Focusing properties of diffractive lenses constructed with the aperiodic *m*-bonacci sequence

Walter D. Furlan[a*], Vicente Ferrando[a,b], and Juan A. Monsoriu[b]
[a]Departamento de Óptica, Universitat de València, 46100 Burjassot, Valencia, Spain.
[b]Centro de Tecnologías Físicas, Universitat Politècnica de València, 46022 Valencia, Spain

ABSTRACT

In this contribution we present a new family of diffractive lenses which are designed using the *m*-bonacci sequence. These lenses are a generalization of the Fibonacci Zone Plates previously reported. Diffractive elements of this type are called aperiodic zone plates because they are characterized by a radial profile that follows a given deterministic aperiodic sequence (Cantor set, Thue-Morse, Fibonacci...). Aperiodic lenses have demonstrated new interesting focusing and imaging properties that have found applications in different fields such as soft X-ray microscopy and spectral domain optical coherence tomography. Here, we show that *m*-bonacci zone plates are inherently bifocal lenses. We demonstrate that the relative separation of their foci depends on the *m*-value of the sequence and also can be correlated with the generalized golden ratio. As a particular case, the properties of the *m*-bonacci sequence with m=2 and m=3, called Fibonacci and Tribonacci Zone Plates respectively are discussed.

Keywords: Aperiodic, *m*-bonacci, zone plates , bifocal diffractive lenses.

INTRODUCTION

In photonics technology, Diffractive Optical Elements (DOEs) have found a large number of new applications in many different areas, covering the whole electromagnetic spectrum from X-ray Microscopy [1], to THz Imaging [2]. Diffractive lenses, such as conventional Fresnel zone plates are essential in many focusing and image forming systems but they have inherent limitations, especially under polychromatic illumination. A Fresnel zone plate can be understood as a periodic element along the squared radial coordinate.

On the other hand, we have shown that also non-periodic and aperiodic structures, can generate diffractive lenses with unique features. In fact. aperiodic order has attracted considerable interest of researchers of several fields. From a theoretical point of view, it is considered a suitable theoretical model to describe the conceptual transition from randomness to periodic order. Besides, from a practical point of view, there is evidence that deterministically ordered aperiodic structures may offer interesting possibilities for technological applications [3]. Our work in this line started with the proposal of the first lenses showing a fractal distriburtion of foci: the Fractal Zone Plates [4, 5], these diffractive lenses, that are generated with the fractal Cantor set, have shown an improved behavior compared with Fresnel zone plates, especially under wide band illumination [6].

Another interesting mathematical generator of aperiodic Zone Plates is the Fibonacci sequence [7], which is generated from the Fibonacci numbers. The Fibonacci sequence is one of the most recurrent mathematical fitting models of different natural phenomena. This sequence has been also employed in the development of different photonic devices [8], such as multilayers and linear gratings [9], circular gratings [10], spiral Zone Plates [11].

In this work we show that Fibonacci zone plates are particular cases of a general set that we have called *m*-bonacci Zone Plates. In particular, we compare the focusing properties of these aperiodic lenses with m=2 (Fibonacci) and m=3 (Tribonacci).

* walter.furlan@uv.es

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M-BONACCI ZONE PLATES DESIGN

We start the design of our aperiodic Zone Plates by obtaining previously the Fibonacci and Tribonacci numbers. Starting with two elements (seeds) F_0=0 and F_0=1 , the Fibonacci numbers, F_j=¨{0,1,1,2,3,5,8,13,21,...} , are obtained by the sequential application of the iterative rule $F_{j+1}= F_j + F_{j-1}$, (j = 1, 2, ...). The golden mean, or golden ratio, is defined as the limit of the factor between two consecutive Fibonacci numbers:

$$\varphi_2 = \lim_{j\to\infty} F_j / F_{j+1} = \left(1+\sqrt{5}\right)/2 \approx 1.618 \quad . \tag{1}$$

On the other hand, the Tribonacci numbers T_j=¨{0, 1, 1, 2, 4, 7, 13, 24, 44,...}, are obtained starting from the first three elements T_0=0 and T_1=1 and T_2=1 by the sequential application of the iterative rule $T_{j+1}= T_j + T_{j-1} + T_{j-2}$, ($j$ = 2, 3, ...). In this case the limit of the factor between two consecutive Tribonacci numbers approaches the value:

$$\varphi_3 = \lim_{j\to\infty} T_j / T_{j+1} \frac{1}{3}\left(1+(19-3\sqrt{33})^{1/3}+(19+3\sqrt{33})^{1/3}\right) \approx 1.839 \cdot \tag{2}$$

In a similar way, a binary aperiodic Fibonacci sequence can also be deterministically generated starting from two seed elements, as for example, $S_1 = \{A\}$ and $S_0 = \{B\}$, and the successive elements of the sequence are obtained simply as the concatenation of the two previous ones: $S_{j+1} = \{S_j , S_{j-1}\}$ for $j\geq 1$. In this way we obtain the Fibonacci based sequence:

$$\begin{aligned}
S_1 &= \{A\} \\
S_2 &= \{AB\} \\
S_3 &= \{ABA\} \\
S_4 &= \{ABAAB\} \\
S_5 &= \{ABAABABA\} \\
S_6 &= \{ABAABABAABAAB\} \\
S_7 &= \{ABAABABAABAABABAABABA\}
\end{aligned} \tag{3}$$

Note that, in any given sequence of order S_j , two successive "B" are separated by either, one or two "A", and that the total number of elements F_{j+1}, results from the sum of F_j elements "A", and F_{j+1} elements "B".

A similar procedure can be followed to create a binary aperiodic structure based on the Tribonacci: Starting from the seed elements: $S_1 = \{A\}$ $S_0 = \{B\}$ and $S_0 = \{AB\}$, the successive elements of the sequence are generated as the concatenation of the three previous ones: $S_{j+1}= S_j + S_{j-1} + S_{j-2}$, for $j \geq 2$. In this way we obtain the Tribonacci based sequence:

$$\begin{aligned}
S_0 &= \{B\} \\
S_1 &= \{A\} \\
S_2 &= \{AB\} \\
S_3 &= \{ABAB\} \\
S_4 &= \{ABABABA\} \\
S_5 &= \{ABABABAABABAB\} \\
S_6 &= \{ABABABAABABABABABABAABAB\} \\
S_7 &= \{ABABABAABABABABABABAABABABABABAABABABABABABA\}
\end{aligned} \tag{4}$$

Note that, as for the Fibonacci based structure, in any given sequence of order S_j, two successive "*B*" are separated by either, one or two "*A*", but the total number of elements T_{j+1}, results from the sum of T_j elements "A", and $T_j + T_{j-1}$ elements "*B*".

The *m*-bonacci binary structure of any order *m* is obtained following the same, conceptually simple, procedure.

When designing a zone plate, each one of these sequences can be used to define the binary generating function $\phi(\xi)$ with compact support on the interval [0,1]. This interval is partitioned in *N* sub-intervals of length $d = 1/N$, and the value that takes at the *j*-th sub-interval S_j is associated to the value of the element for the Fibonacci and the Tribonacci based Zone Plate, being 0 or 1 when its value is "*A*" or "*B*", respectively.

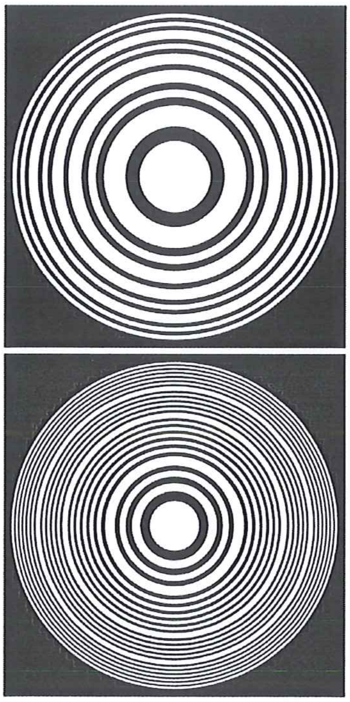

Figure 1. (Top) Zone Plate based on the Fibonacci sequence of order 7.
(Bottom) Zone Plate based on the Tribonacci based sequence of order 7.

From a particular generating function $\phi(\xi)$ the transmittance $q(\xi)$ of the corresponding *m*-bonacci Zone Plate is obtained after performing the following coordinate transformation: $\xi = (r/a)^2$, where *r* is the radial coordinate of the zone plate, and *a* is its maximum value. In Fig. 1 we show the Fibonacci based Zone Plate of order 7 (top) and Tribonacci based Zone Plate of order 7 (bottom). Note that the number of rings in each zone plate is the same to the number of elements of the corresponding sequence (see Eqs 3 and 4).

FOCUSING PROPERTIES

We have computed the axial irradiance along the optical axis provided by the *m*-bonacci zone plates within the Fresnel approximation. Provided that we are dealing with rotationally invariant pupil functions, they can described in terms of a function that only depends on the radial coordinate: $p(r)$. Thus, the axial irradiance at a given axial point under a monochromatic plane wave illumination can be expressed as a function of the axial distance from the pupil plane z:

$$I(z) = \left(\frac{2\pi}{\lambda z}\right)^2 \left| \int_0^a p(r) \exp\left[-i\frac{\pi}{\lambda z} r^2\right] r\, dr \right|^2, \tag{5}$$

In the above equation, a is the maximum extent of the pupil function, and λ is the wavelength of the light. For our purposes it is convenient to express the pupil transmittance as function of a new variable defined as $\xi = (r/a)^2$ in such a way that $q(\xi)=p(r)$ By using the dimensionless axial coordinate $u=a^2/2\lambda z$, the irradiance along the optical axis can be now expressed as

$$I(u) = 4\pi u^2 \left| \int_0^1 q(\xi) \exp[-2\, i\, \pi\, u\, \xi]\; \xi\, d\xi \right|^2, \tag{6}$$

Therefore the axial irradiance can be expressed as the square modulus of the Fourier transform of the pupil function $q(\xi)$

By using the above equation we have computed the axial irradiances provided by the Fibonacci based Zone Plates of order 7 and 9. As it can be seen in Fig. 2, the Fibonacci Zone Plate drive most of the incoming light into two main foci located at $u_1 \approx F_{j-1}$ and $u_2 \approx F_j$ being the ratio of the focal distances $u_2/u_1 \approx \varphi_2 \approx 1.618$.

In the same way we have computed axial irradiances for the Tribonacci Zone Plate of order 7 and order 9. The results are shown in Fig. 3. It can be also observed that the Tribonacci Zone Plate also drive most of the incoming light into two main foci which in this case are located at $u_1 \approx T_{j-1} + T_{j-2}$ and $u_2 \approx T_{j2}$, being the ratio of the focal distances $u_2/u_1 \approx 1/(\varphi_3 - 1) \approx 1.1991$.

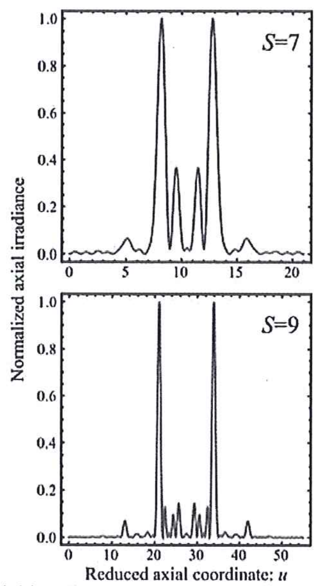

Figure 2. Computed axial irradiance for Fibonacci Zone Plates of different orders.

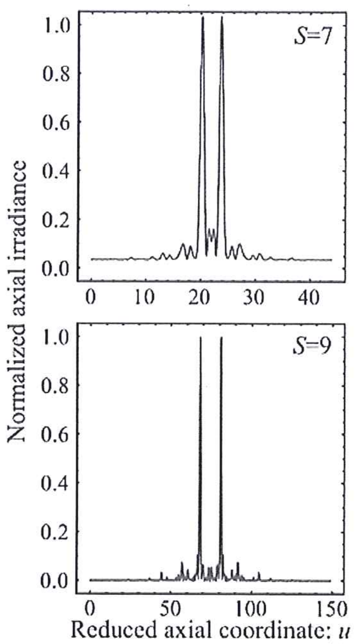

Figure 3. Computed axial irradiance for Tribonacci Zone Plates of different orders.

CONCLUSIONS

Sumarizing, the generalization of the Fibonacci Zone Plate has been presented with the *m*-bonacci sequence. In particular, the focusing properties of Tribonacci Zone Plate has been computed and compared with that provided by the Fibonacci Zone Plate. It was found that the Tribonacci Zone Plates also produces a bifocal axial irradiance which focal positions are related with the Tribonacci numbers. It has been also observed that the pair of foci provided by the Tribonacci based Zone Plate are closer than the two foci of the Fibonacci Zone Plate.

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