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Additional Information

LOG CANONICAL THRESHOLD AND DIAGONAL IDEALS

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Abstract. We characterize the ideals I of $\mathcal{O}_n$ of finite colength whose integral closure is equal to the integral closure of an ideal generated by pure monomials. This characterization, which is motivated by an inequality proven by Demailly and Pham [8], is given in terms of the log canonical threshold of I and the sequence of mixed multiplicities of I.

1. Introduction

Let $\mathcal{O}_n$ denote the ring of analytic function germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. Let I be an ideal of $\mathcal{O}_n$ and let $g_1, \dots, g_r$ be a generating system of I. The *log canonical threshold of* I, denoted by $\mathrm{lct}(I)$, is defined as the supremum of those $s \in \mathbb{R}_{>0}$ such that the function $(|g_1|^2+\cdots+|g_r|^2)^{-s}$ is locally integrable around 0. This number, which does not depend on the chosen generating system of I, is always rational and has a deep relation with other invariants (see for instance [1], [6] or [9]). Moreover, the log canonical threshold can be characterized in several ways and is an object of interest in algebraic geometry, commutative algebra an complex analytic geometry. We refer to [17], [21] and [30] for properties and fundamental results about this number. The *Arnold multiplicity of* I, denoted by $\mu(I)$, is defined as $\mu(I) = \frac{1}{\mathrm{lct}(I)}$.

If no confusion arises, we denote by $\mathbf{m}$ the maximal ideal of $\mathcal{O}_n$. If $i \in \{0, \dots, n\}$, then $e_i(I)$ will denote the mixed multiplicity $e(I, \dots, I, \mathbf{m}, \dots, \mathbf{m})$, where I is repeated i times and $\mathbf{m}$ is repeated $n-i$ times (we refer to [16, §17], [25] and [31] for the definition and basic properties of mixed multiplicities). We recall that $e_1(I) = \mathrm{ord}(I)$, where $\mathrm{ord}(I) = \max\{r \geqslant 1 : I \subseteq \mathbf{m}^r\}$, and $e_n(I) = e(I)$, where $e(I)$ denotes the Samuel multiplicity of I.

If u is the plurisubharmonic function given by $u = \max_j \log|g_j|$, then $e_i(I) = L_i(u)$, where $L_i(u)$ denotes the Lelong number of the current $(dd^c u)^i$ at 0, for $i = 1, \dots, n$ (see for instance the proof of [24, Corollary 4.2] or [6]). Therefore, by the main result of Demailly and Pham in [8], if I denotes any proper ideal of $\mathcal{O}_n$ of finite colength, then

$$\frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} \leqslant \mathrm{lct}(I). \tag{1}$$

Let us denote by $\mathrm{DP}(I)$ the sum that appears in the left hand side of (1).

In Section 2 we show two results relating the mixed multiplicities of I with the initial ideals of the powers of I with respect to a specific local monomial ordering (the negative lexicographical order). We apply these results to fill the gap existing in [8, §3.3], where the proof of inequality (1) is reduced to the monomial case (see Remark 7). This article is motivated by the question of characterizing when equality $\mathrm{DP}(I) = \mathrm{lct}(I)$ holds.

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We recall that, for any ideal $I \subseteq \mathcal{O}_n$, the following chain of inequalities holds

$$\frac{1}{e_1(I)} \geqslant \frac{e_1(I)}{e_2(I)} \geqslant \cdots \geqslant \frac{e_{n-1}(I)}{e_n(I)}, \tag{2}$$

as can be seen, for instance, in [16, Theorem 17.7.2], [26] or [28, p. 41]. As a consequence of the inequality relating the arithmetical and the geometrical means of n positive real numbers, we immediately obtain that, if I is an ideal of $\mathcal{O}_n$ of finite colength, then

$$\frac{n}{e(I)^{1/n}} = n\left(\frac{1}{e_1(I)}\frac{e_1(I)}{e_2(I)}\cdots\frac{e_{n-1}(I)}{e_n(I)}\right)^{1/n} \leqslant \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} \leqslant \operatorname{lct}(I). \tag{3}$$

Then we have that $n^n\mu(I)^n \leqslant e(I)$ and equality holds if and only if $\frac{e_1(I)}{e_2(I)} = \cdots = \frac{e_{n-1}(I)}{e_n(I)}$. It is immediate to see that this last condition is equivalent to saying that $e_i(I) = e_1(I)^i$, for all $i = 1, \dots, n$, which in turn is equivalent to the condition $e(I) = e_1(I)^n = \operatorname{ord}(I)^n$, by (2). We have that $I \subseteq \mathbf{m}^{\operatorname{ord}(I)}$. Then the condition $e(I) = \operatorname{ord}(I)^n$ is equivalent to saying that $\overline{I} = \mathbf{m}^{\operatorname{ord}(I)}$, by the Rees' Multiplicity Theorem (see for instance [14, p. 147] or [16, p. 222]). Therefore it follows that $n^n\mu(I)^n = e(I)$ if and only if $\overline{I} = \mathbf{m}^{\operatorname{ord}(I)}$. This last equivalence was proven previously in [9, Theorem 1.4] by using another procedure.

Inspired by this result, we approach the problem of characterizing the equality $\mathrm{DP}(I) = \operatorname{lct}(I)$ by means of an expression for the integral closure of I. For this purpose, we introduce a class of ideals that we call *diagonal ideals* (see Definition 8). We characterize this class in Theorem 13. This theorem is supported by Corollary 11, where we show a result analogous to Rees' Multiplicity Theorem using $\mathrm{DP}(I)$ instead of $e(I)$. As we will see (Example 15), diagonal ideals are strictly contained in the class of ideals $I \subseteq \mathcal{O}_n$ of finite colength for which the equality $\mathrm{DP}(I) = \operatorname{lct}(I)$ holds.

2. Local monomial orderings and mixed multiplicities

Let us fix a coordinate system $x_1, \dots, x_n \in \mathbb{C}^n$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$, then we denote the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ by x^α. Let $\mathrm{Mon}_n = \{x^\alpha : \alpha \in \mathbb{Z}^n_{\geqslant 0}\}$. Here we recall some definitions taken from [13, Section 1.2] (see also [3, Chapter 4, §3]). A *monomial ordering* in Mon_n is a total ordering $>$ on the set Mon_n such that $x^\alpha > x^\beta$ implies $x^\gamma x^\alpha > x^\gamma x^\beta$, for all $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geqslant 0}$.

Let $>$ be a monomial ordering in Mon_n. We say that $>$ is *local* when $1 > x^\alpha$, for all $\alpha \in \mathbb{Z}^n_{\geqslant 0}$, $\alpha \neq 0$. In the sequel we will consider the local monomial ordering $>$ given by $x^\alpha > x^\beta$ if and only if there exists some $i \in \{1, \dots, n\}$ such that $(\alpha_1, \dots, \alpha_{i-1}) = (\beta_1, \dots, \beta_{i-1})$ and $\alpha_i < \beta_i$, where $\alpha, \beta \in \mathbb{Z}^n_{\geqslant 0}$. In particular $x_n > x_{n-1} > \cdots > x_1$. This monomial ordering is known as the *negative lexicographical order* (see [13, p. 14]).

If $f \in \mathcal{O}_n$, $f \neq 0$, let $f = \sum_k a_k x^k$ be the Taylor expansion of f around the origin. Then we define the *support* of f, denoted by $\mathrm{supp}(f)$, as the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $a_k \neq 0$. Therefore we denote by $\mathrm{in}(f)$ the maximum of the monomials x^k, $k \in \mathrm{supp}(f)$, with respect to the order $>$. Let us remark that, by the definition of the negative lexicographical order, $\mathrm{in}(f)$ exists. We will refer to $\mathrm{in}(f)$ as the *initial monomial of* f (in [3] this monomial is called the *leading monomial of* f and is denoted by $\textsc{lm}(f)$).

If I is an ideal of $\mathcal{O}_n$, then we define the *initial ideal of* I, which we will denote by $\mathrm{in}(I)$, as the ideal of $\mathcal{O}_n$ generated by all monomials $\mathrm{in}(f)$ such that $f \in I$. If I has finite colength, then $\mathrm{in}(I)$ has also finite colength and $\mathrm{in}(I)$ satisfies the following fundamental relation:

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\mathrm{in}(I)}. \tag{4}$$

The above result follows from [3, Theorem 4.3, p. 177] (see also [13, Corollary 7.5.6]). However, the ideals I and $\mathrm{in}(I)$ do not have the same multiplicity in general, as we see in the following easy example.

Example 1. Let us consider the ideal $I = \langle x + y^2, y^3 \rangle \subseteq \mathcal{O}_2$. Using Singular [5] we have that $\mathrm{in}(I) = \langle y^2, xy, x^2 \rangle$ and therefore $e(I) = 3$ and $e(\mathrm{in}(I)) = 4$. We also observe that $e_1(I) = 1$ and $e_1(\mathrm{in}(I)) = 2$.

Proposition 2. *Let I be a proper ideal of $\mathcal{O}_n$ of finite colength. Then $e_j(I) \leqslant e_j(\mathrm{in}(I))$, for all $j = 1, \ldots, n$, and* $\mathrm{lct}(\mathrm{in}(I)) \leqslant \mathrm{lct}(I)$.

Proof. Let us consider the coordinates $(x_1, \ldots, x_n, t)$ in $\mathbb{C}^{n+1}$. Since we suppose that I has finite colength, then I admits a generating system formed by polynomials. In particular, by [13, Corollary 7.4.6] and [13, Corollary 7.5.2], there exists an ideal $J \subseteq \mathcal{O}_{n+1}$ generated by homogeneous polynomials verifying the following properties:

(1) $J_0 = \mathrm{in}(I)$ and $J_1 = I$, where we denote by J_t the ideal of $\mathcal{O}_n$ obtained by fixing the variable t in each element of J;
(2) $\mathcal{O}_{n+1}/J$ is a flat $\mathbb{C}[t]$-algebra;
(3) the rings $\mathcal{O}_n/J_t$ and $\mathcal{O}_n/I$ are isomorphic, for all $t \in \mathbb{C} \setminus \{0\}$.

By the lower semicontinuity of the log canonical threshold (see [7] or [17, Corollary 9.5.39]), we have $\mathrm{lct}(J_0) \leqslant \mathrm{lct}(J_t) = \mathrm{lct}(I)$, for all t small enough, $t \neq 0$, where the equality $\mathrm{lct}(J_t) = \mathrm{lct}(I)$ follows by the existence of a ring isomorphism $\mathcal{O}_n/J_t \simeq \mathcal{O}_n/I$, for all $t \in \mathbb{C} \setminus \{0\}$.

Let us fix an integer $j \in \{1, \ldots, n\}$. We recall that $e_j(J_0) = e(J_0, \ldots, J_0, \mathbf{m}, \ldots, \mathbf{m})$, where J_0 is repeated j times and $\mathbf{m}$ is repeated $n - j$ times. Hence, by [16, Theorem 17.4.9] (see also [29, Corollaire 2.2]), the mixed multiplicity $e_j(I)$ is expressed as

$$e_j(J_0) = e\left(J_0 \frac{\mathcal{O}_n}{\langle h_{j+1}, \ldots, h_n \rangle}\right), \tag{5}$$

for general linear forms $h_{j+1}, \ldots, h_n$ in $\mathbb{C}[x_1, \ldots, x_n]$ (this set of linear forms is empty when $j = n$).

Then, let us fix linear forms $h_{j+1}, \ldots, h_n \in \mathbb{C}[x_1, \ldots, x_n]$ such that relation (5) holds. By the upper semicontinuity of Samuel multiplicity (see [11, p. 547] or [18, p. 126]) we have

$$e_j(J_0) \geqslant e\left(J_t \frac{\mathcal{O}_n}{\langle h_{j+1}, \ldots, h_n \rangle}\right) \geqslant e_j(J_t) \tag{6}$$

where the second inequality follows from [16, Theorem 17.4.9].

The existence of a ring isomorphism $\mathcal{O}_n/J_t \simeq \mathcal{O}_n/I$, for all $t \in \mathbb{C} \smallsetminus \{0\}$, implies that there exists a biholomorphism $\varphi_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $\varphi_t^*(I) = J_t$ (see [10, p. 16] o [12, p. 57]). In particular, we obtain that $e_j(I) = e_j(J_t)$, for all $t \neq 0$. Then, since $J_0 = \mathrm{in}(I)$, we have that $e_j(\mathrm{in}(I)) \geqslant e_j(I)$, for all $j = 1, \dots, n$, by virtue of (6). □

Let $\mathtt{L} \subseteq \{1, \dots, n\}$, $\mathtt{L} \neq \emptyset$. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, then we define $\mathbb{K}^n_{\mathtt{L}} = \{x \in \mathbb{K}^n : x_i = 0$, for all $i \notin \mathtt{L}\}$. Let us denote by $\mathcal{O}_{n,\mathtt{L}}$ the subring of $\mathcal{O}_n$ formed by all functions germs of $\mathcal{O}_n$ depending at most on the variables x_i with $i \in \mathtt{L}$. Let $f \in \mathcal{O}_n$ and let us suppose that the Taylor expansion of f around the origin is given by $f = \sum_k a_k x^k$. Then we denote by $f_{\mathtt{L}}$ the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(f) \cap \mathbb{R}^n_{\mathtt{L}}$. If J is an ideal of $\mathcal{O}_n$ then we denote by $J_{\mathtt{L}}$ the ideal of $\mathcal{O}_{n,\mathtt{L}}$ generated by all elements $f_{\mathtt{L}}$, where $f \in J$.

Lemma 3. *Let J be a proper ideal of $\mathcal{O}_n$ of finite colength and let $\mathtt{L} = \{j, \dots, n\}$, for some $j \in \{1, \dots, n\}$. Then*

$$\mathrm{in}(J_{\mathtt{L}}) = \mathrm{in}(J)_{\mathtt{L}}.$$

Proof. Let us suppose that $\mathtt{L}$ is any non-empty subset of $\{1, \dots, n\}$. Let us take a non-zero element of $\mathrm{in}(J)_{\mathtt{L}}$, that is, let $f \in J$, such that $\mathrm{in}(f)_{\mathtt{L}} \neq 0$. In particular, it follows that $f_{\mathtt{L}} \neq 0$ and $\mathrm{supp}(\mathrm{in}(f)) \subseteq \mathrm{supp}(f_{\mathtt{L}})$, which is equivalent to saying that $\mathrm{in}(f)_{\mathtt{L}} = \mathrm{in}(f) = \mathrm{in}(f_{\mathtt{L}})$. Then $\mathrm{in}(J_{\mathtt{L}}) \supseteq \mathrm{in}(J)_{\mathtt{L}}$.

Let us see the reverse inclusion by assuming that $\mathtt{L} = \{j, \dots, n\}$, for some $j \in \{1, \dots, n\}$. If $j = 1$, there is nothing to prove, so let us suppose that $j > 1$. Let $f \in J$ such that $f_{\mathtt{L}} \neq 0$. By the definition of $f_{\mathtt{L}}$, there exists some element $k \in \mathrm{supp}(f)$ such that $k_1 = \cdots = k_{j-1} = 0$. Hence $x^k > x^{k'}$, for all $k' \in \mathrm{supp}(f)$ such that $k'_i \neq 0$, for some $i \notin \mathtt{L}$, by the definition of the negative lexicographical order. This implies that $\mathrm{supp}(\mathrm{in}(f)) \subseteq \mathrm{supp}(f) \cap \mathbb{R}^n_{\mathtt{L}}$ and hence $\mathrm{in}(f) = \mathrm{in}(f_{\mathtt{L}})$, which means that $\mathrm{in}(f_{\mathtt{L}}) = \mathrm{in}(f)_{\mathtt{L}}$. Therefore $\mathrm{in}(J_{\mathtt{L}}) \subseteq \mathrm{in}(J)_{\mathtt{L}}$. □

If $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a linear change of coordinates and J is an ideal of $\mathcal{O}_n$, then we denote by $\varphi^*(J)$ the ideal of $\mathcal{O}_n$ generated by the elements $g \circ \varphi$, where $g \in J$.

Theorem 4. *Let I be a proper ideal of $\mathcal{O}_n$ of finite colength. Then, for all $j \in \{1, \dots, n\}$, we have*

$$e_j(I) = \lim_{t \to +\infty} \frac{e_j\left(\mathrm{in}(\varphi^*(I)^t)\right)}{t^j} \tag{7}$$

for a general linear change of coordinates $\varphi : \mathbb{C}^n \to \mathbb{C}^n$.

Proof. By [16, Theorem 17.4.9], there exist general linear forms $h_1, \dots, h_n \in \mathbb{C}[x_1, \dots, x_n]$ such that

$$e_j(I) = e\left(I \frac{\mathcal{O}_n}{\langle h_1, \dots, h_{n-j} \rangle}\right)$$

for all $j = 0, \dots, n-1$ (where we consider that this set of linear forms is empty when $j = n$) and $h = (h_1, \dots, h_n)$ is a linear isomorphism. Let $\varphi = h^{-1} : \mathbb{C}^n \to \mathbb{C}^n$. Let us denote by J

the ideal $\varphi^*(I)$. Then

$$e_j(I) = e\left(I\frac{\mathcal{O}_n}{\langle h_1,\dots,h_{n-j}\rangle}\right) = e\left(J\frac{\mathcal{O}_n}{\langle x_1,\dots,x_{n-j}\rangle}\right) = e(J_{\mathtt{L}}) \tag{8}$$

where $\mathtt{L}=\{n-j+1,\dots,n\}$ and $e(J_{\mathtt{L}})$ denotes the Samuel multiplicity of $J_{\mathtt{L}}$ in the ring $\mathcal{O}_{n,\mathtt{L}}$. We remark that $(J_{\mathtt{L}})^t=(J^t)_{\mathtt{L}}$, for all $t\in\mathbb{Z}_{\geqslant 1}$, so we denote this ideal simply by $J^t_{\mathtt{L}}$, for all $t\in\mathbb{Z}_{\geqslant 1}$. By [22, Corollary 1.13] (see also [4, Theorem 1.1]) we have that

$$e(J_{\mathtt{L}}) = \lim_{t\to+\infty}\frac{e\left(\mathrm{in}(J^t_{\mathtt{L}})\right)}{t^j}$$

where $\mathrm{in}(J^t_{\mathtt{L}})$ is the initial ideal of $J^t_{\mathtt{L}}$ with respect to the negative lexicographical ordering in the monomials of $\mathcal{O}_{n,\mathtt{L}}$, for all $t\in\mathbb{Z}_{\geqslant 1}$. By Lemma 3 we have $e(\mathrm{in}(J^t_{\mathtt{L}}))=e\left(\mathrm{in}(J^t)_{\mathtt{L}}\right)$. Moreover $e(\mathrm{in}(J^t)_{\mathtt{L}})\geqslant e_j(\mathrm{in}(J^t))\geqslant e_j(J^t)$, where the first inequality follows from [16, Theorem 17.4.9] and the second inequality is an application of Proposition 2. Putting this information together we obtain the following chain of inequalities:

$$e\left(\mathrm{in}(J^t_{\mathtt{L}})\right) = e\left(\mathrm{in}(J^t)_{\mathtt{L}}\right) \geqslant e_j\left(\mathrm{in}(J^t)\right) \geqslant e_j(J^t) = t^j e_j(J).$$

Then, dividing each term of the previous inequalities by t^j and taking limits, we arrive to

$$\begin{aligned} e_j(I) = e(J_{\mathtt{L}}) &= \lim_{t\to+\infty}\frac{e\left(\mathrm{in}(J^t_{\mathtt{L}})\right)}{t^j} = \lim_{t\to+\infty}\frac{e\left(\mathrm{in}(J^t)_{\mathtt{L}}\right)}{t^j} \\ &\geqslant \lim_{t\to+\infty}\frac{e_j\left(\mathrm{in}(J^t)\right)}{t^j} = \lim_{t\to+\infty}\frac{e_j\left(\mathrm{in}(\varphi^*(I)^t)\right)}{t^j} \\ &\geqslant \lim_{t\to+\infty}\frac{e_j(\varphi^*(I)^t)}{t^j} = e_j(\varphi^*(I)) = e_j(I). \end{aligned}$$

Then the result follows. □

Remark 5. By the argument of the proof of the previous result, if we fix a linear isomorphism $h=(h_1,\dots,h_n):\mathbb{C}^n\to\mathbb{C}^n$ such that $e_j(I)$ coincides with the multiplicity of I in the quotient ring $\mathcal{O}_n/\langle h_1,\dots,h_{n-j}\rangle$, for all $j=0,\dots,n-1$, then relation (7) holds, for all $j=1,\dots,n$, by taking $\varphi:\mathbb{C}^n\to\mathbb{C}^n$ as $\varphi=h^{-1}$.

Corollary 6. *Let I be a proper ideal of finite colength of $\mathcal{O}_n$. Then*

$$\mathrm{DP}(I) = \lim_{t\to+\infty} t\,\mathrm{DP}\left(\mathrm{in}(\varphi^*(I)^t)\right). \tag{9}$$

for a general linear change of coordinates $\varphi:\mathbb{C}^n\to\mathbb{C}^n$.

Proof. Let us fix a general change of coordinates $\varphi:\mathbb{C}^n\to\mathbb{C}^n$ and let us denote the ideal $\varphi^*(I)$ by J. Then, for any $t\in\mathbb{Z}_{\geqslant 1}$, we have

$$\begin{aligned} t\,\mathrm{DP}\left(\mathrm{in}(J^t)\right) &= t\frac{1}{e_1(\mathrm{in}(J^t))} + t\frac{e_1(\mathrm{in}(J^t))}{e_2(\mathrm{in}(J^t))} + \cdots + t\frac{e_{n-1}(\mathrm{in}(J^t))}{e_n(\mathrm{in}(J^t))} \\ &= \frac{1}{e_1(\mathrm{in}(J^t))/t} + \frac{e_1(\mathrm{in}(J^t))/t}{e_2(\mathrm{in}(J^t))/t^2} + \cdots + \frac{e_{n-1}(\mathrm{in}(J^t))/t^{n-1}}{e_n(\mathrm{in}(J^t))/t^n} \end{aligned}$$

By Theorem 4 and the definition of $\mathrm{DP}(I)$ we immediately obtain the desired result. □

Remark 7. Let $I \subseteq \mathcal{O}_n$ be any ideal of finite colength, $I \subseteq \mathbf{m}$. Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be a linear change of coordinates such that relation (9) holds for φ and I. Then, applying relation (1) to the monomial ideal $\mathrm{in}(\varphi^*(I)^t$, for all $t \in \mathbb{Z}_{\geqslant 1}$, we obtain that

$$\mathrm{DP}\left(\mathrm{in}(\varphi^*(I)^t)\right) \leqslant \mathrm{lct}\left(\mathrm{in}(\varphi^*(I)^t)\right)$$

for all $t \in \mathbb{Z}_{\geqslant 1}$. In particular

$$t\,\mathrm{DP}\left(\mathrm{in}(\varphi^*(I)^t)\right) \leqslant t\ \mathrm{lct}\left(\mathrm{in}(\varphi^*(I)^t)\right) \leqslant t\,\mathrm{lct}(\varphi^*(I)^t) = \mathrm{lct}(I) \tag{10}$$

for all $t \in \mathbb{Z}_{\geqslant 1}$, were the second inequality follows from Proposition 2. Therefore, taking limits when $t \to \infty$ in all parts of the previous inequalities, we obtain

$$\mathrm{DP}(I) \leqslant \mathrm{lct}(I) \tag{11}$$

as a consequence of Corollary 6.

To the best of our knowledge, the proof of (11) as a corollary of the analogous result for monomial ideals explained in [8, §3.3] relies on the equality $e_j(I) = e_j(\mathrm{in}(I))$, for all $j = 1, \dots, n$, where $\mathrm{in}(I)$ denotes the initial ideal of I with respect to any monomial order. However, as shown in Example 1, the ideals I and $\mathrm{in}(I)$ do not have the same set of mixed multiplicities in general.

Let us also point out that if $\mathrm{DP}(I) = \mathrm{lct}(I)$, then relation (10) and Corollary 6 show that

$$\mathrm{lct}(I) = \lim_{t \to +\infty} t\,\mathrm{lct}\left(\mathrm{in}(\varphi^*(I)^t)\right).$$

3. Mixed multiplicities and diagonal ideals

Let us fix along the remaining text a coordinate system $x_1, \dots, x_n$ in $\mathbb{C}^n$, unless otherwise stated. Let I be an ideal of $\mathcal{O}_n$. We denote the integral closure of I by $\overline{I}$ and the Newton polyhedron of I by $\Gamma_+(I)$. Let us recall that $\Gamma_+(I)$ is the smallest convex set of $\mathbb{R}^n_+$ containing the supports of the elements of I. Therefore $\Gamma_+(I)$ is equal to the convex hull of the set $\{k + v : k \in \mathrm{supp}(f),\ f \in I,\ v \in \mathbb{R}^n_{\geqslant 0}\}$. In general it holds that $\Gamma_+(I) = \Gamma_+(\overline{I})$ (see [2, p. 399]). If I admits a generating system formed by monomials, then we say that I is a *monomial ideal*.

We define the *term ideal* of I as the ideal generated by all the monomials x^k such that $k \in \Gamma_+(I)$. We will denote this ideal by I^0. If I is a monomial ideal, then $\overline{I}$ is also monomial and therefore $\overline{I} = I^0$ (see [16, p. 11] or [20]); however the converse is not true, as is shown by the ideal I of $\mathcal{O}_2$ given by $I = \langle x^2 + y^2, xy\rangle$. The ideals I for which $\overline{I}$ is generated by monomials are called *Newton non-degenerate ideals* (see [2] or [27]).

Definition 8. Let I be an ideal of $\mathcal{O}_n$. We say that I is *diagonal* when there exist positive integers $a_1, \dots, a_n$ such that $\overline{I} = \overline{\langle x_1^{a_1}, \dots, x_n^{a_n}\rangle}$.

Then any power of the maximal ideal of $\mathcal{O}_n$ is a diagonal ideal. Moreover, any diagonal ideal is Newton non-degenerate. As a consequence of the previous definition, if I is diagonal then I^0

is also, but the converse is not true, as is shown by the ideal of $\mathcal{O}_2$ given by $I = \langle x+y, x^2\rangle \subseteq \mathcal{O}_2$ (in this case I^0 is equal to the maximal ideal).

Let I be a proper ideal of $\mathcal{O}_n$ of finite colength. Then by virtue of (11) and the inclusion $I \subseteq I^0$ we have the inequalities

$$\mathrm{DP}(I) \leqslant \mathrm{lct}(I) \leqslant \mathrm{lct}(I^0). \tag{12}$$

We recall the following result of Howald [15] (see also [20]), where $\mathrm{lct}(I)$ is characterized in terms of a combinatorial characteristic of $\Gamma_+(I)$ if I is a monomial ideal.

Theorem 9. [15] *Let I be a monomial ideal of $\mathcal{O}_n$. Then*

$$\mathrm{lct}(I) = \frac{1}{\min\big\{\mu > 0 : \mu(1,\dots,1) \in \Gamma_+(I)\big\}}.$$

Proposition 10. *Let $n \geqslant 2$ and let $D \subseteq \mathbb{R}^n_{>0}$ be the set defined by*

$$D = \{(t_1,\dots,t_n) \in \mathbb{R}^n_{>0} : t_1^2 \leqslant t_2,\ t_j^2 \leqslant t_{j-1}t_{j+1}, \text{ for all } j = 2,\dots,n-1\}. \tag{13}$$

Let us consider the function $f : \mathbb{R}^n_{>0} \to \mathbb{R}$ given by

$$f(t_1,\dots,t_n) = \frac{1}{t_1} + \frac{t_1}{t_2} + \cdots \frac{t_{n-1}}{t_n},$$

for all $(t_1,\dots,t_n) \in \mathbb{R}^n_{>0}$. Let $a = (a_1,\dots,a_n), b = (b_1,\dots,b_n) \in D$ such that $a_i \leqslant b_i$, for all $i = 1,\dots,n$. Then $f(a) \geqslant f(b)$ and equality holds only if and only if $a = b$.

Proof. Let us see first that D is convex. For all $j = 1,\dots,n$, we define $D_j = \{(t_1,\dots,t_n) \in \mathbb{R}^n_{>0} : t_j^2 \leqslant t_{j-1}t_{j+1}\}$, where we set $t_0 = 1$, for all $t = (t_1,\dots,t_n) \in \mathbb{R}^n_{>0}$. Then it suffices to see that D_j is convex, for all $j = 1,\dots,n$, since $D = D_1 \cap \cdots \cap D_n$.

Let us fix an index $j \in \{1,\dots,n\}$. Let $s = (s_1,\dots,s_n)$ and $t = (t_1,\dots,t_n)$ be elements of D_j and let $\lambda \in [0,1]$. We define $\mathbf{u} = (\sqrt{\lambda s_{j-1}}, \sqrt{(1-\lambda)t_{j-1}})$ and $\mathbf{v} = (\sqrt{\lambda s_{j+1}}, \sqrt{(1-\lambda)t_{j+1}})$. Let us denote by $\mathbf{u}\cdot\mathbf{v}$ the usual scalar product of $\mathbf{u}$ and $\mathbf{v}$. By applying the definition of D_j and the Cauchy-Schwarz inequality we find that

$$(\lambda s_j + (1-\lambda)t_j)^2 \leqslant \big(\lambda\sqrt{s_{j-1}}\sqrt{s_{j+1}} + (1-\lambda)\sqrt{t_{j-1}}\sqrt{t_{j+1}}\big)^2 \tag{14}$$

$$= (\mathbf{u}\cdot\mathbf{v})^2 \leqslant \|\mathbf{u}\|^2\|\mathbf{v}\|^2 = (\lambda s_{j-1} + (1-\lambda)t_{j-1})(\lambda s_{j+1} + (1-\lambda)t_{j+1}). \tag{15}$$

Then $\lambda s + (1-\lambda)t \in D_j$, for all $\lambda \in [0,1]$, and hence D_j is convex. Therefore D is convex.

The proof of the inequality $f(a) \geqslant f(b)$ is contained in the proof of [8, Lemma 3.1], however we reproduce it for the sake of completeness and for its implications in the proof of the second part of the result.

Let us consider the function $g : [0,1] \to \mathbb{R}_{>0}$ defined by $g(\lambda) = f(a + \lambda(b-a))$, for all $\lambda \in [0,1]$. We observe that

$$\frac{\partial f}{\partial t_1}(t) = -\frac{1}{t_1^2} + \frac{1}{t_2}, \qquad \frac{\partial f}{\partial t_j}(t) = -\frac{t_{j-1}}{t_j^2} + \frac{1}{t_{j+1}}, \qquad \frac{\partial f}{\partial t_n}(t) = -\frac{t_{n-1}}{t_n^2} \tag{16}$$

for all $t \in \mathbb{R}^n_{>0}$ and all $j = 2, \dots, n-1$. In particular, we have $\frac{\partial f}{\partial t_j}(t) \leqslant 0$, for all $t \in D$ and all $j = 1, \dots, n$. Then

$$g'(\lambda) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial t_j}(a + \lambda(b-a)) \right) (b_j - a_j) \leqslant 0 \tag{17}$$

for all $\lambda \in]0,1[$. Hence g is a decreasing function, which implies that $f(a) \geqslant f(b)$.

Let us suppose that $f(a) = f(b)$, which means that $g(0) = g(1)$. Then there exists some $\lambda_0 \in]0,1[$ such that $g'(\lambda_0) = 0$, by the Mean Value Theorem. Let $c_0 = a + \lambda_0(b-a) \in D$. By (17) and the fact that $\frac{\partial f}{\partial t_j}(t) \leqslant 0$, for all $t \in D$ and all $j = 1, \dots, n$, we conclude that

$$\frac{\partial f}{\partial t_1}(c_0)(b_1 - a_1) = 0, \qquad \frac{\partial f}{\partial t_j}(c_0)(b_j - a_j) = 0, \qquad \frac{\partial f}{\partial t_n}(c_0)(b_n - a_n) = 0 \tag{18}$$

for all $j = 2, \dots, n-1$.

Let us suppose that $a_n \neq b_n$. Then (18) implies that

$$\frac{\partial f}{\partial t_n}(a + \lambda_0(b-a))(a_n - b_n) = -\frac{a_{n-1} + \lambda_0(b_{n-1} - a_{n-1})}{(a_n + \lambda_0(b_n - a_n))^2}(b_n - a_n) = 0.$$

Then $\lambda = -b_{n-1}/(a_{n-1} - b_{n-1})$, which contradicts the hypothesis that $\lambda \in]0,1[$. Therefore $a_n = b_n$.

If we assume that $a_{n-1} \neq b_{n-1}$, by (16) and (18), we conclude that

$$(\lambda_0 b_{n-1} + (1-\lambda_0)a_{n-1})^2 = (\lambda_0 b_{n-2} + (1-\lambda_0)a_{n-2})(\lambda_0 b_n + (1-\lambda_0)a_n). \tag{19}$$

We observe that, by inequality (14), this condition can not hold if $a_j^2 < a_{j-1}a_{j+1}$ or $b_j^2 < b_{j-1}b_{j+1}$. So (19) forces that

$$a_{n-1}^2 = a_{n-2}a_n \quad \text{and} \quad b_{n-1}^2 = b_{n-2}b_n. \tag{20}$$

By (14), (15) and the characterization of equality in the Cauchy-Schwarz inequality, condition (19) is equivalent to saying that

$$\frac{\sqrt{(1-\lambda_0)a_n}}{\sqrt{(1-\lambda_0)a_{n-2}}} = \frac{\sqrt{(1-\lambda_0)b_n}}{\sqrt{(1-\lambda_0)b_{n-2}}},$$

which in turn is equivalent to saying that $a_n/a_{n-2} = b_n/b_{n-2}$. Then, since $a_n = b_n$, we obtain that $a_{n-2} = b_{n-2}$. Hence $a_{n-1} = b_{n-1}$, by (20), and thus we arrive to a contradiction. The remaining equalities $a_j = b_j$, for all $j = 1, \dots, n-2$, follow analogously. Then the result is proven. $\square$

We recall that a Noetherian local ring is called *quasi-unmixed* when its completion in the topology defined by the maximal ideal is equidimensional. Quasi-unmixed rings are also called *formally equidimensional* (see [16] or [19]).

Corollary 11. *Let R be a Noetherian local ring. Let us suppose that R is quasi-unmixed. Let I_1, I_2 be two proper ideals of finite colength of R such that $I_1 \subseteq I_2$. Then*

$$\mathrm{DP}(I_1) \leqslant \mathrm{DP}(I_2) \tag{21}$$

and equality holds if and only if $\overline{I_1} = \overline{I_2}$.

Proof. If $\overline{I_1} = \overline{I_2}$, then $e_i(I_1) = e_i(I_2)$, for all $i = 1, \dots, n$ (see [16, §17.4] or [29, p. 306]) and hence $\mathrm{DP}(I_1) = \mathrm{DP}(I_2)$. Let D be the set defined in Proposition 10. Let us consider the vectors

$$a = (e_1(I_2), \dots, e_n(I_2)), \quad b = (e_1(I_1), \dots, e_n(I_1)). \tag{22}$$

Since $I_1 \subseteq I_2$, then $e_i(I_2) \leqslant e_i(I_1)$, for all $i = 1, \dots, n$. Moreover, the vectors a and b defined in (22) belong to D, by (2). Then we can apply Proposition 10 to deduce that $\mathrm{DP}(I_1) \leqslant \mathrm{DP}(I_2)$ and equality holds if and only if $e_i(I_1) = e_i(I_2)$, for all $i = 1, \dots, n$. In particular $\mathrm{DP}(I_1) = \mathrm{DP}(I_2)$ implies $e(I_1) = e(I_2)$. The equality $e(I_1) = e(I_2)$ together with the inclusion $I_1 \subseteq I_2$ implies that $\overline{I_1} = \overline{I_2}$ by the Rees' Multiplicity Theorem [16, p. 222]. □

If $f \in \mathcal{O}_n$, then we denote by $J(f)$ the ideal of $\mathcal{O}_n$ generated by $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$. Let us suppose that f has an isolated singularity at the origin, that is, the ideal $J(f)$ has finite colength in $\mathcal{O}_n$. Let $\mu^*(f)$ denote the vector $(\mu^{(1)}(f), \dots, \mu^{(n)}(f))$, where $\mu^{(i)}(f_t)$ denotes the Milnor number of the restriction of f_t to a general plane of dimension i in $\mathbb{C}^n$ passing through the origin, for all $i = 1, \dots, n$ (see [29, § 1]).

We say that a given property (P_t) holds for all $|t| \ll 1$ if there exists an open ball U centered at 0 in $\mathbb{C}$ such that the property (P_t) holds whenever $t \in U$.

Corollary 12. *Let* $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *be an analytic deformation such that* f_t *has an isolated singularity at the origin, for all* $|t| \ll 1$. *Then*

(1) $\mathrm{DP}(J(f_t))$ *is lower semicontinuous, that is,* $\mathrm{DP}(J(f_0)) \leqslant \mathrm{DP}(J(f_t))$, *for all* $|t| \ll 1$
(2) $\mathrm{DP}(J(f_t))$ *is constant, for* $|t| \ll 1$, *if and only if* $\mu^*(f_t)$ *is constant, for* $|t| \ll 1$.

Proof. By the results of Teissier in [29, §1], it is well known that $\mu^{(i)}(f_t) = e_i(J(f_t))$, where $\mu^{(i)}(f_t)$ denotes the Milnor number of the restriction of f_t to a general plane of dimension i in $\mathbb{C}^n$ passing through the origin. Since Milnor numbers are upper semicontinuous (see [12, Theorem 2.6]), we conclude that $e_i(J(f_t)) \leqslant e_i(J(f_0))$, for all $i = 1, \dots, n$. Then both items of the result follow as an immediate consequence of Proposition 10. □

Theorem 13. *Let* I *be a proper ideal of* $\mathcal{O}_n$ *of finite colength. Then the following conditions are equivalent:*

(a) I *is diagonal.*
(b) $\mathrm{lct}(I^0) = \mathrm{DP}(I)$.
(c) $\mathrm{lct}(I) = \mathrm{DP}(I)$ *and* $\mathrm{lct}(I) = \mathrm{lct}(I^0)$.

Proof. Let us see the implication (a) ⇒ (b). Let $a_1, \dots, a_n \in \mathbb{Z}_{\geqslant 1}$ such that $\overline{I} = \overline{\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle}$. In particular $\overline{I} = I^0$. We can assume that $a_1 \leqslant \dots \leqslant a_n$ (by permuting the variables, if necessary). Therefore $e_i(I) = e_i(\overline{I}) = a_1 \cdots a_i$, for all $i = 1, \dots, n$, which implies that $a_i = e_i(I)/e_{i-1}(I)$, for all $i = 1, \dots, n$. Then, by Theorem 9, we obtain that

$$\mathrm{lct}(I^0) = \mathrm{lct}(\overline{I}) = \mathrm{lct}\left(\overline{\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle}\right) = \frac{1}{a_1} + \dots + \frac{1}{a_n} = \mathrm{DP}(I).$$

Let us prove (b) $\Rightarrow$ (a). Let us suppose first that I is an ideal generated by monomials such that $\mathrm{lct}(I^0) = \mathrm{DP}(I)$. Hence $\mathrm{lct}(I) = \frac{1}{\mu_0}$, where $\mu_0 = \min\{\mu > 0 : \mu\mathbf{e} \in \Gamma_+(I)\}$ and $\mathbf{e} = (1, \dots, 1)$, by Theorem 9. Let π denote a supporting hyperplane of $\Gamma_+(I)$ containing the point $\mu_0\mathbf{e}$ and defined by the zeros of a linear form with rational coefficients. Let us write the equation of π as

$$\frac{x_1}{c_1} + \cdots + \frac{x_n}{c_n} = 1$$

where $c_1, \dots, c_n \in \mathbb{Q}_{>0}$. If necessary, we can reorder the variables to obtain $c_1 \leqslant \dots \leqslant c_n$. Let r be a positive integer such that $rc_1, \dots, rc_n \in \mathbb{Z}_{\geqslant 1}$ and let us denote by H the ideal of $\mathcal{O}_n$ generated by $x_1^{rc_1}, \dots, x_n^{rc_n}$. Since π is a supporting hyperplane of $\Gamma_+(I)$ passing through the point $\mu_0\mathbf{e}$, we have $I^r \subseteq \overline{H}$ and $\mathrm{lct}(I^r) = \mathrm{lct}(H)$. Moreover $e_i(H) = r^i c_1 \cdots c_i$, for all $i = 1, \dots, n$, since $c_1 \leqslant \cdots \leqslant c_n$. Therefore

$$\mathrm{lct}(I^r) = \mathrm{lct}(H) = \frac{1}{rc_1} + \frac{1}{rc_2} + \cdots + \frac{1}{rc_n} = \frac{1}{e_1(H)} + \frac{e_1(H)}{e_2(H)} + \cdots + \frac{e_{n-1}(H)}{e_n(H)} = \mathrm{DP}(H). \tag{23}$$

Since $\overline{I} = I^0$, we have that $\mathrm{lct}(I) = \mathrm{DP}(I)$, by hypothesis. Thus

$$\mathrm{lct}(I^r) = \frac{1}{r}\,\mathrm{lct}(I) = \frac{1}{r}\,\mathrm{DP}(I) = \mathrm{DP}(I^r), \tag{24}$$

where the last equality follows from the relation $e_i(I^r) = r^i e_i(I)$, for all $i = 1, \dots, n$ (see [16, Proposition 17.5.1]). Then (23) and (24) show that $\mathrm{DP}(I^r) = \mathrm{DP}(H)$ and, by Corollary 11, we obtain that $\overline{I^r} = \overline{H}$. Thus $r\Gamma_+(I) = \Gamma_+(I^r) = \Gamma_+(H)$, which implies that $\Gamma_+(I)$ has a unique compact face Δ of dimension $n-1$. Since the vertexes of $\Gamma_+(I)$ are contained in $\mathbb{Z}^n_{\geqslant 1}$, we conclude that we can take $r = 1$ and that, in this case, the hyperplane π contains Δ. Consequently $c_i \in \mathbb{Z}_{\geqslant 1}$ and $c_i = e_i(I)/e_{i-1}(I)$, for all $i = 1, \dots, n$. Hence $\overline{I} = \overline{\langle x_1^{c_1}, \dots, x_n^{c_n}\rangle}$, which means that I is diagonal.

Let I be an arbitrary ideal of $\mathcal{O}_n$ of finite colength such that $\mathrm{lct}(I^0) = \mathrm{DP}(I)$. Then, by a direct application of (12) and Corollary 11 we obtain the following chain of inequalities

$$\mathrm{DP}(I) = \mathrm{lct}(I) = \mathrm{lct}(I^0) \geqslant \frac{1}{e_1(I^0)} + \frac{e_1(I^0)}{e_2(I^0)} + \cdots + \frac{e_{n-1}(I^0)}{e_n(I^0)} \tag{25}$$

$$\geqslant \frac{1}{e_1(I)} + \frac{e_1(I)}{e_2(I)} + \cdots + \frac{e_{n-1}(I)}{e_n(I)} = \mathrm{DP}(I). \tag{26}$$

Hence we deduce that $\mathrm{lct}(I^0) = \mathrm{DP}(I^0)$, which implies, by the case analyzed before, that I^0 is a diagonal ideal. Moreover (25) and (26) also show that $\mathrm{DP}(I) = \mathrm{DP}(I^0)$. Then $\overline{I} = \overline{I^0}$, by Corollary 11, and consequently I is a diagonal ideal.

The equivalence between (b) and (c) follows as a direct application of (12). $\square$

Remark 14. (i) We observe that the condition $\mathrm{lct}(I) = \mathrm{DP}(I)$ does not imply that $\mathrm{lct}(I) = \mathrm{lct}(I^0)$ in general and hence it does not force the ideal I to be diagonal, as is shown in Example 15. Obviously, the condition $\mathrm{lct}(I) = \mathrm{lct}(I^0)$ holds if $\overline{I}$ is a monomial ideal. If I is an arbitrary ideal of $\mathcal{O}_n$, let us denote by K_I the ideal of $\mathcal{O}_n$ generated by all the monomials x^k such that

$x^k \in \overline{I}$. Then $\mathrm{lct}(K_I) \leqslant \mathrm{lct}(I) \leqslant \mathrm{lct}(I^0)$. Let $\mathrm{lct}(I^0) = \frac{p}{q}$, $p, q \in \mathbb{Z}_{\geqslant 1}$. Then $\mu(I^0) = \frac{q}{p}$. If we suppose that $(x_1 \cdots x_n)^q \in \overline{I^p}$, then $q \geqslant \mu(K_{I^p})$, by Theorem 9. Since $\overline{K_I^p} \subseteq K_{I^p}$, we have

$$q \geqslant \mu(K_{I^p}) \geqslant \mu(K_I^p) = p\,\mu(K_I) \geqslant p\,\mu(I) \geqslant p\,\mu(I^0) = q. \tag{27}$$

Then all inequalities of (27) become equalities and then $\mathrm{lct}(I) = \mathrm{lct}(I^0)$.

(ii) If I denotes an ideal of $\mathcal{O}_n$ of finite colength generated by monomials, then the equivalence between the conditions $\mathrm{lct}(I) = \mathrm{DP}(I)$ and I is diagonal also follows as a corollary of a more general result stated for multi-circled plurisubharmonic singularities and proved by Rashkovskii in [23, Theorem 1.5] following techniques from pluripotential theory.

Example 15. Let us consider the polynomials of $\mathcal{O}_2$ given by $g_1 = (x+y)^2 + y^4$ and $g_2 = (x+y)y^2$. Let I be the ideal of $\mathcal{O}_2$ generated by g_1 and g_2. Then $e_1(I) = \mathrm{ord}(I) = 2$ and $e(I) = 8$. If we apply to I the linear coordinate change $(x, y) \mapsto (x - y, y)$, then we obtain the ideal $J = \langle x^2 + y^4, xy^2 \rangle$. We observe that J is a Newton non-degenerate ideal (see [2] or [27]), which implies that $\overline{J} = J^0 = \overline{\langle x^2, y^4 \rangle}$. Then J is diagonal and hence $\mathrm{lct}(I) = \mathrm{lct}(J) = \mathrm{lct}(J^0) = \frac{3}{4} = \frac{1}{2} + \frac{2}{8} = \mathrm{DP}(I)$.

We observe that $\Gamma_+(I)$ has a unique compact face Δ of dimension 1, hence I is diagonal if and only if $\overline{I}$ is generated by monomials, which is to say that I is Newton non-degenerate. Following the notation introduced in [2, p. 398] we see that $(g_1)_\Delta = (x+y)^2$, $(g_2)_\Delta = 0$, and hence the solutions of the system $(g_1)_\Delta = (g_2)_\Delta = 0$ are not contained in $\{(x, y) \in \mathbb{C}^2 : xy = 0\}$. Then I is not Newton non-degenerate, by [2, Proposition 3.6] and thus I is not a diagonal ideal, although $\mathrm{lct}(I) = \mathrm{DP}(I)$.

According to Example 15, it seems reasonable to expect that, if we fix coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$ and I is a proper ideal of finite colength of $\mathcal{O}_n$ such that $\mathrm{DP}(I) = \mathrm{lct}(I)$, then there exists a linear coordinate change $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi^*(I)$ is diagonal with respect to $x_1, \dots, x_n$.

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