A note on φ-contractions in probabilistic and fuzzy metric spaces

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Abstract

In a recent paper [Fuzzy Sets and Systems 267 (2015) 86-99], J.X. Fang generalized a crucial fixed point theorem for probabilistic φ-contractions on complete Menger spaces due to J. Jachymski [Nonlinear Analysis 73 (2010) 2199-2203]. In this note we show that actually Fang's theorem is an easy consequence of Jachymski's theorem. We also observe that the proof of a fixed point theorem for complete metric spaces deduced by Fang from his main result is not correct and present a new proof of it.

Key words: Complete Menger space; Probabilistic φ-contraction; Fixed point

Throughout this note we shall use the terminology of [1]. The letters $\mathbb{R}^+$, $\mathbb{R}_0^+$ and $\mathbb{N}$ will denote the sets of all non-negative real numbers, the set of all positive real numbers and the set of all positive integer numbers, respectively.

If (X, F, Δ) is a Menger space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, we say that a mapping $T : X \to X$ is a probabilistic φ-contraction if $F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t)$, for all $x, y \in X$ and $t > 0$.

In [3] Jachymski proved the following nice and elegant fixed point theorem for probabilistic φ-contractions.

Theorem A ([3, Theorem 1]). *Let (X, F, Δ) be a complete Menger space with Δ a triangular norm of H-type, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that*

$$\varphi(t) < t \quad \text{and} \quad \lim_{n\to\infty} \varphi^n(t) = 0 \quad \text{for all } t > 0.$$

If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point x_* and for any $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.

Remark. Actually, Jachymski established Theorem A by assuming that the triangular norm Δ is continuous and that $\varphi(t) > 0$ for all $t > 0$. However, his proof only uses continuity of Δ at $(1, 1)$ which is satisfied for every triangular norm of H-type, and, on the other hand, condition $\varphi(t) > 0$ for all $t > 0$ is automatically satisfied for any probabilistic φ-contraction, as Jachymski's observed in the first lines of Section 2 of [3].

In a recent paper [1], Fang generalized Jachymski's theorem with the help of a certain class of functions from $\mathbb{R}^+$ into iself. To this end, he denoted by Φ the class of all functions

$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$; and by Φ_w the class of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $t > 0$ there exists an $r_t \geq t$ satisfying $\lim_{n\to\infty} \varphi^n(r_t) = 0$.

Obviously $\Phi \subseteq \Phi_w$. In fact, Φ is a proper subclass of Φ_w as it was proved in [1, Example 3.1].

Then, Fang proved the main result of his paper, which is the following generalization of Jachymski's theorem (Theorem A above).

Theorem B ([1, Theorem 3.1]). *Let (X, F, Δ) be a complete Menger space with Δ a triangular norm of H-type, and let $\varphi \in \Phi_w$. If $T : X \to X$ is a probabilistic φ-contraction, then T has a unique fixed point x_* and for any $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.*

The original proof of Theorem B is constructive and quite long. We are going to show that Theorem B can be obtained as an easy consequence of Theorem A.

Indeed, let (X, F, Δ) be a complete Menger space with Δ a triangular norm of H-type, let $\varphi \in \Phi_w$ and let $T : X \to X$ be a probabilistic φ-contraction.

Put $A = \{t > 0 : \lim_{n\to\infty t} \varphi^n(t) = 0\}$.

If $t \in A$, we denote by k_t the first positive integer number such that $\varphi^{k_t-1}(t) \geq t > \varphi^{k_t}(t)$ (recall that $\varphi^0(t) = t$)).

If $t \in \mathbb{R}_0^+ \backslash A$, take an $r_t > t$ such that $r_t \in A$, and, again, denote by k_t the first positive integer number such that $\varphi^{k_t-1}(r_t) > t > \varphi^{k_t}(r_t)$. (Note that in this case the existence of k_t is also guaranteed because $\lim_{n\to\infty} \varphi^n(r_t) = 0$ and $\varphi^0(r_t) = r_t > t$).

Now define a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\phi(0) = 0, \quad \phi(t) = \varphi^{k_t}(t) \text{ if } t \in A, \quad \text{and} \quad \phi(t) = \varphi^{k_t}(r_t) \text{ if } t \in \mathbb{R}_0^+ \backslash A.$$

We first note that, obviously, $\phi(t) < t$ for all $t > 0$. Now we show that $\phi \in \Phi$.

Let $t \in A$. Then $\varphi^k(t) \in A$ for all $k \in \mathbb{N}$, so by the definition of ϕ we deduce that $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is a subsequence of $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ and hence $\lim_{n\to\infty} \phi^n(t) = 0$.

Similarly, if $t \in \mathbb{R}_0^+ \backslash A$, we deduce that $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is a subsequence of $\{\varphi^n(r_t)\}_{n\in\mathbb{N}}$ and hence $\lim_{n\to\infty} \phi^n(t) = 0$.

Finally, we show that T is a probabilistic ϕ-contraction on (X, F, Δ).

Let $x, y \in X$ and $t > 0$. If $t \in A$ we obtain

$$M(Tx, Ty, \phi(t)) = M(Tx, Ty, \varphi^{k_t}(t)) \geq M(x, y, \varphi^{k_t-1}(t) \geq M(x, y, t).$$

If $t \in \mathbb{R}_0^+ \backslash A$ we similarly obtain

$$M(Tx, Ty, \phi(t)) = M(Tx, Ty, \varphi^{k_t}(r_t)) \geq M(x, y, \varphi^{k_t-1}(r_t) \geq M(x, y, t).$$

Hence, we can apply Theorem A and thus T has a fixed point x_* and for any $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.

In **[1]**, Fang deduced from Theorem B the following fixed point result for complete metric spaces which provides an apparent generalization of the celebrated Matkowski's fixed point theorem [4, Theorem 1.2]

Corollary C ([1, Corollary 3.3]). *Let (X,d) be a complete metric space, and let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-decreasing function such that* $\varphi \in \Phi_w$ *and* $\varphi(t) > 0$ *for all* $t > 0$. *If* $T : X \to X$ *satisfies that* $d(Tx, Ty) \leq \varphi(d(x, y))$ *for all* $x, y \in X$, *then* T *has a unique fixed point* x_* *and for any* $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.

The proof of Corollary C strongly depends on the following lemma.

Lemma D ([1, Lemma 3.5]). *Let* (X, d) *be a metric space. Define a mapping* $F : X \times X \to \mathcal{D}^+$ *by*

$$F(x, y)(t) = F_{x,y}(t) = 0 \quad \text{if } t \leq 0 \text{ or } d(x, y) > t > 0, \quad \text{and}$$
$$F(x, y)(t) = F_{x,y}(t) = 1 \quad \text{if } d(x, y) \leq t, \quad (t > 0).$$

Then (X, F, Δ_M) *is a Menger space, and it is complete if and only if* (X, d) *is complete.*

Unfortunately, Lemma D is not true because if (X, d) is a metric space with $x, y \in X$ such that $x \neq y$, we have $d(x, y) > 0$, and thus $F_{x,y}(d(x, y)) = 1$, but for each $t < d(x, y)$ we have $F_{x,y}(t) = 0$, so that $F(x, y) \notin \mathcal{D}^+$.

We conclude this note by showing that, nevertheless, Corollary C is true. To this end we shall apply the following well-known result due to Jachymski.

Theorem E ([2, Corollary of Theorem 2]). *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be such that* $d(Tx, Ty) < d(x, y)$ *for* $x \neq y$, *and* $dTx, Ty) \leq \varphi(d(x, y))$ *for any* $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the condition*

(Ja) *for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that, for each* $t > 0$,

$\varepsilon < t < \varepsilon + \delta$ *implies* $\varphi(t) \leq \varepsilon$.

Then T *has a unique fixed point* x_* *and for any* $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.

Proof of Corollary C. Suppose that φ does not satisfies condition (Ja) above. Exactly as in Remark 1 of [2], there exist $\varepsilon > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} t_n = \varepsilon$, $t_n > \varepsilon$, and $\varphi(t_n) > \varepsilon$ for all $n \in \mathbb{N}$. Since φ is non-decreasing we deduce that $\varphi(t) > \varepsilon$ for all $t > \varepsilon$. Hence $\varphi^n(t) > \varepsilon$ for all $t > \varepsilon$ and $n \in \mathbb{N}$, which contradicts the assumption that $\varphi \in \Phi_w$. So, by Theorem E, T has a unique fixed point x_* and for any $x_0 \in X$, $\lim_{n\to\infty} T^n x_0 = x_*$.

References

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