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Additional Information

Nilpotent matrices and the minus partial order*

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Abstract

In this paper, $\{1\}$ -inverses of a nilpotent matrix as well as matrices above a given nilpotent matrix under the minus partial order are characterized.

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1 Introduction and background

Partial orders and preorders on matrices were studied by several authors. They involve different generalized inverses in their definitions. Some results about matrices, generalized inverses and partial orders on matrices can be found, for instance, in [2, 3, 10, 12, 13].

In [10] the Drazin preorder was defined by means of the Drazin inverse, which does not take into account the nilpotent part of the matrices. For this reason, it is not anti-symmetric and then Drazin preorder is not a partial order. By considering the nilpotent

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part of a matrix, a binary relation on square matrices was defined in [10, Definition 4.4.17] in order to extend the Drazin pre-order to a partial order. This relation compares the core part of both matrices under the sharp partial order and the nilpotent part under the minus order. The sharp partial order, defined for index one matrices, has been widely studied [10]. However, to our acknowledge, the minus partial order for nilpotent matrices has not been investigated and it will be partially considered in this paper. In [8, 9] similar relations to Drazin preorder were studied for rectangular matrices, and then extended to operators on Banach spaces in [4, 11].

Let $\mathbb{C}^{m \times n}$ be the space of complex $m \times n$ matrices and \mathbb{C}^n be the complex vector space of $n \times 1$ vectors. The symbols A^* , A^t , A^{-1} , $\mathcal{R}(A)$, and $\mathcal{N}(A)$ stand for the conjugate transpose, the transpose, the inverse ($m = n$), the range and the null spaces of a matrix $A \in \mathbb{C}^{m \times n}$, respectively. As usual, I_n and O_n denote the $n \times n$ identity and zero matrices, respectively. The subscripts will be omitted when no confusion is caused. The index of a matrix $A \in \mathbb{C}^{n \times n}$, denoted by $\text{ind}(A)$, is the smallest nonnegative integer k such that $\mathcal{R}(A^k) = \mathcal{R}(A^{k+1})$. A matrix $A \in \mathbb{C}^{n \times n}$ is nilpotent if $A^r = O$ for some positive integer r , and the smallest such r is called the nilpotence index of A and coincides with $\text{ind}(A)$.

Let $A \in \mathbb{C}^{m \times n}$. A matrix $A^- \in \mathbb{C}^{n \times m}$ is a $\{1\}$ -inverse of A if $AA^-A = A$. A matrix $X \in \mathbb{C}^{n \times m}$ is the Moore-Penrose inverse of A if it satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$. The Moore-Penrose inverse of a matrix A always exists, is unique and it is denoted by A^\dagger .

On the other hand, the minus partial order, denoted by \leq^- , was introduced by Hartwig in [5] and it was analyzed by several authors, among others [1, 6, 7]. For two given matrices $A, B \in \mathbb{C}^{m \times n}$, it is said that $A \leq^- B$ if and only if $\text{rank}(B - A) = \text{rank}(B) - \text{rank}(A)$. Another equivalent definition for the minus partial order is the following: $A \leq^- B$ if and only if there exists a $\{1\}$ -inverse A^- of A such that $A^-A = A^-B$ and $AA^- = BA^-$.

For every nonzero nilpotent matrix $N \in \mathbb{C}^{n \times n}$ of index k , there exist a block-diagonal matrix $J \in \mathbb{C}^{n \times n}$, called the *Jordan canonical form of N* , and a nonsingular matrix $P \in \mathbb{C}^{n \times n}$ such that

$$N = PJP^{-1} \quad \text{with} \quad J = \text{diag}(J_1, J_2, \dots, J_s) \quad (1)$$

where $s = \dim(\mathcal{N}(N))$ and the matrix J_j is a square matrix of size $n_j \times n_j$ whose entries are equal to 1 along the superdiagonal and zero elsewhere, called *nilpotent Jordan block*, for

each $j \in \{1, 2, \dots, s\}$. Note that if $n_j = 1$ then $J_j = [0]$. When this type of blocks appears, they are located in the lower right corner. If there is more than one, it is considered that J_s is the zero matrix of suitable size. If B is a nilpotent Jordan block then $B^\dagger = B^t$, hence $J^\dagger = J^t$ [2]. The following results are used later.

Theorem 1.1 [2] *Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$, and $C \in \mathbb{C}^{m \times q}$. The following conditions are equivalent:*

- (a) *The matrix equation $AXB = C$ has a solution.*
- (b) *$AA^\dagger C = C$ and $CB^\dagger B = C$.*

In this case, the set of all the solutions of $AXB = C$ is given by $X = A^\dagger CB^\dagger + Y - A^\dagger AYBB^\dagger$, for arbitrary $Y \in \mathbb{C}^{n \times p}$.

Theorem 1.2 [10] *Let $A, B \in \mathbb{C}^{m \times n}$. Then $A \leq^- B$ if and only if*

$$B = A + (I_m - AA^-)W(I_n - A^-A),$$

for some matrix $W \in \mathbb{C}^{m \times n}$ and for some $\{1\}$ -inverse A^- of A .

2 $\{1\}$ -inverses of nilpotent matrices

In this section, $\{1\}$ -inverses of a nilpotent matrix are characterized by using its Jordan canonical form. Notice that for the nilpotent matrix $N = O$, every matrix N^- satisfies $ON^-O = O$, so only nonzero nilpotent matrices will be considered.

Theorem 2.1 *Let $N \in \mathbb{C}^{n \times n}$ be a nonzero nilpotent matrix written in its Jordan canonical form $N = PJP^{-1}$ as in (1). A matrix $N^- \in \mathbb{C}^{n \times n}$ is $\{1\}$ -inverse of N if and only if $N^- = PXP^{-1}$, where the partition of $X = [X_{ij}]_{1 \leq i, j \leq s}$ has been carried out according to the size of the blocks of J and*

$$X_{ij} = \begin{bmatrix} (u^{(ij)})^t & \alpha^{(ij)} \\ R & v^{(ij)} \end{bmatrix}$$

where $u^{(ij)} \in \mathbb{C}^{n_j-1}$, $v^{(ij)} \in \mathbb{C}^{n_i-1}$, and $\alpha^{(ij)} \in \mathbb{C}$, $R \in \mathbb{C}^{(n_i-1) \times (n_j-1)}$ is either the identity matrix (for $i = j$) or the zero matrix (for $i \neq j$), for all $1 \leq i, j \leq s$.

Proof. Let $N \in \mathbb{C}^{n \times n}$ be a nonzero nilpotent matrix such that $N = PJP^{-1}$ where $P \in \mathbb{C}^{n \times n}$ is nonsingular, $J = P^{-1}NP = \text{diag}(J_1, J_2, \dots, J_s)$, $s = \dim(\mathcal{N}(N))$, and J_j is a nilpotent Jordan block of size $n_j \times n_j$, for each $j \in \{1, 2, \dots, s\}$. It is clear that $N^- \in \mathbb{C}^{n \times n}$ is a $\{1\}$ -inverse of N if and only if $X = P^{-1}N^-P$ is a $\{1\}$ -inverse of J .

Let $X = [X_{ij}]_{1 \leq i, j \leq s}$ be a partition of X according to the blocks of J ; i.e., each block X_{ij} has order $n_i \times n_j$. By making the products, it then follows $JXJ = [J_i X_{ij} J_j]_{1 \leq i, j \leq s}$ thus, X is a $\{1\}$ -inverse of J if and only if

$$J_i X_{ii} J_i = J_i, \quad i \in \{1, \dots, s\} \quad (2a)$$

$$J_i X_{ij} J_j = O, \quad i, j \in \{1, \dots, s\}, \quad i \neq j. \quad (2b)$$

In order to solve matrix equations (2a), Theorem 1.1 is applied by setting $A = B = C = J_i$, for $i \in \{1, \dots, s\}$, ($J_i^\dagger = J_i^t$). Hence, the general solution of (2a) is given by

$$X_{ii} = J_i^t J_i J_i^t + Z_i - J_i^t J_i Z_i J_i J_i^t, \quad (3)$$

where $Z_i \in \mathbb{C}^{n_i \times n_i}$ is arbitrary. It is easy to see that

$$J_i^t J_i = I_{n_i} - e_1 e_1^t \quad \text{and} \quad J_i J_i^t = I_{n_i} - e_{n_i} e_{n_i}^t \quad (4)$$

where e_1 and e_{n_i} are the column canonical vectors of \mathbb{C}^{n_i} . By replacing in (3) the expressions for $J_i^t J_i$ and $J_i J_i^t$ obtained in (4), by making some computations, and using some properties of Moore-Penrose inverse, it is obtained

$$X_{ii} = J_i^t + Z_i e_{n_i} e_{n_i}^t + e_1 e_1^t Z_i - e_1 e_1^t Z_i e_{n_i} e_{n_i}^t. \quad (5)$$

Let us denote the entries of matrices Z_i by $z_{hk}^{(i)}$. Then

$$Z_i e_{n_i} e_{n_i}^t = \begin{bmatrix} 0 & 0 & \cdots & z_{1n_i}^{(i)} \\ 0 & 0 & \cdots & z_{2n_i}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & z_{n_i n_i}^{(i)} \end{bmatrix} \quad \text{and} \quad e_1 e_1^t Z_i = \begin{bmatrix} z_{11}^{(i)} & z_{12}^{(i)} & \cdots & z_{1n_i}^{(i)} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Hence, by replacing in expression (5) we obtain

$$X_{ii} = \begin{bmatrix} z_{11}^{(i)} & z_{12}^{(i)} & z_{13}^{(i)} & \cdots & z_{1n_i-1}^{(i)} & z_{1n_i}^{(i)} \\ 1 & 0 & 0 & \cdots & 0 & z_{2n_i}^{(i)} \\ 0 & 1 & 0 & \cdots & 0 & z_{3n_i}^{(i)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & z_{n_i-1n_i}^{(i)} \\ 0 & 0 & 0 & \cdots & 1 & z_{n_in_i}^{(i)} \end{bmatrix}. \quad (6)$$

In order to solve matrix equations (2b), Theorem 1.1 is applied by setting $A = J_i$, $B = J_j$, and $C = O$ for $i, j \in \{1, \dots, s\}$ ($J_k^\dagger = J_k^t$, for $k \in \{i, j\}$). The general solution of (2b) is

$$X_{ij} = W_{ij} - J_i^t J_i W_{ij} J_j J_j^t, \quad (7)$$

for arbitrary $W_{ij} \in \mathbb{C}^{n_i \times n_j}$. By replacing (4) in (7) it is obtained

$$X_{ij} = W_{ij} e_{n_j} e_{n_j}^t + e_1 e_1^t W_{ij} - e_1 e_1^t W_{ij} e_{n_j} e_{n_j}^t. \quad (8)$$

Let $W_{ij} = [w_{hk}^{(ij)}]_{1 \leq h \leq n_i, 1 \leq k \leq n_j}$. By making the multiplications in (8), it results

$$X_{ij} = \begin{bmatrix} w_{11}^{(ij)} & w_{12}^{(ij)} & \cdots & w_{1n_j}^{(ij)} \\ 0 & 0 & \cdots & w_{2n_j}^{(ij)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{n_in_j}^{(ij)} \end{bmatrix}. \quad (9)$$

Both situations (6) and (9) can be written simultaneously as follows:

$$X_{ij} = \begin{bmatrix} (u^{(ij)})^t & \alpha^{(ij)} \\ R & v^{(ij)} \end{bmatrix},$$

where $u^{(ij)} \in \mathbb{C}^{n_j-1}$, $v^{(ij)} \in \mathbb{C}^{n_i-1}$, $\alpha^{(ij)} \in \mathbb{C}$, and $R \in \mathbb{C}^{(n_i-1) \times (n_j-1)}$, for $1 \leq i, j \leq s$, such that

$$(u^{(ij)})^t = \begin{bmatrix} \beta_{11}^{(ij)} & \beta_{12}^{(ij)} & \cdots & \beta_{1n_j-1}^{(ij)} \end{bmatrix} \quad \text{with} \quad \beta_{1k}^{(ij)} = \begin{cases} z_{1k}^{(i)} & \text{if } i = j \\ w_{1k}^{(ij)} & \text{if } i \neq j \end{cases} \quad \text{for } 1 \leq k \leq n_j-1,$$

$$(v^{(ij)})^t = \begin{bmatrix} \gamma_{2n_j}^{(ij)} & \gamma_{3n_j}^{(ij)} & \cdots & \gamma_{n_in_j}^{(ij)} \end{bmatrix} \quad \text{with} \quad \gamma_{hn_j}^{(ij)} = \begin{cases} z_{hn_i}^{(i)} & \text{if } i = j \\ w_{hn_j}^{(ij)} & \text{if } i \neq j \end{cases}, \quad \text{for } 2 \leq h \leq n_i,$$

$$\alpha^{(ij)} = \begin{cases} z_{1n_i}^{(i)} & \text{if } i = j \\ w_{1n_j}^{(ij)} & \text{if } i \neq j \end{cases} \quad \text{and} \quad R = \begin{cases} I_{(n_i-1) \times (n_i-1)} & \text{if } i = j \\ O_{(n_i-1) \times (n_j-1)} & \text{if } i \neq j \end{cases}.$$

This completes the proof. ■

3 Nilpotent matrices and the minus partial order

In this section, a characterization for matrices above a given nilpotent matrix under the minus partial order is obtained. It is based on the form of $\{1\}$ -inverses of a nilpotent matrix as obtained in Theorem 2.1. Notice that if $N = O$ then every matrix Y with the same size as N satisfies $N \leq^- Y$.

Theorem 3.1 *Let $N \in \mathbb{C}^{n \times n}$ be a nonzero nilpotent matrix written in its Jordan canonical form as in (1). The following conditions are equivalent:*

(a) *There exists a matrix $Y \in \mathbb{C}^{n \times n}$ such that $N \leq^- Y$,*

(b) *$Y = N + PVP^{-1}WPUP^{-1}$, for some $W \in \mathbb{C}^{n \times n}$, and $V, U \in \mathbb{C}^{n \times n}$ partitioned according to the size of the blocks of J , $V = [V_{ij}]_{1 \leq i, j \leq s}$, $U = [U_{ij}]_{1 \leq i, j \leq s}$,*

$$V_{ij} = \begin{bmatrix} O & v^{(ij)} \\ 0^t & \delta_{ij} \end{bmatrix}_{n_i \times n_j}, \quad U_{ij} = \begin{bmatrix} \delta_{ij} & (u^{(ij)})^t \\ 0 & O \end{bmatrix}_{n_i \times n_j}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

$v^{(ij)} \in \mathbb{C}^{n_i-1}$, and $u^{(ij)} \in \mathbb{C}^{n_j-1}$, for all $i, j \in \{1, 2, \dots, s\}$.

Proof. (a) \implies (b) Let $N \neq O$ be a nilpotent matrix written as in (1) such that $N \leq^- Y$ for some matrix Y . Then, by Theorem 1.2,

$$Y = N + (I - NN^-)W(I - N^-N), \tag{10}$$

for some matrix W and for some $\{1\}$ -inverse N^- of N . By Theorem 2.1, a $\{1\}$ -inverse of N can be written as $N^- = PXP^{-1}$, where X is partitioned according to the size of the blocks of J such that each block of X has the form

$$X_{ij} = \begin{bmatrix} (u^{(ij)})^t & \alpha^{(ij)} \\ R & v^{(ij)} \end{bmatrix},$$

where $u^{(ij)} \in \mathbb{C}^{n_j-1}$, $v^{(ij)} \in \mathbb{C}^{n_i-1}$ and $\alpha^{(ij)} \in \mathbb{C}$, and $R \in \mathbb{C}^{(n_i-1) \times (n_j-1)}$ is either the identity matrix (for $i = j$) or the zero matrix (for $i \neq j$), for all $i, j \in \{1, 2, \dots, s\}$.

Post-multiplying (10) by the nonsingular matrix P , pre-multiplying it by P^{-1} , and taking into account that $NN^{-} = PJXP^{-1}$ and $N^{-}N = PXJP^{-1}$, then equality (10) can be written equivalently as

$$P^{-1}YP = J + (I - JX)P^{-1}WP(I - XJ). \quad (11)$$

It is easy to see that $[JX]_{ij} = J_iX_{ij}$ and $[XJ]_{ij} = X_{ij}J_j$. Let us write $J_i = \begin{bmatrix} O & I_{n_i-1} \\ 0 & O^t \end{bmatrix}$ according to the blocks of X_{ij} , where $0 \in \mathbb{C}$, $I \in \mathbb{C}^{(n_i-1) \times (n_i-1)}$ and $O \in \mathbb{C}^{n_i-1}$. Then

$$J_iX_{ij} = \begin{bmatrix} R & v^{(ij)} \\ O^t & 0 \end{bmatrix} \quad \text{and} \quad X_{ij}J_j = \begin{bmatrix} 0 & (u^{(ij)})^t \\ O & R \end{bmatrix},$$

with $O \in \mathbb{C}^{n_j-1}$, $0 \in \mathbb{C}$, $u^{(ij)} \in \mathbb{C}^{n_j-1}$, $v^{(ij)} \in \mathbb{C}^{n_i-1}$, $R = I_{n_i-1}$ if $i = j$, and $R = O_{(n_i-1) \times (n_j-1)}$ if $i \neq j$. Some computations give

$$[I - JX]_{ij} = \begin{bmatrix} O & -v^{(ij)} \\ O^t & \delta_{ij} \end{bmatrix} \quad \text{and} \quad [I - XJ]_{ij} = \begin{bmatrix} \delta_{ij} & -(u^{(ij)})^t \\ O & O \end{bmatrix}, \quad \text{where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Set $V_{ij} = [I - JX]_{ij}$, $U_{ij} = [I - XJ]_{ij}$, $V = [V_{ij}]_{1 \leq i, j \leq s}$, and $U = [U_{ij}]_{1 \leq i, j \leq s}$. Hence, from (11), it is clear that $Y = N + PVP^{-1}WPUP^{-1}$, for some matrix W .

(b) \implies (a) It is an easy computation. ■

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