N} : m'_{i+1} < n,\, t \times n \in T(i+1)\}$ verifies that $t \times S'_t \subset T(i+1)\backslash T$ and each element of $t \times S'_t$ does not admit increasing extension in $U$. Now we define $S_t := \emptyset$.

We finish this induction procedure by setting $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$.

By construction $Q_{i+1} \subset T(i+1) \cap (T\backslash U)$, $Q'_{i+1} \subset T(i+1)\backslash T$, and each $t \in Q_{i+1} \cup Q'_{i+1}$ verifies the properties 1., 2. and 3. changing $j$ by $i+1$.

As $T$ does not contain infinite chains we deduce from 1. that for each $(t_1, t_2, \ldots, t_i) \in Q'_i$ there exists $q \in \mathbb{N}$ and $(t_{i+1}, \ldots, t_{i+q}) \in \mathbb{N}^q$ such that $(t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_{i+q}) \in Q_{i+q}$. Whence, for each $i \in \mathbb{N}$, $(\cup_{j>i} Q_j)(i) = Q'_i$ and then $W := \cup\{Q_j : j \in \mathbb{N}\}$ is a subset of $T\backslash U$.

$W$ has the increasing property because from $W(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$, it follows that $|W(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \ldots, t_p) \in W$ then $(t_1, t_2, \ldots, t_i) \in Q'_i$, if $1 < i < p$, and $(t_1, t_2, \ldots, t_p) \in Q_p$, hence the infinite subsets $S'_{t(i-1)}$ and $S_{t(p-1)}$ of $\mathbb{N}$ verify that $t(i-1) \times S'_{t(i-1)} \subset Q'_i \subset W(i)$ and $t(p-1) \times S_{t(p-1)} \subset Q_p \subset W$. □

Next Proposition 8 follows from [15, Propositions 2 and 3] and we give a simplified proof according to our current notation for the sake of completeness.

**Proposition 8.** *Let $\{B, Q_1, \ldots, Q_r\}$ be a subset of the algebra $\mathcal{A}$ of subsets of $\Omega$ and let $M$ be a deep $B$-unbounded absolutely convex subset of $ba(\mathcal{A})$. Then given a positive real number $\alpha$ and a natural number $q > 1$ there exists a finite partition $\{C_1, C_2, \ldots, C_q\}$ of $B$ by elements of $\mathcal{A}$ and a subset $\{\mu_1, \mu_2, \ldots, \mu_q\}$ of $M$ such that $|\mu_i(C_i)| > \alpha$ and $\Sigma_{1\leqslant j\leqslant r} |\mu_i(Q_j)| \leqslant 1$, for $i = 1, 2, \ldots, q$.*

*Proof.* Let $\mathcal{Q} = \{\chi_B, \chi_{Q_1}, \chi_{Q_2}, \ldots, \chi_{Q_r}\}$. The deep $B$-unboundedness of $M$ and the inclusion $M \subset rM$ imply that

$$\sup\{|\mu(D)| : \mu \in rM \cap \mathcal{Q}^\circ,\ D \subset B,\ D \in \mathcal{A}\} = \infty.$$

Hence there exists $P_1 \subset B$, with $P_1 \in \mathcal{A}$, and $\mu \in rM \cap \mathcal{Q}^\circ$ such that $|\mu(P_1)| > r(1+\alpha)$. Clearly $\mu_1 = r^{-1}\mu \in M$, $|\mu_1(P_1)| > 1 + \alpha$ and $|\mu_1(f)| = r^{-1}|\mu(f)| \leqslant r^{-1}$ for each $f \in \mathcal{Q}$, hence $|\mu_1(B)| \leqslant r^{-1} \leqslant 1$ and $\Sigma_{1\leqslant j\leqslant r} |\mu_1(Q_j)| \leqslant r^{-1}r = 1$. The set $P_2 := B\backslash P_1$ verifies that

$$|\mu_1(P_2)| \geqslant |\mu_1(P_1)| - |\mu_1(B)| > 1 + \alpha - 1 = \alpha.$$

From Proposition 5 there exists $i \in \{1, 2\}$ such that $M$ is deep $P_i$-unbounded. To finish the first step of the proof let $C_1 := P_1$ if $M$ is deep $P_2$-unbounded and let $C_1 := P_2$ if $M$ is deep $P_1$-unbounded. Then $M$ is deep $B\backslash C_1$-unbounded.

Apply the same argument in $B\backslash C_1$ to obtain a measurable set $C_2 \subset B\backslash C_1$ and a measure $\mu_2 \in M$ such that $|\mu_2(C_2)| > \alpha$, $|\mu_2(B\backslash(C_1 \cup C_2))| > \alpha$ and $\Sigma\{|\mu_2(Q_j)| : 1 \leqslant j \leqslant r\} \leqslant 1$, being $M$ deep $B\backslash(C_1 \cup C_2)$-unbounded. Hence the proof is provided by applying $q-1$ times this argument. In the last step we define $\mu_q := \mu_{q-1}$ and $C_q = B\backslash(C_1 \cup \cdots \cup C_{q-1})$. □

**Proposition 9.** *Let $B$ be an element of an algebra $\mathcal{A}$ and $\{M_t : t \in T\}$ a family of deep $B$-unbounded subsets of $ba(\mathcal{A})$ indexed by an increasing tree $T$. If $t^j := (t_1^j, t_2^j, \ldots, t_{p_j}^j) \in T$, for each $1 \leqslant j \leqslant k$, and $q = 2 + \Sigma\{p_j : 1 \leqslant j \leqslant k\}$ then for each finite partition $\{C_1, C_2, \ldots, C_q\}$ of $B$ by elements of $\mathcal{A}$ there exists $h \in \{1, 2, \cdots, q\}$ and an increasing tree $T_1$ such that $\{t^1, t^2, \ldots, t^k\} \subset T_1 \subset T$ and $\{M_t : t \in T_1\}$ is a family of deep $B\backslash C_h$-unbounded subsets.*

*Proof.* Let $\{C_1, C_2, \dots, C_q\}$ be a finite partition of $B$ by elements of $\mathcal{A}$ with $q = 2+\Sigma\{p_j : 1 \leqslant j \leqslant k\}$. From Proposition 5 it follows that if $\{M_u : u \in U\}$ is a family of deep $B$-unbounded subsets of $\text{ba}(\mathcal{A})$ indexed by an increasing tree $U$ and $V_i := \{u \in U : M_u \text{ is deep } C_i\text{-unbounded}\}$, $1 \leqslant i \leqslant q$, then $U = \cup_{1\leqslant i\leqslant q} V_i$ and, by Proposition 7, there exists $l$, with $1 \leqslant l \leqslant q$, such that $V_l$ contains an increasing tree $U_l$. Therefore

(a) If $\{M_u : u \in U\}$ is a family of deep $B$-unbounded subsets indexed by an increasing tree $U$ there exists $l \in \{1, 2, \dots, q\}$ and an increasing tree $U_l$ contained in $U$ such that $\{M_u : u \in U_l\}$ is a family of deep $C_l$-unbounded subsets.

In particular, for the increasing tree $T$ and for each element $t^j \in T$, with $1 \leqslant j \leqslant k$, there exist by (a) and Proposition 5:

(1) $i_0 \in \{1, 2, \dots, q\}$ and an increasing tree $T_{i_0}$ contained in $T$ such that $\{M_t : t \in T_{i_0}\}$ is a family of deep $C_{i_0}$-unbounded subsets,
(2) $i^j \in \{1, 2, \dots, q\}$ such that $M_{t^j}$ is deep $C_{i^j}$-unbounded.

Let $S := \{j : 1 \leqslant j \leqslant k,\ t^j \notin T_{i_0}\}$. For each $j \in S$ and each section $t^j(m-1)$ of $t^j = (t^j_1, t^j_2, \dots, t^j_{p_j})$, with $2 \leqslant m \leqslant p_j$, the set $W^j_m := \{v \in \cup_s \mathbb{N}^s : t^j(m-1) \times v \in T\}$ is an increasing tree such that $\{M_{(t^j_1, t^j_2, \dots, t^j_{m-1})\times w} : w \in W^j_m\}$ is a family of deep $B$-unbounded subsets. By (a) there exists:

(3) $i^j_m \in \{1, 2, \dots, q\}$ and an increasing tree $V^j_m$ contained in $W^j_m$ such that

$$\{M_{(t^j_1, t^j_2, \dots, t^j_{m-1})\times v} : v \in V^j_m\}$$

is a family of deep $C_{i^j_m}$-unbounded subsets. Clearly $(t^j_1, t^j_2, \dots, t^j_{m-1}) \times V^j_m \subset T$.

As the number of sets $C_{i_0}$, $C_{i^j}$, $C_{i^j_m}$, with $j \in S$ and $2 \leqslant m \leqslant p_j$, is less or equal than $q-1$, there exists $h \in \{1, 2, \cdots, q\}$ such that

$$D := C_{i_0} \cup (\cup\{C_{i^j} \cup C_{i^j_m} : j \in S,\ 2 \leqslant m \leqslant p_j\}) \subset B \backslash C_h.$$

Let $T_1$ be the union of the sets $T_{i_0}$, $\{t^j : j \in S\}$ and $\{(t^j_1, t^j_2, \dots, t^j_{m-1})\} \times V^j_m$, with $j \in S$ and $2 \leqslant m \leqslant p_j$. Clearly for each $t \in T_1$ the set $M_t$ is deep $D$-unbounded, whence $M_t$ is also deep $B\backslash C_h$-unbounded. By construction $\{t^1, t^2, \dots, t^k\} \subset T_1$ and $T_1$ has the increasing property and it is a subset of the increasing tree $T$. Whence $T_1$ is an increasing tree. □

We finish this section with a combination of Propositions 8 and 9. The obtained Proposition 10 is a fundamental tool for the next section.

**Proposition 10.** *Let $\{B, Q_1, \dots, Q_r\}$ be a subset of an algebra $\mathcal{A}$ of subsets of $\Omega$, and let $\{M_t : t \in T\}$ be a family of deep $B$-unbounded absolutely convex subsets of $\text{ba}(\mathcal{A})$, indexed by an increasing tree $T$. Then for each positive real number $\alpha$ and each finite subset $\{t^j : 1 \leqslant j \leqslant k\}$ of $T$ there exist $\{B_j \in \mathcal{A} : 1 \leqslant j \leqslant k\}$, formed by $k$ pairwise disjoint subsets $B_j$ of $B$, $1 \leqslant j \leqslant k$, a set $\{\mu_j \in M_{t^j},\ 1 \leqslant j \leqslant k\}$ and an increasing tree $T^*$ such that:*

1. *$|\mu_j(B_j)| > \alpha$ and $\Sigma\{|\mu_j(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$, for $j = 1, 2, \dots, k$,*
2. *$\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$ and $\{M_t : t \in T^*\}$ is a family of deep $(B\backslash \cup_{1\leqslant j\leqslant k} B_j)$-unbounded sets.*

*Proof.* Let $t^j := (t^j_1, t^j_2, \dots, t^j_{p_j})$, for $1 \leqslant j \leqslant k$. By Proposition 8 applied to $B$, $\alpha$, $q := 2 + \Sigma_{1\leqslant j\leqslant k} p_j$ and $M_{t^1}$ there exist a partition $\{C^1_1, C^1_2, \dots, C^1_q\}$ of $B$ by elements of $\mathcal{A}$ and $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset M_{t^1}$ such that:

$$\left|\lambda_k(C^1_k)\right| > \alpha \quad \text{and} \quad \Sigma_{1\leqslant i\leqslant r} |\lambda_k(Q_i)| \leqslant 1, \quad \text{for } k = 1, 2, \dots, q, \tag{4}$$

hence Proposition 9 applied to the sets $\{C^1_1, C^1_2, \cdots, C^1_q\}$, $\{M_t : t \in T\}$ and $\{t^j : 1 \leqslant j \leqslant k\}$ gives $h \in \{1, 2, \cdots, q\}$ and a family $\{M_t : t \in T_1\}$ of deep $B\backslash C^1_h$-unbounded subsets indexed by an increasing tree

$T_1$ such that $\{t^1, t^2, \ldots, t^k\} \subset T_1 \subset T$. If $B_1 := C_h^1$ and $\mu_1 := \lambda_h$ then (4) holds with $\lambda_k = \mu_1$ and $C_k^1 = B_1$. Clearly $\{M_t : t \in T_1\}$ is a family of deep $B \backslash B_1$-unbounded subsets.

If we apply again Proposition 8 to $B \backslash B_1$, $\alpha$, $q$ and $M_{t^2}$ we obtain a partition $\{C_1^2, C_2^2, \cdots, C_q^2\}$ of $B \backslash B_1$ by elements of $\mathcal{A}$ and $\{\zeta_1, \zeta_2, \cdots, \zeta_q\} \subset M_{t^2}$ such that

$$\left|\zeta_k(C_k^2)\right| > \alpha \quad \text{and} \quad \Sigma_{1 \leqslant i \leqslant r} \left|\zeta_k(Q_i)\right| \leqslant 1, \quad \text{for } k = 1, 2, \ldots, q,$$

and then by Proposition 9 (applied to $\{C_1^2, C_2^2, \cdots, C_q^2\}$, $\{M_t : t \in T_1\}$ and $\{t^j : 1 \leqslant j \leqslant k\}$ there exists $l \in \{1, 2, \cdots, q\}$ and a family $\{M_t : t \in T_2\}$ of deep $(B \backslash B_1) \backslash C_l^2$-unbounded subsets indexed by an increasing tree $T_2$ such that $\{t^1, t^2, \ldots, t^k\} \subset T_2 \subset T$. Now if $B_2 := C_l^2$ and $\mu_2 := \zeta_l$ then $|\mu_2(B_2)| > \alpha$, $\Sigma\{|\mu_2(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$ and $\{M_t : t \in T_2\}$ is a family of deep $B \backslash (B_1 \cup B_2)$-unbounded subsets. With $k-2$ new repetitions of this procedure we get the proof with $T^* := T_k$. □

## 3. Proof of Theorem 2

With a induction procedure based in Proposition 10 we obtain Proposition 12 that together with the next elementary covering property for families indexed by increasing trees enable to prove Theorem 2.

**Proposition 11.** *If $\mathcal{Y} = \{Y_{m_1, m_2, \ldots, m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ is an increasing web in $Y$ and $T$ is an increasing tree then $Y = \cup\{Y_y : y \in T\}$.*

*Proof.* Let us suppose that $y \in Y \backslash (\cup\{Y_t : t \in T\})$. As $\mathcal{Y}$ is an increasing web and $T$ is an increasing tree then $Y = \cup\{Y_{t(1)} : t \in T\}$, whence there exists $u^1 = (u_1^1, u_2^1, \ldots) \in T$ such that

$$y \in Y_{u_1^1} \backslash (\cup\{Y_t : t \in T\}).$$

Assume that there exists $\{u^2, u^3, \ldots, u^n\} \subset T$ such that $\emptyset \neq u^{j-1}(j-1) = u^j(j-1)$ and $y \in Y_{u^j(j)} \backslash \cup \{Y_t : t \in T\}$, for $2 \leqslant j \leqslant n$. Then $y \in Y_{u^n(n)} \backslash \cup \{Y_t : t \in T\}$, with $u^n(n) = (u_1^n, u_2^n, \ldots, u_n^n)$. As $\mathcal{Y}$ is an increasing web and $T$ is an increasing tree then $Y_{u^n(n)} = \cup\{Y_{u^n(n) \times s} : u^n(n) \times s \in T(n+1)\}$, hence there exists $u^{n+1} \in T$ such that $u^n(n) = u^{n+1}(n)$ and

$$y \in Y_{u^{n+1}(n+1)} \backslash (\cup\{Y_t : t \in T\}).$$

This induction procedure gives the contradiction that $T$ contains the infinite chain $(u^n)_n$. Therefore $Y = \cup\{Y_u : u \in T\}$. □

In Proposition 12 we refer to the sequence $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$, obtained with the first components of $\mathbb{N}^2$ ordered by the diagonal order, i.e., $i_n = n - 2^{-1}h(h+1)$, if $n \in ]\, 2^{-1}h(h+1), 2^{-1}(h+1)(h+2)\,]$ and $h = 0, 1, 2, \ldots$. Let us note that $i_n \leqslant n$, for each $n \in \mathbb{N}$.

**Proposition 12.** *Let $\{\mathcal{B}_{m_1, m_2, \ldots, m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $h \in \mathbb{N}$ such that $\mathcal{B}_{m_1, m_2, \ldots, m_h}$ does not have $sN$-property and let $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$. Then there exist a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$, a sequence $(B_{i_n j_n})_n$ of pairwise disjoints elements of $\mathcal{S}$, a sequence $(\mu_{i_n j_n})_n$ in $ba(\mathcal{S})$ and a covering $(\mathcal{C}_r)_r$ of $\mathcal{S}$ such that for each $n \in \mathbb{N}$*

$$\Sigma_s\{\left|\mu_{i_{n+1} j_{n+1}}(B_{i_s j_s})\right| : 1 \leqslant s \leqslant n\}) < 1, \tag{5}$$

$$|\mu_{i_n j_n}(B_{i_n j_n})| > j_n, \tag{6}$$

$$|\mu_{i_n j_n}(\cup_s\{B_{i_s j_s} : n < s\})| < 1, \tag{7}$$

*and for each $r \in \mathbb{N}$ and each strictly increasing sequence $(n_p)_p$ such that $i_{n_p} = r$, for each $p \in \mathbb{N}$, the set $\{\mu_{i_{n_p} j_{n_p}} : p \in \mathbb{N}\}$ is $\mathcal{C}_r$-pointwise bounded, i.e., for each $H \in \mathcal{C}_r$ we have that*

$$\sup\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty. \tag{8}$$

*Proof.* Let $\{\mathcal{C}_t : t \in \cup_s \mathbb{N}^s\}$ and $T$ be the increasing web in $\mathcal{S}$ and the increasing tree determined in Example 1 such that for each $t \in T$ there exists a deep $\Omega$-unbounded $\tau_s(\mathcal{S})$-closed absolutely convex subset $M_t$ of $\mathrm{ba}(\mathcal{S})$ which is $\mathcal{C}_t$-pointwise bounded, i.e.,

$$\sup\{|\mu(H)| : \mu \in M_t\} < \infty \tag{9}$$

for each $H \in \mathcal{C}_t$.

Then, by induction, we prove that there exist a countable increasing tree $\{t^i : i \in \mathbb{N}\}$ contained in $T$, a strictly increasing sequence of natural numbers $(k_j)_j$, a set $\{B_{ij} : (i,j) \in \mathbb{N}^2, i \leqslant k_j\}$ of pairwise disjoint elements of $\mathcal{S}$ and a set $\{\mu_{ij} \in M_{t^i} : (i,j) \in \mathbb{N}^2, i \leqslant k_j\}$ such that if $(i,j) \in \mathbb{N}^2$ and $i \leqslant k_j$ then

$$\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v < j\} < 1, \tag{10}$$

$$|\mu_{ij}(B_{ij})| > j, \tag{11}$$

and for each $i \in \mathbb{N}$ and each $H \in \mathcal{C}_{t^i}$ we have

$$\sup_j\{|\mu_{ij}(H)| : i \leqslant j\} < \infty. \tag{12}$$

Fix $t^1 \in T$. By Proposition 10 with $B := \Omega$, $\alpha = 1$, $\{Q_1, \ldots, Q_r\} := \emptyset$ and $\{t^i : 1 \leqslant i \leqslant k\} := \{t^1\}$ there exist $B_{11} \in \mathcal{S}$, $\mu_{11} \in M_{t^1}$ and an increasing tree $T_1$ such that

1. $|\mu_{11}(B_{11})| > 1$, $\{M_t : t \in T_1\}$ is a family of deep $\Omega \backslash B_{11}$-unbounded subsets and
2. $t^1 \in T_1 \subset T$.

We define $k_1 := 1$, $S^1 := \{t^1\}$ and $B^1 := B_{11}$.

Suppose that in the following $n-1$ steps of the inductive process we have obtained the finite sequence $k_2 < k_3 < \cdots < k_n$ in $\mathbb{N}\backslash\{1\}$, the increasing trees $T_2 \supset T_3 \supset \cdots \supset T_n$ contained in $T_1$, the subset $\{t^1, t^2, \ldots, t^{k_n}\}$ of $T_n$, the set $\{B_{ij} : i \leqslant k_j,\ j \leqslant n\}$ formed by pairwise disjoint elements of $\mathcal{S}$ and the set $\{\mu_{ij} \in M_{t^i} : i \leqslant k_j,\ j \leqslant n\}$ such that, for each $1 < j \leqslant n$ and each $i \leqslant k_j$:

1. $|\mu_{ij}(B_{ij})| > j$, $\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v < j\} < 1$, the union $B^j := \cup\{B_{sv} : s \leqslant k_v,\ 1 \leqslant v \leqslant j\}$ verifies that $\{M_t : t \in T_j\}$ is a family of deep $\Omega\backslash B^j$-unbounded subsets,
2. $S^j := \{t^i : i \leqslant k_j\} \subset T_j$ and $S^j$ has the increasing property respect to $S^{j-1}$.

To finish the induction procedure let $\{t^{k_n+1}, \ldots, t^{k_{n+1}}\}$ be a subset of $T_n\backslash\{t^i : i \leqslant k_n\}$ that verifies the increasing property with respect to $S^n$. Then applying Proposition 10 to $\Omega\backslash B^n$, $\{B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant n\}$, $T_n$, the finite subset $S^{n+1} := \{t^i : i \leqslant k_{n+1}\}$ of $T_n$ and $n+1$ we obtain a family $\{B_{in+1} : i \leqslant k_{n+1}\}$ of pairwise disjoint elements of $\mathcal{S}$ contained in $\Omega\backslash B^n$, a subset $\{\mu_{in+1} \in M_{t^i} : i \leqslant k_{n+1}\}$ of $\mathrm{ba}(\mathcal{S})$ and an increasing tree $T_{n+1}$ contained in $T_n$ such that for each $i \leqslant k_{n+1}$,

1. $|\mu_{in+1}(B_{in+1})| > n+1$, $\Sigma_{s,v}\{|\mu_{in+1}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v \leqslant n\} < 1$, the union $B^{n+1} := \cup\{B_{sv} : s \leqslant k_s,\ 1 \leqslant v \leqslant n+1\}$ has the property that $\{M_t : t \in T_{n+1}\}$ is a family of deep $\Omega\backslash B^{n+1}$-unbounded subsets,
2. $S^{n+1} \subset T_{n+1}$ and $S^{n+1}$ has the increasing property respect to $S^n$.

By Claim 6, $\cup_n S_n = \{t^i : i \in \mathbb{N}\}$ is an increasing tree, whence, by Proposition 11, the sequence $(\mathcal{C}_{t^i})_i$ is a countable covering of the $\sigma$-algebra $\mathcal{S}$. As $(k_j)_j$ is increasing then $(i,j) \in \mathbb{N}^2$ and $i \leqslant j$ imply that $i \leqslant k_j$, whence $\{\mu_{ij} : j \in \mathbb{N}\backslash\{1, 2, \ldots, i-1\}\} \subset M_{t^i}$ and from this inclusion and (9) with $t = t^i$ it follows (12), i.e., $\sup_j\{|\mu_{ij}(H)| : i \leqslant j\} < \infty$, for each $i \in \mathbb{N}$ and each $H \in \mathcal{C}_{t^i}$.

With a new induction procedure we determine the increasing sequence $(j_n)_n$ such that together with the sequence $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$ verifies (5), (6), (7) and (8).

Let $j_1 := 1$ and suppose that $|\mu_{i_1 j_1}|(\Omega) < s_1$, with $s_1 \in \mathbb{N}$. Let $\{N_u^1, 1 \leqslant u \leqslant s_1\}$ be a partition of $\{m \in \mathbb{N} : m > j_1\}$ in $s_1$ infinite subsets and define $B_u^1 := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N_u^1,\ s \leqslant k_t\}$, $1 \leqslant u \leqslant s_1$. From

$\Sigma\{|\mu_{i_1j_1}|(B^1_u) : 1 \leqslant u \leqslant s_1\} < s_1$ it follows that there exists $u'$, with $1 \leqslant u' \leqslant s_1$, such that $|\mu_{i_1j_1}|(B^1_{u'}) < 1$, whence the sets $N^{(1)} := N^1_{u'}$ and $B^1 := B^1_{u'}$ verify that $N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$ and

$$|\mu_{i_1j_1}|(B^1) < 1.$$

Assume that in the first $l$ steps of this induction we have obtained a finite sequence $j_1 < j_2 < \cdots < j_l$ in $\mathbb{N}$ and a decreasing finite sequence $N^{(1)} \supset N^{(2)} \supset \cdots \supset N^{(l)}$ of infinite subsets of $\mathbb{N}$ such that for each $w \in \mathbb{N}$, $1 \leqslant w \leqslant l$, $N^{(w)} \subset \{n \in \mathbb{N} : n > j_w\}$ and the variation of the measure $\mu_{i_wj_w}$ in the set $B^w := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N^{(w)},\ s \leqslant k_t\}$ verifies the inequality

$$|\mu_{i_wj_w}|(B^w) < 1.$$

Let $j_{l+1}$ be the first element in $N^{(l)}$ and suppose that $\left|\mu_{i_{l+1}j_{l+1}}\right|(\Omega) < s_{l+1}$, with $s_{l+1} \in \mathbb{N}$. Then $j_l < j_{l+1}$ and if $\{N^{l+1}_r, 1 \leqslant r \leqslant s_{l+1}\}$ is a partition of $\{m \in \mathbb{N}^{(l)} : m > j_{l+1}\}$ in $s_{l+1}$ infinite disjoint subfamilies then the subsets $B^{l+1}_r := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N^{l+1}_r, s \leqslant k_t\}$, $1 \leqslant r \leqslant s_{l+1}$, verify that $\Sigma\{\left|\mu_{i_{l+1}j_{l+1}}\right|(B^{l+1}_r) : 1 \leqslant r \leqslant s_{l+1}\} < s_{l+1}$, whence it follows that there exists $r'$, with $1 \leqslant r' \leqslant s_{l+1}$, such that the set $B^{l+1} := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N^{l+1}_{r'},\ s \leqslant k_t\}$ verifies that

$$\left|\mu_{i_{l+1}j_{l+1}}\right|(B^{l+1}) < 1.$$

Set $N^{(l+1)} := N^{l+1}_{r'}$. Then, by induction, we get a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$ and a decreasing sequence $(N^{(n)})_n$ of infinite subsets of $\mathbb{N}$, with $j_2 \in N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$ and $j_{n+1} \in N^{(n)} \subset \{m \in N^{(n-1)} : m > j_n\}$, for each $n > 1$, such that the measurable sets $B^n := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N^{(n)},\ s \leqslant k_t\}$, $n \in \mathbb{N}$, verify that

$$|\mu_{i_nj_n}|(B^n) < 1. \tag{13}$$

The inclusion $j_s \in N^{(s-1)} \subset N^{(n)}$ when $n < s$ and the trivial inequalities $i_s \leqslant s \leqslant k_s \leqslant k_{j_s}$ imply that $\cup\{B_{i_sj_s} : s \in \mathbb{N},\ n < s\} \subset B^n$, hence from (13) it follows that

$$|\mu_{i_nj_n}|(\cup_s\{B_{i_sj_s} : n < s\}) < 1,$$

for each $n \in \mathbb{N}$, and this inequality imply (7) because the variation $|\mu|(B)$ of $\mu$ in a set $B \in \mathcal{S}$ verifies that $|\mu(B)| \leqslant |\mu|(B)$.

From the proved relation $i_s \leqslant k_{j_s}$ and the trivial fact that $s \leqslant n$ implies that $j_s \leqslant j_n < j_{n+1}$ it follows that (10) implies (5). The inequality (6) is a particular case of (11). Finally from (12) with $i = r$ we get (8) because each $(i_{n_p}, j_{n_p})$ verifies that $r = i_{n_p} \leqslant n_p \leqslant j_{n_p}$.

To finish the proposition define $\mathcal{C}_r := \mathcal{C}_{t^r}$, for each $r \in \mathbb{N}$. □

We are at the position to present the proof of Theorem 2. Recall again that $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$.

***Proof of Theorem 2.*** Assume Theorem 2 fails. Then by Proposition 12 there exist a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$, a sequence $(B_{i_nj_n})_n$ of pairwise disjoints elements of the $\sigma$-algebra $\mathcal{S}$, a sequence $(\mu_{i_nj_n})_n$ in $\mathrm{ba}(\mathcal{S})$ and a covering $(\mathcal{C}_r)_r$ of $\mathcal{S}$ such that for each $n \in \mathbb{N}$

$$\Sigma_s\{|\mu_{i_nj_n}(B_{i_sj_s})| : s < n\}) < 1, \tag{14}$$

$$|\mu_{i_nj_n}(B_{i_nj_n})| > j_n, \tag{15}$$

$$|\mu_{i_nj_n}(\cup_s\{B_{i_sj_s} : n < s\})| < 1, \tag{16}$$

and for each strictly increasing sequence $(n_p)_p$ such that $i_{n_p} = r$ for each $p \in \mathbb{N}$ we have that the sequence $(\mu_{i_{n_p}j_{n_p}})_p = (\mu_{rj_{n_p}})_p$ is pointwise bounded in $\mathcal{C}_r$, i.e., for each $H \in \mathcal{C}_r$ we have that

$$\sup\{\left|\mu_{i_{n_p}j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty. \tag{17}$$

As $H_0 := \cup\{B_{i_s j_s} : s = 1, 2, \ldots\} \in \mathcal{S}$ and $(\mathcal{C}_r)_r$ is a covering of the $\sigma$-algebra $\mathcal{S}$ there exists $r' \in \mathbb{N}$ such that $H_0 \in \mathcal{C}_{r'}$. Fix a strictly increasing sequence $(n_q)_q$ in $\mathbb{N}\backslash\{1\}$ such that $i_{n_q} = r'$, for each $q \in \mathbb{N}$. Then, by (17),

$$\sup\{\left|\mu_{i_{n_q} j_{n_q}}(H_0)\right| : q \in \mathbb{N}\} < \infty. \tag{18}$$

The sets $C_q := \cup_s\{B_{i_s j_s} : s < n_q\}$, $B_{i_{n_q} j_{n_q}}$ and $D_q := \cup_s\{B_{i_s j_s} : n_q < s\}$ are a partition of the set $H_0$. By (14), (15) and (16), $\left|\mu_{i_{n_q} j_{n_q}}(C)\right| < 1$, $\mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) > j_{n_q} > n_q$ and $\left|\mu_{i_{n_q} j_{n_q}}(D)\right| < 1$, for each $q \in \mathbb{N}\backslash\{1\}$. Therefore the inequality

$$\left|\mu_{i_{n_q} j_{n_q}}(H_0)\right| > -\left|\mu_{i_{n_q} j_{n_q}}(C)\right| + \mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) - \left|\mu_{i_{n_q} j_{n_q}}\right|(D) > n_q - 2,$$

implies that

$$\lim_p \left|\mu_{i_{n_p} j_{n_p}}(H_0)\right| = \infty,$$

contradicting (18). □

The following corollary extends Theorems 2 and 3 in [14]. Again following [7, 7 Chapter 7, 35.1] a family $\{B_{m_1 m_2 \ldots m_i} : i, m_j \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant p\}$ of subsets of $A$ is an *increasing p-web in A* if $(B_{m_1})_{m_1}$ is an increasing covering of $A$ and $(B_{m_1 m_2 \ldots m_{i+1}})_{m_{i+1}}$ is an increasing covering of $B_{m_1 m_2 \ldots m_i}$, for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i < p$.

**Corollary 13.** *Let $\mathcal{S}$ be a $\sigma$-algebra of subsets of $\Omega$ and let $\{\mathcal{B}_{m_1 m_2 \ldots m_i} : i, m_j \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant p\}$ be an increasing p-web in $\mathcal{S}$. Then there exists $\mathcal{B}_{n_1, n_2, \ldots, n_p}$ such that if $(\mathcal{B}_{n_1, n_2, \ldots, n_p s_{p+1}})_{s_{p+1}}$ is an increasing covering of $\mathcal{B}_{n_1, n_2, \ldots, n_p}$ there exists $n_{p+1} \in \mathbb{N}$ such that each $\tau_s(\mathcal{B}_{n_1, n_2, \ldots, n_p n_{p+1}})$-Cauchy sequence $(\mu_n)_n$ in $ba(\mathcal{S})$ is $\tau_s(\mathcal{S})$-convergent.*

*Proof.* By Theorem 2 there exists $\mathcal{B}_{n_1 n_2 \ldots n_p}$ which has $sN$-property. Hence there exists $\mathcal{B}_{n_1, n_2, \ldots, n_p n_{p+1}}$ which has $N$-property. Then a $\tau_s(\mathcal{B}_{n_1, n_2, \ldots, n_p n_{p+1}})$-Cauchy sequence $(\mu_n)_n$ is $\tau_s(\mathcal{A})$-relatively compact. As $\overline{L(\mathcal{B}_{n_1, n_2, \ldots, n_p n_{p+1}})} = L(\mathcal{S})$ the sequence $(\mu_n)_n$ has no more than one $\tau_s(\mathcal{A})$-adherent point, whence $(\mu_n)_n$ is $\tau_s(\mathcal{A})$-convergent. □

## 4. Applications

We present some applications of Theorem 2 concerning localizations of bounded finitely additive vector measures.

A *finitely additive vector measure*, or simply a *vector measure*, $\mu$ defined in an algebra $\mathcal{A}$ of subsets of $\Omega$ with values in a topological vector space $E$ is a map $\mu\colon \mathcal{A} \to E$ such that $\mu(B \cup C) = \mu(B) + \mu(C)$, for each pairwise disjoint subsets $B$, $C \in \mathcal{A}$. The vector measure $\mu$ is *bounded* if $\mu(\mathcal{A})$ is a bounded subset of $E$, or, equivalently, if the $E$-valued linear map $\mu\colon L(\mathcal{A}) \to E$ defined by $\mu(\chi_B) := \mu(B)$, for each $B \in \mathcal{A}$, is continuous.

A locally convex space $E(\tau)$ is an $(LF)$- or $(LB)$-*space* if it is, respectively, the inductive limit of an increasing sequence $(E_m(\tau_m))_m$ of Fréchet or Banach spaces where the relative topology $\tau_{m+1}|_{E_m}$ induced on $E_m$ is coarser than $\tau_m$, for each $m \in \mathbb{N}$. $(E_m(\tau_m))_m$ is a *defining* sequence for $E(\tau)$ with *steps* $E_m(\tau_m)$, $m \in \mathbb{N}$, and we write $E(\tau) = \Sigma_m E_m(\tau_m)$. If $\tau_{m+1}|_{E_m} = \tau_m$, for each $m \in \mathbb{N}$, then $E(\tau)$ is a *strict* $(LF)$-, or $(LB)$-space. From [7, 19.4(4)] it follows that if $\mu\colon \mathcal{A} \to E(\tau)$ is a vector bounded measure with values in a strict $(LF)$-space $E(\tau) = \Sigma_m E_m(\tau_m)$ then there exists $n \in \mathbb{N}$ such that $\mu(\mathcal{A})$ is a bounded subset of the step $E_n(\tau_n)$. For $\sigma$-algebras the following extension of this result is contained in [14, Theorem 4].

**Theorem 14.** *Let $\mu$ be a bounded vector measure defined in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with values in an $(LF)$-space $E(\tau) = \Sigma_m E_m(\tau_m)$. Then there exists $n \in \mathbb{N}$ such that $\mu(\mathcal{S})$ is a bounded subset of $E_n(\tau_n)$.*

Theorem 2 provides the following proposition that contains Theorem 14 as a particular case.

**Proposition 15.** *Let $\mu$ be a bounded vector measure defined in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with values in a topological vector space $E(\tau)$. Suppose that $\{E_{m_1,m_2,\cdots,m_i} : m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ is an increasing p-web in E. Then there exists $E_{n_1,n_2,\cdots,n_p}$ such that if $E_{n_1,n_2,\cdots,n_p}(\tau_{n_1,n_2,\cdots,n_p})$ is an (LF)-space, the topology $\tau_{n_1,n_2,\cdots,n_p}$ is finer than the relative topology $\tau|_{E_{n_1,n_2,\cdots,n_p}}$ and if $(E_{n_1,n_2,\cdots,n_p,s_{p+1}}(\tau_{n_1,n_2,\cdots,n_p,s_{p+1}}))_{s_{p+1}}$ is a defining sequence for $E_{n_1,n_2,\cdots,n_p}(\tau_{n_1,n_2,\cdots,n_p})$ there exists $n_{p+1} \in \mathbb{N}$ such that $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2,\cdots,n_p,n_{p+1}}(\tau_{n_1,n_2,\cdots,n_p,n_{p+1}})$.*

*Proof.* Let $\mathcal{B}_{m_1,m_2,\ldots,m_i} := \mu^{-1}(E_{m_1,m_2,\ldots,m_i})$ for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i \leqslant p$. By Theorem 2 there exists $(n_1, n_2, \ldots, n_p) \in \mathbb{N}^p$ such that $\mathcal{B}_{n_1,n_2,\ldots,n_p}$ has $sN$-property. Let $(E_{n_1,n_2,\ldots,n_p,s_{p+1}}(\tau_{n_1 n_2 \ldots n_p s_{p+1}}))_{s_{p+1}}$ be a defining sequence for $E_{n_1,n_2,\ldots,n_p}(\tau_{n_1,n_2,\ldots,n_p})$ and let $\mathcal{B}_{n_1,n_2,\ldots,n_p,s_{p+1}} := \mu^{-1}(E_{n_1,n_2,\ldots,n_p,s_{p+1}})$.

As $(\mathcal{B}_{n_1,n_2,\ldots,n_p,s_{p+1}})_{s_{p+1}}$ is an increasing covering of $\mathcal{B}_{n_1,n_2,\ldots,n_p}$ there exists $n_{p+1}$ such that $\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}}$ has $N$-property, whence $L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}})$ is a dense subspace of $L(\dot{\mathcal{S}})$ and then the map with closed graph

$$\mu|_{L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}})} : L(\mathcal{B}_{n_1,n_2,\ldots,n_p,n_{p+1}}) \to E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau_{n_1,n_2,\ldots n_p,n_{p+1}})$$

has a continuous extension $\upsilon : L(\mathcal{S}) \to E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau_{n_1,n_2,\ldots,n_p,n_{p+1}})$ (by [12, 2.4 Definition and ($N_2$)] and [13, Theorems 1 and 14]). The continuity of $\mu : L(\mathcal{S}) \to E(\tau)$ implies that $\upsilon(A) = \mu(A)$, for each $A \in \mathcal{S}$. Whence $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2,\ldots,n_p,n_{p+1}}(\tau_{n_1,n_2,\ldots,n_p,n_{p+1}})$. □

Proposition 15 also holds if we replace $(LF)$-space by an inductive limit of $\Gamma_r$-spaces (see [13, Definition 1] and, taking into account [12, Property ($N_2$) after 2.4 Definition], apply again [13, Theorems 1 and 14]). A particular case of this proposition is the next corollary, which it is also a concrete generalization of Theorem 14.

**Corollary 16.** *Let $\mu$ be a bounded vector measure defined in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with values in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of (LF)-spaces. There exists $n_1 \in \mathbb{N}$ such that for each defining sequence $(E_{n_1,m_2}(\tau_{n_1,m_2}))_{m_2}$ of $E_{n_1}(\tau_{n_1})$ there exists $n_2 \in \mathbb{N}$ which verifies that $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$.*

A sequence $(x_k)_k$ in a locally convex space $E$ is *subseries convergent* if for every infinite subset $J$ of $\mathbb{N}$ the series $\Sigma\{x_k : k \in J\}$ converges and $(x_k)_k$ *is bounded multiplier* if for every bounded sequence of scalars $(\lambda_k)_k$ the series $\Sigma_k \lambda_k x_k$ converges.

A Fréchet space $E$ is Fréchet Montel if each bounded subset of $E$ is relatively compact. Important classes of Montel and Fréchet Montel spaces are considered and studied while Schwartz Theory of Distributions is described, for instance, in [6, Chapter 3, Examples 3, 4, 5 and 6.].

The following corollary is a generalization of [14, Corollary 1.4] and it follows partially from Corollary 16.

**Corollary 17.** *Let $(x_k)_k$ be a subseries convergent sequence in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of (LF)-spaces. Then there exists $n_1 \in \mathbb{N}$ such that for each defining sequence $(E_{n_1,m_2}(\tau_{n_1,m_2}))_{m_2}$ for $E_{n_1}(\tau_{n_1})$ there exists $n_2 \in \mathbb{N}$ such that $\{x_k : k \in \mathbb{N}\}$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$. If, additionally, $E_{n_1,n_2}(\tau_{n_1,n_2})$ is a Fréchet Montel space then the sequence $(x_k)_k$ is bounded multiplier in $E_{n_1,n_2}(\tau_{n_1,n_2})$.*

*Proof.* As the sequence $(x_k)_k$ is subseries convergent then the additive vector measure $\mu : 2^{\mathbb{N}} \to E(\tau)$ defined by $\mu(J) := \Sigma_{k \in J} x_k$, for each $J \in 2^{\mathbb{N}}$, is bounded, because as $(f(x_k))_k$ is subseries convergent for each $f \in E'$ we get that $\Sigma_{k=1}^{\infty} |f(x_k)| < \infty$.

By Corollary 16 there exists $n_1 \in \mathbb{N}$ such that for each defining sequence $(E_{n_1,m_2}(\tau_{n_1,m_2}))_{m_2}$ for $E_{n_1}(\tau_{n_1})$ there exists $n_2 \in \mathbb{N}$ with the property that $\mu(2^{\mathbb{N}}) = \{\Sigma_{k \in J} x_k : J \in 2^{\mathbb{N}}\}$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$. Then $\Sigma_k |\lambda_k f(x_k)| < \infty$ for each continuous linear form $f$ defined on $E_{n_1,n_2}(\tau_{n_1,n_2})$ and each bounded sequence $(\lambda_k)_k$ of scalars, whence $(\Sigma_{j=1}^{k} \lambda_j x_j)_k$ is a bounded sequence in $E_{n_1,n_2}(\tau_{n_1,n_2})$ which has at most one adherent point, because $\Sigma_k \lambda_k f(x_k)$ converges for each $f \in (E_{n_1,n_2}(\tau_{n_1,n_2}))'$. If $E_{n_1,n_2}(\tau_{n_1,n_2})$ is a Montel space then the bounded subset $\{\Sigma_{j=1}^{k} \lambda_j x_j : k \in \mathbb{N}\}$ is relatively compact and then the series $\Sigma_k \lambda_k x_k$ converges in $E_{n_1,n_2}(\tau_{n_1,n_2})$. □

Recall that a vector measure $\mu$ defined in an algebra $\mathcal{A}$ of subsets of $\Omega$ with values in a Banach space $E$ is *strongly additive* whenever given a sequence $(B_n)_n$ of pairwise disjoint elements of $\mathcal{A}$ the series $\Sigma_n \mu(B_n)$ converges in norm [2, I.1. Definition 14]. Each strongly additive vector measure $\mu$ is bounded [2, I.1. Corollary 19].

**Corollary 18.** *Let $\mu$ be a bounded vector measure defined in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with values in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of $(LB)$-spaces such that each $E_m(\tau_m)$ admit a defining sequence $(E_{m,m_2}(\tau_{m,m_2}))_{m_2}$ of Banach spaces which does not contain a copy of $l^\infty$. If $H$ is a dense subset of $E'(\tau_s(E))$ such that $f\mu$ is countably additive for each $f \in H$, then there exists $(n_1, n_2) \in \mathbb{N}^2$ such that $\mu$ is a $E_{n_1,n_2}(\tau_{n_1,n_2})$-valued countably additive vector measure.*

*Proof.* By Corollary 16 there exists $(n_1, n_2) \in \mathbb{N}^2$ such that $\mu(\mathcal{S})$ is a bounded subset of $E_{n_1,n_2}(\tau_{n_1,n_2})$. As $E_{n_1,n_2}(\tau_{n_1,n_2})$ does not contain a copy of $l^\infty$ then, by ([2, I.4. Theorem 2]), the measure $\mu$ is strongly additive, hence if $(B_n : n \in \mathbb{N})$ is a sequence of pairwise disjoint subsets of $\mathcal{S}$ then $\Sigma_n \mu(B_n)$ converges to the vector $x$ in $E_{n_1,n_2}(\tau_{n_1,n_2})$. Therefore $f(x) = \Sigma_n f\mu(B_n)$ for each $f \in E'$ and, by countably additivity of $f\mu$ when $f \in H$, we have that $f(x) = \Sigma_n f\mu(B_n) = f\mu(\cup_n B_n)$ for each $f \in H$. By density $x = \mu(\cup_n B_n)$, whence $\Sigma_n \mu(B_n) = \mu(\cup_n B_n)$ in $E_{n_1,n_2}(\tau_{n_1,n_2})$. □

**Proposition 19.** *Let $\mu$ be a bounded* vector measure *defined in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with values in a topological vector space $E(\tau)$. Suppose that $\{E_{m_1,m_2,...,m_i} : m_j \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant p\}$ is an* increasing $p$-web *in $E$. There exists $E_{n_1,n_2,...,n_p}$ such that if $(E_{n_1,n_2,...,n_p,m_{p+1}})_{m_{p+1}}$ is an increasing covering of $E_{n_1,n_2,...,n_p}$ with the property that each relative topology $\tau|_{E_{n_1,n_2,...,n_p,m_{p+1}}}$, $m_{p+1} \in \mathbb{N}$, is sequentially complete then there exists $n_{p+1} \in \mathbb{N}^p$ such that $\mu(\mathcal{S}) \subset E_{n_1,n_2,\cdots,n_p,n_{p+1}}$.*

*Proof.* Let $\mathcal{B}_{m_1,m_2,...,m_i} := \mu^{-1}(E_{m_1,m_2,...,m_i})$ for each $m_j \in \mathbb{N},\ \ 1 \leqslant j \leqslant i \leqslant p+1$. By Theorem 2 there exists $(n_1, n_2, \ldots, n_p) \in \mathbb{N}^p$ such that $\mathcal{B}_{n_1,n_2,...,n_p}$ has $sN$-property, whence there exists $n_{p+1} \in \mathbb{N}^p$ such that $\mathcal{B}_{n_1,n_2,...,n_p,n_{p+1}}$ has $N$-property, therefore $E_{n_1,n_2,...,n_p n_{p+1}}(\tau|_{E_{n_1,n_2,...,n_p,n_{p+1}}})$ is a dense subspace of $E(\tau)$, hence density and sequential completeness imply that the continuous restriction of $\mu$ to $L(\mathcal{B}_{n_1,n_2,...,n_p,n_{p+1}})$ has a continuous extension $v$ to $L(\mathcal{S})$ with values in the space $E_{n_1,n_2,...,n_p,n_{p+1}}(\tau|_{E_{n_1,n_2,...,n_p,n_{p+1}}})$. As $\mu\colon L(\mathcal{S}) \to E(\tau)$ is continuous then $v = \mu$ and we get that $\mu(\mathcal{S}) \subset E_{n_1,n_2,...,n_p,n_{p+1}}$. □

**Corollary 20.** *Let $\mu$ be a bounded additive vector measure defined in a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ with values in an inductive limit $E(\tau) = \Sigma_{m_1} E_{m_1}(\tau_{m_1})$ of an increasing sequence $(E_{m_1}(\tau_{m_1}))_{m_1}$ of countable dimensional topological vector spaces. Then there exists $n_1$ such that span$\{\mu(\mathcal{S})\}$ is a finite dimensional subspace of $E_{n_1}(\tau_{n_1})$.*

*Proof.* For each $m_1 \in \mathbb{N}$ let $(E_{m_1,m_2})_{m_2}$ be an increasing covering of $E_{m_1}$ by finite dimensional vector subspaces. $\{E_{m_1,m_2} : m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant 2\}$ is an *increasing* 2-*web in* $E$. As the relative topology $\tau|_{E_{m_1,m_2}}$ induced on $E_{m_1,m_2}$ is complete then, by Proposition 19, there exists $(n_1, n_2) \in \mathbb{N}^2$ such that $\mu(\mathcal{S}) \subset E_{n_1,n_2}$. □


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