Completely simple endomorphism rings of modules

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ABSTRACT

It is proved that if $A_p$ is a countable elementary abelian $p$-group, then:
(i) The ring $\text{End} (A_p)$ does not admit a nondiscrete locally compact ring topology. (ii) Under (CH) the simple ring $\text{End} (A_p)/I$, where $I$ is the ideal of $\text{End} (A_p)$ consisting of all endomorphisms with finite images, does not admit a nondiscrete locally compact ring topology. (iii) The finite topology on $\text{End} (A_p)$ is the only second metrizable ring topology on it. Moreover, a characterization of completely simple endomorphism rings of modules over commutative rings is obtained.

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1. INTRODUCTION

The notion of associative simple ring can be extended for associative topological rings in several ways:

(i) simple abstract ring endowed with a nondiscrete ring topology (for instance, the classification of nondiscrete locally compact division rings, see [25, Chapter IV] and [4, 15, 16]; we refer to some historical notes about locally compact division rings to [29]);
(ii) topological ring without nontrivial closed ideals (see [22, 31]).

(iii) topological ring $R$ with the property that if $f : R \to S$ is a continuous homomorphism in a topological ring $S$, then either $f = 0$ or $f$ is a topological embedding of $R$ into $S$ (see [24]).

In all cases it is assumed that multiplication is not trivial.

I. Kaplansky has mentioned (see [20], p. 56) that the classification of locally compact simple rings in positive characteristic $p$ is difficult. He proved that every simple nondiscrete locally compact simple torsion-free ring is a matrix ring over a locally compact division ring. However in [26] (see also [30]) has been constructed a nondiscrete locally compact simple ring of positive characteristic which is not a matrix ring over a division ring. Thereby the program of classification of nondiscrete locally compact simple rings was finished. Nevertheless it is interesting to look for new examples of locally compact simple rings.

If $A_p$ is a countable elementary abelian $p$-group and $I$ is the ideal of the ring $\text{End}(A_p)$ consisting of endomorphisms with finite images, then the factor ring $\text{End}(A_p)/I$ is a simple von Neumann regular ring. We prove that under (CH) this ring does not admit a nondiscrete locally compact ring topology.

S. Ulam (see [23, Problem 96, p. 181]) posed the following problem: "Can the group $S_\infty$ of all permutations of integers so metrized that the group operation (composition of permutations) is a continuous function and the set $S_\infty$ becomes, under this metric, a compact space? (locally compact?)". E.D. Gaughan (see [10]) has solved this problem in the negative.

We study in §3 an analogous problem for the endomorphism ring of a countable elementary abelian $p$-group, namely: "Does the endomorphism ring $\text{End}(A_p)$ of a countable elementary $p$-group $A_p$ admit a nondiscrete locally compact ring topology?". Similarly to the Ulam’s problem we obtain a negative answer to this question. Moreover, we prove that $T_{\text{fin}}$ is the only ring topology $T$ on $\text{End}(A_p)$ such that $(\text{End}(A_p), T)$ is complete and second metrizable.

We classify in §4 the completely simple rings $(\text{End}(M), T_{\text{fin}})$ of vector spaces $M$ over division rings. Corollary 4.4 gives a characterization of semisimple left linearly compact minimal rings. It should be mentioned that Corollary 4.4 is related to a result from [3] stating that any semisimple ring admits at most one linearly compact topology.

Furthermore, we obtain in §5 a description of completely simple rings of the form $(\text{End}(M_R), T_{\text{fin}})$ of modules $M$ over a commutative ring $R$. We extend the result of [28] to topological rings $(\text{End}(M_R), T_{\text{fin}})$.

2. Notation, Conventions and Preliminary Results

Rings are assumed to be associative, not necessarily with identity. Topological spaces are assumed to be completely regular. The weight (see [8], p.12) of the space $X$ is denoted by $w(X)$. The pseudocharacter of a point $x \in X$ (see [8], p.135) is the smallest cardinal of the form $|U|$, where $U$ is a family

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of open subsets of \( X \) such that \( \cap \mathcal{U} = \{ x \} \). The closure of a subset \( A \) of the topological space \( X \) is denoted by \( \overline{A} \) and the interior by \( \text{Int}(A) \) (see [8], p.14).

A topological space \( X \) is called a Baire space (see [8], p.198) if for each sequence \( \{ X_1, X_2, \ldots \} \) of open dense subsets of \( X \) the intersection \( \cap_{i=1}^{\infty} G_i \) is a dense set.

An abelian group \( A \) is called elementary abelian \( p \)-group (\( p \) prime) if for all \( a \in A \). Such a group is a direct sum of copies of the cyclic group \( \mathbb{Z}(p) \).

The subring of a ring \( R \) generated by a subset \( S \), is denoted by \( \langle S \rangle \). A ring \( R \) is called locally finite if every its finite subset is contained in a finite subring.

A topological ring \((R, T)\) is called metrizable if its underlying additive group satisfies the first axiom of countability. A ring \( R \) with \( 1 \) is called Dedekind-finite if each equality \( xy = 1 \) implies \( yx = 1 \). It is well-known that every finite ring with identity is Dedekind-finite. Since every compact ring with identity is a subdirect product of finite rings, it follows that every compact ring with identity is Dedekind-finite. If \( A \subseteq R \), then \( \text{Ann}_R(A) := \{ x \in R \mid xA = 0 \} \). If \( X, Y \) are the subsets of \( R \), then \( X \cdot Y := \{ xy \mid x \in X, y \in Y \} \). A topological ring \( R \) is called compactly generated (see [27, Chapter II]) if there exists a compact subset \( K \) such that \( R = \langle K \rangle \). If \( (R, T) \) is a topological ring and \( I \) is an ideal of \( R \), then the quotient topology of the factor ring \( R/I \) is denoted by \( T/I \). If \( K \) is an ideal of an abelian group \( A \), then set \( T(K) = \{ \alpha \in \text{End}(A) \mid \alpha(K) = 0 \} \).

When \( K \) runs over all finite subsets of \( A \), the family \( \{ T(K) \} \) defines a ring topology \( T_{\text{fin}} \) on \( \text{End}(A) \). This topology is called the finite topology.

Lemma 2.1. For any abelian group \( A \) the ring \((\text{End}(A), T_{\text{fin}})\) is complete.

Proof. See [27, Theorem 19.2]. \( \square \)

Lemma 2.2 (Cauchy’s criterion). In a Hausdorff complete commutative group \( G \), in order that a family \( (x_\alpha)_{\alpha \in \Omega} \) should be summable it is necessary and sufficient that, for each neighborhood \( V \) of zero in \( G \), there is a finite subset \( \Omega_0 \) of \( \Omega \) such that \( \Sigma_{\alpha \in K} x_\alpha \in V \) for all finite subsets \( K \) of \( \Omega \) which do not meet \( \Omega \).

Proof. See [5], p.263. \( \square \)

Lemma 2.3. If \( (x_\alpha)_{\alpha \in \Omega} \) is a summable subset in \((\text{End}(A), T_{\text{fin}})\) then every subset \( \Delta \) of \( \Omega \) the family \( (x_\beta)_{\beta \in \Delta} \) is summable.

Proof. Let \( V \) be a neighborhood of zero of \((\text{End}(A), T_{\text{fin}})\). We can consider without loss of generality that \( V \) is a left ideal of \( \text{End}(A) \). There exists a finite subset \( \Omega_0 \) of \( \Omega \) such that \( \Sigma_{\alpha \in K} x_\alpha \in V \) for every finite subset \( K \) of \( \Omega \) for which \( K \cap \Omega_0 = \varnothing \). Let \( F \) be a finite subset of \( \Delta \) such that \( F \cap (\Omega_0 \cap \Delta) = \varnothing \). If \( \alpha \in F \), then \( \alpha \notin \Omega_0 \), hence \( \Sigma_{\alpha \in F} x_\alpha \in V \). By Cauchy’s criterion the family \( (x_\beta)_{\beta \in \Delta} \) is summable. \( \square \)

A topological ring \((R, T)\) is called minimal (see, for instance, [7]) if there is no ring topology \( \mathcal{U} \) such that \( \mathcal{U} \leq T \) and \( \mathcal{U} \neq T \). A topological ring \((R, T)\) is called simple if \( R \) is simple as a ring without topology. A topological ring \((R, T)\) is called weakly simple if \( R^2 \neq 0 \) and every its closed ideal is either 0.
A topological ring \((R, T)\) is called completely simple (see [24]) if \(R^2 \neq 0\) and for every continuous homomorphism \(f : (R, T) \to (S, U)\) in a topological ring \((S, U)\) either \(\ker f = R\) or \(f\) is a homeomorphism of \((R, T)\) on \(\text{Im}(f)\). Equivalently, \(R^2 \neq 0\) and \((R, T)\) is weakly simple and minimal. Let \(M\) be a unitary right \(R\)-module over a commutative ring \(R\) with 1. The module \(M\) is called divisible if \(Mr = M\) for every \(0 \neq r \in R\). A right \(R\)-module \(M\) is called faithful if \(Mr = 0\) implies \(r = 0\) (\(r \in R\)). A right \(R\)-module \(M\) is called torsion-free if \(mr = 0\) implies that either \(m = 0\) or \(r = 0\), where \(m \in M\) and \(r \in R\). Recall that a submodule \(N\) of an \(R\)-module \(M\) is called fully invariant if \(\alpha(N) \subseteq N\) for every endomorphism \(\alpha\) of \(M\).

Remark 2.4. If \(R\) is a von Neumann regular ring, then \(R^2 = R\).

Lemma 2.5. An ideal \(I\) of a von Neumann regular ring is von Neumann regular.

Proof. Let \(i \in I\). Thus there exists \(x \in R\) such that \(ixi = i\). It follows that \(ixixi = i\) and \(xix \in I\). \(\square\)

Corollary 2.6. If \(I\) is an ideal of a von Neumann regular ring \(R\), then any ideal \(H\) of \(I\) is an ideal of \(R\), too.

Proof. \(RH = RH^2 \subseteq IH \subseteq H\). Similarly, \(HR \subseteq H\). \(\square\)

If \(A_p\) is a \(p\)-elementary countable group, then

\[
I = \{ \alpha \in \text{End}(A_p) \mid |\text{Im}(\alpha)| < \aleph_0\}.
\]

Fix a linear basis \(\{v_i \mid i \in \mathbb{N}\}\) of \(A_p\) over the field \(\mathbb{F}_p\). Using this fixed basis, we define the map \(e_i : A \to A\) such that

\[
e_i(v_j) = \delta_{ij}v_j, \quad (i, j \in \mathbb{N})
\]

where \(\delta_{ij}\) is the Kronecker delta.

Lemma 2.7. We have for \(\text{End}(A_p)\):

(i) \(I\) is a von Neumann regular ring.

(ii) \(I\) is a simple ring.

(iii) The factor ring \(\text{End}(A_p)/I\) is simple von Neumann regular.

(iv) \(I\) is a locally finite ring.

Proof. (i): The ring \(\text{End}(A_p)\) is regular (see [21, Theorem 4.27, p. 63]), so \(I\) is von Neumann regular by Lemma 2.5.

(ii), (iii): The ideal \(I\) is the only nontrivial ideal of the ring \(\text{End}(A_p)\) (see [17, §17, Theorem 1, p. 93]). This means that \(\text{End}(A)/I\) is simple. It is regular by the part (i).

(iv) Since \(I\) is simple (see [17, §12, Proposition 1]), it suffices to show that \(I\) contains a nonzero locally finite right ideal.

Let us show that the left ideal \(Ie_1\) of \(I\) is locally finite as a ring (equivalently, as a \(\mathbb{F}_p\)-algebra). We have \(0 \neq e_1 \in Ie_1\). If \(H\) is the left annihilator of \(Ie_1\), then,
obviously, $H$ is a locally finite ring, hence it is locally finite as a $F_p$-algebra. We claim that $Ie_1/H$ is finite. Define $\beta_n \in H$ ($n \geq 2$) in the following way

$$\beta_n(v_i) = \begin{cases} v_n, & \text{for } i = 1; \\ 0, & \text{for } i \neq 1. \end{cases}$$

Let us prove that $Ie_1 = F_p e_1 + \sum_{n=2}^\infty F_p \beta_n$.

If $\alpha \in I$, then $\alpha(v_1) = r_1 v_1 + \cdots + r_n v_n$, where $r_i \in F_p$ and $n \in \mathbb{N}$, so

$$\alpha e_1(v_1) = r_1 e_1(v_1) + r_2 \beta_2(v_1) + \cdots + r_n \beta_n(v_1)$$

$$= (r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n)(v_1);$$

$$\alpha e_1(v_j) = (r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n)(v_j) \quad (j \neq 1).$$

This yields

$$\alpha e_1 = r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n$$

and so $Ie_1 = F_p e_1 + \sum_{n=2}^\infty F_p \beta_n$.

In particular, $Ie_1 = F_p e_1 + H$, and so $H$ has a finite index in $Ie_1$. Clearly, $Ie_1$ is a locally finite $F_p$-algebra (see [17, Proposition 1, p. 241]) and $I$ is a locally finite $F_p$-algebra (see [17, Proposition 2, p. 242]).

The next result can be deduced from [27, Lemma 36.11].

**Lemma 2.8.** Let $A$ be a locally compact, compactly generated, and totally disconnected ring. If $A$ contains a dense locally finite subring $B$, then $A$ is compact.

**Proof.** Let $A = \langle V \rangle$, where $V$ is a compact symmetric neighborhood of zero. Since $V$ is compact, the subset $V + V + V \cdot V$ also is compact. Since $B$ is dense, $A = B + V$. By compactness of $V + V + V \cdot V$ there exists a finite subset $H \subseteq B$ such that $V + V + V \cdot V \subseteq H + V$. Since $B$ is a locally finite ring, we can assume without loss of generality that $H$ is a subring. Let $H \setminus \{0\} = \{h_1, \ldots, h_k\}$. The subset

$$H + h_1 V + \cdots + h_k V + V$$

is an open subgroup of $R(+)$. Indeed, this subset is symmetric and

$$(H + h_1 V + \cdots + h_k V + V) + (H + h_1 V + \cdots + h_k V + V)$$

$$\subseteq H + h_1 (V + V) + \cdots + h_k (V + V) + V + V$$

$$\subseteq H + h_1 V + \cdots + h_k V + V.$$

We prove by induction on $m$ that

$$V^{[m]} \subseteq H + h_1 V + \cdots + h_k V + V, \quad (m \in \mathbb{N})$$

where $V^{[1]} = V$ and $V^{[m]} = V^{[m-1]} \cdot V$ for all $m$.

The inclusion is obvious for $m = 1$.

Assume that the assertion has been proved for $m \geq 1$. Clearly,

$$V^{[m+1]} = V^{[m]} \cdot V \subseteq H \cdot V + h_1 (V \cdot V) + \cdots + h_k (V \cdot V) + V \cdot V \subseteq$$

$$h_1 V + \cdots + h_k V + h_1 (H + V) + \cdots + h_k (H + V) + H + V \subseteq$$

$$H + h_1 V + \cdots + h_k V + V.$$

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Consequently, \( A = H + h_1V + \cdots + h_kV + V \), therefore \( A \) is compact. \( \Box \)

An element \( x \) of a topological ring is called discrete if there exists a neighborhood \( V \) of zero such that \( xV = 0 \) (i.e., the right annihilator of \( x \) is open).

**Lemma 2.9.** The set of all discrete elements of a topological ring is an ideal. A simple ring with identity does not contain nonzero discrete elements.

3. **Locally compact ring topologies on \( \text{End}(A) \) of a countable elementary abelian \( p \)-group \( A \)**

**Theorem 3.1.** Let \( R \) be a simple, nondiscrete and locally compact ring of \( \text{char}(R) = p > 0 \) and \( 1 \in R \). If \( V \) is a compact open subring of \( R \) and \( \{ e_\alpha \mid \alpha \in \Omega \} \) is a set of orthogonal idempotents in \( R \), then 
\[
|\Omega| \leq w(V).
\]

**Proof.** The ring \( R \) does not contain nonzero discrete elements by Lemma 2.9. Since \( R \) is locally compact and \( \text{char}(R) = p \), it is totally disconnected. Additionally, \( R \) has a fundamental system of neighborhoods of zero consisting of compact open subrings by [19, Lemma 9].

If \( V \) is a compact open subring of \( R \), then by continuity of the ring operations for each \( \alpha \in \Omega \) there exists an open ideal \( V_\alpha \) of \( V \) such that \( e_\alpha V_\alpha \subseteq V \). Clearly, there exists \( y_\alpha \in V_\alpha \) for which \( e_\alpha y_\alpha \neq 0 \) since \( R \) has no nonzero discrete elements.

We claim that hold the following two properties:

(i) \( e_\alpha y_\alpha \notin \{ e_\beta y_\beta \mid \beta \neq \alpha \} \) for each \( \alpha \in \Omega \);

(ii) the set \( X = \{ e_\alpha y_\alpha \mid \alpha \in \Omega \} \) is a discrete subspace of \( V \).

Indeed, if \( e_\alpha y_\alpha \in \{ e_\beta y_\beta \mid \beta \neq \alpha \} \) for some \( \alpha \in \Omega \), then
\[
e_\alpha y_\alpha = e_\alpha e_\alpha y_\alpha \in e_\alpha \{ e_\beta y_\beta \mid \beta \neq \alpha \} \subseteq \{ e_\alpha e_\beta y_\beta \mid \beta \neq \alpha \} = \{ 0 \},
\]
so \( e_\alpha y_\alpha = 0 \), a contradiction. The part (i) is proved.

(ii) Now, for each \( \alpha \in \Omega \) we have \( V \setminus \{ e_\beta y_\beta \mid \beta \neq \alpha \} \) is open and, consequently,
\[
(V \setminus \{ e_\beta y_\beta \mid \beta \neq \alpha \}) \cap X = \{ e_\alpha y_\alpha \},
\]
by (i). Therefore the point \( e_\alpha y_\alpha (\alpha \in \Omega) \) of \( X \) is isolated. In other words, the subspace \( X \) of \( V \) is discrete.

Since \( X \) is discrete, \( |\Omega| = |X| = w(X) \leq w(V) \) (see [1, Exercises 98-99, p. 72]). \( \Box \)

**Theorem 3.2.** Let \( A_p \) be a countable elementary abelian \( p \)-group. Then the ring
\[
I = \{ \alpha \in \text{End}(A_p) \mid |\text{Im}(\alpha)| < \aleph_0 \}
\]
does not admit a nondiscrete ring topology \( \mathcal{U} \) such that \( (I, \mathcal{U}) \) is a Baire space.
Proof. Put \( S_n = \{ \alpha \in I \mid \alpha(A) \subseteq F_p v_1 + \cdots + F_p v_n \} \), where \( n \in \mathbb{N} \). Clearly, \( I = \bigcup_{n \in \mathbb{N}} S_n \) and
\[
S_n = \{ \alpha \in I \mid e_i \alpha = 0 \text{ for } i > n \} = \text{Ann}_r(\{ e_k \mid k > n \}).
\]
This yields that the subset \( S_n \) is closed due the continuity of the ring operations.

Since \( I \) is a Baire space, there exists \( n \in \mathbb{N} \) such that \( \text{Int}(S_n) \neq \emptyset \), hence \( S_n \) is an open subgroup.

Set \( \beta \in I \) such that
\[
\beta(v_i) = \begin{cases} v_{n+i}, & \text{for } i = 1, \ldots, n; \\ 0, & \text{for } i > n. \end{cases}
\]
Let \( W \subseteq S_n \) be a neighborhood of zero of \( (I, U) \) such that \( \beta W \subseteq S_n \). If \( w \in W \setminus \{0\} \), then there exist \( a \in A \) and \( r_1, \ldots, r_n \in F_p \) such that
\[
0 \neq w(a) = \sum_{i=1}^{n} r_i v_i \quad \text{and} \quad \beta(w(a)) = \sum_{i=1}^{n} r_i v_{n+i}.
\]
There exists \( j \in 1, \ldots, n \) such that \( r_j \neq 0 \). Then
\[
eq w(a) = \sum_{i=1}^{n} r_i v_i \quad \text{and} \quad \beta(w(a)) = \sum_{i=1}^{n} r_i v_{n+i},
\]
hence \( e_{n+j} \beta w(a) = r_j v_{n+j} \neq 0 \), and so \( \beta w \notin S_n \), a contradiction. \( \square \)

Corollary 3.3. Under the notation of Theorem 3.2 the ring \( I \) does not admit a nondiscrete locally compact ring topology.

Proof. This follows from the fact that each locally compact space is a Baire space (see [6, Theorem 1, p. 117]). \( \square \)

Our main result is the following.

Theorem 3.4. The endomorphism ring \( \text{End}(A_p) \) of a countable elementary abelian \( p \)-group \( A_p \) does not admit a nondiscrete locally compact ring topology.

Proof. We use the notation and results from section 2. Denote \( R = \text{End}(A_p) \).

Assume on the contrary that there exists on \( R \) a nondiscrete locally compact ring topology \( T \).

Fact 1. The ring \( (R, T) \) has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Since the additive group of the ring \( R \) has exponent \( p \), it is totally disconnected (this follows from \([12, \text{Theorem 9.14, p. 95}]\)). By I. Kaplansky’s result (see \([19, \text{Lemma 9}]\)), the ring \( (R, T) \) has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Fact 2. The group \( Rv_i \) is countable for each \( i \in \mathbb{N} \).

We claim that \( Rv_i \) is infinite. Indeed, for each \( j \in \mathbb{N} \) put \( \beta_j \in R \) such that
\[
\beta_j(v_k) = \begin{cases} v_j, & \text{for } k = i; \\ 0, & \text{for } k \neq i. \end{cases}
\]
If \( j \neq s \), then \( \beta_j e_i(v_i) = \beta_j(v_i) = v_j \) and \( \beta_s e_i(v_i) = \beta_s(v_i) = v_s \), hence \( \beta_j e_i \neq \beta_s e_i \), so \( Rv_i \) is infinite.
The ring $Re_i$ is countable. Indeed, consider the mapping $\psi : Re_i \to \mathbb{F}_{p^{v_i}}$, where $$\psi(\alpha e_i)(rv_i) = \alpha(rv_i) \quad \text{for all} \quad r \in \mathbb{F}_p.$$ If $\alpha e_i \neq \beta e_i$ ($\alpha, \beta \in R$), then there exists an element $x = \sum_j r_j v_j \in A_p$ such that $\alpha e_i(x) \neq \beta e_i(x)$, hence, $\alpha(r_i v_i) \neq \beta(r_i v_i)$. Thus $$\psi(\alpha e_i)(r_i v_i) = \alpha(r_i v_i) \neq \beta(r_i v_i) = \psi(\beta e_i)(r_i v_i).$$ The latter means that $\psi$ is an injective mapping of $Re_i$ into $\mathbb{F}_{p^{v_i}}$. Since $\mathbb{F}_{p^{v_i}}$ is countable, $Re_i$ is countable, too.

Fact 3. $I$ is a closed ideal of $R$. We claim that $I$ is not dense in the topological ring $(R, T)$. Assume the contrary. Since $I$ is locally finite and is a maximal ideal, $(R, T)$ is topologically locally finite by Lemma 2.8. The ring $R$ contains two elements $x, y$ such that $xy = 1$ and $yx \neq 1$. The subring $(x, y)$ is compact, hence Dedekind-finite, a contradiction. We obtained that $(R/I, T/I)$ is a nondiscrete metrizable locally compact ring.

Fact 4. $I$ is a discrete ideal of $R$.

This follows from Theorem 3.2.

Fact 5. $Re_i$ is a discrete left ideal of $R$ for every $i \in \mathbb{N}$.

Indeed, $Re_i \subseteq I$ and $I$ is discrete by Fact 4 for every $i \in \mathbb{N}$.

Fact 6. $\text{Ann}_l(e_i)$ is open in $R$ for every $i \in \mathbb{N}$.

Indeed, the group homomorphism $q : R \to Re_i, r \mapsto re_i$, is continuous. Since $Re_i$ is discrete $q^{-1}(0) = \text{Ann}_l(e_i)$ is open.

Fact 7. $\cap_i \text{Ann}_l(e_i) = 0$.

Obvious.

Fact 8. $T \geq T_{\text{fin}}$.

We notice that $\text{Ann}(e_i) = T(\{v_i\})$ for every $i \in \mathbb{N}$. For, if $\alpha e_i = 0$, then $\alpha(v_i) = \alpha e_i(v_i) = 0$, i.e., $\alpha \in T(\{v_i\})$. Conversely, if $\alpha \in T(\{v_i\})$, then $\alpha e_i(v_i) = \alpha(v_i) = 0$. If $j \neq i$ then $\alpha e_i(v_j) = 0$. Therefore $\alpha e_i = 0$. Moreover $$T(\{v_1, \ldots, v_n\}) = \cap_{i=1}^n T(\{v_i\}) = \cap_{i=1}^n \text{Ann}_l(e_i) \in T \quad (\forall n \in \mathbb{N}).$$ Since the family $\{T(\{v_1, \ldots, v_n\})\}$ forms a fundamental system of neighborhoods of zero of $(R, T_{\text{fin}})$, we get that $T_{\text{fin}} \leq T$.

Fact 9. The ring $(R, T)$ is metrizable.

Since $\cap_{i \in \mathbb{N}} \text{Ann}_l(e_i) = 0$, the pseudocharacter of $(R, T)$ is $\mathbb{R}_0$. If $V$ is a compact open subring of $(R, T)$ (see Fact 1), then the pseudocharacter of $V$ also is $\mathbb{R}_0$. However in every compact space the pseudocharacter of a point coincides with its character. Therefore $(R, T)$ is metrizable.

Fact 10. $(R/I, T/I)$ has an open compact subring.

Indeed, it is well-known (see [19]) that every totally disconnected ring has a fundamental system of neighborhood of zero consisting of compact open subrings. Henceforth $V$ is a fixed open compact subring of $(R/I, T/I)$.

Fact 11. $(R/I)$ contains a family of orthogonal idempotents of cardinality $2^{\mathbb{R}_0}$.

Indeed, the family $\{e_i\}_{i \in \mathbb{N}}$ of idempotents of the ring $(R, T_{\text{fin}})$ is summable and $1_A = \Sigma_{n \in \mathbb{N}} e_n$, where $1_A$ is the identity of $R$. 

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The first ordinal number of cardinality \( \mathfrak{c} \) of continuum is denoted by \( \omega(\mathfrak{c}) \). Let \( \{ N(\alpha) \mid \alpha < \omega(\mathfrak{c}) \} \) be a family of infinite almost disjoint subsets of \( \mathbb{N} \) (see [8, Example 3.6.18, p. 175–176]). Put \( f_{N(\alpha)} = \sum_{i \in N(\alpha)} e_i \) for each \( \alpha < \omega(\mathfrak{c}) \). The element \( f_{N(\alpha)} \) exists by Lemma \( 2.3 \). Then:

(i) \( f_{N(\alpha)} \notin I \) for every \( \alpha < \omega(\mathfrak{c}) \);
(ii) \( f_{N(\alpha)} f_{N(\beta)} \in I \) for each \( \alpha, \beta < \omega(\mathfrak{c}) \) and \( \alpha \neq \beta \).

If \( g_\alpha = f_{N(\alpha)} + I \) for each \( \alpha < \omega(\mathfrak{c}) \), then \( \{ g_\alpha \mid \alpha < \omega(\mathfrak{c}) \} \) is the required system of orthogonal idempotents.

The subring \( V \) is metrizable (by Fact 9). Since \( V \) is compact and \( R/I \) is a simple von Neumann regular ring by Lemma \( 2.7 \) and \( \omega(V) \leq \aleph_0 \), we obtain a contradiction to Theorem \( 3.1 \). □

**Theorem 3.5.** (CH) Under the notation of Theorem \( 3.4 \), the ring \( R/I \) does not admit a nondiscrete locally compact ring topology.

**Proof.** Assume on the contrary that the factor ring \( R/I \) admits a nondiscrete locally compact ring topology \( T \), so \( (R/I, T) \) contains an open compact subring \( V \). Since the cardinality of \( R/I \) is continuum and \( V \) is infinite, the power of \( V \) is continuum. Since we have assumed (CH), the subring \( V \) is metrizable, hence second metrizable (see [14, 18]). However we have proved in Theorem 3.4 that the ring \( R/I \) contains a family of orthogonal idempotents of cardinality \( \mathfrak{c} \), a contradiction with Theorem 3.1. □

**Theorem 3.6.** The finite topology \( T_{fin} \) is the only second metrizable ring topology \( T \) on \( R \) for which \( (R, T_{fin}) \) is complete.

**Proof.** Let \( K = \langle F \rangle \), where \( F \) is a finite subset of \( A \). Clearly, there exists a subgroup \( A' \) of \( A \) such that \( A = K \oplus A' \). Choose \( e_F \in R \) such that \( e_F \mid K = \text{id}_K \) and \( e_F(A') = 0 \). Clearly,

\[ T(K) = R(1 - e_F) \]

and \( \alpha K = 0 \) if and only if \( \alpha \in R(1 - e_F) \), so the family \( \{ R(1 - e_F) \} \), where \( F \) runs over all finite subset of \( A \), forms a fundamental system of neighborhoods of zero for \( (R, T_{fin}) \).

There exists an injective map of \( Re_F \) to \( \text{Hom}(K, A) \), so the left ideal \( Re_F \) is countable, due to countability \( \text{Hom}(K, A) \). Since \( e_F^2 = e_F \), the Peirce decomposition

\[ R = Re_F \oplus R(1 - e_F) \]

of \( R \) with respect to the idempotent \( e_F \) is a decomposition of the topological group \( (R, +, T) \). It follows that \( Re_F \) is discrete, hence \( R(1 - e_F) \) is open (in the topology \( T \)). Hence \( T \supseteq T_{fin} \), so \( T = T_{fin} \) (see [9, Theorem 30] or [11]). □

4. Completely simple topological endomorphism rings of vector spaces

**Theorem 4.1.** Let \( Af \) be a right vector space over a division ring \( F \) and \( S = \text{End}(Af) \). The following conditions are equivalent:

(i) \( (S, T_{fin}) \) is a completely simple topological ring.
(ii) $\dim(A_F) = \infty$ or $\dim(A_F) < \infty$ and $F$ does not admit a nondiscrete ring topology.

Proof. (i) $\Rightarrow$ (ii): If $A_F$ is finite-dimensional, then $S$ is discrete and isomorphic to the matrix ring $M(n, F)$, where $n$ is the dimension of $A_F$. Then, obviously, $F$ does not admit a nondiscrete ring topology.

(ii) $\Rightarrow$ (i): If $\dim(A_F) = n < \infty$, then $S \cong M(n, F)$. Since $F$ does not admit nondiscrete ring topologies, the same holds for $M(n, F)$.

Let $A_F$ be infinite dimensional. Fix a basis $\{x_\alpha\}_{\alpha \in \tau}$ over $F$, where $\tau$ is an infinite ordinal number. It is well-known that the topological ring $(S, T_{fin})$ is weakly simple (see [22, Satz 12, p. 258]) and the family $\{T(x_\alpha)\}_{\alpha \in \tau}$ is a prebase at zero for the finite topology $T_{fin}$ of $S$.

Assume on the contrary that there exists a Hausdorff ring topology $T$, coarser than $T_{fin}$ and different from it. Let $e_\alpha \in S$ such that $e_\alpha^2 = e_\alpha$ and $e_\alpha(x_\beta) = \delta_{\alpha, \beta}x_\alpha$ for each $\alpha < \tau$, where $\delta_{\alpha, \beta}$ is the Kronecker delta.

Fact 1. $T(x_\alpha) = \text{Ann}_F(e_\alpha)$ for each $\alpha < \tau$.

Indeed, if $p \in T(x_\alpha)$, then $pe_\alpha(x_\alpha) = p(x_\alpha) = 0$. If $\beta \neq \alpha$, then $e_\alpha(x_\beta) = 0$, hence $pe_\alpha = 0$, i.e. $p \in \text{Ann}_F(e_\alpha)$. Conversely, if $pe_\alpha = 0$, then we have $p(x_\alpha) = pe_\alpha(x_\alpha) = 0$, i.e. $p \in T(x_\alpha)$.

Fact 2. There exists $\alpha_0 < \tau$ for which $Se_{\alpha_0}$ is nondiscrete in $(S, T)$.

Assume on the contrary that for every $\alpha < \tau$ there exists a neighborhood $V_\alpha$ of zero of $(S, T)$ such that $Se_{\alpha} \cap V_\alpha = 0$. If $U_\alpha$ is a neighborhood of zero of $(S, T)$ such that $U_\alpha e_\alpha \subseteq V_\alpha$, then $U_\alpha e_\alpha = 0$, hence $\text{Ann}_F(e_\alpha) = T(x_\alpha)$ is open in $(S, T)$. Hence $T_{fin} \subseteq T$ and $T = T_{fin}$, a contradiction.

Fact 3. $(Se_{\alpha_0} \cap V)x_{\alpha_0} \nsubseteq \bigoplus_{\beta \in K} x_{\beta} F$ for any neighborhood $V$ of zero of $(S, T)$ and any finite subset $K$ of the set $[0, \tau)$ of all ordinal numbers less than $\tau$.

Assume on the contrary that there exists a finite subset $K$ of $[0, \tau)$ and a neighborhood $V$ of zero of $(S, T)$ such that

$$
(S e_{\alpha_0} \cap V)x_{\alpha_0} \subseteq \bigoplus_{\beta \in K} x_{\beta} F.
$$

Fix $\gamma \in [0, \tau) \setminus K$. For each $\beta \in K$ define $q_\beta \in S$ such that $q_\beta(x_\beta) = x_\gamma$ and $q_\beta(x_\delta) = 0$ for $\delta \neq \beta$.

Let $V_0$ be a neighborhood of zero of $(S, T)$ such that $V_0 \subseteq V$ and $q_\beta V_0 \subseteq V$ for all $\beta \in K$. There exists $0 \neq h \in Se_{\alpha_0} \cap V_0$ by Fact 2 and $hx_{\alpha_0} \neq 0$ by Fact 1. Since $Se_{\alpha_0} \cap V_0 \subseteq Se_{\alpha_0} \cap V$, we obtain that $hx_{\alpha_0} = \Sigma_{\beta \in K} x_{\beta} f_{\beta}$, $(f_{\beta} \in F)$ by (4.1). There exists $\beta_0 \in K$ such that $f_{\beta_0} \neq 0$ (because $hx_{\alpha_0} \neq 0$), so

$$
q_{\beta_0} h = q_{\beta_0}(\Sigma_{\beta \in K} x_{\beta} f_{\beta}) = \xi_{\beta_0} x_\gamma \notin \bigoplus_{\beta \in K} x_{\beta} F,
$$

a contradiction. Therefore Fact 3 is proved.

Now let $V$ be a neighborhood of zero of $(S, T)$. Pick up a neighborhood $V_0$ of zero of $(S, T)$ such that $V_0 \cdot V_0 \subseteq V$. Since $T \leq T_{fin}$, there exists a finite subset $K$ of $[0, \tau)$ such that

$$
T(\{x_{\beta} \mid \beta \in K\}) \subseteq V_0.
$$
We have \((S e_{\alpha} \cap V_0) x_{\alpha} \not\subseteq \oplus_{\beta \in K} x_{\beta} F\) by Fact 3. It follows that there exists \(q \in S e_{\alpha} \cap V_0\) such that
\[
q(x_{\alpha}) \not\subseteq \oplus_{\beta \in K} x_{\beta} F.
\]
Clearly, \(q(x_{\alpha}) \in A_F\), so it can be written as \(q(x_{\alpha}) = \sum_{\alpha < \tau} x_{\alpha} f_{\alpha}\), where \(f_{\alpha} \in F\) and there exists \(\beta_0 \notin K\) such that \(f_{\beta_0} \neq 0\).

Consider the element \(s \in S\) such that \(s(x_{\beta_0}) = x_{\alpha} f_{\beta_0}^{-1}\) and \(s(x_{\lambda}) = 0\) for \(\lambda \neq \beta_0\). Evidently, \(s \in T(K)\), hence
\[
sq \in T(K) \cdot V_0 \subseteq V_0 \cdot V_0 \subseteq V.
\]
Moreover, \(sq(x_{\alpha}) = s(x_{\beta_0} f_{\beta_0} + \cdots) = x_{\alpha}\). Since \(q \in S e_{\alpha}\), we obtain that \(sq(x_{\alpha}) = 0\) for \(\beta \neq \alpha_0\). Consequently, \(e_{\alpha_0} = sq \in V\) for every neighborhood \(V\) of zero of \((S, T)\), a contradiction. \(\square\)

**Remark 4.2.** The question of existence of an uncountable division ring which does not admit a nondiscrete Hausdorff ring topology is open. Several results on this topic can be found in Chapter 5 of [2].

**Theorem 4.3.** Let \(\prod_{\alpha \in \Omega} R_\alpha\) be a family of compact rings with identity. Then the product \(\prod_{\alpha \in \Omega} R_\alpha\) is a minimal ring if and only if every \((R_\alpha, T_\alpha)\) is a minimal topological ring. (Here \(\prod_{\alpha \in \Omega} T_\alpha\) is the product topology on the ring \(\prod_{\alpha \in \Omega} R_\alpha\).)

**Proof.** \(\Rightarrow\): Assume on the contrary that there exists \(\beta \in \Omega\) and a ring topology \(T'\) on \(R_\beta\) such that \(T' \leq T_\beta\) and \(T' \neq T_\beta\). Consider the product topology \(U\) on \(\prod_{\alpha \in \Omega} R_\alpha\), where \(R_\alpha\) is endowed with \(T_\alpha\) when \(\alpha \neq \beta\) and \(R_\beta\) is endowed with \(T'\). Obviously, \(U \leq \prod_{\alpha \in \Omega} T_\alpha\) and \(U \neq \prod_{\alpha \in \Omega} T_\alpha\), a contradiction.

\(\Leftarrow\): Denote by \(\pi_\alpha(\alpha \in \Omega)\) the projection of \(\prod_{\alpha \in \Omega} R_\alpha\) on \(R_\alpha\). By definition of the product topology, \(\prod_{\alpha \in \Omega} T_\alpha\) is the coarsest topology on \(\prod_{\alpha \in \Omega} R_\alpha\) for which the projections \(\pi_\alpha(\alpha \in \Omega)\) are continuous.

Let \(U\) be a ring topology on \(\prod_{\alpha \in \Omega} R_\alpha\), \(U \leq \prod_{\alpha \in \Omega} T_\alpha\) and \(\beta \in \Omega\). Since
\[
U \mid_{R_\beta \times \prod_{\gamma \neq \beta} R_\gamma(\{\gamma\})},
\]
it follows that \(U \mid_{R_\beta \times \prod_{\gamma \neq \beta} R_\gamma(\{\gamma\})}) = (\prod_{\alpha \in \Omega} T_\alpha) \mid_{R_\beta \times \prod_{\gamma \neq \beta} R_\gamma(\{\gamma\})}\) by minimality of \((R_\beta, T_\beta)\).

Then the family \(\{V \times \prod_{\gamma \neq \beta} \{\gamma\}\}\) when \(V\) runs all neighborhoods of zero of \((R_\beta, T_\beta)\) is a fundamental system of neighborhoods of zero of
\[
(R_\beta \times \prod_{\gamma \neq \beta} \{\gamma\}), \quad U \mid_{R_\beta \times \prod_{\gamma \neq \beta} R_\gamma(\{\gamma\})}).
\]
Since \(R_\beta \times \prod_{\gamma \neq \beta} \{\gamma\}\) is an ideal with identity of \(\prod_{\alpha \in \Omega} R_\alpha\), the topological ring \((\prod_{\alpha \in \Omega} R_\alpha, U)\) is a direct sum of ideals \(R_\beta \times \prod_{\gamma \neq \beta} \{\gamma\}\) and \(\{0\} \times \prod_{\gamma \neq \beta} R_\gamma\). Let \(V\) be a neighborhood of zero of \((R_\beta, T_\beta)\). Then \(V \times \prod_{\gamma \neq \beta} R_\gamma\) be a neighborhood of zero of \((\prod_{\alpha \in \Omega} R_\alpha, U)\) and \(\pi_\beta(V \times \prod_{\gamma \neq \beta} R_\gamma) = V\).

We have proved that \(\pi_\beta\) is a continuous function from \((\prod_{\alpha \in \Omega} R_\alpha, U)\) to \((R_\beta, T_\beta)\). It follows that \(\prod_{\alpha \in \Omega} T_\alpha \leq U\) and so \(U = \prod_{\alpha \in \Omega} T_\alpha\). \(\square\)
Corollary 4.4. A left linearly compact semisimple ring is minimal if and only if has no direct summands of the form $M(n, \Delta)$, where $\Delta$ is a division ring which does not admit a nondiscrete Hausdorff ring topology.

Proof. This follows from Theorems 4.1, 4.3 and the Theorem of Leptin (see [234]) about the structure of left linearly compact semisimple rings.

Corollary 4.5. A semisimple linearly compact ring $(R, T)$ having no ideals isomorphic to matrix rings over infinite division rings is minimal.

5. Completely simple endomorphism rings of modules

The endomorphism ring of a right $R$-module $M$ is denoted by End $(M_R)$.

Lemma 5.1. Let $M$ be a divisible, torsion-free module over a commutative domain $R$ and $K$ the field of fractions of $R$. The additive group of $M$ has a structure of a vector $K$-space such that $R$-endomorphisms of $M$ are exactly the $K$-linear transformations.

Proof. We define a structure of a right vector $K$-space as follows: if $\frac{m}{n} \in K$ and $m \in M$, then there exists a unique $x \in M$ such that $ma = xb$: set $m \circ \frac{a}{n} = x$. Moreover, if $\frac{a}{n} = \frac{b}{n}$ and $0 \neq m \in M$, then $m \circ \frac{a}{n} = m \circ \frac{b}{n}$. Indeed, if $m \circ \frac{a}{n} = x$ and $m \circ \frac{b}{n} = y$, then $mad = xbd$ and $mbe = ybd$ which means that $xbd = ybd$, hence $x = y$.

Let $\alpha \in \text{End} (M_R)$, $\frac{a}{n} \in K$, $m \in M$. By definition, $am = b(\frac{a}{n} \circ m)$, hence, $\alpha \circ m = b(\frac{a}{n} \circ m)$, which means that $\alpha(\frac{a}{n} \circ m) = \frac{a}{n} \circ \alpha(m)$, so $\alpha$ is a $K$-linear transformation. Note that, if $a \in R$ and $m \in M$, then $m \circ \frac{a}{n} = ma$.

Conversely, if $\alpha$ is a $K$-linear transformation, $a \in R$, $m \in M$, then $\alpha(\frac{a}{n} \circ m) = \frac{a}{n} \circ \alpha(m)$, i.e. $\alpha(am) = \alpha(m)$. We have proved that every $K$-linear transformation is an right $R$-module homomorphism.

Remark 5.2. The center $Z(R)$ of a weakly simple ring $R$ is a domain.

Remark 5.3. For every right $R$-module $M$ the underlying group $M(+)$ is a discrete left topological $(\text{End} (M_R), T_{fin})$-module.

Indeed, $T(m)(m) = 0$ for every $m \in M$. Moreover, $\text{End} (M_R)(\{0\}) = \{0\}$, so $M$ is a discrete left topological $(\text{End} (M_R), T_{fin})$-module.

Theorem 5.4. Let $M_R$ be a module over a commutative ring $R$.

If the topological ring $(\text{End} (M_R), T_{fin})$ is weakly simple, then:

(i) $P = \{ r \in R \mid Mr = 0 \}$ is a prime ideal of $R$.

(ii) $M$ is a vector space over the field $K$ of fractions of $R/P$ and the $R$-endomorphisms of $M$ are exactly the $K$-linear transformations.

Conversely, if $M_R$ is an $R$-module and are satisfied (i) and (ii), then the ring $(\text{End} (M_R), T_{fin})$ is a weakly simple topological ring.
Completely simple endomorphism rings of modules

Proof. $\Rightarrow$: If $(\End(M_R), T_{fin})$ is weakly simple, then the mapping:

$$\alpha_r : M \to M, \quad m \mapsto mr \quad (r \in R)$$

is an $R$-module homomorphism and $\alpha_r \in Z(= \text{the center of } \End(M_R))$.

First we show that the part (i) holds. Indeed, if $a, b \in R$ and $ab = 0$, then $\alpha_a \alpha_b = 0$ (see (5.1)). Thus $(\End(M_R)\alpha_a) \cdot (\End(M_R)\alpha_b) = 0$, so

$$(\End(M_R)\alpha_a) \cdot (\End(M_R)\alpha_b) = 0.$$ 

Since $\End(M_R)$ is weakly simple, one of them, say $\End(M_R)\alpha_a$, is zero. This implies that $\alpha_a = 0$, hence $a \in P$.

(ii) The structure of $R/P$-module on $M$ is defined as follows: if $r \in R$ and $m \in M$, then put $M(r + P) = mr$.

Note that $M$ is a torsion-free right $R/P$-module. Assume that $m(r + P) = 0$, where $0 \neq r + P \in R/P$ and $0 \neq m \in M$. Then $mr = 0 = \alpha_r(m)$ (see (5.1)). Thus $\End(M_R)\alpha_r(m) = 0$. It follows that $(\End(M_R)\alpha_r)(m) = 0$ by Remark 5.3. Since $\End(M_R)$ is weakly simple

$$\End(M_R)\alpha_r = \End(M_R).$$

We obtained that $\End(M_R)(m) = 0$, so $m = 0$, a contradiction.

Under this convention $R$-submodules are exactly $R/P$-submodules and $R$-endomorphisms are exactly $R/P$-endomorphisms.

The module $M$ is a divisible $R/P$-module. Indeed, if $0 \neq r + P \in R/P$, then $0 \neq M(r + P) = Mr$. Suppose that $Mr \neq M$. Consider

$$I = \{ \alpha \in \End(M_R) \mid \alpha(M) \subseteq Mr \}.$$ 

Since $Mr$ is a fully invariant submodule, $I$ is a two-sided ideal of the ring $(\End(M_R), T_{fin})$.

The ideal $I$ is closed. Indeed, let $\alpha \in \overline{T}$. If $m \in M$, then there exists $\beta \in I$ such that $\alpha - \beta \in T(m)$. Clearly, $\alpha(m) = \beta(m) \in Mr$ and so $\alpha \in I$. We have proved that $I$ is closed.

Since $1_M \notin I$, $I = 0$. It follows that $\alpha_r = 0$ (see (5.1)), a contradiction.

The module $M$ has a structure of a right $K$-vector space and $\End(M_R)$ is exactly the ring of endomorphisms of $M$ by Lemma 5.1.

The converse follows from Theorem 4.1. $\Box$

A characterization of completely simple topological ring $\End(M_R)$ is given by the following.

Theorem 5.5. Let $M_R$ be a module over a commutative ring $R$. The topological ring $(\End(M_R), T_{fin})$ is completely simple if and only are satisfied the conditions (i) and (ii) of Theorem 5.4 and either

(i) $M$ is finite or

(ii) $M$ is infinite and the dimension of $M$ over the field $K$ is infinite.

Proof. $\Rightarrow$: According to Theorem 5.4, the ideal $P$ is prime and the topology of $\End(M_R)$ coincide with the finite topology of $\End(M_K)$, where $K$ is the field of fractions of $R/P$. If $M$ is finite, we have the part (i). Assume that
$M$ is infinite. If $R/P$ is finite, then the dimension of $M$ over $K$ is infinite. Suppose that $R/P$ is infinite and $\dim_K(M) = n < \aleph_0$. Then $M$ is isomorphic to $M(n,K)$. Since $K$ is an infinite field, it admits a nondiscrete ring topology (see [13]) and we obtain a contradiction because $\text{End}(M_R)$ is a discrete ring. Consequently $\dim_K(M)$ is infinite.

$\Leftarrow$ This follows from Theorems 4.1 and 5.4. $\square$

**Corollary 5.6.** The topological ring $(\text{End}(A), T_{fin})$ of an abelian group $A$ is completely simple if and only one of the following conditions holds:

(i) $A$ is a elementary abelian $p$-group.

(ii) $A$ is a divisible torsion-free group of infinite rank.

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**References**


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