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Robust Control Strategies for Unstable Systems with Input/Output Delays

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Ricardo Sanz

Abstract

Time-delay systems are ubiquitous in many engineering applications, such as mechanical or fluid transmissions, metallurgical processes or networked control systems. Time-delay systems have attracted the interest of control researchers since the late 50's. A wide variety of tools for stability and performance analysis has been developed, specially over the past two decades.

This thesis is focused on the problem of stabilizing systems that are affected by delays on the actuator and/or sensing paths. More specifically, the contributions herein reported aim at improving the performance of existing controllers in the presence of external disturbances. Time delays unavoidably degrade the control loop performance. Disturbance rejection has been a matter of concern since the first predictive controllers for time-delay systems emerged. The key idea of the strategies presented in this thesis is the combination of predictive controllers and disturbance observers. The latter have been successfully applied to improve the disturbance rejection capabilities of conventional controllers. However, the application of this methodology to time-delay systems is rarely found in the literature. This combination is extensively investigated in this thesis.

Another handicap of predictive controllers has to do with their implementation, which can induce instability if not done carefully. This issue is related to the fact that predictive control laws take the form of integral equations. An alternative control structure that avoids this problem is also reported in this thesis, which employs an infinite-dimensional observer, governed by a hyperbolic partial differential equation.

Resumen

Los sistemas con retardo temporal aparecen con frecuencia en el ámbito de la ingeniería, por ejemplo en transmisiones hidráulicas o mecánicas, procesos metalúrgicos o sistemas de control en red. Los retardos temporales han despertado el interés de los investigadores en el ámbito del control desde finales de los años 50. Se ha desarrollado una amplia gama de herramientas para el análisis de su estabilidad y prestaciones, especialmente durante las dos últimas décadas.

Esta tesis se centra en la estabilización de sistemas afectados por retardos temporales en la actuación y/o la medida. Concretamente, las contribuciones que aquí se incluyen tienen por objetivo mejorar las prestaciones de los controladores existentes en presencia de perturbaciones. Los retardos temporales degradan, inevitablemente, el desempeño de un bucle de control. No es de extrañar que el rechazo de perturbaciones haya sido motivo de estudio desde que emergieron los primeros controladores predictivos para sistemas con retardo. Las estrategias presentadas en esta tesis se basan en la combinación de controladores predictivos y observadores de perturbaciones. Estos últimos han sido aplicados con éxito para mejorar el rechazo de perturbaciones de controladores convencionales. Sin embargo, la aplicación de esta metodología a sistemas con retardo es poco frecuente en la literatura, la cual se investiga exhaustivamente en esta tesis.

Otro inconveniente de los controladores predictivos está relacionado con su implementación, que puede llevar a la inestabilidad si no se realiza cuidadosamente. Este fenómeno está relacionado con el hecho de que las leyes de control predictivas se expresan mediante una ecuación integral. En esta tesis se presenta una estructura de control alternativa que evita este problema, la cual utiliza un observador de dimensión infinita, gobernado por una ecuación en derivadas parciales de tipo hiperbólico.

Resum

Els sistemes amb retard temporal es troben amb freqüència a l'enginyeria, per exemple en transmissions hidràuliques o mecàniques, processos metal·lúrgics o sistemes de control en red. Els retards temporals han despertat el interés dels investigadors de control des de finals dels anys 50. S'han desenvolupat una àmplia gamma de ferramentes per a l'anàlisi de la seua estabilitat i prestacions, especialment durant les dos últimes dècades.

Aquesta tesi s'ocupa de l'estabilització de sistemes afectats per retards temporals en l'actuació o la mesura. Concretament, les contribucions que ací s'inclouen tenen com a objectiu millorar les prestaciones dels controladors existents en presència de pertorbacions. Els retards temporals degraden, inevitablement, les prestacions d'un bucle de control. No és d'extranyar doncs, que el rebuig de pertorbacions haja sigut motiu d'estudi d'ençà que van emergir els primer controladors predictius per a sistemes amb retard. Les estratègies presentades en aquesta tesi es basen en la combinació de controladors predictius amb observadors de pertorbacions. Aquests últims han sigut aplicats amb èxit per a millorar el rebuig de pertorbacions de controladors convencionals. No obstant això, l'aplicació d'aquesta metodologia a sistemes amb retard es poc freqüent a la literatura, la qual s'investiga exhaustivament en aquesta tesi.

Un altre inconvenient dels controladors predictius està relacionat amb la seua implementació, la qual pot ocasionar inestabilitat si no es realitza amb compte. Aquest fenòmen està relacionat amb el fet que les lleis de control predictives s'expressen mitjançant una eqüació integral. En aquesta tesi es presenta una estructura de control alternativa que evita aquest problema, la qual fa servir un observador de dimensió infinita, governat per una ecuació en derivades parcials de tipus hiperbòlic.

Motivation and thesis outline

One of the main difficulties when designing a control system is, undoubtedly, the presence of time delays, specially if the system being controlled is open-loop unstable. It should be remarked that delays are not necessarily intrinsic to the process itself, but they can be originated when sensing and/or actuating. Therefore, even if small, delays arise in any digitally implemented control system.

This thesis is focused on the design of robust control strategies for systems with input and/or output delays. During the past two decades, the study of time-delay has advanced significantly, so that ample tools for analysis and design are now available. Most of the effort has been put towards finding less conservative stability conditions. Regarding control strategies, predictive controllers are still the cornerstone to stabilize exponentially unstable time-delay systems. Since they emerged in the late 50's, disturbance rejection has been an issue, which is still a matter of research. Recall that most industrial processes operate on regulatory basis, making disturbance rejection the central goal. Another concern with predictive controllers has to do with their implementations, which may lead to instability if not done carefully.

The objectives of this thesis can be summarized as follows:

- Improve disturbance attenuation of predictive controllers by using disturbance observers
- Develop stability conditions and design procedures that can be used to tune the proposed controllers intuitively
- Explore control techniques that avoid distributed integral terms and see how they can be combined with disturbance observers

Thesis outline

This thesis is structured into three parts. Part I provides an introduction to time-delay systems. Fundamental concepts regarding their solution, stability notions and analysis tools are revisited. A special emphasis is put on the review of control strategies. Results reported in Part II are concerned with disturbance rejection improvement of predictor-based controllers. This part starts with an introductory chapter that reviews disturbance observers and its limitations when applied to time-delay systems. The rest of the chapters are based on publications derived from this thesis, which are detailed in Section 3.3. Finally, Part III takes a slightly different perspective, as it is based on the modeling of the delay phenomenon by means of partial differential equations. An introductory chapter to this approach is also given. The content of the rest of the chapters, which are concerned with avoiding the implementation issues of predictive control laws, is detailed in Section 8.3.

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Glossary

ADRC	Active Disturbance Rejection Control
DOB	Disturbance Observer
DOBC	Disturbance Observer Control
DTC	Dead-Time Compensator
ESO	Extended State Observer
FSA	Finite Spectrum Assignment
FSP	Filtered Smith Predictor
GM	Gain Margin
GP	Generalized Predictor
GSP	Generalized Smith Predictor
IAE	Integral Absolute Error
LHS	Left-Hand Side
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant
LTV	Linear Time-Varying
MIMO	Multiple-Input Multiple-Output
MSP	Modified Smith Predictor
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
PID	Proportional Integral Derivative
PM	Phase Margin
RHS	Right-Hand Side
SISO	Single-Input Single-Output
SP	Smith Predictor
TDS	Time Delay System
UAV	Unmanned Aerial Vehicle
UDE	Uncertainty and Disturbance Estimator

Part I

Time-delay systems

Chapter 1

Introduction

Time-delay systems (TDSs) belong to a class of functional differential equations that are infinite-dimensional, as opposed to ordinary differential equations (ODEs). Delays often arise in many engineering applications. They may be intrinsic to the process being controlled (chemical reactions, drilling or milling processes) or originated when implementing the control strategy at the sensor or actuator paths (communication delays, jitter, slow actuators). Stability analysis of TDSs involves several challenges. In spite of being linear systems, they possess in general an infinite number of poles, which makes their analysis in the frequency domain complicated. This chapter is focused on the time-domain approach. The generalization of the direct Lyapunov method for TDSs is presented, known as the Lyapunov-Krasovskii theorem. This approach leads to stability conditions in terms of linear matrix inequalities, for which a short introduction is also given. This chapter ends with some simple stability conditions, derived upon Lyapunov-Krasovskii functions via the so-called descriptor method.

1.1 Solution concept and state of a TDS

In many physical and biological phenomena, the rate of variation in the system state depends on past states. Time delays have been also observed to occur in many engineering systems, such as mechanical and fluid transmissions, metallurgical processes, exhaust gas recirculation in diesel engines or networked control systems (Sipahi et al. 2011). Even if the process being controlled has no inherent delays, they often appear in the digital implementation of the controller, where sampling and actuation introduce delays. Delays are unavoidable when performing measurements because of the sampling process, which can be significant if additional filtering techniques have to be computed. The computational time that it takes for the processing unit to compute the control algorithm may be also relevant in some cases. The communication between the processing unit and the actuator is another source of delays (see Fig. 1.1).

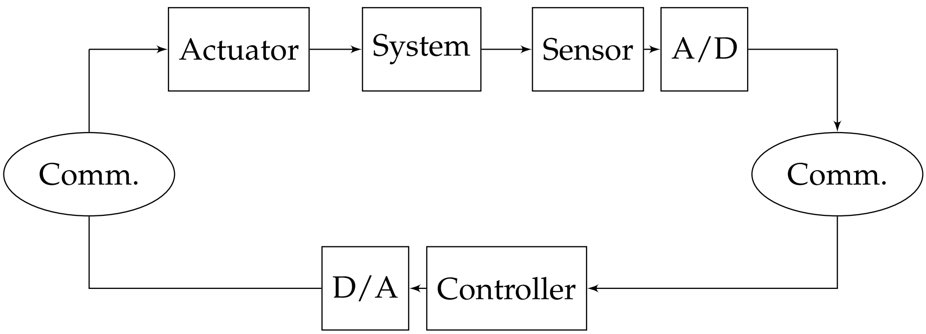


Figure 1.1: Structure of a digital controller implementation

A general linear time-invariant (LTI) system with a single constant delay $h > 0$ has the form

$$\dot{x}(t) = Ax(t) + A_1 x(t-h), \qquad t \geq 0 \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $A, A_1 \in \mathbb{R}^{n\times n}$ are constant matrices. A fundamental difference with respect to ODEs shows up when one tries to find the solution of (1.1). Instead of an initial value $x(0)$, an initial value function $x(s) = \phi(s)$, $\forall s \in [-h, 0]$ is needed to compute the solution of (1.1) over the interval $[0, h]$. Therefore, a proper state for a TDS is given by a function

$$x_t : [-h, 0] \to \mathrm{R}^n : \qquad x_t(\theta) = x(t+\theta), \forall\theta \in [-h, 0].$$

TDSs are then infinite-dimensional. Indeed, partial differential equations of transport type can be used to model the delay phenomenon. The reader is referred to

Part III of this thesis for further details about this approach. Another feature that distinguish a TDS from an ODE is the fact that the solution of the former may converge to zero in finite time (Fridman 2014).

The solution to (1.1) over the interval $t \in [0, h]$ needs of the initial condition $x_0(\theta) = \phi(\theta)$. Once it has been obtained, the same procedure can be performed for $t \in [h, 2h]$, $t \in [2h, 3h]$ and so forth.

Example 1.1. *(Fridman 2014) Consider the TDS $\dot{x}(t) = -x(t-h)$ with $\phi(\theta) = 1$, $\theta \in [-h, 0]$ and $h = 1$. The solution can be obtained by solving*

$$t \in [0, h], \qquad \dot{x}(\theta) = -\phi(t-h) = -1, \qquad x(0) = 1,$$

which yields $x(t) = 1 - t$, $\forall t \in [0, h]$. The solution over the next interval can be obtained by solving

$$t \in [h, 2h], \qquad \dot{x}(\theta) = -x(\theta - h) = -(1 - \theta + h), \qquad x(h) = 0,$$

which yields $x(t) = t^2/2 - 2t + 3/2$, $\forall t \in [h, 2h]$. The method can be repeated leading to polynomial in t solutions.

In order to obtain a general solution to (1.1), let us define the fundamental matrix $\Phi(t)$ that satisfies the equation

$$\dot{\Phi}(t) = A\Phi(t) + A_1\Phi(t-h), \tag{1.2}$$

with the initial condition $\Phi(t) = 0$, $\forall t < 0$ and $\Phi(t) = I$, $t = 0$. Then, the solution to (1.1) is given by (Bellman et al. 1963)

$$x(t) = \Phi(t)\phi(0) + \int_{-h}^{0} \Phi(t - \theta - h)A_1\phi(\theta)\,\mathrm{d}\theta. \tag{1.3}$$

Among other ways, (1.3) can be proved using the Laplace transform, which is defined by

$$F(s) = \mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

Taking the Laplace transform of (1.1), yields

$$sX(s) - \phi(0) = AX(s) + A_1[e^{-sh}X(s) + \int_{-h}^{0} e^{-s(\theta+h)}\phi(\theta)\,\mathrm{d}\theta].$$

which one can solve for $X(s)$, leading to

$$X(s) = \Delta^{-1}(s)[\phi(0) + \int_{-h}^{0} e^{-s(\theta+h)}\phi(\theta)\,\mathrm{d}\theta], \tag{1.4}$$

where $\Delta(s) = sI - A - A_1 e^{-sh}$. From (1.2), it can be seen that $X(t) = \mathcal{L}^{-1}\{\Delta^{-1}\}(t)$ and thus $\mathcal{L}\{X(t-\theta-h)\} = e^{-s(\theta+h)}\Delta^{-1}(s)$. Taking the inverse Laplace transform of (1.4) yields (1.3).

1.2 Systems with input and output delays

Delays often arise in practice when sensing and/or actuating. An LTI system with a single input delay $h > 0$, can be represented in state-space form as

$$\dot{x}(t) = Ax(t) + Bu(t-h), \tag{1.5}$$
$$y(t) = Cx(t). \tag{1.6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$ and A, B, C are constant matrices of appropriate dimensions. Most of the contributions reported in this thesis deal with systems in such form. Systems with measurement delays can be represented in a similar fashion,

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1.7}$$
$$y(t) = Cx(t-h). \tag{1.8}$$

Both (1.5)-(1.6) and (1.7)-(1.8) are input-output equivalent. An alternative representation in the Laplace domain is given by

$$y(s) = G(s)e^{-sh}u(s), \tag{1.9}$$

where $G(s) = C(sI-A)^{-1}B$ is the delay-free part of the plant. Also, the variable $\bar{y}(s) = G(s)u(s)$ is referred to as the non-delayed output.

1.3 Stability notions

A general retarded differential equation can be represented as

$$\dot{x}(t) = f(t, x_t), \qquad \forall t \geq t_0, \tag{1.10}$$

where $f : \mathbb{R} \times C[-h, 0] \to \mathbb{R}^n$ is continuous in both arguments, locally Lipschitz in the second argument and $C[-h, 0]$ denotes the Banach space of continuous functions equipped with the norm

$$\|x_t\|_C = \max_{\theta \in [-h,0]} |x(t+\theta)|.$$

It is assumed that $f(t, 0) = 0$, which guarantees that (1.10) has a trivial solution $x(t) \equiv 0$. The trivial solution of (1.10) is (Fridman 2014):

- *uniformly stable* if $\forall t_0 \in \mathbb{R}$ and $\forall \epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that $\|x_{t_0}\|_C < \delta(\epsilon)$ implies $|x(t)| < \epsilon$ for all $t \geq t_0$
- *uniformly asymptotically stable* if it is uniformly stable and there exists a $\delta_a > 0$ such that for any $\eta > 0$ there exists a $T((\delta_a, \eta)$ such that $\|x_{t_0}\|_C < \delta_a$ implies $|x(t)| < \eta$ for all $t \geq t_0 + T(\delta_a, \eta)$ and $t_0 \in \mathbb{R}$
- *globally uniformly asymptotically stable* if δ_a can be an arbitrarily large finite number

For autonomous systems, that is, those which do not depend explicitly on time, the term "uniform" can be dropped.

1.4 Frequency domain vs time domain

Frequency domain stability analysis is useful for linear systems. When applied to TDSs, one faces the problem of studying the roots of a quasipolynomial, which is a polynomial that contains exponential functions. In contrast to ODEs, the characteristic equation of TDSs has generally an infinite number of solutions, which points out their infinite-dimensional nature, as discussed above. This approach is not used throughout this thesis. The reader is referred to (Gu et al. 2003) and the references therein for additional information on this methodology.

As in ordinary (non-delayed) systems, a powerful method for stability and performance analysis of TDS is based on the direct Lyapunov method. There are two main direct Lyapunov methods: (Krasovskii 1963) and (Razumikhin 1956).

All the stability analysis reported in this thesis are based on the Lyapunov-Krasovskii approach. For additional information on the technique developed by Razumikhin and its applications, the reader is referred to (Fridman 2014).

Time delays have various effects on stability. Most commonly, delays have a destabilizing effect. Consider the simple time-delay system $\dot{x}(t) = -x(t-h)$, where $h \geq 0$ is the time delay, whose solutions converge to zero only for $h \in [0, \pi/2)$. However, delays may also have a stabilizing effect. For example, the system $\ddot{y}(t) + y(t) + y(t-h) = 0$ is unstable for $h = 0$ but asymptotically stable for $h = 1$. The first scenario is the one considered in this thesis, where delays are assumed to jeopardize stability. For delay-stabilizing applications, the reader is referred to (Niculescu et al. 2004; Fridman et al. 2017) and the references therein.

1.5 Lyapunov-Krasovskii approach

A criterion for the stability of the trivial solution is given in the following theorem.

Theorem 1.1 (Lyapunov-Krasovskii). *Suppose that $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous non-decreasing functions, $u(s), v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$. The trivial solution of (1.10) is uniformly stable if there exists a continuous functional $V : \mathbb{R} \times C[-h, 0] \to \mathbb{R}_+$, which is positive definite, i.e.,*

$$u(|x(t)|) \leq V(t, x_t) \leq v(\|x_t\|_C), \tag{1.11}$$

such that its derivative along (1.10) is non-positive in the sense that

$$\dot{V}(t, x_t) \leq -w(|x(t)|). \tag{1.12}$$

If $w(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable. Furthermore, if $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

In some cases, functionals $V(t, x_t, \dot{x}_t)$, which depend also on the state derivative, are useful to prove stability (Fridman 2014). Let us introduce the Banach space $W[-h, 0]$ of absolutely continuous functions with the norm

$$\|x\|_W = \|x\|_C + \int_{-h}^{0} |\dot{x}(s)|^2 \, \mathrm{d}s.$$

Theorem 1.1 is then extended to continuous functionals $V : \mathbb{R} \times W[-h, 0] \times L_2(-h, 0) \to \mathbb{R}_+$ as follows:

Theorem 1.2. *Suppose that $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous non-decreasing functions, $u(s), v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$. The trivial solution of (1.10) is uniformly stable if there exists a continuous functional $V : \mathbb{R} \times W[-h, 0] \times L_2(-h, 0) \to \mathbb{R}_+$, which is positive definite, i.e.,*

$$u(|x(t)|) \leq V(t, x_t, \dot{x}_t) \leq v(\|x_t\|_W),$$

such that its derivative along (1.10) is non-positive in the sense that

$$\dot{V}(t, x_t, \dot{x}_t) \leq -w(|x(t)|).$$

If $w(s) > 0$ for $s > 0$, then the trivial solution is uniformly asymptotically stable. Furthermore, if $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

1.6 Linear matrix inequalities

The application of the direct Lyapunov method to the stability analysis of linear systems leads to linear matrix inequalities (LMIs). The first LMI was obtained by Lyapunov in about 1890, when he showed that the ODE $\dot{x}(t) = Ax(t)$ was asymptotically stable if and only if there exists a positive definite matrix P such that $A^T P + PA < 0$. The solution of LMIs by computer via convex programming was recognized in the early 80's, while efficient interior-point methods were developed a few years later. See (Boyd et al. 1994) for a detailed historical perspective and extensive information about LMIs.

1.6.1 Standard LMI problems

An LMI has the form

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{1.13}$$

where $x = [x_1, ..., x_m]^T$ is the vector of decision variables and the symmetric matrices $F_i \in \mathbb{R}^{n\times n}$ are given. In control problems the decision variables are usually matrices.

Example 1.2. *Consider for example the LMI $A^T P + PA < 0$. Let $E_1, ..., E_m$ be a basis for symmetric n by n matrices, where $m = n(n+1)/2$. Then, $P = \sum_{i=1}^{m} x_i E_i$, and the LMI can be expressed as $A^T(\sum_{i=1}^{m} x_i E_i) + (\sum_{i=1}^{m} x_i E_i)A < 0$, which is in the form of (1.13) with $F_0 = 0$ and $F_i = -A^T E_i - E_i A$. Therefore, one may refer to the matrix P as the decision (matrix) variable.*

In most cases, one will be interested simply on studying the feasibility of (1.13). The feasibility problem consists on finding x such that (1.13) holds. There are several toolboxes available online to solve LMIs, which are actually convex optimization problems. In this thesis, all LMIs have been solved using the toolbox Yalmip for MATLAB®, which employs an optimization solver called Sedumi (Sturm 1999; Lofberg 2004).

1.6.2 Schur complement

Nonlinear inequalities can be converted into LMIs using the Schur complement, stated in the following lemma.

Lemma 1.1. *Given matrices $A = A^T, B, C = C^T$ of appropriate dimensions, the following holds:*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0 \Longleftrightarrow C > 0 \quad \& \quad A - BC^{-1}B^T > 0. \tag{1.14}$$

Proof. This result follows from the congruent transformation (which does not change the sign of the eigenvalues)

$$\begin{bmatrix} I & -BC^{-1} \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & 0 \\ -C^{-1}B^T & I \end{bmatrix} = \begin{bmatrix} A - BC^{-1}B^T & 0 \\ 0 & C \end{bmatrix}.$$

□

Example 1.3. *Consider for example the nonlinear matrix inequality $A^TP + PA + PBR^{-1}B^TP + Q < 0$, which is quadratic in P. Using Schur complement, it is equivalent to the LMI*

$$\begin{bmatrix} -A^TP - PA - Q & PB \\ B^TP & R \end{bmatrix} > 0.$$

1.6.3 S-procedure

Sometimes one encounters the constraint that some quadratic function be negative whenever other quadratic functions are all negative. This is the case, for example, when dealing with norm-bounded uncertainties.

Lemma 1.2. *(Yakubovich 1977) Let $F_0 \in \mathbb{R}^{n \times n}$ and $L_1, ..., L_p \in \mathbb{R}^{n \times n}$. If there exist real scalars $\lambda_i \geq 0$, such that*

$$F_0 - \sum_{i=1}^{p} \lambda_i L_i > 0,$$

then $x^T F_0 x >$ for all $0 \neq x \in \mathbb{R}^n$ satisfying $x^T L_i x \geq 0$ for all $i = 1, ..., p$.

Example 1.4. *Consider the system $\dot{x}(t) = Ax(t) + g(x)$ where $|g(x)|^2 \leq x^T Mx$. The Lyapunov analysis with $V = x^T Px$ shows that asymptotic stability is guaranteed if there exists a matrix $P > 0$ such that*

$$\begin{bmatrix} x & g \end{bmatrix} \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ g \end{bmatrix} < 0,$$

where g satisfies $g^T g \leq x^T Mx$. Applying the $\mathcal{S}$-procedure the latter is equivalent to the existence of a $\lambda \geq 0$ such that

$$\begin{bmatrix} A^T P + PA + \lambda M & PB \\ B^T P & -\lambda I \end{bmatrix} < 0.$$

1.7 Delay-independent stability conditions

The Lyapunov-Krasovskii approach (Theorem 1.1) is used here to develop delay-independent stability conditions. These are obviously conservative conditions, which guarantee the stability of (1.1) independently of the size of h. Let us consider the LK functional

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Qx(s)\,\mathrm{d}s, \tag{1.15}$$

where $P, Q > 0$. Note that (1.15) satisfies (1.11) with $u(s) = \underline{\lambda}(P)s^2$ and $v(s) = (\overline{\lambda}(P) + h\overline{\lambda}(Q))s^2$. The derivative of (1.15) is given by

$$\dot{V}(t, x_t) = 2x^T(t)P[Ax(t) + A_1 x(t-h)] + x^T(t)Qx(t) - x^T(t-h)Qx(t-h), \tag{1.16}$$

and thus (1.12) holds if

$$\begin{bmatrix} A^T P + PA + Q & PA_1 \\ A_1^T P & -Q \end{bmatrix} < 0, \tag{1.17}$$

which guarantees (1.1) to be asymptotically stable, by Theorem 1.1. This simple delay-independent condition implies that A and $A \pm A_1$ are Hurwitz. This is a severe limitation, as it implies that this method cannot be applied, for example, to prove stability of an open-loop unstable system with delayed feedback.

1.8 Delay-dependent stability conditions

Delay-dependent stability conditions are derived using the relation

$$x(t-h) = x(t) - \int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s,$$

which transforms (1.1) into

$$\dot{x}(t) = (A + A_1)x(t) - A_1 \int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s. \tag{1.18}$$

First delay-dependent conditions were derived by substituting the term $\dot{x}(t)$ in (1.18) by (1.1), leading to the so-called first model transformation (Li et al. 1995; Dambrine et al. 1995)

$$\dot{x}(t) = (A + A_1)x(t) - A_1 \int_{t-h}^{t} [Ax(s) + A_1 x(s-h)]\,\mathrm{d}s. \tag{1.19}$$

However, this led to conservative conditions because (1.1) and (1.19) are not equivalent in terms of stability (Kharitonov et al. 2000). The descriptor approach, introduced in (Fridman 2001), mitigates this problem.

1.8.1 Descriptor approach

The descriptor model transformation is obtained by defining a new variable $y(t) = \dot{x}(t)$ and rewriting (1.18) as

$$\dot{x}(t) = y(t), \tag{1.20}$$

$$0 = -y(t) + (A + A_1)x(t) - A_1 \int_{t-h}^{t} y(s)\,\mathrm{d}s. \tag{1.21}$$

Also, when computing the derivative of the LK functional, the term $\dot{x}(t)$ is not substituted by the right hand side of (1.1). Furthermore, an additional term, which is identically zero, is added to the functional derivative. To compensate for the integral term, the following functional is proposed (Fridman et al. 2003) [1]

$$V(t, x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{-h}^{0} \int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}s. \tag{1.22}$$

[1] The double integral term in (1.22) can be alternatively expressed as $\int_{t-h}^{t}(h+s-t)\dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}x$, which is sometimes used in Part II.

whose derivative is given by

$$\begin{aligned}\dot{V}(t, x_t, \dot{x}_t) &= 2x^T(t)P\dot{x}(t) \\ &\quad + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][-\dot{x}(t) + (A + A_1)x(t) - A_1 \int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s] \\ &\quad + h\dot{x}^T(t)R\dot{x}(t) - \int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}s. \end{aligned} \tag{1.23}$$

Note that the second term in (1.23) can be added because it is identically zero, as it can be seen from (1.21). Jensen's inequality is further used to bound

$$-\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}s \leq -\frac{1}{h}\int_{t-h}^{t} \dot{x}^T(s)\,\mathrm{d}sR\int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s. \tag{1.24}$$

Using (1.24) into (1.23) yields $\dot{V}(t, x_t, \dot{x}_t) \leq \eta^T(t)\Psi\eta(t) < 0$ if $\Psi < 0$, where

$$\Psi = \begin{bmatrix} P_2^T(A + A_1) + (A + A_1)P_2 & P - P_2^T + (A + A_1)^T P_3 & -hP_2^T A_1 \\ (*) & -P_3 - P_3^T + hR & -hP_3^T A_1 \\ (*) & (*) & -hR \end{bmatrix}, \tag{1.25}$$

and $\eta(t) = \mathrm{col}\left\{x(t), \dot{x}(t), \int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s\right\}$. The matrices P_2, P_3 are referred to as "slack variables". This methodology leads to less conservative results for uncertain systems, even for systems without delay. The important feature in (1.25) is that the matrix P does not multiply to the system matrices, which allows dealing with polytopic uncertainties and also an efficient design procedure.

1.8.2 Improved conditions

The conditions derived above are sufficient and thus they can always be improved. Consider a Lyapunov-Krasovskii functional with an additional term

$$V(t, x_t, \dot{x}_t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(s)Sx(s)\,\mathrm{d}s + h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)R\dot{x}(s)\,\mathrm{d}s,$$

whose derivative along (1.1) is given by

$$\begin{aligned}\dot{V}(t, x_t, \dot{x}_t) &= 2x^T(t)P\dot{x}(t) \\ &\quad + 2[x^T(t)P_2^T + \dot{x}^T(t)P_3^T][-\dot{x}(t) + (A + A_1)x(t) - A_1 \int_{t-h}^{t} \dot{x}(s)\,\mathrm{d}s]\end{aligned}$$

$$
\begin{aligned}
&+ h^2 \dot{x}^T(t) R \dot{x}(t) - h \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s)\, \mathrm{d}s \\
&+ x^T(t) S x(t) - x^T(t-h) S x(t-h),
\end{aligned} \tag{1.26}
$$

where the descriptor method was applied. Jensen's inequality (1.24) can be rewritten as

$$
-h \int_{t-h}^{t} \dot{x}^T(s) R \dot{x}(s)\, \mathrm{d}s \leq -[x(t) - x(t-h)]^T R [x(t) - x(t-h)]. \tag{1.27}
$$

Using (1.27) into (1.26) yields $\dot{V}(t, x_t, \dot{x}_t) \leq \eta^T(t) \Xi \eta(t) < 0$ if $\Xi < 0$, where

$$
\Xi = \begin{bmatrix} P_2^T A + A P_2 + S - R & P - P_2^T + A^T P_3 & P_2^T A_1 + R \\ (*) & -P_3 - P_3^T + h^2 R & P_3^T A_1 \\ (*) & (*) & -S - R \end{bmatrix}, \tag{1.28}
$$

and $\eta(t) = \mathrm{col}\{x(t), \dot{x}(t), x(t-h)\}$. Note that (1.28) has more decision variables than (1.25).

1.9 H_∞-norm of TDSs

Consider the following LTI system

$$
\begin{aligned}
\dot{x}(t) &= A x(t) + B w(t), \\
z(t) &= C x(t) + D w(t),
\end{aligned}
$$

where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^m$ is a disturbance, $z \in \mathbb{R}^q$ is a controlled output, A, B, C and D are real matrices with appropriate dimensions and $G(s) = C(sI - A)^{-1} B + D$ is the transfer function matrix of the system.

Assuming that A is Hurwitz, the H_∞-norm of the proper transfer function matrix $G(s)$ is defined as

$$
\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(G(j\omega)),
$$

where $\sigma_{\max} = \sqrt{\lambda_{\max}(G^T(-j\omega) G(j\omega))}$, is the maximum singular value of $G(j\omega)$. When $G(s)$ is a scalar transfer function, the H_∞-norm is given by

$$
\|G\|_\infty = \sup_{\omega \in \mathbb{R}} |G(j\omega)|.
$$

The H_∞-norm has a nice interpretation in the time domain. Let $w(t)$ be a square integrable input signal. Then, one can show that the following equality holds (Skogestad et al. 2007)

$$\|G\|_\infty = \sup_{w\neq 0} \frac{\|z\|_{L_2}^2}{\|w\|_{L_2}^2}.$$

The right-hand side of the previous expression is called the induced L_2-gain of the system $G(s)$. Consider now the linear perturbed TDS

$$\dot{x}(t) = Ax(t) + A_1 x(t-h) + Bw(t), \tag{1.29}$$

$$z(t) = C_0 x(t) + C_1 x(t-h), \tag{1.30}$$

with a constant delay $h \geq 0$. The H_∞-norm of this system is defined by $\|G\|_\infty$, where $G(s) = (C_0 + C_1 e^{-sh})(sI - A - A_1 e^{-sh})^{-1} B$.

1.10 Lyapunov-based L_2-gain analysis

The Lyapunov method is also a powerful technique for performance analysis. Consider the linear perturbed TDS (1.29)-(1.30), where $w \in L_2[0,\infty)$. For a given $\gamma > 0$, let us introduce the performance index,

$$J = \int_0^\infty (z^T(s)z(s) - \gamma^2 w^T(s)w(s)\,\mathrm{d}s. \tag{1.31}$$

Conditions are sought such that $J < 0$ for all $x(t)$ satisfying (1.29)-(1.30) with zero initial condition $x_0 \equiv 0$ and for all $0 \neq w(t) \in L_2[0, \inf)$. The following lemma provides the basis to derive a Lyapunov-based L_2-gain analysis (Fridman 2014).

Lemma 1.3. *Given $\gamma > 0$, if along (1.29) the inequality*

$$\dot{V}(t, x_t, \dot{x}_t) + z^T(t)z(t) - \gamma^2 w^T(t)w(t) \leq -\alpha(|x(t)|^2 + |w(t)|^2), \tag{1.32}$$

holds for all $t \geq 0$ with some $\alpha > 0$, then the system (1.29)-(1.30) is internally asymptotically stable and has L_2-gain (H_∞-norm) less than γ.

Using the same Lyapunov-Krasovskii functional as in Section 1.8.2 above, and applying Schur complement to the quadratic term $z^T(t)z(t)$, one finds that (1.32) holds along the trajectories of (1.29) if

$$\begin{bmatrix} & \Xi & & \vert & P_2^T B & C_0^T \\ & & & \vert & P_3^T B & 0 \\ & & & \vert & 0 & C_1^T \\ -- & -- & -- & -- & -- & -- \\ (*) & (*) & (*) & \vert & -\gamma^2 I & 0 \\ (*) & (*) & (*) & \vert & (*) & -I \end{bmatrix} < 0. \tag{1.33}$$

Therefore, by Lemma 1.3, the following has been proved (Fridman 2014):

Theorem 1.3 (Bounded real lemma)**.** *Given scalars $\gamma > 0$ and $h > 0$, let there exist positive definite matrices P, R, S and matrices P_2, P_3 that satisfy the LMI* (1.33) *where Ξ is given by* (1.28)*. Then, the system* (1.29)-(1.30) *is internally stable and has an L_2-gain less than γ.*

Chapter 2

Control strategies

Control of time-delay systems has been of great interest among researchers since the late 50's. One of the paradigms when controlling time-delay systems, introduced by Otto J.M Smith in 1957, consists of removing the effect of the delay in the feedback loop. In this way, all the design techniques available for delay-free systems can be applied. Although this methodology seems ideal, it has some limitations and drawbacks, regarding disturbance rejection performance and internal stability issues for unstable systems. Many modifications and alternative control strategies have been proposed to overcome with these problems, some of which are reviewed in this chapter.

2.1 Predictor-based controllers

An ideal (but unfeasible) scenario is depicted in Fig. 2.1, where the non-delayed output $\bar{y}$, is not accessible. In this sense, predictor-based controllers are probably the most logic way to deal with stabilization of input-delayed systems. Indeed, it was pointed out in (Mirkin et al. 2003) that state prediction is a fundamental concept for time-delay systems, much like state observation for conventional systems. The idea is to obtain a prediction of the output $\bar{y}(s) = y(s)e^{sh}$ in (1.9), or the state $x(t+h)$ in (1.5), which are then used to control the system as if there was no delay. Obviously, the latter expressions are non-causal and they cannot be computed in a straightforward way. However, they can be reformulated into causal expressions using the information of the input over the past h units of time. Several approaches to do so have been reported in the literature, which are reviewed next.

2.1.1 *Smith Predictor (SP)*

The Smith Predictor, introduced in (Smith 1957), aims to counteract the time delay in the feedback loop so that a controller can be designed for the equivalent delay-free system. The result is a delayed response of the delay-free system, as if the delay was pushed out of the feedback loop. The structure of the Smith Predictor is depicted in Fig. 2.2, where

$$Z(s) = G(s) - G(s)e^{-sh}, \tag{2.1}$$

is commonly referred to as the predictor block. Note that if $h = 0$, then $Z(s) = 0$, resulting in a conventional feedback control structure. The purpose of the SP is highlighted if one looks at the new controlled variable, denoted by $\bar{y}$, which is given by $\bar{y}(s) = Z(s)u(s) + y(s)$. This is the new controlled variable because the controller acts on this signal, i.e., the control is computed as $u(s) = C(s)[r(s) -$

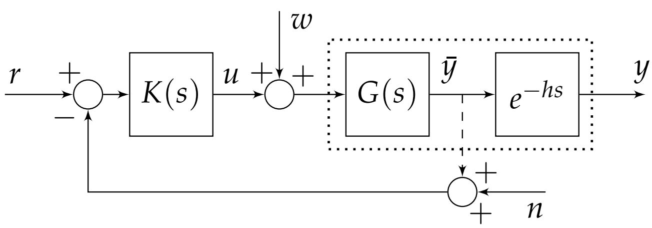


Figure 2.1: Structure of an ideal control loop (unfeasible)

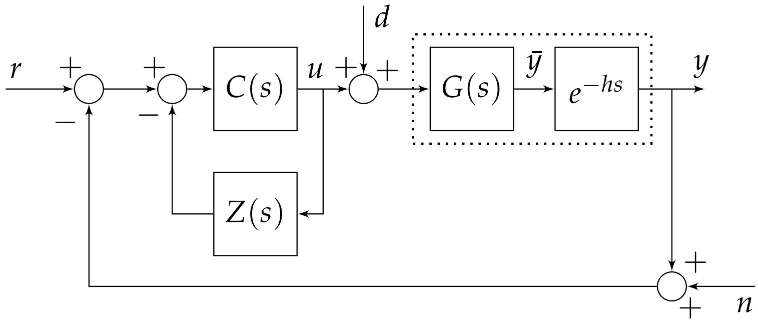


Figure 2.2: Structure of the Smith Predictor

$\bar{y}(s)]$. It is easy to show, using (1.9) and (2.1), that the relation $\bar{y}(s) = G(s)u(s)$ holds. Therefore, the SP setup allows reconstructing a non-delayed output, so that the controller can be designed simply for the delay-free plant $G(s)$. This is the celebrated feature of the SP. It also implies that the delay element is removed from the denominators of all input-output transfer functions, given by

$$G_{ry}(s) = \frac{G(s)C(s)}{1+G(s)C(s)}e^{-sh}, \tag{2.2}$$

$$G_{dy}(s) = \frac{G(s)}{1+G(s)C(s)}e^{-sh} + \frac{G(s)C(s)}{1+G(s)C(s)}Z(s)e^{-sh}, \tag{2.3}$$

$$G_{nu}(s) = \frac{-C(s)}{1+C(s)G(s)} \tag{2.4}$$

The SP has several limitations. It is only applicable to open-loop stable plants because the unstable poles of $G(s)$ cannot be removed from the disturbance response $G_{dy}(s)$ since they appear in $Z(t)$. Integrative processes can be handled but then constant disturbances cannot be rejected even if the controller has integral action. See Section 3.1 below for further details.

2.1.2 *Dead-time compensators (DTCs)*

The numerous modifications of the Smith Predictor are commonly referred to as dead-time compensators (DTCs). These modifications aim at mitigating some of the limitations of the SP. For example, it was well-known that the SP was not able to reject constant disturbances for integrative processes. The modified schemes introduced in (Watanabe et al. 1981a; Astrom et al. 1994; Matausek et al. 1999; Zhong et al. 2002; García et al. 2008) solved this problem in different ways. The treatment of open-loop unstable systems was also addressed in several works (Majhi et al. 2000; Lu et al. 2005; Liu et al. 2005b; García et al. 2006; Normey-Rico et al. 2009). A survey on dead-time compensators is given in (Normey-Rico et al. 2008). See also the monograph (Normey-Rico 2007).

2.1.3 *Finite spectrum assignment (FSA)*

It was shown in (Manitius et al. 1979) that the control law

$$u(t) = Kx(t+h) = K\left[e^{Ah}x(t) + \int_0^h e^{A\xi}Bu(t-\xi)\,\mathrm{d}\xi\right], \tag{2.5}$$

applied to (1.5) yields a finite spectrum assignment of the closed-loop system. This strategy was also developed in (Kwon et al. 1980) and (Artstein 1982) with slightly different perspectives. The controller (2.5) has some drawbacks as well. The predicted variable does not converge to the actual one in the presence of a disturbance and thus it cannot be rejected in a straightforward manner. Another handicap of state predictors lies on the fact that their implementation requires the computation of a distributed integral term. Even more, the control law (2.5) becomes an integral equation in $u(t)$, provided that it appears on the lhs but also on the rhs under an integral sign. For unstable systems, obtaining this integral term as the solution to a differential equation must be discarded because it involves unstable pole-zero cancellations when A is not Hurwitz. This has been a matter of concern for some researchers (Mondié et al. 2002; Mondié et al. 2003; Zhong 2004), as the discretization of the integral may lead to instability of the closed loop.

2.2 PID-based controllers

In the previous section it was shown how predictive controllers achieve outstanding performance for a time-delay system as they eliminate the effect of the delay in the feedback loop. However, DTC structures are more complex and are usually less intuitive to tune than traditional PIDs, which are used in about 90 % of industrial applications (Åström et al. 2001). Let us consider a conventional control loop as depicted in Fig. 2.3, where the controller is assumed to be a PID in series form, given by

$$K(s) = \frac{u(s)}{e(s)} = K_0(s)(1+T_d s), \qquad K_0(s) = k\frac{1+T_i s}{T_i s}. \tag{2.6}$$

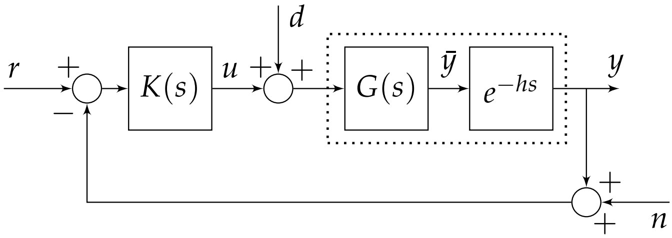


Figure 2.3: Structure of a conventional control loop

The controller (2.6) has been written in such an unconventional form because the term $(1 + T_d s)$ acting on $e(s)$ may be interpreted as an approximate prediction of the error T_d units of time ahead. Therefore, selecting $T_d = h$, the PI part of the controller $K_0(s)$, can be tuned as if there was no delay. The approximation $e^{sh}e(s) \approx (1 + hs)e(s)$ is only valid when the delay is smaller than the closed-loop characteristic time constant (see Section 4.2 in (Normey-Rico 2007)).

Tuning of PID controller has been also pursued by approximating prediction-based controllers. The Smith Predictor controller in Section 2.1.1 can be cast into the control structure depicted in Fig. 2.3 with $K(s) = \frac{C(s)}{1+G(s)C(s)(1-e^{-sh})}$. Therefore, one can always approximate the non-rational term by a Padé approximation $e^{-sh} \approx (1 - 0.5hs)/(1 + 0.5hs)$. For low-order systems, a standard PID controller can be obtained (Normey-Rico 2007). For higher-order systems, a truncated Maclaurin series expansion of $K(s)$ can be used to obtain a PID-like controller (Lee et al. 2000).

Some authors have pointed out DTCs should be used when the dead-time of the process dominates over its characteristic time (Ingimundarson et al. 2002). A different analysis was presented in (Normey-Rico 2007), where it was shown the improvement has more to do with the dead-time uncertainty rather than its size.

2.3 Integral-free prediction-based controllers

It was discussed above that a careless implementation of the state predictor (2.5) may lead to instability . Over the past years, some effort has been put towards finding control laws that handle large input delays but avoid integral terms.

2.3.1 Predictor in observer form

An approach that avoids the use of distributed terms was given in (Besancon et al. 2007), introducing the so-called sequential predictors. This idea was further developed in (Najafi et al. 2013) where an LMI-based H_∞ controller design is reported. The idea is described next in detail as it plays a key role in some of the contributions of this thesis. Let us denote by $z(t)$ the predicted state h units of time ahead, i.e., $z(t) = x(t+h)$. Differentiating $z(t)$ and using (1.5)-(1.6), one can easily derive that the following holds

$$\dot{z}(t) = Az(t) + Bu(t), \tag{2.7}$$

$$y(t) = Cz(t-h). \tag{2.8}$$

Then, in the light of (2.7)-(2.8), one can design an observer

$$\dot{\hat{z}}(t) = A\hat{z}(t) + Bu(t) + L(y(t) - C\hat{z}(t-h)), \tag{2.9}$$

where L is the observer gain, which should be chosen such that the error dynamics is stable. Differentiating the error $e(t) = x(t) - \hat{z}(t-h)$ and using (1.5) and (2.9) yields

$$\dot{e}(t) = Ae(t) - LCe(t-h). \tag{2.10}$$

If (2.10) is stable, then $\lim_{t\to\infty}[x(t+h) - \hat{z}(t)] = 0$ and consequently, one could use the control law $u(t) = K\hat{z}(t)$ into (1.5), leading to

$$\dot{x}(t) = (A+BK)x(t) - BKe(t). \tag{2.11}$$

The closed-loop is composed of (2.10)-(2.11). The latter is stable simply by choosing K such that $A+BK$ is Hurwitz. If L is chosen such that $A-LC$ is Hurwitz, it is well known that (2.10) is stable for a sufficiently small h. From a conceptual point of view, this is a novel method to obtain the predicted state via an observer and thus, avoiding distributed integral terms. From a practical point of view, there is not much benefit, since one could have also chosen $u(t) = Kx(t)$ in the first place, leading to $\dot{x}(t) = Ax(t) + BKx(t-h)$, which is also stable for small enough h if $A+BK$ is Hurwitz.

The strategy described above is actually useful only when sequential observers come into play. Recall that the problem of stabilizing (1.5)-(1.6) has been reduced to finding an asymptotically stable observer for (2.7)-(2.8), in which the measurement is delayed. The idea of sequential observers was originally introduced in (Germani et al. 2002). It consists of constructing a chain of observers so that each of the them estimates a prediction of the state over a period of time equal to a fraction of the delay. For example, let us define

$$z_1(t) = x(t + h/2) \quad \text{and} \qquad z_2(t) = x(t + h).$$

Differentiating these variables and using (1.5)-(1.6), one can show that the following holds

$$\begin{aligned}
\dot{z}_1(t) &= Az_1(t) + Bu(t - \frac{h}{2}), \\
\dot{z}_2(t) &= Az_2(t) + Bu(t), \\
y(t) &= Cz_2(t - h) = Cz_1(t - \frac{h}{2}).
\end{aligned}$$

Then, one can design the following chain of observers

$$\dot{\hat{z}}_1(t) = A\hat{z}_1(t) + Bu(t - \frac{h}{2}) + L_1(y(t) - C\hat{z}_1(t - \frac{h}{2})), \tag{2.12}$$

$$\dot{\hat{z}}_2(t) = A\hat{z}_2(t) + Bu(t) + L_2(\hat{z}_1(t) - C\hat{z}_2(t - \frac{h}{2})). \tag{2.13}$$

Differentiating the errors $e_1(t) = x(t) - \hat{z}_1(t - h/2)$ and $e_2(t) = x(t) - \hat{z}_2(t - h)$ and using (1.5) and (2.12)-(2.13) yields

$$\dot{e}_1(t) = Ae_1(t) - L_1Ce_1(t - \frac{h}{2}), \tag{2.14}$$

$$\dot{e}_2(t) = Ae_2(t) - L_2e_2(t - \frac{h}{2}) + L_2e_1(t - \frac{h}{2}). \tag{2.15}$$

The application of the control law $u(t) = K\hat{z}_2(t)$ into (1.5) yields

$$\dot{x}(t) = (A + BK)x(t) - BKe_2(t). \tag{2.16}$$

Using a two-element sequential observer, the closed-loop is composed of (2.14)-(2.16). The key difference with respect to the single-element observer is that the delayed terms in the error dynamics are affected by a fraction of the delay. In this case, a half. Then, the allowable delay is enlarged. This methodology can be generalized to an arbitrarily large number of elements in the observer chain, which allows dealing with large delays. Since the observers are actually generating predictions of the state, this overall technique has been referred to as sequential predictors in the literature (Najafi et al. 2013). This technique has been exploited recently by many researchers (Léchappé et al. 2016; Cacace et al. 2016; Vafaei et al. 2016; Mazenc et al. 2016; Mazenc et al. 2017a; Zhou et al. 2017).

2.3.2 *Other techniques*

It was shown in (Mazenc et al. 2003) that a chain of integrators with input delay can be stabilized by a saturated controller. This approach was also extended to general feedforward systems in (Mazenc et al. 2004). Sliding mode control for delay systems was studied in (Richard et al. 2001) and a robust design via LMIs was introduced in (Gouaisbaut et al. 2002). Continuous pole placement was also investigated in (Michiels et al. 2002) for systems with delay. In (Zhou et al. 2012), a truncated predictor that ignores the infinite-dimensional part of the controller (2.5) was proposed. Time-varying delays were considered and an observer-based controller is also proposed for output-feedback stabilization of systems without exponentially unstable modes. This requirement was removed in the analysis reported in (Yoon et al. 2013), although the delay had to be upper bounded by a sufficiently small constant. The truncated predictor has been also reformulated to include higher-order terms (Zhou et al. 2014).

Part II

Disturbance observers and predictive feedback

Chapter 3

Introduction

Among traditional control objectives, disturbance rejection in time-delay systems deserves special attention because delays impose fundamental limitations no matter what controller is used. Indeed, if a disturbance reaches the input at time t_0, the information lag will cause the system to run in an open-loop fashion over the time window $t \in [t_0, t_0 + h]$, where h is the time delay. The results reported in this part of the thesis are based on the combination of disturbance observers for and predictor to deal with perturbed time-delay systems. An introduction to disturbance observer based control is given in this chapter, with special emphasis on its drawbacks and limitations when delays are present.

3.1 Limitations of the Smith Predictor

The most celebrated strategy to compensate delays was proposed in 1957 with the introduction of the Smith Predictor (SP), applicable to SISO open-loop stable plants (Smith 1957). The SP exhibited drawbacks that were early detected. Let us look at the disturbance response of the SP, given by (2.3). First of all, one can see that the SP is only applicable to open-loop stable systems, provided that $Z(s)$, which contains the poles of $G(s)$, appears explicitly in (2.3). Integrative processes are a special case of unstable plants. These systems can be handled by the SP because when $G(s)$ has a pole at $s = 0$, it is canceled out by the zero at the origin of $1 - e^{-sh}$ and thus that pole does not appear in (2.3). Regarding the disturbance rejection issues, let us assume that the controller contains an integrator. Then, the steady-state response to a unit load disturbance is given by

$$\lim_{t\to\infty} y_d(t) = \lim_{s\to 0} G_{dy}(s) = \lim_{s\to 0} G(s)[1 - e^{-sh}], \tag{3.1}$$

where it was used that, because of the integrator in $C(s)$, the first term in (2.3) vanishes in the limit and $G(0)C(0)/(1 + G(0)C(0)) = 1$. From (3.1), for asymptotically stable processes, $\lim_{t\to\infty} y_d(t) = 0$, which is satisfactory. However, if an integrative plant, such as $G(s) = G_0(s)/s$ with $G_0(s)$ stable, is considered, then using L'Hôpital's rule in (3.1) one has that

$$\lim_{t\to\infty} y_d(t) = G_0(0) \lim_{s\to 0} \frac{1 - e^{-sh}}{s} = hG_0(0),$$

which implies that load disturbances are not rejected, even if the controller contains integral action.

3.1.1 Modifications of the Smith Predictor

Many structures, commonly referred to as dead-time compensators (DTCs), were developed to mitigate these issues (Wang et al. 2004; Normey-Rico et al. 2008), either to achieve load disturbance rejection for pure integrating processes with long dead-time (Watanabe et al. 1981a; Astrom et al. 1994; Matausek et al. 1999; Normey-Rico et al. 2002; García et al. 2008; Uma et al. 2010; Chakraborty et al. 2017), or to control unstable time-delay systems (Majhi et al. 2000; Tan et al. 2003; Hang et al. 2003; Liu et al. 2005a; Lu et al. 2005; García et al. 2006; Normey-Rico et al. 2009; Tan 2010; Begum et al. 2017). These schemes commonly have an inner stabilizing loop and employ more controllers. Furthermore, most solutions are highly specific on the control goals and/or the plant structure, and they fail in completely removing the delay element from the feedback loop, making the design process more complicated.

The underlying idea of the SP was extended to MIMO stable/unstable systems with the finite spectrum assignment (FSA) technique (Manitius et al. 1979), also known as the reduction-based approach (Artstein 1982). In contrast to the SP, this strategy was formulated in the time domain with the introduction of a state predictor. However, the disturbance rejection issues persisted, provided that the predicted state would not converge to the actual one. A modified predictor is given in (Léchappé et al. 2015) where additional delayed feedback was considered to reject constant disturbances (see Chapter 6 for details). The idea of additional feedback had already been used in (García et al. 2008), although in the frequency domain.

3.2 Disturbance observer based control

Many branches of control theory have emerged to handle the effects of unknown disturbances and uncertainties, such as robust control, adaptive control, sliding mode control or internal model control. Disturbance observer based control differs from the aforementioned approaches in the fact that it counteracts the disturbance actively in a feedforward fashion. During the past decades, disturbance observers (also referred to as unknown input estimators) have gained popularity through a variety of applications such as robotic manipulators (Chen et al. 2000; Katsura et al. 2007), high speed XY positioning (Kempf et al. 1999), missile control (Chen 2003), or magnetic bearings (Chen et al. 2004), among others (Li et al. 2016). Disturbance observers have been formulated both in the frequency and state-space domains. See (Chen et al. 2016) for a review of the available methods. The transfer function approach was the first proposed and its is often used because of its simple logic when tuning the so-called Q-filter. In this thesis, the state-space formulation is preferred. This is because, when applied to time-delay systems, a wide variety of tools is available for analysis and design. Both formulations are briefly reviewed next.

3.2.1 *Frequency domain formulation*

Let us consider a minimum-phase SISO LTI system subject to an input disturbance, which is represented in the frequency domain by

$$Y(s) = G(s)[U(s) + D(s)], \tag{3.2}$$

where U is the control input, Y is the output, D is the disturbance and $G(s)$ is a model of the plant. From (3.2), an estimation of the disturbance can be obtained as

$$\hat{D}(s) = Q(s)[\underbrace{G^{-1}(s)Y(s) - U(s)}_{D(s) \text{ from } (3.2)}] = Q(s)U(s) - \frac{Q(s)}{G(s)}Y(s), \tag{3.3}$$

where the filter $Q(s)$ is introduced to make the quotient $Q(s)/G(s)$ realizable. Then, the relative degree of $Q(s)$ has to be greater or equal than that of $G(s)$. A diagram of the overall disturbance observer based control setup is depicted in Fig. 3.1, where the control law is $u(s) = C(s)[r(s) - y(s)] - \hat{d}(s)$, being r the set-point reference signal. It should be remarked that this scheme, as depicted in the figure, is not implementable. The input-output transfer functions are easily obtained as

$$G_{ry} = \frac{G(s)C(s)}{1 + G(s)C(s)},$$
$$G_{dy} = \frac{G(s)}{1 + G(s)C(s)}[1 - Q(s)].$$

From the expressions above it is evident that: i.) the set-point and disturbance responses are decoupled, and ii.) the disturbance rejection performance mainly depends on the design of the filter $Q(s)$. The disturbance observer (DOB) developed above was first introduced in (Nakao et al. 1987).

3.2.2 *Time-domain formulation*

Let us consider a multiple-input LTI system, represented by

$$\dot{x}(t) = Ax(t) + B[u(t) + d(t)], \tag{3.4}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control action, $d \in \mathbb{R}^m$ is the disturbance and A, B are matrices of appropriate dimensions. A simple time-domain formulation of a disturbance observer can be obtained as follows. Consider a

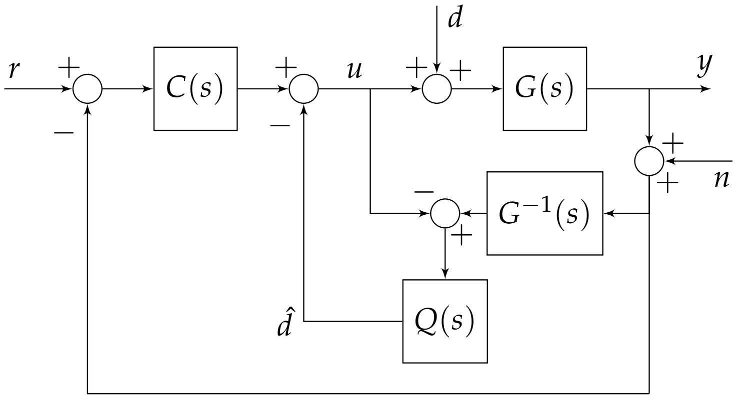


Figure 3.1: DOB structure in the frequency domain

disturbance estimation defined by

$$\dot{\hat{d}}(t) = -\omega_o \hat{d}(t) + \omega_o \underbrace{[B^+ \dot{x}(t) - B^+ Ax(t) - u(t)]}_{d(t) \text{ from } (3.4)} \tag{3.5}$$

where B^+ denotes the left pseudo-inverse of B such that $B^+B = I_m$ and ω_o is to be understood as the bandwidth, provided that (3.5) has the structure of a low-pass filter driven by $d(t)$. Now, to avoid the state derivative term, let us define an auxiliary variable $z(t) = \hat{d}(t) - \omega_o B^+ x(t)$, which can be used to rewrite (3.5) as

$$\dot{z}(t) = -\omega_o[z(t) + \omega_o B^+ x(t)] + \omega_o B^+[Ax(t) - Bu(t)], \tag{3.6}$$

$$\hat{d}(t) = z(t) + \omega_o B^+ x(t). \tag{3.7}$$

Note that, in contrast to (3.5), the state derivative does not appear in the expressions above and thus they are implementable. Selecting the control law $u(t) = Kx(t) - \hat{d}(t)$, defining the estimation error as $\tilde{d}(t) = d(t) - \hat{d}(t)$ and using (3.4)-(3.5), the closed-loop is governed by

$$\dot{x}(t) = (A + BK)\hat{x}(t) + B\tilde{d}(t), \tag{3.8}$$

$$\dot{\tilde{d}}(t) = -\omega_o \tilde{d}(t) + \dot{d}(t). \tag{3.9}$$

From (3.8)-(3.9), one can see that a separation principle holds also for the disturbance observer based control in the time domain, provided that K and ω_o can be selected independently. The observer developed here is equivalent to the uncertainty and disturbance estimator (UDE), introduced in (Zhong et al. 2004) from a frequency-domain perspective. The following observer

$$\dot{z}(t) = -LB[z(t) + Lx(t)] + L[Ax(t) - Bu(t)], \tag{3.10}$$

$$\hat{d}(t) = z(t) + Lx(t), \tag{3.11}$$

was also reported in (Yang et al. 2010), which is equivalent to (3.6)-(3.7) if $L = B^{+}\omega_o$ is chosen.

3.2.3 *Disturbance observers for time-delay systems*

The first application of disturbance observers for time-delay systems seems to be the one reported in (Kempf et al. 1996). There, it was pointed out that the standard DOB structure offers poor tolerance to time delays, which should be then taken into account. Let us illustrate this point with a simple case.

Example 3.1. *Consider a nominal plant $G_n(s) = G(s)e^{-sh}$ and the conventional DOB structure depicted in Fig. 3.1, with the simple choice $Q(s) = \omega_o^r/(s+\omega_o)^r$, where r is the relative degree of $G(s)$. The delay can be modeled as a multiplicative uncertainty $G(s) = G_n(s)(1+\Delta(s))$, with $\Delta(s) = e^{-sh} - 1$. From Fig. 3.1, the open-loop gain is given by $L(s) = Q(s)/(1-Q(s))$, which leads to a complementary sensitivity function $T(s) = L(s)/(1+L(s)) = Q(s)$. Therefore, a robust stability criterion is given by $\|Q\|_\infty \|\Delta\|_\infty < 1$. One can show that $|\Delta(j\omega)| < 1, \forall\omega < \omega_\Delta$, where $\omega_{\gtrsim} 1/h$. Then, a simple robust criterion for stability is given by $\omega_o < 1/h$.*

This example shows that a non-modeled time delay in the loop determines an upper bound on ω_o to guarantee stability and, consequently, it limits the achievable performance of this technique. A suitable modification of the DOB is proposed in (Kempf et al. 1996), by adding a delay block to the input signal.

Later, this modification was used in the context of disturbance rejection improvement for time-delay systems (Zhong et al. 2002). Therein, integral processes $G(s) = G_0(s)/s$ with $G_0(s)$ stable, were considered. This work was one of the many others that aimed at improving the unsatisfactory disturbance rejection performance of the Smith Predictor for integrating processes (Watanabe et al. 1981a; Astrom et al. 1994; Matausek et al. 1999). The overall strategy consisted of an open-loop controller combined with the modified DOB, as depicted in Fig. 3.2. The input-output transfer functions are obtained as

$$G_{ry} = \frac{G(s)C(s)}{1+G(s)C(s)} e^{-sh},$$

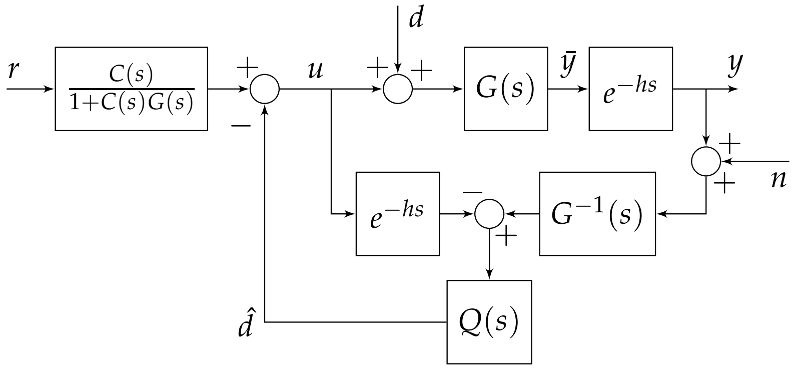


Figure 3.2: DOB structure for time-delay systems (Zhong et al. 2002)

$$G_{dy} = G(s)e^{-sh}[1 - Q(s)e^{-sh}].$$

Rejection of different types of disturbances could be achieved by imposing conditions on the term $1 - Q(s)e^{-sh}$, which yields requirements on $Q(s)$. For example, for constant disturbances to be rejected, $Q(0) = 1$ and $\dot{Q}(0) = h$ must hold.

Other than the contribution discussed above, the DOB technique in the context of time-delay systems did not raise much interest among researchers for many years. A state-space DOB was used in (Chen et al. 2010) to deal with systems with state delays. The UDE was also adapted to this scenario in (Stobart et al. 2011). However, input delays were not considered in any case. Special mention deserves the work developed in (Kim et al. 2010), whose underlying idea is the same as in most of the contributions reported throughout the next chapters, namely, the combination of a predictor and a disturbance observer. The authors of (Kim et al. 2010) seemed to be unaware of the modified DOB introduced in (Kempf et al. 1996) for time-delay systems. Instead, they handle the delay using first order Padé approximation. Since such approach yields non-minimum phase zeros, the inverse of the plant cannot be computed and a conveniently modified DOB needs to be adopted.

3.3 Contributions in this part

The contributions reported in the following chapters are summarized here.

Chapter 4 is based on (Sanz et al. 2017), published in *ISA Transactions*, where a Generalized Smith Predictor (GSP) is introduced. The proposed control scheme applies to stable/unstable minimum/non-minimum phase systems and it can be easily tuned to yield an exact output prediction even in the presence of disturbances. The design of the primary controller is then straightforward. This is a unified solution in contrast to most previous approach that based ad-hoc structures for specific plants and/or goals, as discussed in Section 3.1.1.

Chapter 5 is based on (Sanz et al. 2016), which was published in *Automatica*. In this work, a new prediction is defined that makes use of a future estimation of the disturbance. Such estimation is obtained by a truncated Taylor expansion upon estimates of the disturbance and its derivatives, which are provided by a modified disturbance observer. Disturbance attenuation improvement over previous strategies is proved using a Lyapunov analysis.

Chapter 6 is based on (Sanz et al. 2018), which was published in *International Journal of Robust and Nonlinear Control*. In contrast to previous contributions, this work deals with output-feedback stabilization and mismatched disturbances. An extended state observer is combined with a predictor in observer form. Rejection of mismatch disturbances with known dynamics is achieved while ensuring attenuation of unmodeled components.

Chapter 7 is based on (Sanz et al. 2017), which was published in *Transactions on Industrial Electronics*. This work addressed the problem discussed in Section 3.2.3, regarding the poor tolerance of disturbance observers to delays. This is a slightly different solution, which consists of running the disturbance observer upon a predicted state. The results, both in simulations and experiments, show that this approach allows a more aggressive tuning of the observer and thus an improved disturbance rejection.

Chapter 4

A Generalized Smith Predictor (GSP)

In this work, a generalization of the Smith Predictor (SP) is proposed to control linear time-invariant (LTI) time-delay single-input single-output (SISO) systems. Similarly to the SP, the combination of any stabilizing output-feedback controller for the delay-free system with the proposed predictor leads to a stabilizing controller for the delayed system. Furthermore, the tracking performance and the steady-state disturbance rejection capabilities of the equivalent delay-free loop are preserved. In order to place this contribution in context, some modifications of the SP are revisited and recast under the same structure. The features of the proposed scheme are illustrated through simulations, showing a comparison with respect to the corresponding delay-free loop, which is here considered to be the ideal scenario. In order to emphasize the feasibility of this approach, a successful experimental implementation in a laboratory platform is also reported.

4.1 Introduction

An LTI time-delay SISO process subject to input disturbances can be described by

$$y(s) = G(s)e^{-hs}[u(s) + w(s)] \triangleq \bar{y}(s)e^{-hs} \tag{4.1}$$

where $y \in \mathbb{R}$ is the measurable output, $\bar{y} \in \mathbb{R}$ is the unmeasurable non-delayed output, $u \in \mathbb{R}$ is the control input, $w \in \mathbb{R}$ is an input disturbance, $h \geq 0$ is a constant time delay and $G(s) = C(sI - A)^{-1}B$ is referred to as the delay-free system.

When controlling a time-delay system, an ideal scenario is depicted in Fig. 4.1. It is "ideal" in the sense that the delay is pushed out of the feedback loop, the non-delayed output $\bar{y}$ is available, and thus the controller $K(s)$ can be simply designed for the rational part of the model, $G(s)$, using conventional techniques. Since $\bar{y}$ is not accessible, a reasonable approach consists of constructing an output prediction $\hat{y}$, so that it can be used to control the system as in the ideal scenario. The prediction should be based on the available input/output information, having the following structure:

$$\hat{y}(s) \triangleq F_1(s)u(s) + F_2(s)y(s) \tag{4.2}$$

where the filters $F_1(s)$ and $F_2(s)$ must be stable and derived from the plant model.

In the seminal work (Smith 1957), the Smith Predictor (SP) makes use of the filter $F_1^{\text{SP}}(s) \triangleq G(s) - G(s)e^{-sh}$, sometimes referred to as the SP block, whereas $F_2^{\text{SP}}(s) \triangleq 1$. It is easy to verify that the prediction $\hat{y}_{\text{SP}}(s) \triangleq F_1^{\text{SP}}(s)u(s) + F_2^{\text{SP}}(s)y(s)$ satisfies $\hat{y}_{\text{SP}}(s) = G(s)u(s) = \bar{y}(s)$, if there are no disturbances. Indeed, the SP removes the delay element from the denominators of all the closed-loop sensitivity functions, reducing the control problem to that of a delay-free system. The methodology described above has been referred to as the "Smith's Principle" in the literature. However, the SP cannot be applied to open-loop unstable plants and regardless of the main controller, only constant disturbances can be rejected (Guzmán et al. 2008). Some of the many SP modifications proposed in the literature have been reviewed in Section 3.1.1. For integrating and unstable systems, none of them except those proposed in (Watanabe et al. 1981a; García et al. 2006; Normey-Rico et al. 2009), fulfill the Smith's Principle. Next, these schemes are reviewed and recast under the same structure, in order to place the present work in context.

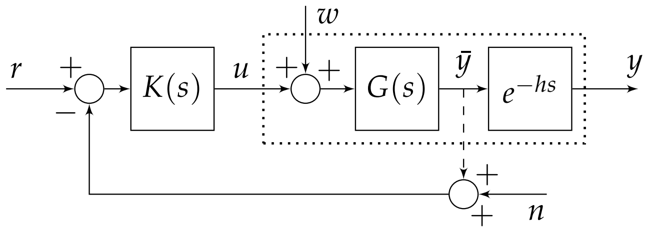


Figure 4.1: An ideal control loop (unfeasible)

4.1.1 The Smith's Principle

As aforementioned, few schemes have been proposed to generalize the Smith Predictor to unstable systems, avoiding the instability of the predictor block while fulfilling the Smith's Principle. The first attempt in this direction can be found in (Watanabe et al. 1981a). In that work, the SP block was modified by choosing $F_1^{\mathrm{MSP}}(s) \triangleq G^{\dagger}(s) - G(s)e^{-sh}$ and $F_2^{\mathrm{MSP}}(s) = 1$, where $G^{\dagger}(s) \triangleq Ce^{-Ah}(sI - A)^{-1}B$. However, it was later when this approach was generalized and named as the Modified Smith Predictor (MSP) (Palmor 1996). The key feature of this scheme is that the MSP block can be computed in the time domain as[1] $\mathcal{L}^{-1}\{F_1^{\mathrm{MSP}}(s)u(s)\} = Ce^{-Ah}\int_0^h e^{A\xi}Bu(t-\xi)\,\mathrm{d}\xi$, which is a definite integral and therefore, stable. Regarding disturbance rejection, the MSP alters the low frequency gain of the primary controller because it has non-zero static gain, that is, $F_1^{\mathrm{MSP}}(0) \neq 0$. Consequently, constant disturbances cannot be rejected even if the primary controller contains integral action. This drawback was already addressed in (Watanabe et al. 1981a) by choosing $G^{\dagger}(s) = -C\int_0^h e^{-A\xi}\,\mathrm{d}\xi B + Ce^{-Ah}(sI - A)^{-1}B$, with the inconvenient that $G^{\dagger}(s)$ is no longer strictly-proper and the corresponding controller may be more complicated.

Other proposals were developed inspired on the discrete-time framework. In (Normey-Rico et al. 2009), the SP was complemented with an additional filter, $F_2^{\mathrm{FSP}}(s) \triangleq F_r(s)$, leading to the Filtered Smith Predictor (FSP). The resulting predictor block was $F_1^{\mathrm{FSP}}(s) \triangleq G(s) - G(s)F_r(s)e^{-sh}$, where the new filter $F_r(s)$ played a key role, being used to avoid the unstable modes in $F_1^{\mathrm{FSP}}(s)$. In continuous-time, this pole-zero cancellation cannot be performed by the use of polynomial division because the numerator of $F_1^{\mathrm{FSP}}(s)$ is a non-rational expression. However, in the discrete-time framework, this can be done analytically

[1]Here $\mathcal{L}^{-1}\{\cdot\}$ denotes the inverse Laplace transform operator.

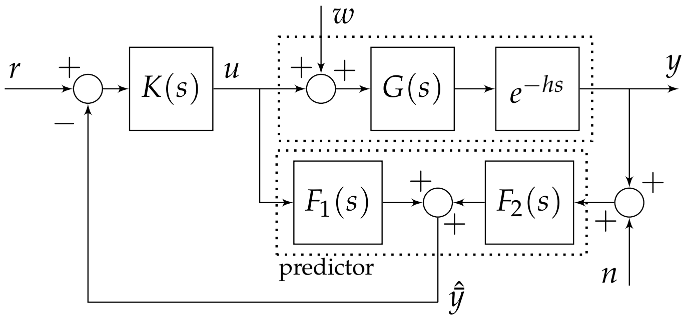


Figure 4.2: A general structure for predictor-based control schemes

Table 4.1: Filters in Fig. 4.2 for each scheme reviewed in Section 4.1.1

Scheme	$F_1(s)$	$F_2(s)$	Proposed in
SP	$G(s) - G(s)e^{-sh}$	1	(Smith 1957)
MSP	$G^{\dagger}(s) - G(s)e^{-sh}$	1	(Watanabe et al. 1981a)
FSP	$G(s) - G(s)F_r(s)e^{-sh}$	$F_r(s)$	(Normey-Rico et al. 2009)
GP	$G(s) - G^*(s)e^{-sh}$	$G^*(s)/G(s)$	(García et al. 2006)

by solving a Diophantine equation. In the same process, the block can be also adjusted to reject any class of disturbances (Santos 2016; Santos et al. 2017).

The Generalized Predictor (GP), originally proposed in (García et al. 2006), was originated from a discrete-time reasoning. However, the formulation next presented is developed in continuous-time for the sake of comparison. Similarly to the MSP, the instability of the GP block was avoided by selecting $F_1^{\text{GP}}(s) \triangleq G(s) - G^*(s)e^{-sh}$ with $G^*(s) \triangleq Ce^{Ah}(sI - A)^{-1}B$, whereas $F_2^{\text{GP}}(s) = G^*(s)/G(s)$. As a result, the GP block can be computed by $\mathcal{L}^{-1}\{F_1^{\text{GP}}(s)u(s)\} = C\int_0^h e^{A\xi}Bu(t-\xi)\,\mathrm{d}\xi$, which is a stable block. In order to cancel the effect of constant disturbances, the GP made use of an extra loop, making the analysis more complicated (García et al. 2013).

The schemes previously reviewed lead to a control structure as depicted in Fig. 4.2, with filters given in Table 1.

In what follows, with special emphasis on transparency and design simplicity of the resulting control strategy, a generalization of the SP is proposed to solve the following problem:

Problem 4.1. *Consider a controller K designed to meet some requirements based on the delay-free loop depicted in Fig. 4.1. Then, find a predictor, that is, design filters F_1 and F_2, such that the same controller K in Fig. 4.2:*
A) guarantees internal stability
B) achieves the same nominal tracking performance
C) achieves rejection of the same type of disturbances

4.2 Problem reformulation

As already mentioned, a celebrated feature of the SP is that it exactly reduces the control problem to its delay-free counterpart, by constructing an "exact" prediction. In what follows, a prediction $\hat{y}$ for the system (4.1) is said to be exact if $\hat{y}(s) = \bar{y}(s)$ hold in the nominal case, that is, with known plant and no disturbances. It is easy to show that a prediction computed by (4.2) is exact if and only if

$$F_1(s) = \left(1 - F_2(s)e^{-hs}\right)G(s). \tag{4.3}$$

The main advantage of obtaining an exact prediction is that the design and analysis of the resulting control-loop are drastically simplified, which is a highly appreciated feature of the original SP. This is formally stated by the following proposition:

Proposition 4.1. *If the output prediction computed by (4.2) is exact, then the input-output transfer functions of the predictor-based control loop depicted in Fig. 4.2 satisfy:*

$$G_{r,y}(s) = \bar{G}_{r,y}(s) \tag{4.4}$$
$$G_{w,y}(s) = \bar{G}_{w,y}(s) + \bar{G}_{r,y}(s)F_1(s) \tag{4.5}$$
$$G_{n,y}(s) = \bar{G}_{n,y}(s)F_2(s) \tag{4.6}$$
$$G_{n,u}(s) = \bar{G}_{n,u}(s)F_2(s) \tag{4.7}$$

where $\bar{G}_{r,y}$, $\bar{G}_{w,y}$, $\bar{G}_{n,y}$ and $\bar{G}_{n,u}$ are the input-output transfer functions of the ideal loop in Fig. 4.1, given by

$$\bar{G}_{r,y} = \frac{G(s)K(s)e^{-hs}}{1+G(s)K(s)}, \qquad \bar{G}_{w,y} = \frac{G(s)e^{-hs}}{1+G(s)K(s)},$$
$$\bar{G}_{n,y} = -\frac{G(s)K(s)e^{-hs}}{1+G(s)K(s)}, \qquad \bar{G}_{n,u} = -\frac{K(s)e^{-hs}}{1+G(s)K(s)}.$$

Proof. The proposition follows simply by solving the block diagrams in Figs. 4.1-4.2 and using (4.3). □

Remark 4.1. *Inspecting Table 1, the condition (4.3) holds for all the schemes reviewed in Section 4.1.1 but the MSP, in which the controller has to be designed for the modified plant $G^{\dagger}(s)$. This can be an inconvenient in some cases, as already mentioned.*

Now, Problem 1 is translated into finding a predictor with some constraints. To that purpose, let us introduce the following assumption:

Assumption 4.1. *The external disturbance can be expressed as $w(s) = \bar{w}w_0(s)$, with unknown amplitude $\bar{w} \in \mathbb{R}$ and known dynamics $w_0(s)$*

Assumption 1 implies that the type of disturbance to be rejected should be known, e.g., step, ramp, or sine wave with a given frequency. Although this may seem restrictive, some attenuation of disturbances not described by $w_0(s)$ is also expected. This can be analyzed in detail by looking at the bode plot of (4.5).

Lemma 4.1. *Let us consider the control loop in Fig. 4.2 with a predictor such that:*
i.) the prediction $\hat{y}$ is exact
ii.) the filters $F_1(s)$, $F_2(s)$ are stable
iii.) the following equivalent conditions hold (assuming that the limits involved exist)

$$\lim_{t\to\infty} \left(\bar{y}(t) - \hat{y}(t)\right) = 0 \iff \lim_{s\to 0} sF_1(s)w_0(s) = 0$$

Then, that predictor solves Problem 1.

Proof. If i.) holds then Proposition 1 is valid and the control loop in Fig. 4.2 is internally stable iff (4.4)-(4.7), the so-called "gang of four", are stable . Recall that $K(s)$ is designed such that all transfer functions of the ideal loop, denoted with an upper bar are stable, whereas the filters are stable by ii.). Then A) in Problem 1 is fulfilled. Also, the set-point responses in (4.4) for both loops are the same and thus B) is fulfilled. Finally, using (4.1)-(4.3), the output prediction error due to the input disturbance satisfies

$$e(s) \triangleq \bar{y}(s) - \hat{y}(s) = F_1(s)w(s) \tag{4.8}$$

The equivalence in iii.) follows from (4.8) and Assumption 1. Furthermore, notice that if iii.) holds, from (4.5), the load disturbance response of the ideal loop is recovered in steady-state, and then, C) is fulfilled. This completes the proof. ☐

4.3 Proposed generalized SP

The main contribution of this chapter, a generalized SP, is presented next.

Lemma 4.2 (GSP). *Let us consider an arbitrary decomposition of the delay-free plant such that*

$$G(s) = \Gamma(s)\tilde{G}(s) \tag{4.9}$$

where $\Gamma(s)$ *is proper, stable and may have non-minimum phase zeros; and* $\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B} = \tilde{N}(s)/\tilde{D}(s)$ *is strictly proper, minimum phase and may have unstable poles. Then, the computation of* (4.2) *with the stable filters*

$$F_1(s) = \Phi_{\tilde{G}}(s)\Gamma(s) \qquad F_2(s) = \frac{\tilde{N}^*(s)}{\tilde{N}(s)} \tag{4.10}$$

where $\tilde{G}^*(s) = \tilde{C}e^{\tilde{A}h}(sI - \tilde{A})^{-1}\tilde{B} = \tilde{N}^*(s)/\tilde{D}(s)$ *and*

$$\Phi_{\tilde{G}}(s) = \tilde{C}\left(I - e^{-(sI-\tilde{A})h}\right)(sI - \tilde{A})^{-1}\tilde{B},$$

provides an exact output prediction.

Proof. Let us consider the system (4.1) with $w(t) = 0$ and the decomposition (4.9). The auxiliary variable $v(t)$ is defined such that $v(s) = \Gamma(s)u(s)$, which implies that $y(s) = \tilde{G}(s)e^{-sh}v(s)$. Let us also introduce the following internal representation

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t) + \tilde{B}v(t) \\ y(t) = \tilde{C}x(t-h) \end{cases}$$

so that $\tilde{G}(s) = \tilde{C}(sI - \tilde{A})^{-1}\tilde{B}$. A non-delayed state prediction is given by

$$\hat{x}(t) = e^{\tilde{A}h}x(t-h) + \int_0^h e^{\tilde{A}\xi}\tilde{B}u(t-\xi)\,\mathrm{d}\xi \tag{4.11}$$

Since the state is not accessible, an output prediction $\hat{y}(t) = \tilde{C}\hat{x}(t)$, is sought instead. Using (4.11), it follows that

$$\hat{y}(t) = \tilde{C}e^{\tilde{A}h}x(t-h) + \tilde{C}\int_0^h e^{\tilde{A}\xi}\tilde{B}v(t-\xi)\,\mathrm{d}\xi \tag{4.12}$$

It can be shown that (Zhong 2004)

$$\tilde{C}\int_0^h e^{\tilde{A}\xi}\tilde{B}v(t-\xi)\,\mathrm{d}\xi = \mathcal{L}^{-1}\{\Phi_{\tilde{G}}(s)v(s)\} \tag{4.13}$$

where $\Phi_{\tilde{G}}(s) \triangleq \tilde{C}\left(I - e^{-(sI-\tilde{A})h}\right)(sI-\tilde{A})^{-1}\tilde{B}$. Also, since $\mathcal{L}\{x(t-h)\} = (sI - \tilde{A})^{-1}\tilde{B}e^{-sh}v(s)$ and $v(s) = \tilde{G}^{-1}(s)e^{sh}y(s)$, one can write

$$Ce^{\tilde{A}h}x(t-h) = \mathcal{L}^{-1}\left\{\frac{\tilde{G}^*(s)}{\tilde{G}(s)}y(s)\right\} \tag{4.14}$$

where $\tilde{G}^*(s) = \tilde{C}e^{\tilde{A}h}(sI-\tilde{A})^{-1}\tilde{B}$. Plugging (4.13)-(4.14) and $v(s) = \Gamma(s)u(s)$ into (9.120) yields

$$\hat{y}(s) = \mathcal{L}\{\hat{y}(t)\} = \frac{\tilde{G}^*(s)}{\tilde{G}(s)}y(s) + \Phi_{\tilde{G}}(s)\Gamma(s)u(s) \tag{4.15}$$

The lemma follows by using the fact that $\tilde{G}^*(s)/\tilde{G}(s) = \tilde{N}^*(s)/\tilde{N}(s)$ because the transfer functions $\tilde{G}^*(s)$ and $\tilde{G}(s)$ have the same denominator. □

Intuitively, Lemma 2 implies that, regarding prediction, the plant can be decomposed into: $\tilde{G}(s)$, which is projected h units of time ahead by the operator $\Phi_{\tilde{G}}(s)$; and $\Gamma(s)$, which appears explicitly in the predictor. The usefulness of the GSP introduced above lies on the fact that the prediction is exact no matter what decomposition is chosen. Therefore, $\Gamma(s)$ can be appropriately selected so that $F_1(s)$ has some desired properties.

Theorem 4.1. *Let us consider the following decomposition*

$$\Gamma(s) = \frac{N^+(s)N_\Gamma^-(s)}{D_\Gamma^-(s)}\frac{Q(s)}{w_0(s)} \qquad \tilde{G}(s) = \frac{N_{\tilde{G}}^-(s)}{D^+(s)D_{\tilde{G}}^-}\frac{w_0(s)}{Q(s)} \tag{4.16}$$

where: i.) the unstable poles and non-minimum phase zeros of $G(s)$ are collected in $D^+(s)$ and $N^+(s)$, respectively; ii.) its stable poles $D^-(s)$, and minimum phase zeros $N^-(s)$, are arbitrarily partitioned so that $D_\Gamma^-(s)D_{\tilde{G}}^-(s) = D^-(s)$ and $N_\Gamma^-(s)N_{\tilde{G}}^-(s) = N^-(s)$; and iii.) $Q(s)$ is a strictly-proper filter such that $\tilde{G}(s)$ is strictly-proper and $\Gamma(s)$ is at least proper.

Then, the GSP introduced in Lemma 2 with $\Gamma(s)$, $\tilde{G}(s)$ given above, solves Problem 1.

Proof. Recall that Problem 1 is solved if the conditions in Lemma 1 are fulfilled. By Lemma 2, the items i.) and ii.) hold. Using (4.8), the limit of the output prediction error (if it exists) can be computed as $\lim_{t\to\infty} e(t) = \lim_{s\to 0} sF_1(s)w(s)$.

Using (4.10), (4.16) and Assumption 1, it follows that

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\Phi_{\tilde{G}}(s)\frac{N^+(s)N_\Gamma^-(s)}{D_\Gamma^-(s)}\frac{Q(s)}{w_0(s)}\bar{w}w_0(s)$$
$$= \lim_{s\to 0} s\Phi_{\tilde{G}}(s)\frac{N^+(s)N_\Gamma^-(s)}{D_\Gamma^-(s)}Q(s)\bar{w} = 0 \tag{4.17}$$

Notice that the limit always exists because all transfer functions in (4.17) are stable, including $\Phi_{\tilde{G}}(s)$. This completes the proof. □

Remark 4.2. *It should be remarked that although there are infinite choices for $Q(s)$, its relative degree, denoted by r_q, is constrained. In order to fulfill the third condition in Theorem 1 , it can be shown that $r_q \in [\underline{r}_q, \overline{r}_q] \triangleq [z^+ + r_{w_0} - p^-, p + r_{w_0} - 1]$, where r_{w_0} is the relative degree of $w_0(s)$, z^+ is the number of non-minimum phase zeros, p is the number of total poles and p^- is the number of stable poles. A simple choice is $Q(s) = (s+\omega)^{-r_q}$, $r_q \in [\underline{r}_q, \overline{r}_q]$, which leaves only two parameters to be adjusted.*

4.4 Predictor-based control

The main features of the control-loop resulting from applying the proposed predictor are discussed in this section. First, performance and robustness indices are derived in comparison to those of the ideal delay-free loop. Later, the stability of the digital implementation is thoroughly analyzed as it is a critical issue, specially for unstable plants.

4.4.1 Performance and robustness

From (4.4) in Proposition 1, the proposed strategy recovers the tracking performance of the ideal loop. Also, it can be verified from (4.5) that

$$S(s) = \bar{S}(s)\big(1 + K(s)F_1(s)\big) \tag{4.18}$$

where $\bar{S}(s)$ and $S(s)$ are the sensitivity functions of the control loops in Fig. 4.1 and Fig. 4.2, respectively. On the other hand, let us consider a multiplicative uncertainty such that $G(s) = G_0(s)\big(1 + \Delta(s)W_\Delta(s)\big)$ with $\|\Delta\|_\infty \leq 1$, and thus the robust stability condition for Fig. 4.1 is $\|\bar{M}(s)\|_\infty \leq 1$ where $\bar{M} = \bar{L}(s)W_\Delta(s)/\big(1 + \bar{L}(s)\big)$ and $\bar{L}(s) = G_0(s)K(s)$. Then, denoting by $\|M(s)\|_\infty \leq 1$ the new robust stability condition for the loop Fig. 4.2, it can be show that

$$M(s) = \bar{M}(s)F_2(s) \tag{4.19}$$

The robustness with respect to a delay mismatch is also a matter of concern (Michiels et al. 2003). Considering $h = h_0 + \delta h$, the robust stability can be checked by modeling it as multiplicative uncertainty (very conservative), or the analytic bounds on δh can be computed by

$$\max\left\{-h_0, \frac{\phi_i^-}{\omega_i}\right\} < \delta h < \min \frac{\phi_i^+}{\omega_i}, \tag{4.20}$$

where ω_i are crossover frequencies at which $L(j\omega_i) = 1$, $\phi_i^+ > 0, \phi_i^- < 0$ are the corresponding signed phase margins, and $L(s)$ is the loop transfer function of Fig. 4.2, given by

$$L(s) = \frac{G(s)K(s)F_2(s)e^{-hs}}{1 + K(s)F_1(s)}.$$

Since $K(s)$ should be designed for the delay-free system, one can use (4.18)-(4.19) along with Theorem 1 to design a filter $Q(s)$ so that the desired disturbance rejection performance and robustness are achieved, if possible. However, it is worth mentioning that there are fundamental limitations (Normey-Rico 2007). For example, it is not possible to achieve $\bar{S}(s) = S(s)$ at all frequencies because $F_1(s) = 0$ implies $F_2(s) = e^{sh}$, which is not realizable. Also, $\bar{M}(s) = M(s)$ can only be achieved for stable systems, because it implies that $F_2(s) = 1$ and thus $F_1(s) = G(s)\left(1 - e^{-hs}\right)$, which is the conventional Smith Predictor. It should be also remarked that, although predictors achieve nominal stability for any $h > 0$, there is a limitation on the achievable delay margin for unstable systems using LTI controllers (Middleton et al. 2007). Therefore, above that value, an infinitesimal delay would lead to instability, making the controller unfeasible in practice.

4.4.2 Discrete-time implementation

Some details regarding the digital implementation will be discussed next. Since predictive schemes are sensitive to non-minimum phase zeros and unstable poles, it is important to consider a discretized plant from the beginning. Let us define the sampling period $T_s > 0$, and the discretized process model $G(z) = \mathcal{Z}\{G(s)\}$, where $\mathcal{Z}\{\cdot\}$ is the Z-transform operator. The delay is assumed to be a multiple of the sampling period, i.e., $h = T_s d$ for some $d \in \mathbb{N}$. Then, the discrete-time counterpart of (4.1) is given by

$$y(z) = G(z)z^{-d}[u(z) + w(z)] \tag{4.21}$$

where $G(z) = C_z(zI - A_z)^{-1}B_z$. The discrete form of the proposed predictor for the system (4.21) is introduced in the following lemma:

Lemma 4.3 (Discrete-time GSP)**.** *Let us consider an arbitrary decomposition of the delay-free plant such that*

$$G(z) = \Gamma(z)\tilde{G}(z) \tag{4.22}$$

where $\Gamma(z)$ is proper, stable and may have non-minimum phase zeros; and $\tilde{G}(z) = \tilde{C}_z(zI - \tilde{A}_z)^{-1}\tilde{B}_z = \tilde{N}(z)/\tilde{D}(z)$ is strictly proper, minimum phase and may have unstable poles. Then, the computation of

$$\hat{y}(z) = F_1(z)u(z) + F_2(z)y(z) \tag{4.23}$$

with the stable filters

$$F_1(z) = \Phi_{\tilde{G}}(z)\Gamma(z) \qquad F_2(z) = \frac{\tilde{N}^*(z)}{\tilde{N}(z)} \tag{4.24}$$

where $\tilde{G}^*(z) = \tilde{C}_z\tilde{A}_z^d(zI - \tilde{A}_z)^{-1}\tilde{B}_z = \tilde{N}^*(z)/\tilde{D}(z)$ and

$$\Phi_{\tilde{G}}(z) = \tilde{C}_z \sum_{j=1}^{d} \tilde{A}_z^{j-1} z^{-j} \tilde{B}_z,$$

provides an exact output prediction.

Proof. It can be readily seen that the following identity holds $\sum_{j=1}^{d} \tilde{A}_z^{j-1} z^{-j} = \left(I - \tilde{A}_z^d z^{-d}\right)(zI - \tilde{A}_z)^{-1}$. Pre-multiplying by $\tilde{C}_z$ and post-multiplying by $\tilde{B}_z$ in the previous identity yields $\Phi_{\tilde{G}}(z) = \tilde{G}(z) - \tilde{G}^*(z)z^{-d}$. Therefore, from (4.24), if follows that $F_1(z) = \left(\tilde{G}(z) - \tilde{G}^*(z)z^{-d}\right)\Gamma(z)$. Also, since $\tilde{G}(z)$ and $\tilde{G}^*(z)$ have the same poles, then $F_2(z) = \tilde{N}^*(z)/\tilde{N}(z) = \tilde{G}^*(z)/\tilde{G}(z)$. Plugging these expressions into (4.23) and after some manipulations, it follows that $\hat{y}(z) = G(z)u(z)$ for the nominal case, i.e., $w = 0$. This completes the proof. □

Remark 4.3. *Notice that the decomposition introduced in Theorem 1 applies to the discrete-time case, simply replacing the argument s of the transfer functions by z. It is important to perform the decomposition of the plant using the discretized model, because additional zeros may be introduced during the discretization process.*

Remark 4.4. *The digital implementation of the distributed terms arisen in $F_1(s)$ has been a major concern for many years (Mondié et al. 2003). The implementation structure given in Lemma 3 mitigates this problem. Another implementation issue was devised in (Zhong et al. 2004) for the case of systems with fast stable modes, where the Unified Smith Predictor (USP) was proposed. This problem can be avoided in the proposed scheme by placing the fast stable modes in $\Gamma(s)$ and selecting $Q(s)$ accordingly.*

For the convenience of potential users, the set-up of the proposed strategy can be summarized as follows: **1.)** Obtain a discrete-time model of the plant as shown in (16); **2.)** Design a primary controller $K(z)$, for the delay-free system (using conventional design techniques); **3.)** Find a suitable decomposition $G(z) = \Gamma(z)\tilde{G}(z)$ satisfying the conditions in Theorem 1 (choosing $Q(z)$ as simple as possible, e.g., a low-pass filter, for design simplicity); **4.)** Construct the filters $F_1(z)$, $F_2(z)$ and implement the output predictor $\hat{y}(z)$ as described in Lemma 3; **5.)** Use $\hat{y}(z)$ as the input to the controller; and **6.)** Adjust the parameters in $Q(z)$ to reach a trade-off between performance and robustness.

4.5 Simulations

In this section, simulations are carried out to validate the proposed strategy. Let us consider (4.1), being

$$G(s) = \frac{1}{s-1} \qquad \text{and} \qquad h = 1.5\text{ s}.$$

Remark 4.5. *It should be remarked that this is a rather challenging example. To the best of the authors' knowledge, this system has not been robustly controlled with such a large delay (Wang et al. 2004). Furthermore, it is pointed out in (Middleton et al. 2007) that no LTI controller can stabilize this system for delays $h > 2$ s.*

The equivalent ZOH sampled system (4.21), with a sampling period $T_s = 0.01$ s is obtained as $G(z) = b_z/(z - a_z)$ with $a_z = e^{T_s}$, $b_z = \int_0^{T_s} e^{\theta}\, d\theta = 1 - e^{T_s}$ and a discrete delay $d = 150$. The controllers below are designed taking into account only the delay-free system, using conventional procedures. The predictor is adjusted to yield zero steady-state prediction error for some type of disturbances. Then it is showed how the straightforward combination of the predictor with the main controller stabilizes the delayed system, keeping the same tracking performance while maintaining the disturbance rejection capabilities.

Table 4.2: Absolute performance and robustness for different values of τ_q

	GM	PM	IAE/IAE^*	ω_c	δh
delay-free	$[0.79, \text{inf}]$	$35.9°$	1	1.3 rad/s	-
$\tau_q = 0.1$ s	$[0.83, 1.20]$	$11.5°$	1.04	51.3 rad/s	± 0.02 s
$\tau_q = 0.75$ s	$[0.87, 1.18]$	$7.71°$	1.07	11.4 rad/s	± 0.09 s
$\tau_q = 1.5$ s	$[0.89, 1.18]$	$6.4°$	1.12	8.5 rad/s	± 0.12 s
$\tau_q = 2$ s	$[0.91, 1.18]$	$5.9°$	1.20	7.8 rad/s	± 0.13 s
$\tau_q = 5$ s	$[0.94, 1.18]$	$5.0°$	2.10	6.5 rad/s	± 0.15 s

4.5.1 *Constant disturbance rejection*

Rejection of constant disturbances is a typical requirement in practice, and it is here chosen to illustrate the main features of the proposed strategy. *Primary controller design:* A simple 2-DoF PI-controller $K(s)$ with a set-point filter $F_r(s)$ is designed, in the Laplace domain for convenience, for the equivalent delay-free system as follows

$$K(s) = k\frac{t_i s + 1}{t_i s} \qquad F_r(s) = G_r(s)/T(s)$$

with $k = 2\beta + 1$ and $t_i = k/\beta^2$, which yields a closed-loop characteristic polynomial $(s + \beta)^2$. For the prefilter,

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}, \qquad G_r(s) = \frac{1}{\tau_r s + 1},$$

leading to a characteristic response time for the set-point tracking, τ_r. For the simulations below, the parameters $\beta = 1/8$ and $\tau_r = 2$ s are arbitrary selected. Notice that this controller is able to reject constant disturbances in the delay-free case and it has been designed without considering the time delay. *Predictor design:* The predictive scheme is implemented according to Lemma 3, with $N_\Gamma^-(z) = b_z$, $N_{\tilde{G}}^-(z) = 1$, $N^+(z) = 1$, $D^+(z) = (z + a_z)$, $D_\Gamma^-(z) = 1$, $D_{\tilde{G}}^-(z) = 1$ and $w_0(s) = 1/s$, to reject constant disturbances. Notice that according to Remark 1, in this case $r_q \in [1, 1]$ and thus the simplest possible choice for the filter is taken, $Q(s) = 1/(\tau_q s + 1)$, being τ_q an adjustable parameter. The following discretization can be obtained

$$\mathcal{Z}\left\{\frac{Q(s)}{w_0(s)}\right\} = \mathcal{Z}\left\{\frac{s}{\tau_q s + 1}\right\} = \frac{z - 1}{\tau_q z + (T_s - \tau_q)} \tag{4.25}$$

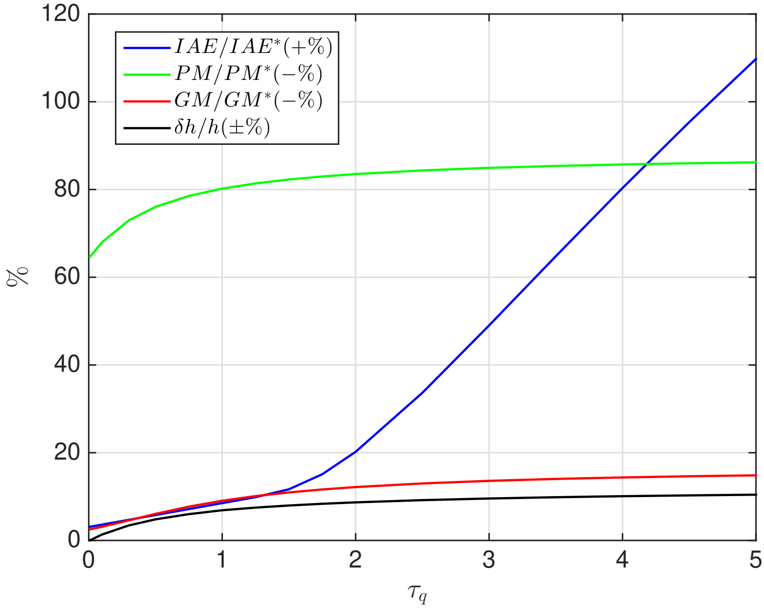


Figure 4.3: Relative robustness and performance indices as a function of τ_q

and thus, applying the decomposition in Theorem 1 yields

$$\Gamma(z) = \frac{b_z(z-1)}{\tau_q z + (T_s - \tau_q)} \quad \text{and} \quad \tilde{G}(z) = \frac{\tau_q z + (T_s - \tau_q)}{(z-1)(z-a_z)}.$$

A state-space realization of $\tilde{G}(z) = \tilde{C}_z(zI - \tilde{A}_z)^{-1}\tilde{B}_z$ is given by

$$\tilde{C}_z = [T_s - \tau_q,\ \tau_q], \quad \tilde{A}_z = \begin{bmatrix} 0 & 1 \\ 1 + a_z & -a_z \end{bmatrix} \quad \text{and} \quad \tilde{B}_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

According to Lemma 3, the filter $F_1(z)$ can be computed by

$$F_1(z) = \Phi_{\tilde{G}}(z) \frac{b_z(z-1)}{\tau_q z + (T_s - \tau_q)}$$

with $\Phi_{\tilde{G}}(z) = \tilde{C}_z \sum_{j=1}^{d} \tilde{A}_z^{j-1} z^{-j} \tilde{B}_z$ and $\tilde{A}_z, \tilde{B}_z, \tilde{C}_z$ given above. In order to compute $F_2(z)$, notice that, because of the canonical form of the state-space representation of $\tilde{G}(z) = \tilde{N}(z)/\tilde{D}(z)$, it follows that $\tilde{N}(z) = \tau_q z + (T_s - \tau_q) = \langle \tilde{C}_z, [1, z] \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the dot product. In other words, the coefficients of the nu-

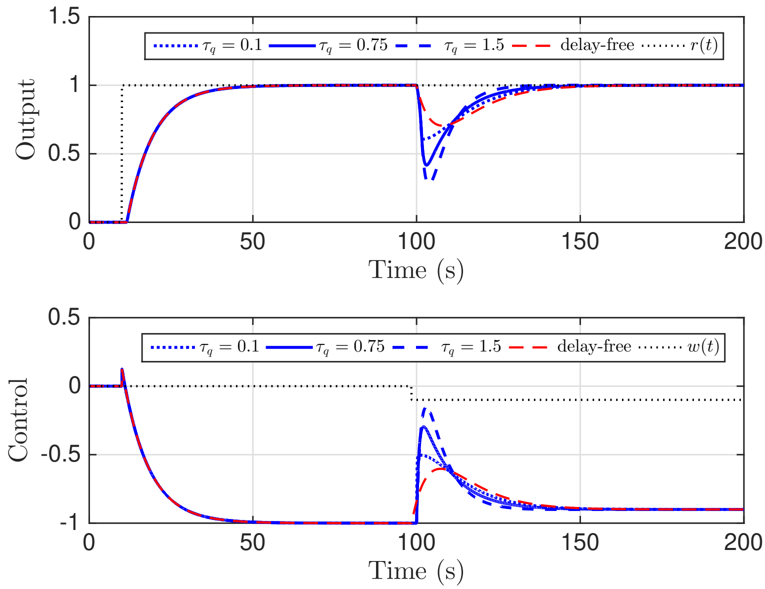


Figure 4.4: Nominal performance for different values of τ_q

merator of the transfer function are simply given by $\tilde{C}_z$. Then, according to Lemma 3, the numerator of $\tilde{G}^*(z)$ is obtained as $\tilde{N}^*(z) = \left\langle \tilde{C}_z \tilde{A}_z^d, [1, z] \right\rangle$ and thus the filter $F_2(z)$ can be computed by

$$F_2(z) = \frac{\langle \tilde{C}_z, [1, z] \rangle}{\langle \tilde{C}_z \tilde{A}_z^d, [1, z] \rangle}$$

In the current configuration, there is only one parameter left to be tuned, namely, τ_q. Its influence can be illustrated through a plot like the one depicted in Fig. 4.3. In this representation, one can see the phase margin reduction (green), the gain margin reduction (red) and the integral absolute error (IAE) increment for a load step disturbance (blue), all of them expressed as relative values over the corresponding delay-free loop characteristics. According to the previous indicators, lower values of τ_q increase both robustness and performance. However, the allowable delay uncertainty (black) approaches to zero as $\tau_q \to 0$. The data shown in Table 2 (absolute GM, absolute PM, relative IAE and absolute delay mismatch, δh) illustrates with more details the same behavior depicted in Fig. 4.3. Furthermore, an additional measure is added, namely, the crossover frequency of the transfer function (4.6), denoted by ω_c. It can be concluded that the improvement

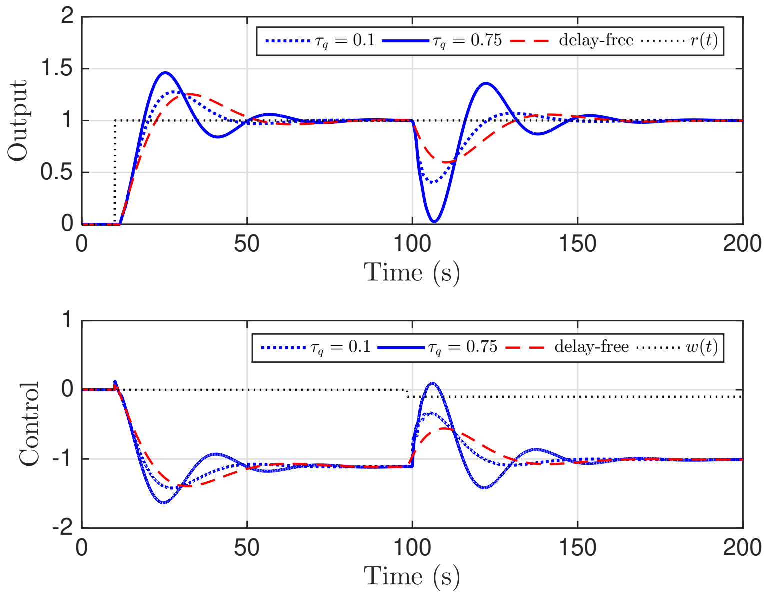


Figure 4.5: Robustness to -10% gain variation for different values of τ_q

in GM, PM and IAE comes at the expense of more noise and less tolerance to delay mismatch.

Therefore, a trade-off must be reached, which is mainly constrained by how accurate the delay is known. From Fig. 4.3, for this particular example, values $\tau_q > 2$ should be discarded, as they reduce performance substantially (fast grow of relative IAE) whereas the other indices are barely improved. On the other hand, values in the range $\tau_q < 0.5$ offer a tolerance of less than 5% to a delay mismatch, and thus they should also be ruled out. An interval of interest in practice for this example is hence given by $0.5 < \tau_q < 2$.

The discussion above is illustrated next through some simulations. The system is driven by a step reference $r(t) = 1, \forall t \geq 10$ s; and an input disturbance $w(t) = -0.1, \forall t \geq 100$ s is applied. The first scenario is shown in Fig. 4.4, where the different values of τ_q are simulated in nominal conditions. One can clearly see how the disturbance rejection performance improves as τ_q is reduced. A second simulation shows the effect of τ_q in the robust performance. The gain of the actual plant is decreased by 10% while the delay is kept with its nominal value. The results are shown in Fig. 4.5. It is verified that, as discussed before, if the delay is accurately known, lower values of τ_q provide better robustness. The

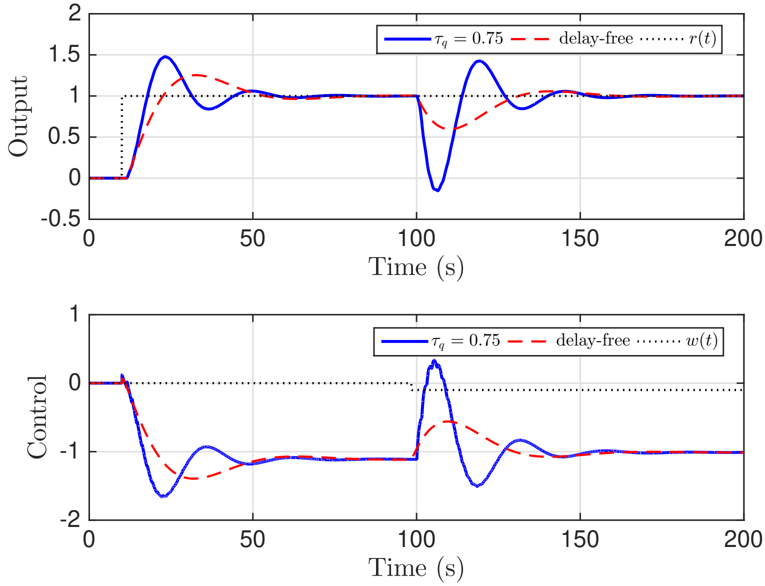


Figure 4.6: Robustness to -10% gain and $+5\%$ delay variations

output corresponding to $\tau_q = 1.5$ s is not included here because it is almost unstable. The third simulation considers the same scenario as before but the actual delay is increased by 5%. In this case, the value $\tau_q = 0.1$ s leads to instability, as it has a very small tolerance to a delay mismatch (see Table 2). However, the performance for $\tau_q = 0.75$ s, shown in Fig. 4.6, is satisfactory.

4.5.2 Sinusoidal disturbance rejection

The design process to reject a sinusoidal input disturbance is illustrated next. A resonant controller $K(s)$ with a set-point filter $F_r(s)$ is designed for the equivalent delay-free system as follows

$$K(s) = k\frac{(s+\omega)^2}{s^2+\omega^2} \qquad F_r(s) = G_r(s)/T(s) \tag{4.26}$$

with $k = 5$, $\omega = 0.2$ rad/s which yields a stable closed-loop with phase margin 60°. The prefilter is the same as in the previous example, with $\tau_r = 2$ s. Notice that this controller is able to reject sinusoidal disturbances of frequency $\omega = 0.5$ rad/s in the delay-free case. The predictive scheme is implemented according to Lemma 3, with the same decomposition as in the previous example

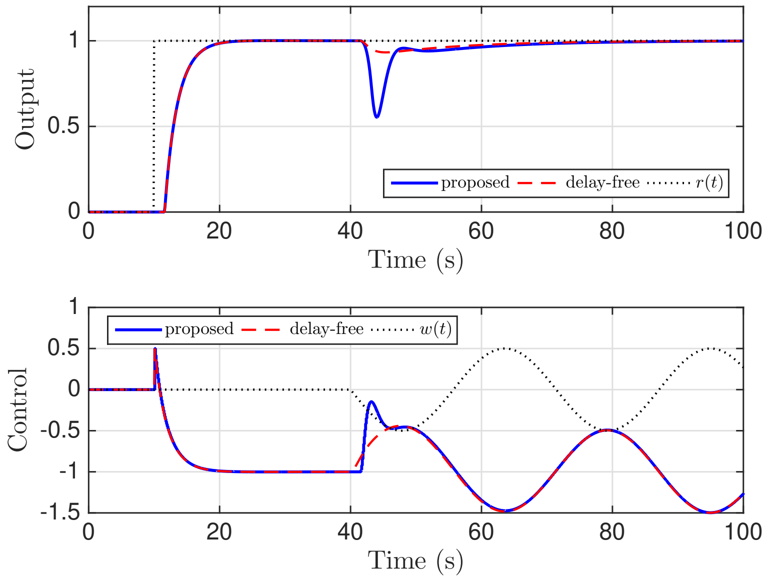


Figure 4.7: Rejection of a sinusoidal input disturbance

but choosing $w_0(s) = \omega/(s^2 + \omega^2)$. Again, in this case $r_q \in [2, 2]$ according to Remark 1, and thus the simple choice $Q(s) = 1/(\tau_q s + 1)^2$ is made, being τ_q an adjustable parameter.

The simulation is shown in Fig. 4.7, where the adjustable parameter is selected as $\tau_q = 1$ s; the system is driven by a step reference $r(t) = 1, \forall t \geq 10$ s; and an input disturbance $w(t) = -0.2 \sin 0.2(t - 40), \forall t \geq 40$ s is applied. It can be seen how the disturbance is finally rejected in spite of the large time delay, as it happens in the delay-free case. Regardless of whether the frequency of sinusoidal disturbances is accurately known in practice, the main purpose of the previous simulation is to show that the proposed predictor can be easily adjusted to keep the same steady-state disturbance rejection capabilities of the corresponding delay-free loop.

Figure 4.8: Experimental device

4.6 Experimental results

An experimental validation is reported using the 3D Hover laboratory platform manufactured by Quanser©, depicted in Fig. 4.8. It consists of a quadrotor-like structure mounted on a 3-DoF pivot joint so that the body can freely rotate in roll, pitch and yaw. The angular position is the controlled variable, which is measured by a encoder with a resolution of 0.04 deg, while the input is the voltage sent to the motors. The experiment is performed in one of the roll/pitch axes, which is modeled as a double integrator $G(s) = 0.1/s^2$. The control loop is implemented at $T_s^{-1} = 100$ Hz, where T_s is the sampling period, using a POSIX thread in a computer running Linux with a soft real-time patched kernel. An artificial delay of $h = 250$ ms (or $d = 25$ sampling periods) is introduced by software. The resulting controller is required to reject step disturbances. It should be remarked that the double integrator model is just an approximation of the real plant. In fact, the experimental device has a large uncertainty due to the motor dynamics, which is neglected in the design process. Furthermore, although the number of samples in the artificial delay is known, the actual delay depends on the computational time, which is slightly varying and thus, another source of uncertainty.

Following the Step 1, described at the end of Section 4, a discretization of the plant, $G(z) = 0.5T_s^2(z+1)/(z-1)^2$, is obtained. In the Step 2, a primary PID-controller is designed

$$K(s) = k_p \left(1 + \frac{t_d s}{\tau_f s + 1} + \frac{1}{t_i s} \right)$$

with a set-point filter $F_r(s) = 1/(\tau_r s + 1)$ and $\tau_r = 2$ s, $k_p = 50$ V/rad, $t_d = t_i = 0.5$ s, with $\tau_f = 0.2$, which leads to a large phase margin of 80° and fast step disturbance rejection for the delay-free system. Following the Steps 3-4, the predictor is implemented according to Lemma 3, with $N_\Gamma^-(z) = 0.5T_s^2$, $N_{\tilde{G}}^-(z) = 1$, $N^+(z) = z + 1$, $D^+(z) = (z-1)^2$, $D_\Gamma^-(z) = D_{\tilde{G}}^-(z) = 1$. Notice that the zero $(z + 1)$ is treated as a non-minimum phase term in order to avoid numerical issues. According to Remark 2, the relative degree of $Q(s)$ needs to be $r_q = 2$, and thus

$$\frac{Q(z)}{w_0(z)} = \mathcal{Z} \left\{ \frac{s}{(\tau_q s + 1)^2} \right\} = \frac{T(z-1)e^{-T/\tau_q}}{\tau_q^2 (z - e^{-T/\tau_q})^2}.$$

The parameter τ_q is tuned online and finally set to $\tau_q = 0.25$ s. Two experiments with the same pattern are carried out. A step reference of 5 deg is commanded at $t = 1$ s and an input disturbance of -4 V is applied at $t = 20$ s. One can see in Fig. 4.9 that, the designed PID-controller in combination with the proposed predictor stabilizes the system and rejects load disturbances (full blue). An experiment without the predictor is also reported, simply to illustrate that this delay is large enough to be considered, as the designed PID-controller cannot stabilize the system by itself (dashed red).

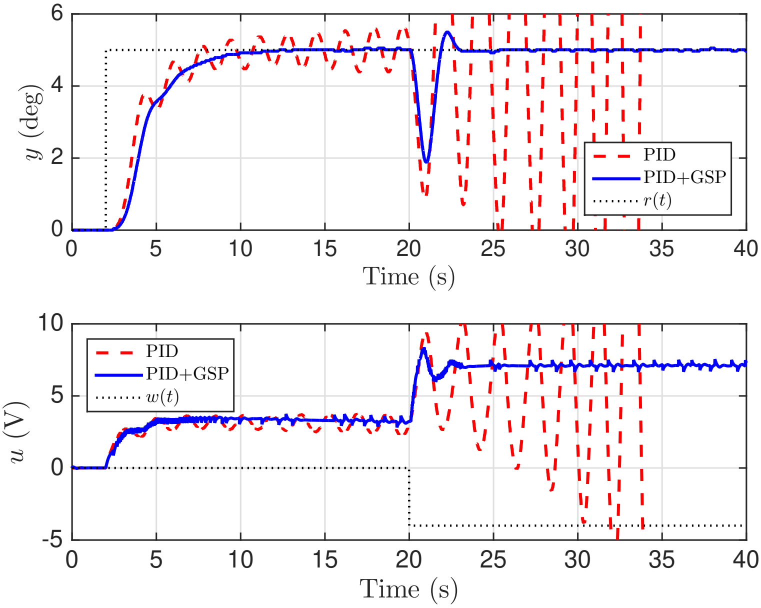


Figure 4.9: Experimental results of the proposed scheme applied to an unstable plant with a delay of $h = 250$ ms

Chapter 5

Enhanced disturbance rejection for LTI systems with input delay

In this chapter, a new predictor-based control strategy for LTI systems with input delay and unknown disturbances is proposed. The disturbing signal and its derivatives up to the r-th order are estimated by means of an observer, which are then used to construct a prediction of the disturbance. Such prediction allows defining a new predictive scheme taking into account its effect. Also, a suitable transformation of the control input is presented and a performance analysis is carried out to show that, for a given controller, the proposed solution leads to better disturbance attenuation than previous approaches in the literature for smooth enough perturbations.

5.1 Introduction

The problem considered in this chapter deals with possibly open-loop unstable disturbed LTI systems, defined by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t-h) + d(t)] \\ u(t) &= u_0(t) \qquad t \in [-h, 0) \\ x(0) &= x_0 \end{aligned} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}$ is the control input, $d(t) \in \mathbb{R}$ is an unknown input disturbance, $h > 0$ is a known and constant input delay, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$ are known matrices.

The Smith Predictor (Smith 1957), can be considered as the first predictor-based control for open-loop stable linear systems. Later, the same concept was extended for open-loop unstable systems by introducing an h units of time ahead state predictor, (Artstein 1982; Manitius et al. 1979):

$$\hat{x}_1(t+h) \triangleq e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-s)}Bu(s)\,\mathrm{d}s, \tag{5.2}$$

referred to as the conventional prediction throughout this chapter. The variable $\hat{x}_1(t+h)$ is understood as the projection of the state starting at $x(t)$ driven by the control history $u(t+s)$, $s \in [-h, 0]$. In the absence of disturbances, the feedback law $u(t) = K\hat{x}_1(t+h)$ achieves asymptotic stabilization for any $h > 0$ with a proper choice of K.

However, in a disturbed system, an error is introduced in the prediction $\hat{x}_1$. Since there is always an error between the exact and the approximated predictions, it is not possible to remove constant disturbances even using integral action. Although it is an interesting topic from a practical point of view (Krstic 2010a), only few articles have addressed this problem. In an effort to predict the evolution of the disturbances, adaptive algorithms have drawn the attention of some researchers. For example, sinusoidal disturbances of unknown frequency are identified and rejected in (Pyrkin et al. 2010) for LTI systems with known delay, and more recently in (Basturk et al. 2015) for systems with matched uncertainties (see also the references therein). Also, adaptive schemes are used to estimate and reject constant disturbances for unknown input delay in (Bresch-Pietri et al. 2012), and for known distributed delays in (Bekiaris-Liberis et al. 2013). Other works avoid any a priori knowledge of the disturbance structure. For example, a filtered version of the predicted state (5.2) is proved to minimize a cost functional involving the disturbance in (Krstic 2008a; Krstic 2010a). Recently in (Léchappé et al. 2015), a simple solution is considered, where additional feedback from the difference between the measured, $x(t)$, and delayed predicted state, $\hat{x}_1(t)$, is used to define a new prediction

$$\hat{x}_2(t+h) \quad = \quad \hat{x}_1(t+h) + [x(t) - \hat{x}_1(t)] \tag{5.3}$$

With this simple modification, it is proved that for a certain class of disturbing signals, the new prediction leads to better attenuation than the conventional one. However, perfect cancellation is only possible for constant disturbances, and the attenuation depends entirely on the characteristics of the disturbance.

The main contribution reported in this chapter is a new predictive scheme that takes into account a prediction of the disturbing signal, denoted by $\hat{d}(t+h)$. Such prediction is constructed from estimates of the disturbance and its derivatives up to the r-th order, which are obtained by means of a tracking differentiator. The predicted disturbance is used to define a new state prediction, allowing to compensate the effect of the disturbance in the overall system. A performance analysis based on Lyapunov's theory is carried out to prove that the proposed scheme performs better than previous proposals in the literature, in the presence of smooth enough time-varying disturbances, achieving perfect cancellation in some particular cases.

5.2 Problem statement

Let us consider the system (5.1). Other than the accessibility to the full state, the following assumptions are taken.

Assumption 5.1. *The pair (A,B) is controllable.*

Assumption 5.2. *The unknown disturbance $d(t)$ is uniformly bounded by $|d(t)| \leq D_0$ and it is $(r+1)$-times continuously differentiable with $|d^{(r+1)}(t)| \leq D_{r+1}$, $\forall t \geq 0$.*

From (5.1), it can be seen that the actual projection of the state at time $t+h$ is given by

$$x(t+h) \quad = \quad e^{Ah}x(t) + \int_{t-h}^{t} e^{A(t-s)}B[u(s) + d(s+h)]\,\mathrm{d}s. \tag{5.4}$$

Although (5.4) cannot be used in practice because the disturbance is unknown, an approximated prediction of the state h units of time ahead for the system (5.1), denoted by $\hat{x}(t+h)$, can be obtained by computing the conventional prediction

(5.2). From (5.2) and (5.4), the prediction error is given by

$$x(t+h) - \hat{x}_1(t+h) = \int_{t-h}^{t} e^{A(t-s)} B d(s+h)\, \mathrm{d}s. \tag{5.5}$$

In the disturbance-free case, $d(t) \equiv 0$, it can be seen from (5.5) that stabilizing $\hat{x}_1$ is equivalent to stabilize x because the prediction is exact. However, when $d(t) \neq 0$, the predicted state $\hat{x}_1$ is corrupted. In such case, if the control law is designed so that $\hat{x}_1$ tends to zero, then x will not tend to zero even for constant perturbations. This fact is illustrated by the following proposition, taken from (Léchappé et al. 2015):

Proposition 5.1. *The asymptotic convergence of $\hat{x}_1$ to zero implies the asymptotic convergence of x to $\int_{t-h}^{t} e^{A(t-s)} B d(s)\, \mathrm{d}s$.*

For constant disturbances, the prediction $\hat{x}_2$ in (5.3) avoids this problem as stated by the following proposition, also taken from (Léchappé et al. 2015):

Proposition 5.2. *For constant disturbances, the asymptotic convergence of $\hat{x}_2$ to zero implies the asymptotic convergence of x to zero.*

However, both predictions share some drawbacks: their accuracy is only determined by the characteristics of the disturbance signal, and perfect cancellation of time-varying disturbances is never possible. In the next section, a new prediction that mitigates these problems, denoted by $\hat{x}_3$, is proposed.

5.3 Proposed Predictor-based control

Let us assume that a future estimation of the disturbance $\hat{d}(t+h)$, is available. Then, a new predicted state which considers the effect of the disturbance can be computed by

$$\hat{x}_3(t+h) \triangleq e^{Ah} x(t) + \int_{t-h}^{t} e^{A(t-s)} B[u(s) + \hat{d}(s+h)]\, \mathrm{d}s. \tag{5.6}$$

The disturbance prediction error is defined as

$$\sigma(t) \triangleq d(t) - \hat{d}(t). \tag{5.7}$$

From (5.4), (5.6) and using the definition (5.7), the error of the new prediction is given by

$$x(t+h) - \hat{x}_3(t+h) = \int_{t-h}^{t} e^{A(t-s)} B\sigma(s+h)\,\mathrm{d}s. \tag{5.8}$$

Proposition 5.3. *If $\sigma \to 0$, then the asymptotic convergence of the new prediction $\hat{x}_3$ to zero implies the asymptotic convergence of x to zero.*

Proof. If $\hat{x}_3$ tends to zero, from (5.8) it can be seen that x tends to $\int_{t-h}^{t} e^{t-s} B\sigma(s+h)\,\mathrm{d}s$, and the proposition follows. □

Therefore, with the proposed predictive scheme, the disturbance attenuation will depend on the accuracy of the disturbance prediction estimation. To this purpose, the methodology adopted here is based on (Zhong et al. 2004) where, for a delay-free system, an estimation of the unknown uncertainties and disturbances is obtained using the system model. Considering the input-delayed system (5.1), the disturbance can be written as[1]

$$d(t) = B^{+}\left[\dot{x}(t) - Ax(t)\right] - u(t-h), \tag{5.9}$$

which cannot be computed because the state derivative is unknown. However, following the ideas in (Zhong et al. 2004), a filtered disturbance can be obtained as $\hat{d}(t) = \mathcal{L}^{-1}\left\{G(s)D(s)\right\}$, where $D(s) = B^{+}\left[sX(s) - AX(s)\right] - e^{-hs}U(s)$, and $G(s)$ is a strictly-proper unity-gain filter. In the simplest case, $G(s)$ can be chosen as a first-order low-pass filter.

Note that because of the input delay, even if the disturbance was perfectly identified, it could not be counteracted until h units of time later, which would lead to poor performance. To mitigate this problem, a strictly-proper filter $H(s)$ is designed such that an estimated prediction of the disturbance h units of time ahead is obtained as

$$\hat{d}(t+h) \triangleq \mathcal{L}^{-1}\left\{H(s)D(s)\right\}. \tag{5.10}$$

The underlying idea behind the filter $H(s)$ is to make an estimation $\hat{d}_0(t)$ of the disturbance, and its derivatives up to the r-th order, gathered in

$$\hat{\delta}(t) \triangleq [\hat{d}_0(t), \hat{d}_1(t), \ldots, \hat{d}_r(t)], \tag{5.11}$$

[1] Here $B^{+} \triangleq (B^T B)^{-1} B^T$ denotes the pseudoinverse of B

which are then used to construct a prediction h units of time ahead by using a truncated Taylor series expansion

$$\hat{d}(t+h) \triangleq \sum_{j=0}^{r} \frac{h^j}{j!}\hat{d}_j(t) \triangleq C_H\hat{\delta}(t), \tag{5.12}$$

with $C_H \triangleq \left[1, h, \ldots, \frac{h^r}{r!}\right]$. The following lemma introduces a linear tracking differentiator which is used to prove the main result.

Lemma 5.1. *Let us consider a signal $\xi(t)$ and its derivatives up to the r-th order gathered in the vector $\Xi \triangleq [\xi(t), \dot{\xi}(t), \ldots, \xi^{(r)}(t)]^T$ satisfying $|\xi^{(r+1)}| < M$, and an estimation $\hat{\Xi}(t) \triangleq [\hat{\xi}_0(t), \hat{\xi}_1(t), \ldots, \hat{\xi}_r(t)]^T$ given by the following dynamic system $\xi(t)$:*

$$\dot{\hat{\Xi}}(t) = \underbrace{\begin{bmatrix} -c_0 & 1 & 0 & \ldots & 0 \\ -c_1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -c_r & 0 & 0 & \ldots & 0 \end{bmatrix}}_{A_H} \hat{\Xi}(t) + \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_r \end{bmatrix}}_{B_H} \xi(t), \tag{5.13}$$

with $c_j = \binom{r+1}{j+1}\omega_o^{j+1}$ and $\omega_0 > 0$. Then (5.13) is exponentially stable and the following holds $\lim_{t\to\infty} |\xi^{(j)}(t) - \hat{\xi}_j(t)| \leq \frac{c_{j-1}}{c_r}M$.

Proof. The system (5.13) can be alternatively expressed as

$$\begin{aligned} \dot{\hat{\xi}}_j(t) &= -c_j\hat{\xi}_0(t) + \hat{\xi}_{j+1}(t) + c_j\xi(t), \; j = 0, 1, \ldots, r-1 \\ \dot{\hat{\xi}}_r(t) &= -c_r\hat{\xi}_0(t) + c_r\xi(t). \end{aligned} \tag{5.14}$$

Let us denote the estimation error of the j-th derivative of the input signal as $e_j(t) = \xi^{(j)}(t) - \hat{\xi}_j(t)$, which allows to rewrite (5.14) as

$$\begin{aligned} \dot{e}_j(t) &= -c_je_0(t) + e_{j+1}(t), \; j = 0, 1, \ldots, r-1, \\ \dot{e}_r(t) &= -c_re_0(t) + \xi^{(r+1)}, \end{aligned} \tag{5.15}$$

or, in matrix form, $\dot{e}(t) = A_He(t) + [0_{r-1}\, 1]^T\xi^{(r+1)}$. Notice that A_H has a unique eigenvalue, $-\omega_o$, with multiplicity $r+1$. Computing the analytic expression for the transfer function of each channel, one can see that $|e_j(s)/\xi^{(r+1)}(s)|_\infty = c_{j-1}/c_r$, which corresponds to the maximum amplification of the input on each channel when $t \to \infty$. ☐

Theorem 5.1. *Let us consider a disturbance $d(t)$ satisfying the Assumption 2 with $D_{r+1} > 0$, a filter given by*

$$H(s) = C_H(sI - A_H)^{-1}B_H, \tag{5.16}$$

with bandwidth $\omega_o > 0$, and A_H,B_H and C_H as defined in (5.12)-(5.13). The disturbance prediction (5.10) can be implemented through the following dynamic system

$$\dot{\hat{z}}(t) = A_H\hat{z}(t) + \left[A_H B_H B^+ - B_H B^+ A\right] x(t) - B_H u(t-h), \tag{5.17}$$

$$\hat{d}(t+h) = C_H \left[\hat{z}(t) + B_H B^+ x(t)\right], \tag{5.18}$$

with $\hat{z}(t)$ being an auxiliary variable. Then, the disturbance prediction error (5.7) is ultimately bounded by $\sigma_\infty \triangleq D_{r+1}\left(\beta(\omega_o) + \frac{h^{r+1}}{(r+1)!}\right)$, where $\beta : (0, +\infty) \to \mathbb{R}^+$ satisfies $\lim\limits_{\omega_o\to\infty} \beta(\omega_o) = 0$.

Proof. Let us apply the tracking differentiator (5.13) to the disturbance $d(t)$. Using the vector defined in (5.11), it follows that

$$\dot{\hat{\delta}}(t) = A_H\hat{\delta}(t) + B_H d(t). \tag{5.19}$$

Plugging (5.9) into (5.19), and performing the change of variable $\hat{z}(t) \triangleq \hat{\delta}(t) - B_H B^+ x(t)$ in (5.19) and (5.12), yields (5.17) and (5.18), respectively. The transfer function (5.16) follows directly as a realization of (5.12) and (5.19).

Now, using (5.7), (5.12) and the complete Taylor series representation $d(t+h) = \sum_{j=0}^{r} \frac{h^j}{j!} d^{(j)}(t) + \epsilon_r$, one has that $\sigma(t+h) = \sum_{j=0}^{r} \frac{h^j}{j!}[d^{(j)}(t) - \hat{d}_j(t)] + \epsilon_r$ where ϵ_r is the Taylor remainder that is known to be bounded by $|\epsilon_r| \leq D_{r+1} \cdot h^{r+1}/(r+1)!$. Using the Lemma 5.1 one can bound $\lim\limits_{t\to\infty} |d^{(j)}(t) - \hat{d}_j(t)| \leq (c_{j-1}/c_r)D_{r+1}$, and thus $\lim\limits_{t\to\infty} |\sigma(t)| \leq D_{r+1}\left(\sum_{j=0}^{r} \frac{h^j}{j!}(c_{j-1}/c_r) + h^{r+1}/(r+1)!\right)$. Using the factorial expression for c_j, the theorem follows with $\beta(\omega_o) = \sum_{j=0}^{r} \frac{h^j}{j!} \frac{(r+1)!}{j!(r+1-j)!} \omega_0^{j-r-1}$. □

Although the previous results regarding the new prediction $\hat{x}_3$ are rather general, a particular control transformation is also proposed. Since a prediction of the disturbance is already available, a suitable transformation is given by

$$v(t) \triangleq u(t) + \hat{d}(t+h), \tag{5.20}$$

where $v(t)$ is the new control input to the system.

5.4 Performance analysis

The Artstein's reduction (Artstein 1982), is a useful tool to analyze time delay systems as it transforms the original system into a delay-free one. It is easy to show that the reduction of system (5.1) with the conventional predicted variable $z_1(t) \triangleq \hat{x}_1(t+h)$ leads to

$$\dot{z}_1(t) = Az_1(t) + Bu(t) + e^{Ah}Bd(t), \tag{5.21}$$

while the reduced system using the alternative prediction $z_2(t) \triangleq \hat{x}_2(t+h)$, proposed in (Léchappé et al. 2015), is derived as

$$\dot{z}_2(t) = Az_2(t) + Bu(t) + Bd(t) + e^{Ah}B[d(t) - d(t-h)]. \tag{5.22}$$

Similarly, considering the proposed prediction (5.6) and the control transformation (5.20), the reduction with $z_3(t) \triangleq \hat{x}_3(t+h)$ is given by

$$\dot{z}_3(t) = Az_3(t) + Bv(t) + e^{Ah}B\sigma(t). \tag{5.23}$$

An improvement of the proposal is already highlighted by Proposition 5.3, that is, the new predictive scheme will cancel time-varying disturbances if $\sigma(t)$ tends to zero. From Theorem 5.1, $\sigma(t)$ tends to zero if $D_{r+1} = 0$. Hence, constant disturbances can be perfectly canceled for $r = 0$; the same applies for $r = 1$ and disturbances with linear growth; and so on.

In order to evaluate the attenuation for other time-varying disturbances, note that all three reduced systems (5.21)-(5.23) have the generic form $\dot{\chi}(t) = A\chi(t) + B\vartheta(t) + g(t)$, that is, a nominal system with a perturbation term. Since the pair (A, B) is controllable, there exists a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ such that the feedback law $\vartheta(t) = f(\chi(t))$ makes the origin of the nominal system ($g(t) \equiv 0$) globally exponentially stable. Furthermore, if $|g(t)| \leq \bar{g}$, $\forall t \geq t_0$, there exist $\alpha, \beta, \gamma > 0$ such that $|\chi(t)| \leq \beta|\chi(0)|e^{-\alpha t} + \gamma\bar{g}$, $\forall t \geq 0$. Hence, the following ultimate bounds hold

$$\lim_{t\to\infty} |z_1(t)| \leq \gamma|B||e^{Ah}|D_0, \tag{5.24}$$

$$\lim_{t\to\infty} |z_2(t)| \leq \gamma|B|\left[|e^{Ah}|hD_1 + D_0\right], \tag{5.25}$$

$$\lim_{t\to\infty} |z_3(t)| \leq \gamma|B||e^{Ah}|\sigma_\infty. \tag{5.26}$$

In order to obtain the bound on the actual state, the transformation has to be undone. In (Léchappé et al. 2015), it is shown that $|x(t)| \leq |\hat{x}_1(t)| + \eta|B|D_0$, and also that $|x(t)| \leq |\hat{x}_2(t)| + \eta|B|hD_1$, with $\eta = |\int_{-h}^{0} e^{As}\,\mathrm{d}s|$. Similarly, from (5.8) and Theorem 5.1, it is obtained that $\lim_{t\to\infty} |x(t)| \leq \lim_{t\to\infty} |\hat{x}_3(t)| + \eta|B|\sigma_\infty$. Gathering these results and (5.24)-(5.26), the different predictive schemes lead to the following ultimate bounds of the state:

$$\lim_{t\to\infty} |x(t)| \leq \left[\eta + \gamma|e^{Ah}|\right] |B|D_0 \triangleq r_1, \tag{5.27}$$

$$\lim_{t\to\infty} |x(t)| \leq \left[\eta + \gamma|e^{Ah}|\right] |B|hD_1 + \gamma|B|D_0 \triangleq r_2, \tag{5.28}$$

$$\lim_{t\to\infty} |x(t)| \leq \left[\eta + \gamma|e^{Ah}|\right] |B|\sigma_\infty \triangleq r_3. \tag{5.29}$$

Lemma 5.2. *Consider the conventional prediction $\hat{x}_1$ leading to the bound (5.27) and the proposed scheme leading to the bound (5.29). There exists a sufficiently large $\omega_o > 0$ such that $r_3 < r_1$, if*

$$\frac{D_{r+1}}{D_0}\frac{h^{r+1}}{(r+1)!} < 1 \tag{5.30}$$

Proof. From (5.27), (5.29), the condition $r_3 < r_1$ is implied by $\sigma_\infty < D_0$. The previous condition is fulfilled if the observer bandwidth satisfies $\beta(\omega_o) < \frac{D_0}{D_{r+1}} - \frac{h^{r+1}}{(r+1)!}$. Because of the properties of $\beta(\omega_o)$, this is always possible with a sufficiently large $\omega_o > 0$ if (5.30) holds. □

Lemma 5.3. *Consider the alternative prediction $\hat{x}_2$ leading to the bound (5.28) and the proposed scheme leading to the bound (5.29). There exists a sufficiently large $\omega_o > 0$ such that $r_3 < r_2$, if*

$$\frac{D_{r+1}}{D_1}\frac{h^{r}}{(r+1)!} < 1 \tag{5.31}$$

Remark 5.1. *Let us consider sinusoidal disturbances $d(t) = D_0 \sin \omega t$. From (5.30), the new proposal can lead to better attenuation than the conventional prediction simply with $r = 0$ if $\omega < 1/h$. Similarly, from (5.31), the proposal in (Léchappé et al. 2015) can be outperformed with $r = 1$ if $\omega < 2/h$. Notice also that in the limit $r \to \infty$, the new prediction improves attenuation for sinusoidal disturbance with arbitrarily large frequency.*

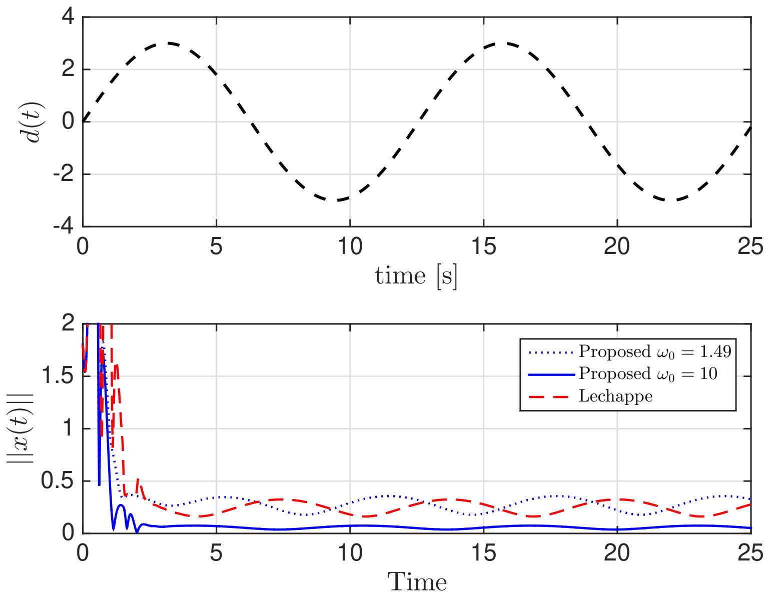


Figure 5.1: Simulations for $x(0) = [1.5 \quad 1]^T$, $h = 0.5$ s and sinusoidal perturbation

5.5 Numerical validation

In order to validate the bounds derived in the previous section, let us consider the system (Léchappé et al. 2015),

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -9 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-h) + \begin{bmatrix} 0 \\ d(t) \end{bmatrix}. \tag{5.32}$$

The simulation considers the same scenario in (Léchappé et al. 2015), with an input delay $h = 0.5$ s, the system starting from $x(0) = [1.5,\, 1]^T$ and a sinusoidal disturbance $d(t) = 3\sin(0.5t)$. The predictor-based control law in (Léchappé et al. 2015) is $u(t) = -[k_p,\, k_d]\, \hat{x}_2(t+h)$ with $k_p = 45$, $k_d = 18$. The same control law is selected for the proposed scheme by computing (5.20) with $v(t) = -[k_p,\, k_d]\, \hat{x}_3(t+h)$. The observer is calculated according to the Theorem 5.1 with $r = 1$. In this case, the attenuation can be improved because the condition (5.31) is fulfilled for $h = 0.5$ s and $\omega = 0.5$ rad/s. The simulation in Fig. 5.1 shows the limit case (same attenuation), along with a larger value $\omega_o = 10$ rad/s (better attenuation).

Chapter 6

Rejection of mismatched disturbances for input-delayed systems

The problem of output stabilization and disturbance rejection for input-delayed systems is tackled in this chapter. First, a suitable transformation is introduced to translate mismatched disturbances into an equivalent input disturbance. Then, an extended state observer is combined with a predictive observer structure to obtain a future estimation of both the state and the disturbance. A disturbance model is assumed to be known but attenuation of unmodeled components is also considered. Stability is proved via Lyapunov-Krasovskii functionals, leading to sufficient conditions in terms of linear matrix inequalities for the closed-loop analysis and parameter tuning. The proposed strategy is illustrated through a numerical example.

6.1 Introduction

In this chapter, the asymptotic stabilization of linear time delay systems in the presence of external mismatched disturbances is considered. In order to estimate the disturbance, a structure similar to the one presented in (Guo et al. 2005) is adopted, consisting of an extended state observer (ESO) that contains both the plant and disturbance models. The main contribution of the present work lies on extending the applicability of the ESO to input-delayed systems using a predictor in observer form (Najafi et al. 2013). As a result, a prediction h units of time ahead of both the state and the disturbance is obtained. Also, attenuation of unmodeled components of the disturbance is considered, which is a departure from (Guo et al. 2005). Furthermore, the proposed strategy is designed to deal with mismatched uncertainties and partial state measurement, in contrast to (Krstic 2008a; Léchappé et al. 2015; Sanz et al. 2016; Basturk et al. 2015). The regulation problem is translated into a conventional H_∞ stabilization problem and sufficient stability conditions in terms of linear matrix inequalities are derived.

6.2 Problem statement

The developments presented in this chapter consider a class of disturbed single-input time delay systems given by

$$\dot{\mathcal{X}}(t) = A\mathcal{X}(t) + Bu(t-h) + \Delta_l d(t) \tag{6.1}$$
$$y(t) = C\mathcal{X}(t) \tag{6.2}$$
$$z(t) = D\mathcal{X}(t), \tag{6.3}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{q\times n}$ are known matrices, $\mathcal{X} \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the measured output, $z \in \mathbb{R}^q$ is the regulated variable, $d : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an unknown external disturbance and $u \in \mathbb{R}$ the actuator signal, affected by a delay of h units of time. The vector $\Delta_l \in \mathbb{R}^n$ is defined such that its lth entry is equal to one while the rest are zero, being $l \in [1, n]$. The following assumptions are made:

Assumption 6.1. *The pair (A, C) is detectable and the pair (A, B) is controllable.*

Assumption 6.2. *The external disturbance can be represented by* $d(t) = \nu(t) + \eta(t)$*, where*

$$\begin{aligned} \dot{\xi}_d(t) &= A_\xi \xi_d(t) && (6.4) \\ \nu(t) &= C_\xi \xi_d(t), && (6.5) \end{aligned}$$

the matrices $A_\xi \in \mathbb{R}^{r\times r}, C_\xi \in \mathbb{R}^{1\times r}$ *are known (the so-called exogenous system) and form a completely observable pair,* $\xi_d \in \mathbb{R}^r$ *is the generator vector with unknown initial condition* $\xi_d(0)$*, and* $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *is an unknown bounded signal that represents the unmodeled disturbance components and satisfies* $\eta(t) \in L_2[0, \infty)$.

Assumption 6.3. *The pair* $\left(\begin{bmatrix} A & BC_\xi \\ 0 & A_\xi \end{bmatrix}, [C, \ 0] \right)$ *is detectable.*

Assumption 6.4. *The matrix* $D \in \mathbb{R}^{q\times n}$ *has the structure* $D = [\bar{D}, 0]$*, with* $\bar{D} \in \mathbb{R}^{q\times l}$.

The first part of Assumption 6.1 is necessary for the stabilization of the system via error feedback while the second part is assumed for simplicity. Assumption 6.2 is similar to that of the output regulation theory (Fridman 2003; Isidori et al. 1990). The eigenvalues of the matrix A_ξ usually lie on the imaginary axis, which means that for $\eta(t) = 0$, the model (6.4)-(6.5) can represent sinusoidal disturbances or piecewise-continuous signals of polynomial growth. Assumption 6.3 does not imply loss of generality because it can always be fulfilled if (C, A) is detectable, by changing the dimension of the exogenous model (Isidori et al. 1990). Finally, Assumption 6.4 simply points out that the effect of mismatched disturbances cannot be completely removed from all states if $l \neq n$.

The goal is to find a control strategy that, in spite of the input delay, achieves cancellation of mismatched disturbances accurately modeled by (6.4)-(6.5), that is, when $\eta(t) = 0$. Also, some attenuation level characterized by the L_2-gain (denoted by $\gamma > 0$) should be guaranteed when there are unmodeled components in the disturbance, that is, when $\eta(t) \neq 0$. This is cast into an H_∞ problem as follows:

Problem 6.1. *Under Assumptions 6.1-6.4, find a dynamic output control law that internally stabilizes (6.1)-(6.2) and guarantees* $\|z(t)\|_2 \leq \gamma \|\eta(t)\|_2$ *for all* $0 \neq \eta(t) \in L_2[0, \infty)$ *and some* $\gamma > 0$*, assuming* $\mathcal{X}_0 = 0$.

Before introducing the proposed strategy to solve this problem, the system (6.1)-(6.3) is reformulated in a more convenient form. By virtue of Assumption 6.1 and without loss of generality, let us consider the pair (A, B) to be given in the canonical controllable form, that is, with

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}. \tag{6.6}$$

The disturbance $d(t)$ in (6.1) can be mismatched (if $l \neq n$), meaning that it affects the state through channels in which the input has no influence. In order to obtain an equivalent input disturbance, let us consider the following change of variable (Ding 2003):

$$\begin{aligned} x_j(t) &= \mathcal{X}_j(t), \quad \forall j \in \{1, \dots, l\}, \\ x_j(t) &= \mathcal{X}_j(t) + \nu^{(j-l-1)}(t), \quad \forall j \in \{l+1, \dots, n\}, \end{aligned} \tag{6.7}$$

which can be used to transform the system (6.1)-(6.3) into

$$\dot{x}(t) = Ax(t) + B[u(t-h) + w(t)] + \Delta_l \eta(t), \tag{6.8}$$

$$y(t) = Cx(t), \tag{6.9}$$

$$z(t) = Dx(t), \tag{6.10}$$

where

$$w(t) = \frac{1}{b}\left(\nu^{(n-l)}(t) - \sum_{j=l+1}^{n} a_j \nu^{(j-l-1)}(t)\right). \tag{6.11}$$

Proposition 6.1. *The exogenous model (6.4)-(6.5) is also a generator of the equivalent input disturbance defined in (6.11), i.e., it can be represented by*

$$\dot{\xi}_w(t) = A_\xi \xi_w(t), \tag{6.12}$$

$$w(t) = C_\xi \xi_w(t), \tag{6.13}$$

where $\xi_w \in \mathbb{R}^r$ is a generator vector with unknown initial condition $\xi_w(0)$.

Proof. Let us rewrite (6.11) as $w(t) = \sum_{k=0}^{\bar{k}} c_j \nu^{(k)}(t)$ where $j = k + l + 1$ being k a new summation index with $\bar{k} = n - l$, and the coefficients $c_j = -a_j/b, \forall j \in [l+1, n]$, $c_{n+1} = 1/b$ have been defined for convenience. From, (6.4)-(6.5), the following identities hold

$$w(t) = \sum_{k=0}^{\bar{k}} c_j \nu^{(k)}(t) = \sum_{k=0}^{\bar{k}} c_j C_\xi A_\xi^k \xi_d(t) = C_\xi \xi_w(t),$$

where the definition $\xi_w(t) \triangleq \sum_{k=0}^{\bar{k}} c_j A_\xi^k \xi_d(t)$ has been used in the last equality. Differentiating $\xi_w(t)$ and using (6.4) it is easy to see that (6.12) holds, which completes the proof. □

The term $w \in \mathbb{R}$ should be understood as an equivalent input disturbance. Note that the triplet (A, B, C) is not modified by this transformation and thus controllability and detectability are preserved. Intuitively, the components of the disturbance are pushed through the chain of integrators by considering their higher derivatives. It should be remarked that the change of variable (6.7) is only used for analysis purposes and it is not needed for the implementation of the proposed control strategy. In what follows, the equivalent system (6.8)-(6.10) along with the generator model (6.12)-(6.13) are considered.

Remark 6.1. *The transformation* (6.7) *is not well defined if $l = n$ because in such case the model* (6.1)-(6.3) *is already in the form of* (6.8)-(6.10) *and the subsequent analysis can be directly applied.*

Remark 6.2. *Although the generalization to MIMO systems seems feasible, it cannot be derived in an easy way from the proposed strategy. On one hand, having multiple inputs usually implies having multiple time delays as well. In that case, the extension of the predictor-observer introduced in the next section is not trivial. On the other hand, the derivation of the transformation* (6.7) *is not straightforward for the general MIMO case, as it would require using the concept of Brunowski canonical form and vector relative degree.*

6.3 Proposed control strategy

The proposed solution to Problem 6.1 is given in this section. The observer-based controller structure is introduced and the closed-loop equations are derived. Then, sufficient stability conditions are given in terms of linear matrix inequalities.

6.3.1 Observer-based predictive controller

Let us represent the system dynamics by defining an augmented state $\zeta(t) = [x^T(t),\ \xi_w(t)]^T$, which includes the exosystem model. Using (6.8)-(6.9) and (6.12)-(6.13), the dynamics in terms of $\zeta(t)$ is derived as

$$\dot{\zeta}(t) = \underbrace{\begin{bmatrix} A & BC_\xi \\ 0 & A_\xi \end{bmatrix}}_{A_z} \zeta(t) + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_z} u(t-h) + \underbrace{\begin{bmatrix} \Delta_l \\ 0 \end{bmatrix}}_{B_\eta} \eta(t), \tag{6.14}$$

$$y(t) = \underbrace{\begin{bmatrix} C & 0 \end{bmatrix}}_{C_z} \zeta(t), \tag{6.15}$$

where $A_z \in \mathbb{R}^{(n+r)\times(n+r)}$, $B_z \in \mathbb{R}^{(n+r)}$, $C_z \in \mathbb{R}^{q\times(n+r)}$. The main idea introduced in this work is to construct an observer to obtain an estimation of the augmented state h units of time ahead $\zeta(t+h)$, denoted by $\bar{\zeta}(t) \triangleq [\bar{x}^T(t),\ \bar{\xi}_w(t)]^T$. In this way, the observer forecasts both the state and the disturbance, which can be computed as $\bar{w}(t) = C_\xi \bar{\xi}_w(t)$. Note that, because of the input delay, the latter is needed to effectively counteract the disturbance, as pointed out in (Sanz et al. 2016). Following the ideas in (Najafi et al. 2013), a plausible observer is given by

$$\dot{\bar{\zeta}}(t) = A_z\bar{\zeta}(t) + B_z u(t) + L\left(y(t) - C_z\bar{\zeta}(t-h)\right), \qquad \bar{\zeta}_0 = 0. \tag{6.16}$$

The estimation error is defined by[1]

$$e(t) \triangleq [e_x^T(t), e_\xi^T(t)]^T \triangleq \zeta(t) - \bar{\zeta}(t-h), \tag{6.17}$$

where $e_x \in \mathbb{R}^n$ and $e_\xi \in \mathbb{R}^r$. Differentiating (6.17) and using (6.14)-(6.15), the error dynamics is given by

$$\dot{e}(t) = A_z e(t) - LC_z e(t-h) + B_\eta \eta(t). \tag{6.18}$$

Assuming that L is chosen such that (6.18) is stable, the control law can be selected analogous to that of conventional controllers compensating for matched uncertainties (Chen et al. 2016):

$$u(t) = -K\bar{x}(t) - \bar{w}(t) = -K\bar{x}(t) - C_\xi\bar{\xi}_w(t) = -[K,\ C_\xi]\bar{\zeta}(t). \tag{6.19}$$

[1] Intuitively, the estimation error should be defined as $\zeta(t+h) - \bar{\zeta}(t)$, provided that $\bar{\zeta}(t)$ is supposed to be a future estimation of $\zeta(t)$. However, the definition (6.17) is arbitrarily chosen to avoid non-causal terms in subsequent derivation.

Plugging (6.19) into (6.8) and using (6.13), (6.17), leads to

$$\dot{x}(t) = (A - BK)x(t) + [BK,\ BC_\xi]e(t). \tag{6.20}$$

For convenience, let us define $\mu(t) \triangleq [x^T(t), e^T(t)]^T$ and rewrite the dynamics (6.18) and (6.20) along with the regulated variable as

$$\dot{\mu}(t) = \underbrace{\begin{bmatrix} A - BK & [BK,\ BC_\xi] \\ 0 & A_z \end{bmatrix}}_{A_0} \mu(t) + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -LC_z \end{bmatrix}}_{A_1} \mu(t-h) + \underbrace{\begin{bmatrix} \Delta_l \\ B_\eta \end{bmatrix}}_{B_0} \eta(t), d \tag{6.21}$$

$$z(t) = \underbrace{\begin{bmatrix} D & 0 \end{bmatrix}}_{D_0} \mu(t). \tag{6.22}$$

Then Problem 6.1 has been then translated into an H_∞ stabilization problem of the closed-loop defined by (6.21)-(6.22), which is tackled next.

6.3.2 *Closed-loop disturbance rejection*

Lemma 6.1. *Given gains K, L and $\gamma > 0$, $\bar{h} > 0$, let there exist symmetric positive definite matrices $P, Q, R \in \mathbb{R}^{(2n+r)\times(2n+r)}$ and matrices $P_2, P_3 \in \mathbb{R}^{(2n+r)\times(2n+r)}$ that satisfy the LMI*

$$\Phi_1 = \begin{bmatrix} \Phi_1(1,1) & P - P_2^T + A_0^T P_3 & Re^{-2\delta\bar{h}} + P_2^T A_1 & P_2^T B_0 & D_0^T \\ (*) & -P_3 - P_3^T + \bar{h}^2 R & -P_3^T A_1 & P_3^T B_0 & 0 \\ (*) & (*) & -(S+R)e^{-2\delta\bar{h}} & 0 & 0 \\ (*) & (*) & (*) & -\gamma^2 I & 0 \\ (*) & (*) & (*) & (*) & -I \end{bmatrix} < 0, \tag{6.23}$$

where $\Phi_1(1,1) = A_0^T P_2 + P_2^T A_0 + 2\delta P + Q - Re^{-2\delta\bar{h}}$. Then the system (6.21)-(6.22) is exponentially stable with a decay rate $\delta > 0$ for any delay $0 \le h \le \bar{h}$ and achieves $\|z(t)\|_2 \le \gamma \|\eta(t)\|_2$ for any $0 \ne \eta(t) \in L_2[0,\infty)$.

Proof. The proof is derived using the Lyapunov-Krasovskii functional (LKF)

$$\begin{aligned} V(\mu_t, \dot{\mu}_t) &= \mu^T(t)P\mu(t) + h\int_{t-h}^{t} e^{2\delta(s-t)}\mu^T(s)S\mu(s)\,\mathrm{d}s \\ &\quad + h\int_{-h}^{0}\int_{t+\theta}^{t} e^{2\delta(s-t)}\dot{\mu}^T(s)R\dot{\mu}(s)\,\mathrm{d}s\mathrm{d}\theta, \end{aligned} \tag{6.24}$$

which is a slightly simplified version of the one presented in (Fridman et al. 2009). The statement of Lemma 6.1 holds true if it can be shown that (Fridman 2014)

$$\dot{V}(\mu_t, \dot{\mu}_t) + 2\delta V(\mu_t, \dot{\mu}_t) + z^T(t)z(t) - \gamma^2|\eta(t)|^2 \leq 0. \tag{6.25}$$

Differentiating (6.24), one finds

$$\begin{aligned} \dot{V}(\mu_t, \dot{\mu}_t) + 2\delta V(\mu_t, \dot{\mu}_t) \leq \; & 2\mu^T(t)P\dot{\mu}(t) + 2\delta\mu^T(t)P\mu(t) + h^2\dot{\mu}^T(t)R\dot{\mu}(t) \\ & -he^{-2\delta h}\int_{t-h}^{t} \dot{\mu}^T(s)R\dot{\mu}(s)\,\mathrm{d}s + \mu^T(t)S\mu(t) \\ & -e^{-2\delta h}\mu^T(t-h)S\mu(t-h) \\ & +2[\mu^T(t)P_2^T + \dot{\mu}^T(t)P_3^T]\cdot[\text{RHS of (6.21)} - \dot{\mu}(t)]. \end{aligned} \tag{6.26}$$

The last term in (6.26), which is identically zero, follows from the application of the descriptor method (Fridman 2014). The Jensen's inequality is employed to bound

$$-h\int_{t-h}^{t} \dot{\mu}^T(s)R\dot{\mu}(s) \leq -[\mu(t) - \mu(t-h)]^T R\,[\mu(t) - \mu(t-h)]. \tag{6.27}$$

Let us define $q(t) = [\mu^T(t),\, \dot{\mu}^T(t),\, \mu^T(t-h),\, w(t)]^T$. Using (6.26)-(6.27), it follows that (6.25) holds if (6.23) is satisfied, completing the proof. □

Theorem 6.1. *Given $\gamma > 0$, $\bar{h} > 0$ and tuning parameters $\alpha > 0$, $\delta > 0$, $\epsilon > 0$, let there exist symmetric positive definite matrices $P, Q, R \in \mathbb{R}^{(2n+r)\times(2n+r)}$, $S \in \mathbb{R}^{n\times n}$ and matrices $P_{21} \in \mathbb{R}^{n\times n}$, $P_{22} \in \mathbb{R}^{(n+r)\times(n+r)}$, $X \in \mathbb{R}^{1\times n}$ that satisfy the following LMIs:*

$$\Phi_2 = \begin{bmatrix} \Phi_2(1,1) & P - P_2^T + A_0^T\epsilon P_2 & Re^{-2\delta\bar{h}} + \begin{bmatrix} 0 & 0 \\ 0 & YC_z \end{bmatrix} & P_2^T B_0 & D_0^T \\ (*) & -\epsilon P_2 - \epsilon P_2^T + \bar{h}^2 R & -\epsilon\begin{bmatrix} 0 & 0 \\ 0 & YC_z \end{bmatrix} & \epsilon P_2^T B_0 & 0 \\ (*) & (*) & -(S+R)e^{-2\delta\bar{h}} & 0 & 0 \\ (*) & (*) & (*) & -\gamma^2 I & 0 \\ (*) & (*) & (*) & (*) & -I \end{bmatrix} < 0, \tag{6.28}$$

$$SA^T + X^T B^T + AS + BX + 2\alpha S < 0, \tag{6.29}$$

where $\Phi_2(1,1) = A_0^T P_2 + P_2^T A_0 + 2\delta P + Q - Re^{-2\delta\bar{h}}$ *and* $P_2 = diag\{P_{21}, P_{22}\}$. *Then, the control law* (6.19) *computed by means of the observer* (6.16) *with* $K = XS^{-1}$ *and* $L = (P_{22}^T)^{-1}Y$, *solves Problem* 6.1.

Proof. In order to partially linearize the LMI (6.23), let us assume a diagonal structure for $P_2 = diag\{P_{21}, P_{22}\}$ and, following (Fridman 2014), $P_3 = \epsilon P_2$. Defining $Y = P_{22}^T L$, and after some straightforward calculations, the LMI (6.23) is transformed into (6.28). From the triangular structure of the state matrices in (6.21), $A - BK$ needs to be Hurwitz to ensure the stability of the overall system, which is guaranteed by (6.29). □

Remark 6.3. *The problem posed in Theorem 6.1 has to be solved sequentially, obtaining first K from* (6.29) *and then L from* (6.28)*. The parameter* $\alpha > 0$ *is user-supplied and determines how aggressive the resulting controller will be. The value of* γ *can be supplied or, alternatively, defining* $\beta = \gamma^2$*, the problem can be cast into minimizing* β *subject to* (6.28)*. The parameter* $\epsilon > 0$ *needs to be supplied and it should be iteratively adjusted to reach the best value of* β *in the minimization problem just described (there is a convex behavior of* β *with respect to* ϵ *as explained in (Fridman et al. 2002)).*

6.4 Trajectory tracking

In this section, it is shown how the proposed method can be easily adapted to solve the problem of trajectory tracking. First, the following assumption is made:

Assumption 6.5. *The desired trajectory* $r(t)$ *is bounded and sufficiently smooth so that* $r(t) \in \mathcal{C}^n$.

Problem 6.2. *Under Assumptions 6.1-6.5, find a dynamic output control law that internally stabilizes* (6.1)-(6.2) *and guarantees* $\|z(t) - r(t-h)\|_2 \leq \gamma \|\eta(t)\|_2$ *for all* $0 \neq \eta(t) \in L_2[0, \infty)$*, assuming* $\mathcal{X}_0$.

In what follows, the tracking problem is reduced to a stabilization problem so that the methodology described in Section 6.3 can be directly applied. To that end, let us introduce an auxiliary reference system

$$\dot{x}_r(t) = Ax_r(t) + Bu_r(t), \qquad y_r(t) = Cx_r(t-h), \qquad z_r(t) = Dx_r(t-h), \tag{6.30}$$

where the auxiliary state $x_r \in \mathbb{R}^n$ has zero initial condition $x_{r_0} = 0$, the matrices A, B, D are the same as in (6.1)-(6.2) and $u_r \in \mathbb{R}$ is to be designed such that the auxiliary system is internally stable and $\lim_{t\to\infty} \left(z_r(t) - r(t)\right) = 0$. It can be easily verified, because of the canonical structure of (A, B), that the control signal

$$u_r(t) = -\frac{1}{b}\sum_{j=1}^{n} a_j x_{r_j} + \frac{1}{b}\sum_{j=1}^{n}\left(k_{r_j}\left(r^{(j-1)}(t) - x_j(t)\right) + r^{(n)}(t)\right) \tag{6.31}$$

achieves that goal for any set of gains $k_{r_j} > 0$. Now, let us define the following variables

$$\tilde{x}(t) \triangleq x(t) - x_r(t-h), \qquad \tilde{y}(t) \triangleq y(t) - y_r(t), \qquad \tilde{z}(t) \triangleq z(t) - z_r(t). \tag{6.32}$$

Differentiating $\tilde{x}(t)$ and using (6.8)-(6.9), (6.30) and (6.32) leads to

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B[\tilde{u}(t-h) + w(t)] + \Delta_l \eta(t), \tag{6.33}$$

$$\tilde{y}(t) = C\tilde{x}(t), \qquad \tilde{z}(t) = D\tilde{x}(t), \tag{6.34}$$

where the variable $\tilde{u}(t) \triangleq u(t) - u_r(t)$ has been defined. After this transformation, the new control objective is to drive $\tilde{z}(t)$ to zero. Note that (6.33)-(6.34) has the same structure as (6.8)-(6.9). Therefore, the tracking problem has been reduced to the stabilization problem solved in Section 6.3. This result is summarized in the following theorem, which is given without proof:

Theorem 6.2. *Under conditions of Theorem 6.1, let K, L be given, which are found from LMIs (6.28)-(6.29). Then, given any set of positive gains $k_{r_j} > 0$, $j = 1, \ldots, n$, the control law*

$$u(t) = \tilde{u}(t) + u_r(t) = -[K, C_\zeta]\bar{\zeta}(t) + u_r(t), \tag{6.35}$$

with $u_r(t)$ given by (6.31) and $\bar{\zeta}(t)$ computed by means of the observer

$$\dot{\bar{\zeta}}(t) = A_z\bar{\zeta}(t) + B_z\tilde{u}(t) + L\left(\tilde{y}(t) - C_z\bar{\zeta}(t-h)\right), \qquad \bar{\zeta}_0 = 0, \tag{6.36}$$

solves Problem 6.2.

Remark 6.4. *The auxiliary control signal $u_r(t)$ can be chosen arbitrarily as long as $\lim_{t\to\infty}\left(z_r(t) - r(t)\right) = 0$. Therefore, alternative expressions to (6.31) are plausible. The tuning of the resulting strategy is intuitive because the tracking performance is decoupled from the stability, as it happens with a conventional two degrees of freedom PID controller. This can be seen from the control law (6.35), where the feedback term depends only on K, and u_r is a feed-forward term generated by the auxiliary system (6.30), which has no influence on the stability. The gains k_{r_j} can be thus arbitrarily adjusted without jeopardizing the stability.*

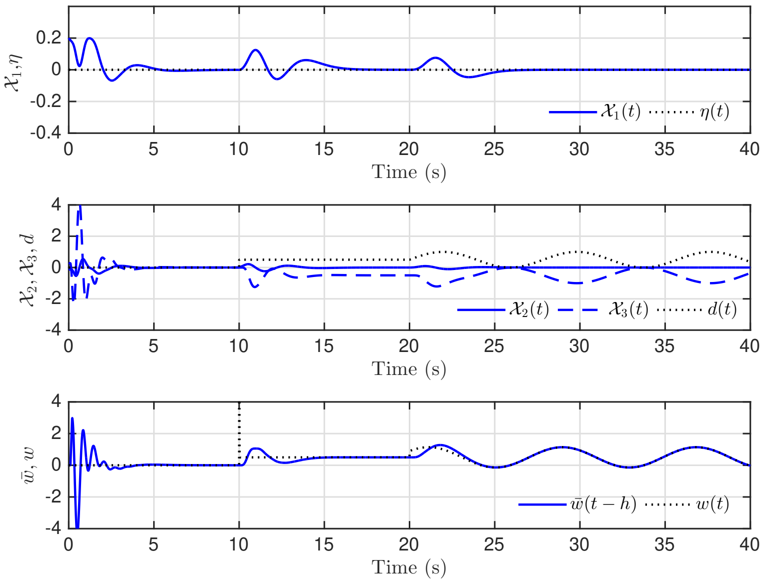


Figure 6.1: Simulation with accurate disturbance model: output and unmodeled component (top), internal states and mismatched disturbance (center), equivalent input disturbance and its delayed prediction (bottom)

6.5 Simulations

Let us consider an electromechanical system described by

$$\dot{\mathcal{X}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -k \end{bmatrix} \mathcal{X}(t) + \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} u(t-h) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} d(t), \tag{6.37}$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathcal{X}(t), \qquad z(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathcal{X}(t) \tag{6.38}$$

where $\mathcal{X}_1, \mathcal{X}_2$ are position and velocity, $\mathcal{X}_3$ represents the actuator dynamics, u is the control input and d can be generated by an external force or torque. In this example, the delay is taken as $h = 0.1$ s, and it is assumed that $d(t)$ is a biased sinusoidal disturbance with known frequency, $\omega = 0.8$ rad/s. The controller is designed according to Theorem 6.1. Choosing $\alpha = 1$ and solving (6.29) yields

$$S = \begin{bmatrix} 0.47 & -0.55 & 0.31 \\ -0.55 & 0.89 & -1.06 \\ 0.31 & -1.06 & 3.15 \end{bmatrix}, \quad X = \begin{bmatrix} -0.75 & 2.10 & 0.50 \end{bmatrix}, \tag{6.39}$$

and thus $K = XS^{-1} = \begin{bmatrix}15.66 & 17.18 & 4.37\end{bmatrix}$. Choosing $\delta = 0.8$ the LMI (6.28) is found to be feasible with a minimum $\gamma = 11.5$ for $\epsilon = 0.3$. In this configuration, the observer gain is given by

$$L^T = \begin{bmatrix}9.14 & 36.83 & 76.42 & 53.89 & 89.20 & 44.12\end{bmatrix}. \tag{6.40}$$

The results of the first simulation are shown in Fig. 6.1. It can be seen in the top plot how the proposed strategy achieves cancellation of the disturbance effect in the output, as stated in Theorem 6.1. The disturbance signal is given by $\{d(t) = 0, \forall t \in [0, 10);\ d(t) = 1, \forall t \in [10, 20);\ d(t) = 1 + \sin 0.8t, \forall t \geq 20\}$. One can see in the bottom plot that the equivalent input disturbance (dashed black) is accurately predicted by the observer (blue).

Remark 6.5. *As mentioned above, the linearized LMI in Lemma 6.1 leads to a very conservative value of γ. If the resulting system is analyzed using Lemma 6.1 with K and L given by (6.39)-(6.40), a tighter value $\gamma = 0.74$ is obtained. The exact minimum can be obtained by inspecting the magnitude plot of the transfer function $T_{\eta\to z}(s) \triangleq D_0\left(sI - A_0 - A_1 e^{-sh}\right)^{-1} B_0$, which reveals that $|T_{\eta\to z}(s)|_\infty = 0.63$. The system is thus contractive, i.e., unmodeled components are attenuated at all frequencies.*

The second simulation shows the effect of adding an unmodeled disturbance component. In this case, a sinusoidal of higher frequency and smaller amplitude, $\eta(t) = 0.5 \sin 5t, \forall t \geq 30$, is added to the previous disturbance signal. The simulation results are shown in Fig. 6.2 where it can be seen that the known components of the disturbance are still canceled out while the unmodeled component is attenuated by a factor of $|T_{\eta\to z}(5i)| = 0.13$. In this case the equivalent input disturbance cannot be exactly predicted, as expected.

Finally, the third simulation demonstrates the trajectory tracking capabilities of the proposed strategy. The signal u_r is computed using (6.31), with $k_{r_1} = \omega_r^3/b$, $k_{r_2} = 3\omega_r^2/b$, $k_{r_3} = 3\omega_r/b$ and $\omega_r = 10$ rad/s. The tracking signal is chosen as $r(t) = \sin t$. The results are shown in Fig. 6.3, where the system starts from the origin and the same disturbance signal as in the first simulation, depicted in Fig. 6.1, is used. One can see how the output of the system tracks the reference in spite of the disturbance.

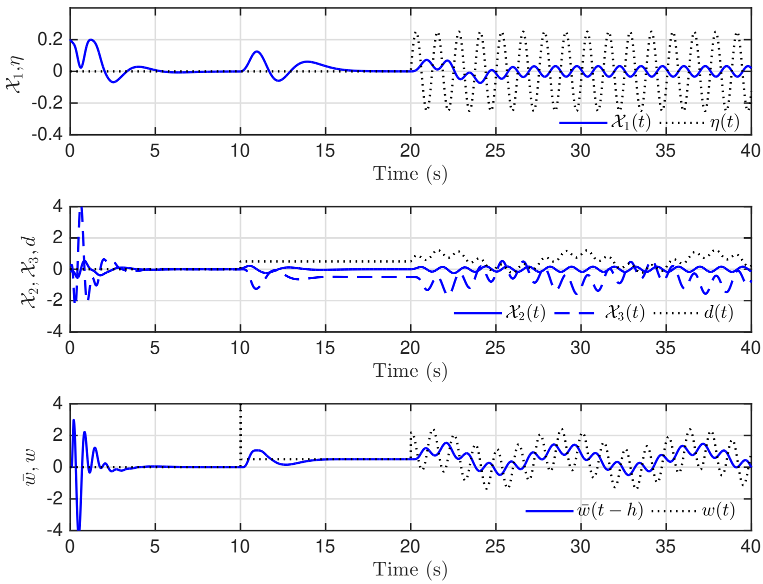


Figure 6.2: Simulation with inaccurate disturbance model: output and unmodeled component (top), internal states and mismatched disturbance (center), equivalent input disturbance and its delayed prediction (bottom)

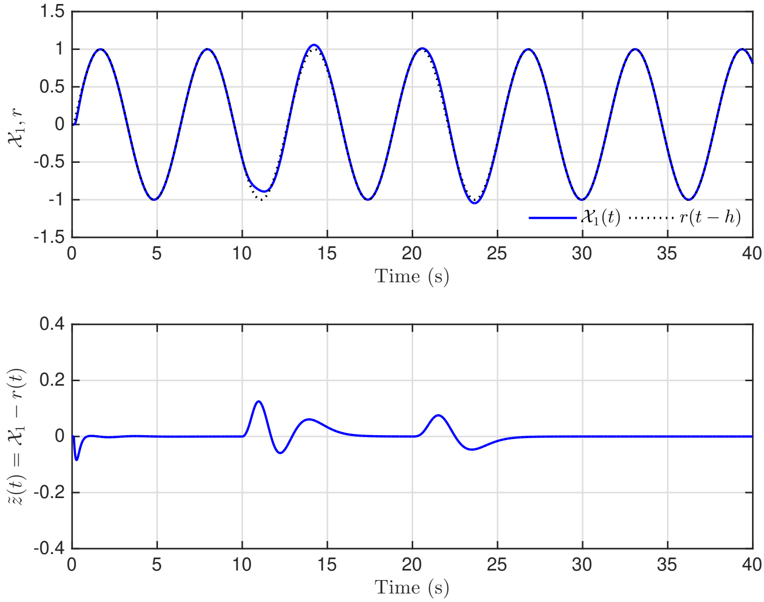


Figure 6.3: Simulation with trajectory tracking: output and reference (top), tracking error (bottom)

Chapter 7

Improving performance of disturbance observers via predictive feedback

In this chapter, a new robust control strategy based on a predictor and the uncertainty and disturbance estimator (UDE) is developed for a class of uncertain nonlinear systems with input/output delays. The closed-loop system is analyzed and sufficient stability conditions are derived based on Lyapunov analysis. The proposed strategy is applied to the particular case of quadrotor systems and validated through extensive simulations to evaluate performance and robustness. The controller is also implemented in a quadrotor prototype and validated in flight tests.

7.1 Introduction

Aerial Vehicles (UAVs) have gained an enormous interest for their civil potential applications. Among different UAVs, quadrotors are remarkably popular and have been used extensively in research over the past decade (Castillo et al. 2004; Mahony et al. 2012). A high-performance attitude control is a prerequisite for developing any other high-level control tasks (Bouabdallah et al. 2007). The quadrotor dynamics involves challenges such as parametric uncertainties, nonlinearity, coupling and external disturbances. Although many solutions have been proposed in the literature, very few of them have been validated in real flight tests and the most popular techniques are still based on classical control strategies (Bouabdallah et al. 2004; Castillo et al. 2005; Sanchez et al. 2008; Lim et al. 2012). This is mainly due to the constraints imposed by the limited computational resources of the embedded systems, which are typically micro-controllers. Also, and perhaps more importantly, because of the unstable nature of quadrotors, controllers must run typically at very high frequencies (Tomic et al. 2012).

Robust control for quadrotors is still an active field of research (Zheng et al. 2011; Islam et al. 2015; Zhao et al. 2015) because the aerodynamic effects are extremely hard to be accurately modeled (Hoffmann et al. 2007) and, specially in outdoor applications, a UAV is constantly perturbed by wind gusts (Waslander et al. 2009). Disturbance observers have drawn the attention of many researchers (Dong et al. 2014; Islam et al. 2015; Lee et al. 2014) as a tool for facing these problems.

Several approaches exist in the literature related to control based on the estimation of uncertainties and external disturbances, for example, adaptive robust control (Sun et al. 2013), uncertainty and disturbance estimator (UDE) (Zhong et al. 2004; Zhong et al. 2011), extended state observer based control (Yao et al. 2014) , disturbance observer based control (DOBC) (Ohishi et al. 1987), active disturbance rejection control (ADRC) (Han 2009), etc. The UDE strategy has demonstrated remarkable performance in handling uncertainties and disturbances in practical applications (Kolhe et al. 2013; Zhu et al. 2015; Ren et al. 2015; Kuperman et al. 2015).

Among the different problems that must be overcome in real implementations, time delays (Zhong 2006) deserve a special attention. In a micro-aerial vehicle, the angular position and velocity are typically estimated by means of filters resulting in delayed measurements (Sanz et al. 2014). Most control strategies can fail even for very small delays, which unavoidably appear in practical implementations due to the computational time, communications or actuator delays, mainly if fast disturbances are expected. In order to extend the applicability of these strategies to time delay systems, some modifications are needed.

The main contribution of this chapter is a new robust control strategy for a class of nonlinear time-delay systems, with a particular application to real-time quadrotor attitude control. The proposed control law combines a modified UDE with a state predictor that can be applied to control systems with measurement or actuation delays. The proposed method not only remains stable under the presence of large time delays, but it also results in a much better performance when small delays are present, as it is the case of any digital control system (Lozano et al. 2004). Sufficient conditions for the closed-loop stability are derived. The control law is validated through simulations and in real-time experiments with quadrotors.

7.2 Problem formulation and preliminaries

Consider the following class of nonlinear systems

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t-h_1) + d(t)] + f(x(t)) \\ y(t) &= x(t-h_2) \end{aligned} \tag{7.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and control variables respectively, $f : \mathbb{R}^n \to \mathbb{R}^n$ is an unknown possibly non-linear function, and $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is the vector of external disturbances. It is assumed that there is a constant input delay h_1, and also that the state of the plant is fully accessible with a measurement delay h_2. The total delay in the loop is denoted by $h = h_1 + h_2$. A representation of such system is depicted in Fig. 7.1(a). The following assumptions are taken:

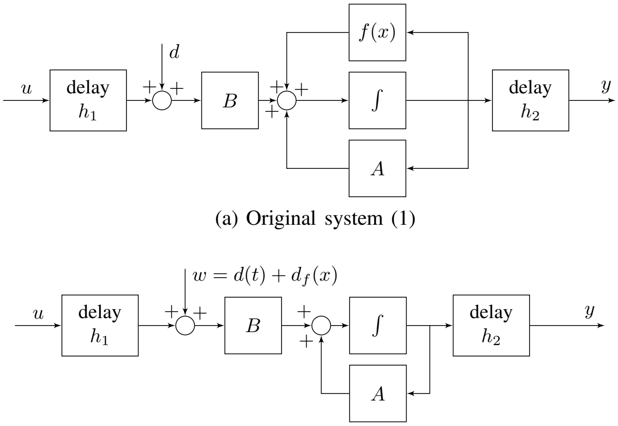
Assumption 7.1. *The pair (A, B) is controllable*

Assumption 7.2. *The time delay $h \geq 0$ is constant and known*

Assumption 7.3. *The uncertainty $f(x)$ belongs to the column space of B, i.e., there exists a vector $d_f(x) \in \mathbb{R}^m$ such that $f(x) = Bd_f(x)$*

Assumption 7.4. *There is a region $\mathcal{D} = \{x \in \mathbb{R}^n : |x| \leq r_x\}$ where: i.) $d_f(x)$ is locally bounded, ii.) $d_f(0) = 0$ and iii.) its derivative is locally bounded by $|\nabla d_f(x)| \leq c_x$*

Assumption 7.5. *The initial condition for (7.1) given by $x(s) = \varphi(s)$, $\forall s \in [-h, 0]$ with $\varphi : [-h, 0] \to \mathbb{R}^n$, is entirely contained in $\mathcal{D}$, that is, $|\varphi|_\infty < \delta$ for some $\delta < r_x$*



(a) Original system (1)

(b) Equivalent system (3)

Figure 7.1: System representations

Assumption 7.6. *The unforced system* (7.1) *starting from* $x(s) = \varphi(s)$, $\forall s \in [-h, 0]$ *satisfies*[1] $|x(\xi)| < \infty, \forall \xi \in [0, h]$

Assumption 7.7. *The input disturbance* $d(t)$ *is uniformly bounded and its derivative is bounded by* $||\dot{d}(t)|| \leq c_d$, $\forall t \geq 0$

Let us define the lumped term $w(x(t), t) \in \mathbb{R}^m$, to contain all the model uncertainties and external disturbances as follows

$$w(x(t), t) \triangleq d_f(x(t)) + d(t). \tag{7.2}$$

Using (7.2) and the Assumption 7.3, the model (7.1) can be represented as shown in Fig. 7.1(b), that is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t - h_1) + w(x(t), t)] \\ y(t) &= x(t - h_2). \end{aligned} \tag{7.3}$$

The underlying idea behind the original UDE (Zhong et al. 2004) is that the unknown lumped signal $w(x(t), t)$ along the solutions of (7.3), denoted hereafter simply by $w(t)$, can be accurately estimated and counteracted. However, as aforementioned, in the presence of input/output delays this strategy has limi-

[1] For nonlinear systems that exhibit finite escape time, there is a limit above which the system cannot be controlled (Krstic 2008b). The Assumption 7.6 prevents the unforced system from exhibiting finite escape time smaller than h.

tations and needs to be improved. The extension of this methodology to time delay systems is presented in the next section. To this end, a conventional state prediction (see for example (Krstic 2010a)) is computed by using the nominal model as

$$\hat{x}(t+h_1) = e^{Ah}x(t-h_2) + \int_0^h e^{A\xi}Bu(t-\xi)\,\mathrm{d}\xi. \tag{7.4}$$

However, the model used to predict the state may be inaccurate because of model uncertainties or external disturbances, as stated in the following proposition.

Proposition 7.1. *The error between the nominal prediction and the actual projection of the state is given by*

$$x(t+h_1) - \hat{x}(t+h_1) = \int_0^h e^{A\xi}Bw(t-\xi+h_1)\,\mathrm{d}\xi \tag{7.5}$$

Proof. Using the actual model (7.3), the actual projected state is given by

$$\begin{aligned} x(t+h_1) = e^{Ah}x(t-h_2) + \int_0^h e^{A\xi}Bu(t-\xi)\,\mathrm{d}\xi \\ + \int_0^h e^{A\xi}Bw(t-\xi+h_1)\,\mathrm{d}\xi. \end{aligned} \tag{7.6}$$

The proposition follows subtracting (7.6) and (7.4). □

7.3 Proposed control strategy

7.3.1 Control law development

The goal is to regulate the state $x(t)$ of the closed-loop system so that it asymptotically tracks the state of the reference model with the desired dynamics given by

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t-h_1), \tag{7.7}$$

where $A_m \in \mathbb{R}^{n\times n}$, is Hurtwitz, $B_m \in \mathbb{R}^{n\times m}$, $x_m \in \mathbb{R}^n$ and $r(t) \in \mathbb{R}^m$.

Assumption 7.8. *The input reference command $r(t)$ is bounded by $|r(t)| \leq \rho$, $\forall t \geq 0$*

Now, the following feedback law is proposed (see Fig. 7.2)

$$u(t) \triangleq -F\hat{x}(t+h_1) + Mr(t) + u_w(t), \tag{7.8}$$

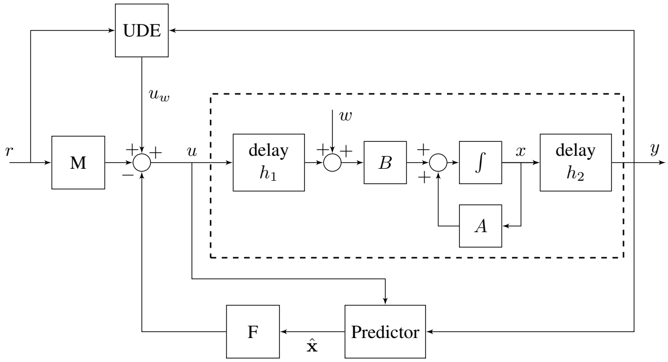


Figure 7.2: Proposed control strategy

where the matrices are chosen such that

$$A - BF = A_m, \qquad BM = B_m, \tag{7.9}$$

and the term $u_w(t)$, defined further below, will be used to compensate the uncertainties. Introducing (7.8) into the system (7.3) and using (7.9) yields

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + B_m r(t - h_1) + B[u_w(t - h_1) + \Delta(t)], \\ y(t) &= x(t - h_2), \end{aligned} \tag{7.10}$$

where the term

$$\Delta(t) \triangleq w(t) + F \int_0^h e^{A\xi} B w(t - \xi) \, \mathrm{d}\xi, \tag{7.11}$$

has been introduced to gather the original uncertainties of the system along with the error introduced by the predictor, both of them unknown. Once the effect of the delay has been counteracted, an observer based on the UDE is adopted in an outer loop to handle the overall uncertainties and disturbances. This approach might resemble the one-loop-at-a-time design procedure widely used in aircraft control, where an inner loop referred to as stability augmentation system (SAS) is used to increase stability, and outer loops are used to provide additional features for maneuvering (Schmidt 2012).

The new control input $u_w(t)$ should be chosen to cancel out the term $\Delta(t)$ which, despite being unknown, can be expressed using (7.10) as

$$\Delta(t) = B^{+} \left[\dot{x}(t) - A_m x(t) - B_m r(t - h_1) - B u_w(t - h_1) \right]. \tag{7.12}$$

Equation (7.12) is not implementable because the state $x(t)$ is not accessible at time t. Instead of (7.12), consider the signal

$$\Delta(t-h_2) = B^{+}\left[\dot{y}(t) - A_m y(t) + B_m r(t-h) - B u_w(t-h)\right], \tag{7.13}$$

which is the result of delaying $\Delta(t)$ by h_2 units of time. In this way, the measurement $y(t) = x(t-h_2)$ appears in the signal to be estimated. Another handicap is that the term $\dot{y}(t)$ is not realizable, but it can be approximated in the frequency domain by using a strictly-proper low-pass filter[2] $G_f(s) = 1/(T_f s + 1)$ (Zhong et al. 2004). Then, the estimated uncertainty can be defined as

$$\hat{\Delta}(t-h_2) \triangleq \mathcal{L}^{-1}\{G_f(s)\} * \Delta(t-h_2), \tag{7.14}$$

where $G_f(s) = G_f(s) I_m$. And thus, selecting

$$u_w(t) \triangleq -\hat{\Delta}(t-h_2), \tag{7.15}$$

and plugging it into (7.10) results in the closed-loop system

$$\dot{x}(t) = A_m x(t) + B_m r(t-h_1) + B e(t), \tag{7.16}$$

where $e(t)$ is the cancellation error defined as

$$e(t) \triangleq \Delta(t) - \hat{\Delta}(t-h). \tag{7.17}$$

Remark 7.1. *In the nominal disturbance-free case $e(t) \equiv 0$ and thus the closed-loop system* (7.16) *has the desired dynamics specified by the reference model* (7.7).

In order to analyze the stability in the presence of uncertainties, the dynamics of the error (7.17) has to be derived. According to (7.14), the estimator dynamics can be expressed as

$$\dot{\hat{\Delta}}(t-h) = -\frac{1}{T_f}\hat{\Delta}(t-h) + \frac{1}{T_f}\Delta(t-h). \tag{7.18}$$

Differentiating (7.17), using (7.18), and adding and subtracting $\frac{1}{T_f}\Delta(t)$, the dynamics of the cancellation error can be written as

$$\dot{e}(t) = -\frac{1}{T_f}e(t) + g(t), \tag{7.19}$$

[2] The Laplace transformation is introduced to facilitate the manipulation of expressions.

where

$$g(t) \triangleq \dot{\Delta}(t) + \frac{1}{T_f}[\Delta(t) - \Delta(t-h)]. \tag{7.20}$$

The initial condition for (7.19) is $e(s) = \Delta(s)$, $\forall s \in [-h, 0]$ by (7.17), assuming that the observer starts from zero, that is $\hat{\Delta}(s) = 0$, $\forall s \in [-h, 0]$.

The closed-loop system is thus composed of (7.16) and (7.19), which can be expressed altogether as

$$\dot{\eta}(t) = A_\eta \eta(t) + \begin{bmatrix} 0 \\ g(t) \end{bmatrix} + \begin{bmatrix} B_m \\ 0 \end{bmatrix} r(t - h_1), \tag{7.21}$$

where $\eta(t) = [x(t),\, e(t)]^T$ is an augmented state and the matrix A_η is given by

$$A_\eta = \begin{bmatrix} A_m & B \\ 0 & -\frac{1}{T_f} I_m \end{bmatrix}.$$

7.3.2 *Closed-loop stability*

As A_η is Hurtwitz, the main issue to analyze the stability is the boundedness of the term $g(t)$ that drives the error dynamics. This is addressed by the following lemma.

Lemma 7.1. *Under Assumptions 7.4 and 7.7, the term $g(t)$ defined in (7.20) satisfies $|g(t)| \leq \gamma_x |x(t)| + \gamma_e |e(t)| + \gamma_0, \forall x \in \mathcal{D}$, with constants γ_x, γ_e, γ_0 subsequently defined.*

Proof. Let us first introduce the following notation $\alpha_m \triangleq |A_m|$, $\beta_m \triangleq |B_m|$ and $\beta \triangleq |B|$. According to the definition (7.20), $g(t)$ can be bounded as

$$|g(t)| \leq |\dot{\Delta}(t)| + \frac{1}{T_f}|\Delta(t) - \Delta(t-h)|. \tag{7.22}$$

The first term in (7.22) is obtained by differentiating (7.11) as

$$\begin{aligned} \dot{\Delta}(t) = \nabla d_f \cdot \dot{x}(t) + F \int_0^h e^{A\xi} B \nabla d_f \cdot \dot{x}(t-\xi)\, \mathrm{d}\xi \\ + \dot{d}(t) + F \int_0^h e^{A\xi} B \dot{d}(t-\xi)\, \mathrm{d}\xi. \end{aligned} \tag{7.23}$$

By Assumptions 7.4 and7.7, (7.23) can be bounded by

$$|\dot{\Delta}(t)| \leq c_x|\dot{x}(t)| + \mu c_x \sup_{\xi\in[t-h,t]} |\dot{x}(\xi)| + c_d + \mu c_d, \tag{7.24}$$

where $\mu \triangleq |F| \left(\int_0^h |e^{A\xi}| \, d\xi \right) \beta$. From (7.16), one has that $|\dot{x}(t)| \leq \alpha_m|x(t)| + \beta|e(t)| + \beta_m\rho$. Assume now that there exist $q_e, q_x > 1$ such that $|x(t+\xi)| \leq q_x|x(t)|$ and $|e(t+\xi)| \leq q_e|e(t)| \, \forall\xi \in [-h, 0]$, (Trinh et al. 1997). Note that this assumption does not imply a priori the stability of the system. Then, it follows that $\sup_{\xi\in[t-h,t]} |\dot{x}(\xi)| \leq \alpha_m q_x|x(t)| + \beta q_e|e(t)| + \beta_m\rho$ and thus (7.24) is finally bounded by

$$\begin{aligned} |\dot{\Delta}(t)| \leq{}& c_x\alpha_m (1+\mu q_x) |x(t)| + c_x\beta (1+\mu q_e) |e(t)| \\ &+ (1+\mu)(c_d + c_x\beta_m\rho). \end{aligned} \tag{7.25}$$

The second term in (7.22) can be bounded using the Leibniz-Newton formula and the Young's inequality as follows

$$\begin{aligned} \frac{1}{T_f}|\Delta(t) - \Delta(t-h)| &\leq \frac{1}{T_f} \int_{t-h}^{t} |\dot{\Delta}(\xi)| \, d\xi \\ &\leq \frac{h}{T_f} \sup_{\xi\in[t-h,t]} |\dot{\Delta}(\xi)|. \end{aligned} \tag{7.26}$$

Proceeding similarly as above, from (7.25) it is easy to obtain that

$$\begin{aligned} \sup_{\xi\in[t-h,t]} |\dot{\Delta}(\xi)| \leq{}& c_x\alpha_m (1+\mu q_x) q_x|x(t)| \\ &+ c_x\beta (1+\mu q_e) q_e|e(t)| \\ &+ (1+\mu)(c_d + c_x\beta_m\rho). \end{aligned} \tag{7.27}$$

Gathering (7.22) and (7.25)-(7.27) yields

$$
\begin{aligned}
|g(t)| \leq\ & c_x\alpha_m\left(1+\mu q_x\right)\left(1+\frac{hq_x}{T_f}\right)|x(t)| \\
& + c_x\beta\left(1+\mu q_e\right)\left(1+\frac{hq_e}{T_f}\right)|e(t)| \\
& + \left(c_d + c_x\beta_m\rho\right)\left(1+\mu\right)\left(1+\frac{h}{T_f}\right) \\
& \triangleq \gamma_x|x(t)| + \gamma_e|e(t)| + \gamma_0.
\end{aligned}
\tag{7.28}
$$

□

This allows stating the following result:

Theorem 7.1. *Under Assumptions 1-6, the system (1) having no external inputs, i.e, $d(t) = r(t) \equiv 0$, controlled by (7.8) is asymptotically stable for some $\delta > 0$ and for any delay $0 \leq h \leq h^*$ if there exist a positive definite symmetric matrix P such that $PA_\eta + A_\eta^T P = -I_{m+n}$ and positive constants $q_x, q_e > 1$ such that $2|P_2|(\gamma_x + \gamma_e) < 1$ where $\gamma_x = c_x\alpha_m\left(1+\mu q_x\right)\left(1+\frac{h^* q_x}{T_f}\right)$, $\gamma_e = c_x\beta\left(1+\mu q_e\right)\left(1+\frac{h^* q_e}{T_f}\right)$ and $\mu = |F|\left(\int_0^{h^*}|e^{A\xi}|\,\mathrm{d}\xi\right)\beta$.*

Proof. Let us choose the Lyapunov candidate function $V(\eta) = \eta^T P\eta$ whose derivative along the trajectories of (7.21), with $r(t) \equiv 0$, is given by

$$
\begin{aligned}
\dot{V}(\eta) &= \eta^T(PA_\eta + A_\eta^T P)\eta + 2\eta^T P_2 g(t) \\
&\leq -|\eta|^2 + 2|\eta||P_2||g|.
\end{aligned}
\tag{7.29}
$$

Setting $d(t) = r(t) \equiv 0$ in Lemma 1, that is $c_d = \rho = 0$, the term γ_0 vanishes and, provided that $\gamma_x, \gamma_e > 0$, then $|g(t)| \leq \gamma_x|x(t)| + \gamma_e|e(t)| \leq (\gamma_x + \gamma_e)|\eta(t)|$. It follows then that for any $|\eta| < r_x$ (which implies $x \in \mathcal{D}$), (7.29) is bounded by

$$
\begin{aligned}
\dot{V}(\eta) &\leq -|\eta|^2 + 2|P_2|(\gamma_x + \gamma_e)|\eta|^2 \\
&\leq 0, \qquad \text{if} \qquad 2|P_2|(\gamma_x + \gamma_e) < 1,
\end{aligned}
\tag{7.30}
$$

and thus the system is asymptotically stable for h^* if $2|P_2|(\gamma_x + \gamma_e) < 1$ hold. Note that $V(t)$ is always decreasing because its derivative is negative in $|\eta| < r_x$ by (7.30), and hence $|\eta(t)|$ is always decreasing because $V(\eta) = |P||\eta|^2$. That implies that $\lim_{t\to\infty} |\eta(t)| = 0$ if $|\eta(0)| < r_x$, which can always be achieved by choosing a small enough δ in Assumption 7.5. Furthermore, the terms γ_x, γ_e grow monotonically with the delay an hence any $0 \leq h \leq h^*$ will also satisfy the condition, which completes the proof. □

Corollary 7.1. *If there is no delay, that is $h = 0$, it is always possible to find a choice for $A_m \prec 0$, $T_f > 0$ such that the system is asymptotically stable.*

Proof. Setting $h = 0$, the constants in Lemma 1 are simplified to $\gamma_x = c_x \alpha_m$ and $\gamma_e = c_x \beta$, and thus the stability condition of Theorem 1 is simply given by $2|P_2|c_x(\beta + \alpha_m) < 1$. Note that P is the solution to $PA_\eta + A_\eta^T P = -I_{m+n}$. The matrix A_η is upper triangular and its eigenvalues are the collection of those of A_m and $-1/T_f I_m$. Hence, $|P|$ can be arbitrarily reduced by choosing A_m, T_f properly. □

Remark 7.2. *Note that, for given a controller tuning A_m and T_f, which satisfy the conditions of Theorem 1, the admissible delay is upper bounded by h^*. If the delay was larger, one would try to reduce $|P|$ as indicated by Corollary 1 to keep the system stable, but doing so would have the opposite effect on $(\gamma_x + \gamma_e)$. Hence there is a maximum tolerable delay above which the system cannot be stabilized (this is well known for uncertain time-delay LTI systems).*

Theorem 7.2. *The system (1) controlled by (7.8), with an external disturbance $d(t) \neq 0$ satisfying Assumption 7.7, and a reference command satisfying Assumption 7.8, will be stable for some $\delta > 0$ and any delay $0 \leq h \leq h^*$ if the conditions in Theorem 1 hold and $\frac{2|P_2|(\gamma_0+\rho\beta_m)}{1-2|P_2|(\gamma_x+\gamma_e)} < r_x$ where $\gamma_0 = (c_d + c_x\beta_m\rho)(1+\mu)\left(1+\frac{h^*}{T_f}\right)$ and $\mu = |F|\left(\int_0^{h^*} |e^{A\xi}|\,\mathrm{d}\xi\right)\beta$.*

Proof. Considering external inputs, the term γ_0 in Lemma 1 does not vanish. Choosing the same Lyapunov function as in the proof of Theorem 1, the derivative along the new trajectories of (21) is given by

$$\begin{aligned}\dot{V}(\eta) &\leq (2|P_2|(\gamma_x + \gamma_e) - 1))\,|\eta|^2 \\ &\quad + 2|P_2|(\gamma_0 + \rho\beta_m)|\eta| \\ &\leq 0, \qquad \text{if} \qquad |\eta| > r_\eta,\end{aligned} \tag{7.31}$$

with $r_\eta \triangleq \frac{2|P_2|(\gamma_0+\rho\beta_m)}{1-2|P_2|(\gamma_x+\gamma_e)}$. Note that $1 - 2|P_2|(\gamma_x + \gamma_e) > 0$ if the conditions of Theorem 1 are met. Then, according to (7.31), the region defined by $\Omega_\eta = \{\eta \in \mathbb{R}^{n+m} : |\eta| \leq r_\eta\}$ is positively invariant, which means that any trajectory starting from outside will eventually reach Ω_η and remain inside for all future time (Khalil et al. 1996). The parameter r_η should be understood as how far from the origin the system is steered because of the inputs. Using Theorem 4.18 in (Khalil et al. 1996), if $r_\eta < r_x$, there exists a class $\mathcal{KL}$ function γ and a finite $T \geq 0$ for the initial state $|\eta(0)| \leq r_x$, such that the solution satisfies $|\eta(t)| \leq \gamma(|\eta(0)|, t), \forall\, 0 \leq t \leq T$ and $|\eta(t)| \leq r_\eta$, $\forall t \geq T$. □

7.3.3 Digital implementation

Recall that the proposed control law is given by (7.8). Regarding the computation of $u_w(t)$, using (15)-(17), the following expression for the UDE control action can be obtained (see Fig. 7.2)

$$\mathbf{U}_w(s) = [I - G_f(s)]^{-1} G_f(s) B^+ [(sI - A_m)Y(s) + B_m R(s)],$$

which can be easily discretized and implemented in a digital micro-controller.

The other key variable to be computed is the predicted state $\hat{x}(t + h_1)$. Its analytic expression is given in (7.4). The implementation of the distributed integral requires some attention (Zhong 2004; Zhong 2005). The predictor is implemented in discrete-time form as in (Lozano et al. 2004),

$$x_{k+d_1} = A_k x_{k-d_2} + \sum_{j=0}^{d-1} A_k^{h-j-1} B_k u_{k+j-h},$$

where (A_k, B_k) is a discretization of (A, B) and $d_1, d_2, d_3 \in \mathbb{N}$ are defined as $d_1 = h_1/T_s$, $d_2 = h_2/T_s$, $d = d_1 + d_2$, being T_s the discretization time.

To summarize the tuning procedure, four decisions are considered: the sample time T_s, the filter time constant T_f, the prediction horizon h, and the desired reference model.

7.4 Application to Quadrotor Aircraft

In this section, the performance and robustness of the proposed strategy are illustrated through several simulations using a quadrotor model. These results are validated experimentally in flight tests with a quadrotor prototype.

7.4.1 Modeling of quadrotor systems

A fairly accurate[3] model of a quadrotor is given by the following set of nonlinear equations (Bouabdallah et al. 2004)

$$\left.\begin{aligned}\ddot{\phi}(t) &= \frac{I_y - I_z}{I_x}\dot{\theta}(t)\dot{\psi}(t) - \frac{J}{I_x}\Omega\dot{\theta}(t) + \frac{u_\phi(t-h)}{I_x},\\ \ddot{\theta}(t) &= \frac{I_z - I_x}{I_y}\dot{\psi}(t)\dot{\phi}(t) + \frac{J}{I_y}\Omega\dot{\phi}(t) + \frac{u_\theta(t-h)}{I_y},\\ \ddot{\psi}(t) &= \frac{I_x - I_y}{I_z}\dot{\phi}(t)\dot{\theta}(t) + \frac{u_\psi(t-h)}{I_z},\end{aligned}\right\} \tag{7.32}$$

$$\ddot{z}(t) = \cos\phi(t)\cos\theta(t)\frac{u_z(t-h)}{m} - g, \tag{7.33}$$

where ϕ, θ and ψ are the roll, pitch and yaw Euler angles, I_i, $i = \{x, y, z\}$ are the moments of inertia and u_i, $i = \{\phi, \theta, \psi\}$ are the input torques, all of them defined along the axes of a body-fixed reference frame, z is the height, m is the mass of the vehicle, g is the gravity acceleration, u_z is the input total thrust, J is the inertia of the propellers and Ω is the sum of the angular velocities of the motors (taking the sign into account). An input delay h is also considered. This delay may be caused by the communications with the Electronic Speed Controller of the motors, their response time and also because of the digital implementation of the control law.

The uncertainty in the rotational subsystem (7.32) satisfies Assumption 7.3 because it can be written in terms of (7.3) with

$$A = \begin{bmatrix} 0_3 & I_3 \\ 0_3 & 0_3 \end{bmatrix} \quad B = \begin{bmatrix} 0_3 \\ I \end{bmatrix} \quad d_f(x) = \begin{bmatrix} (I_y - I_z)\dot{\theta}\dot{\psi} - J\Omega\dot{\theta} \\ (I_z - I_x)\dot{\phi}\dot{\psi} + J\Omega\dot{\phi} \\ (I_x - I_y)\dot{\phi}\dot{\theta} \end{bmatrix}$$

being $x = [\phi,\ \theta,\ \psi,\ \dot{\phi},\ \dot{\theta},\ \dot{\psi}]^T$ the state vector, $J \triangleq diag\{I_x^{-1}, I_y^{-1}, I_z^{-1}\}$ and $u = [u_\phi, u_\theta, u_\psi]^T$. The matched uncertainty is locally bounded and vanishes at the origin. Furthermore, its gradient is given by $\nabla d_f(x) = [0_3\ \nabla_{12}]$ with

$$\nabla_{12} \triangleq \begin{bmatrix} 0 & (I_y - I_z)\dot{\psi} - J\Omega & (I_y - I_z)\dot{\theta} \\ (I_z - I_x)\dot{\psi} + J\Omega & 0 & (I_z - I_x)\dot{\phi} \\ (I_x - I_y)\dot{\theta} & (I_x - I_y)\dot{\phi} & 0 \end{bmatrix} \tag{7.34}$$

[3] The rotational kinematic model used to derive (32) is linearized around the origin. Note that because of the singularity of the Euler representation, the linearization is only valid for $|\theta| < \pi/2$.

which is also locally bounded, thus satisfying the Assumption 7.4. Similarly, the height subsystem (7.33) satisfies the Assumption 7.3 because it can be written in terms of (7.3) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \qquad d(t) = mg$$

and $u = u_z \cos\theta \cos\phi$. Note that the disturbance $d(t)$ is constant and satisfies Assumption 7.7, while in this case $d_f(x) \equiv 0$ thus satisfying Assumption 7.4. Regarding the finite-escape condition, the unforced system (7.32) is reduced to the Euler's equations that describe the free rotation of a rigid body. As the consequence of conservation of the energy and angular momentum, the solution to this set of equations is bounded in time (Fowles et al. 2004). Also, setting $u_z = 0$ in (7.33) yields a linear equation. Therefore, there is not finite escape time and Assumption 6 holds.

Remark 7.3. *From the arguments presented above, the proposed controller applied to quadrotor systems can only be proven to be locally stabilizing around the origin (in spite of uncertainties, external disturbances and input delay)*

7.4.2 *Simulations*

For the sake of clarity, the performance and robustness are first illustrated using a SISO model. At the end of this section the control strategy is validated using the full nonlinear MIMO quadrotor model. The performance and robustness of the proposed strategy are illustrated next using one of the axes. Any of the equations in (7.32) can be seen as $\ddot{y} = bu(t-h) + w(t)$ or alternatively, in state-space form as

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t-h) + \begin{bmatrix} 0 \\ w(t) \end{bmatrix}, \tag{7.35}$$

where $w(t)$ represents the interaction with the missing states, uncertainties in the parameter b or external disturbances like wind gusts.

The controller is implemented as suggested in Section 7.3.3, with $T_s = 10$ ms and the reference model given by

$$\dot{x}_m = \begin{bmatrix} 0 & 1 \\ -\omega_c^2 & -2\omega_c \end{bmatrix} x_m + \begin{bmatrix} 0 \\ \omega_c^2 \end{bmatrix} y^{\text{ref}}(t), \tag{7.36}$$

where ω_c is the desired closed-loop bandwidth. The parameters T_f and h will be changed throughout this section.

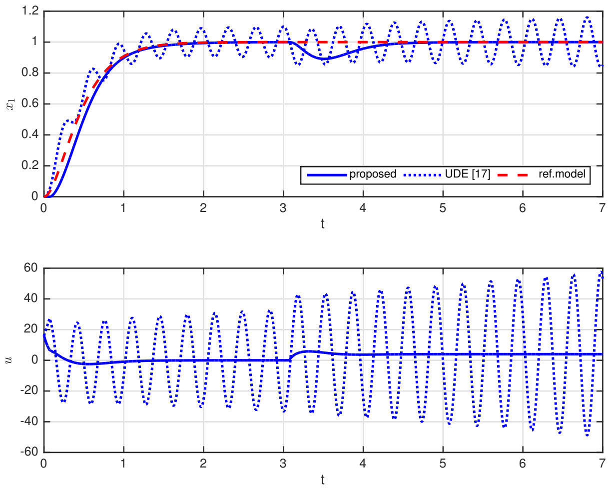


Figure 7.3: Simulations comparing the original UDE (Zhong et al. 2004) and the proposed strategy for a delay $h = 70$ ms.

Stability: Fig. 7.3 shows a comparison of the original UDE-based control without considering the delays (Zhong et al. 2004) and the proposed strategy. The filter is tuned with $T_f = 0.1$ s and the delay is $h = 70$ ms. One can see how the proposed strategy can deal with that delay, ensures the matching of the reference model, and rejects input load disturbances. *Performance:* the proposed strategy is compared with a PID controller $u(s) = K_c\left(1 + \frac{T_d s}{\alpha T d s} + \frac{1}{T_i s}\right) e(s)$, with a prefilter such that $e(s) = F_r(s)r(s) - y(s)$, with $F_r(s) = \frac{1}{T_d T_i s^2 + T_i s + 1}$, and $K_c = \frac{0.17}{bh^2}$, $T_i = 8.51h$, $T_d = 2.87h$, $\alpha = 0.1$, as suggested by (Ali et al. 2010). These are, to the best of our knowledge, the simplest PID tuning rules for time delay systems in terms of performance. The comparison is shown in Fig. 7.4 for a delay $h = 100$ ms, where it can be seen that the proposed control law outperforms the PID controller. In order to test the robustness against modeling errors in the delay, if an increment in the time delay of 50% is assumed, the proposal in (Ali et al. 2010) becomes unstable whereas the proposed strategy remains stable, as shown in Fig. 7.5.

Another simulation shows the influence of the filter time constant T_f in Fig. 7.6, where a wind disturbance is simulated between $t = 2$ s and $t = 5$ s. The wind disturbance is simulated by passing a white noise signal through a low-pass filter. It can be clearly seen how the lower the T_f, the better the disturbance rejection performance. It is also important that the choice of T_f does not affect the reference tracking performance, which is always desirable for an easier tuning.

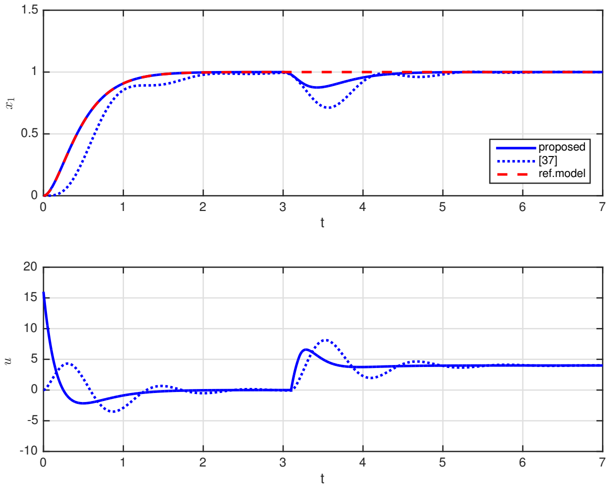


Figure 7.4: Performance comparison with a conventional PID controller (Ali et al. 2010) with a delay $h = 100$ ms ($T_f = 0.1$ s)

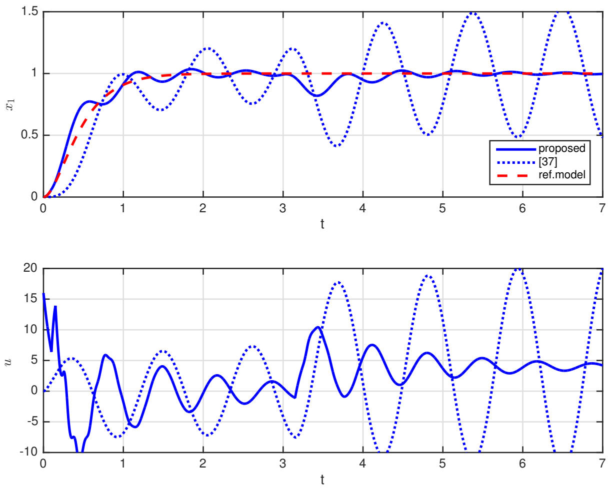


Figure 7.5: Robustness conventional PID controller (Ali et al. 2010) with a +50% delay error (filter tuned with $T_f = 0.1$ s)

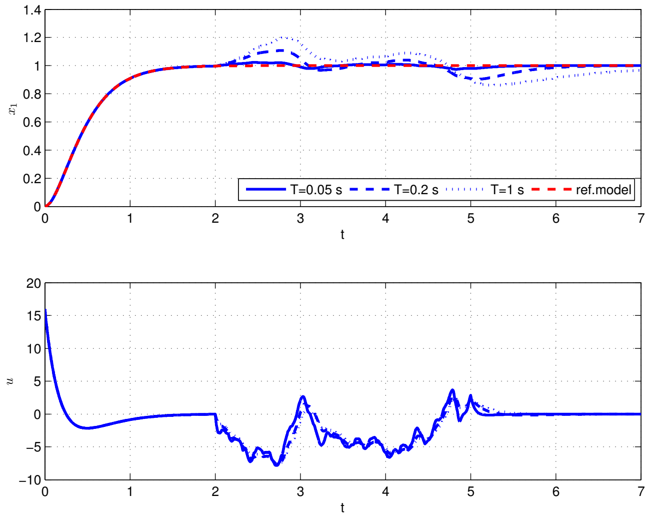


Figure 7.6: Simulations showing the influence of the parameter T_f in external disturbance rejection performance

Robustness: robustness with respect to uncertainties in the delay h is investigated. Fig. 7.7 shows different simulations with a percentage of uncertainty in the delay h around a nominal value of $h = 200$ ms.

Nonlinear multivariable model: once the properties of the proposed strategy have been illustrated, the full control of a nonlinear quadrotor model in (7.32) is presented next. Comparing to the model in (7.7), it can be seen that for this particular case $x = [\phi,\ \theta,\ \psi,\ \dot{\phi},\ \dot{\theta},\ \dot{\psi}]^T$,

$$A = \begin{bmatrix} 0_3 & I_3 \\ 0_3 & 0_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0_3 \\ I_3 \end{bmatrix}, \tag{7.37}$$

and

$$f(x,t) = \begin{bmatrix} 0, & 0, & 0, & \dot{\theta}\dot{\psi} - \dot{\theta}, & \dot{\psi}\dot{\phi} + \dot{\phi}, & \dot{\phi}\dot{\theta} \end{bmatrix}^T. \tag{7.38}$$

For simplicity, every axis is designed with the same closed-loop dynamics, by specifying the following reference model

$$A = \begin{bmatrix} 0_3 & I_3 \\ -\omega_c^2 I_3 & -2\omega_c I_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0_3 \\ \omega_c^2 I_3 \end{bmatrix}, \tag{7.39}$$

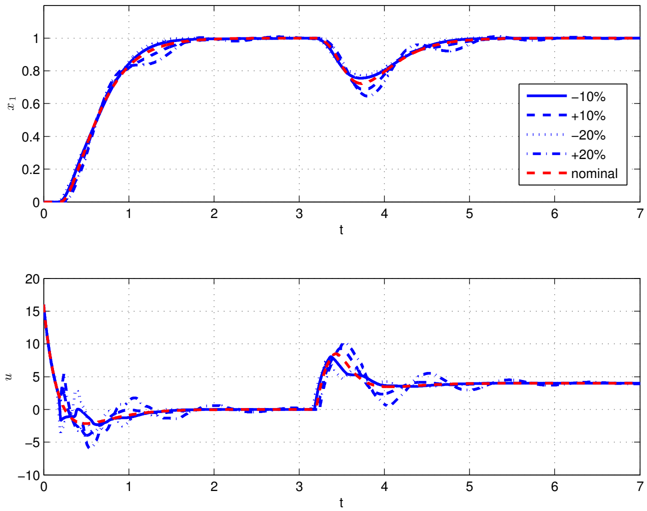


Figure 7.7: Simulations showing robustness with respect to the uncertainty in the delay around a nominal value of $h = 200$ ms (filter tuned with $T_f = 0.1$ s)

and selecting $\omega_c = 5$ rad/s. A delay of $h = 150$ ms is considered and a filter time constant $T_f = 0.1$ s is chosen. The simulation in Fig. 7.8 reproduces a real situation where the vehicle is perturbed by wind gusts and the goal is to keep it at hovering (zero reference in roll and pitch) while tracking a square reference signal in the yaw axis. The proposed controller is compared with that in (Sanahuja et al. 2010), with $u_x(t) = -\sigma_{x_1}(k_p x) - \sigma_{i_2}(k_d \dot{x})$ for each axis $x = \{\phi, \theta, \psi\}$ and the saturation function is defined as

$$\sigma_{b_i}(s) = \begin{cases} -b_i, & s < b_i \\ s, & -b_i \leq s \leq b_i \\ b_i, & s > b_i \end{cases} .$$

The controller is tuned with $k_p = \omega_c^2$ and $k_d = 2\omega_c$, for the sake of comparison, and the saturation bounds are chosen as $b_{\phi_1} = b_{\theta_1} = 4$, $b_{\phi_2} = b_{\theta_2} = 5$ and $b_{\psi_1} = 50$, $b_{\psi_2} = 60$. The bounds on the yaw axis have to be larger to allow good tracking performance. The benefit of this controller is that it allows to bound some of the states, thus limiting the size of the nonlinearities which depend on those states. However, in practice, small saturations make the system convergence very slow, leading to poor performance, as shown in Fig. 7.8, while the proposed controller is able to achieve satisfactory tracking and disturbance rejection in spite of the delay and nonlinearities.

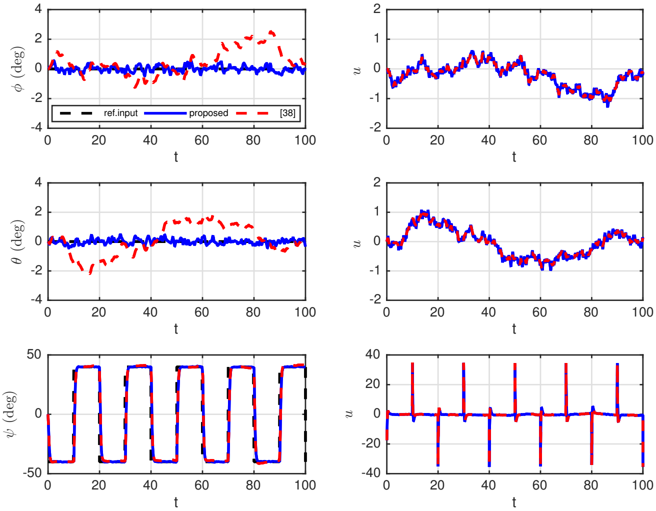


Figure 7.8: Simulations showing hover flight and tracking performance in the yaw axis under the presence of wind disturbances (nonlinear MIMO system time delay $h = 150$ ms and filter tuned with $T_f = 0.1$ s)

Figure 7.9: Quadrotor prototype used in experiments

7.4.3 *Experimental validation in flight tests*

There are some handicaps to overcome in real flights, e.g., large model uncertainties, measurement noise, flapping and ground effects, wind gusts, etc. The quadrotor prototype used for the experiments has a distance of 41 cm between

Table 7.1: Controller Parameters in Flight Tests

	b [deg/s^2]	ω_c [rad/s]	T_f [s]	h [ms]
ϕ	1500	2.5	0.6	25
θ	1500	2.5	0.6	25
ψ	1000	4	1	25
z	3000	2.5	0.6	250

rotors, weighting about 1.3 kg without battery. The basic hardware consists of a MikroKopter frame, YGE 25i electronic speed controllers, RobbeRoxxy 2827-35 brushless motors and 10x4.5 plastic propellers. All the computations are carried out on-board at 400 Hz using an Igep v2 board running Xenomai real-time operating system. In this system, the delays in the control loop come from: i) the Kalman filter algorithm which introduces a small delay due in the measurements, and ii) the response time of the motors drivers.

For the controller synthesis, each axis is modeled by a double integrator as in (7.35). The controller is implemented with a sample time $T_s = 2.5$ ms and a reference model as in (7.36) is proposed. The controller tuning is shown in Table 7.1.

The first experiment consists of applying yaw and height step references during stationary flight (roll and pitch references set to zero). The result of this experiment is shown in the Fig. 7.10. One can see how the roll and pitch angles are kept very close to zero. Their root mean squared errors are 0.6 deg and 0.7 deg for roll and pitch, respectively. The tracking performance of the yaw angle is very good. The performance in the height control is also remarkable, because it is more challenging due to the large delay of the ultrasonic sensor and the huge mass of the vehicle.

In the second experiment, disturbances are applied to the quadrotor in stationary flight. These disturbances are generated by hitting the vehicle in the pitch axis. The result of this experiment is shown in the Fig. 7.11 where it can be seen that the vehicle recovers successfully. It is remarkable that the quadrotor is deviated more than 30 deg from its equilibrium point and yet it remains stable.

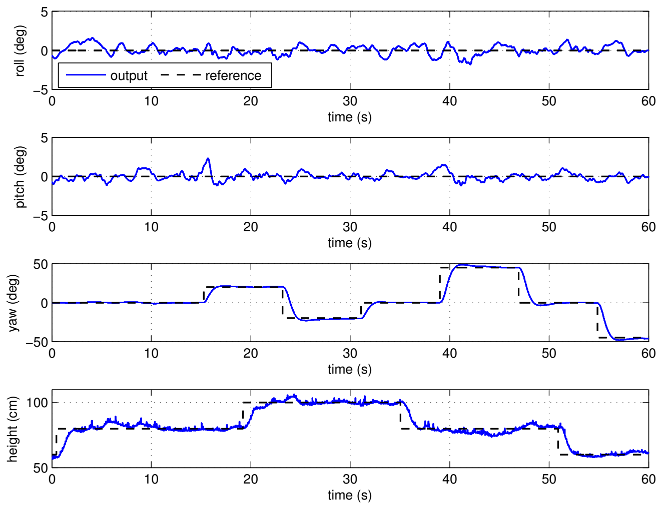


Figure 7.10: Tracking performance in a real flight.

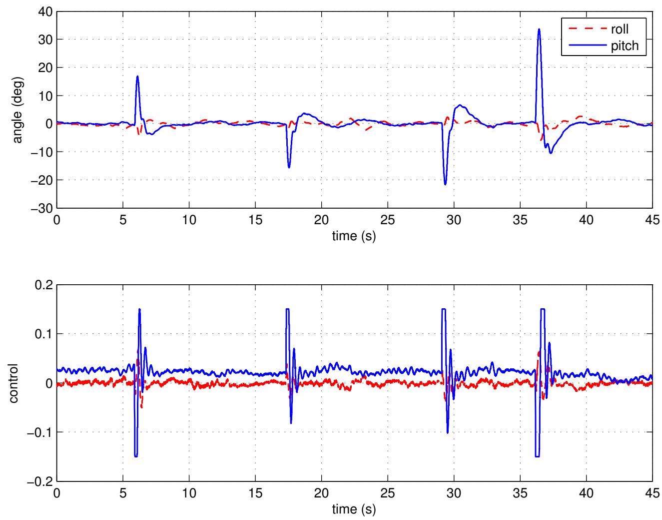


Figure 7.11: Disturbance rejection in a real flight. A video of this experiment is available at https://youtu.be/AaCOYglzBao .

Part III

Time-delay systems in the PDE framework

Chapter 8

Introduction

This chapter serves as an introduction to the third part of the thesis, which is based on the study of time delay systems within the framework of partial differential equations (PDEs). Indeed, delay systems belong to the broad class of distributed parameter systems. In particular, delay is a phenomenon that can be represented by a first-order hyperbolic PDE. The PDE framework provides ample and rigorous tools for analysis and design, not only for time-delay systems, but also for other systems with infinite-dimensional actuator or sensor dynamics. Some of those tools are reviewed in this chapter.

8.1 ODEs with transport dynamics

In this section it shown how time-delay systems can be modeled as the interconnection of an ODE (plant dynamics) and a PDE of transport type (delay dynamics) Krstic 2010a. Then the backstepping approach developed in (Krstic et al. 2008) for first-order hyperbolic PDEs is applied, showing that a predictor controller is obtained. For educational purposes, a finite backstepping example is presented and a link with its infinite-dimensional counterpart is established. Finally, an infinite-dimensional observer for systems with measurement delay, reported also in (Krstic et al. 2008), is introduced.

8.1.1 Time-delay systems as ODE-PDE cascades

The delay phenomenon can be represented as a partial differential equation of transport type. For example, a delayed signal $\phi(t-D)$ can be represented by $\phi(t-D) = u(0,t)$, where $u(x,t)$ is the PDE state variable satisfying

$$u_t(x,t) = u_x(x,t), \tag{8.1}$$

$$u(D,t) = \phi(t), \tag{8.2}$$

for all $x \in [0, D]$ and all $t \geq 0$. The equation (8.1) is a first-order hyperbolic PDE with unity propagation speed and whose boundary condition is given by (8.2). The variable x is commonly referred to as the spatial coordinate. In this context, however, it should be understood as a time-like variable. To see this, observe that the solution of (8.1)-(8.2) is given by

$$u(x,t) = \phi(t + x - D), \tag{8.3}$$

which can be alternatively written as $u(x,t) = \phi(t - h(x))$, with $h(x) = D - x$. Therefore, for a given t, the solution (8.3) consists of a distribution of the signal $\phi(t - h(x))$, $\forall x \in [0, D]$, that is, over the time window $[t-D, t]$. This is why x is referred to a time-like variable. Evaluating (8.3) at $x = 0$ yields $u(0,t) = \phi(t-D)$, which is the result claimed above.

This representation can be now applied to the input-delay system

$$\dot{X}(t) = AX(t) + BU(t-D), \tag{8.4}$$

with $D \geq 0$, which can be alternatively represented as the ODE-PDE cascade

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{8.5}$$

$$u_t(x,t) = u_x(x,t), \tag{8.6}$$

$$u(D,t) = U(t). \tag{8.7}$$

8.1.2 Backstepping controller design

The central idea when designing controllers for infinite-dimensional systems is to find a backstepping transformation

$$w(x,t) = u(x,t) - \int_0^x q(x,y)u(y,t)\,\mathrm{d}y - \gamma(x)^T X(t), \tag{8.8}$$

that maps the system (8.5)-(8.7) into the target system

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{8.9}$$
$$w_t(x,t) = w_x(x,t), \tag{8.10}$$
$$w(D,t) = 0. \tag{8.11}$$

The functions $q(x,t)$ and $\gamma(x)$ in (8.8) must be computed so that (8.5)-(8.7) hold. After some calculations, the following functions are obtained

$$q(x,y) = Ke^{A(x-y)}B \qquad \text{and} \qquad \gamma(x)^T = Ke^{Ax}, \tag{8.12}$$

and thus (8.8) becomes

$$w(x,t) = u(x,t) - K\left(e^{Ax}X(t) + \int_0^x e^{A(x-y)}Bu(y,t)\,\mathrm{d}y\right). \tag{8.13}$$

Evaluating (8.13) at $x = D$, the control law is obtained as

$$U(t) = u(D,t) = K\left(e^{AD}X(t) + \int_0^D e^{A(D-y)}Bu(y,t)\,\mathrm{d}y\right). \tag{8.14}$$

The controller (8.14) derived with the backstepping approach is equivalent to the one early derived in (Manitius et al. 1979; Kwon et al. 1980; Artstein 1982). This can be better seen by plugging the PDE solution $u(y,t) = U(t + y - D)$, into (8.14) and using the change of variable $\theta = t + y - D$ to obtain

$$U(t) = K\left(e^{AD}X(t) + \int_{t-D}^t e^{A(t-\theta)}BU(\theta)\,\mathrm{d}\theta\right).$$

8.1.3 Relation with finite backstepping

In order to understand the infinite-backstepping transformation, a finite example taken from (Krstic 2010a) is exposed here. The idea of backstepping control applies to systems in strict-feedback form. Let us consider the following simple system

$$\dot{X}(t) = AX(t) + Bu(t), \tag{8.15}$$

$$\dot{u}(t) = U(t), \tag{8.16}$$

where the actuator is modeled by a simple integrator. Although it may be confusing, the notation $u(t), U(t)$ is chosen to keep the analogy with the system (8.5)-(8.7). A reasonable backstepping transformation is given by $w(t) = u(t) - KX(t)$, which is desired to map (8.15)-(8.16) into

$$\dot{X}(t) = (A + BK)X(t) + Bw(t), \tag{8.17}$$

$$\dot{w}(t) = -cw(t), \tag{8.18}$$

with $c > 0$. Differentiating $w(t)$, using (8.15)-(8.16), imposing (8.18), and solving for $U(t)$ yields

$$U(t) = K(cX(t) + \dot{X}(t)) - cu(t),$$

which completes the backstepping design.

One may wonder, however, where is strict-feedback structure hidden in (8.5)-(8.7). The following example is derived with the aim of providing some insight into this fact. Let us consider a discretization of the x domain such that $u_j(t) = u(jh, t)$ with $h = D/N$ and $j \in \{0, 1, \ldots, N\}$. For this example, let us choose $N = 2$, which leads to a partition $\{u_0(t), u_1(t), u_2(t)\}$ and $h = D/2$. Note that $u_2(t) = u(D, t) = U(t)$ by (8.7). Discretizing the transport PDE (8.6) in the spatial variable x with a first-order approximation of the derivative and defining a vector $z(t) = [X^T(t), u_0(t), u_1(t)]^T$, the following is obtained

$$\dot{z}(t) = \begin{bmatrix} A & B & 0 \\ 0 & -\frac{1}{h} & \frac{1}{h} \\ 0 & 0 & -\frac{1}{h} \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{h} \end{bmatrix} U(t). \tag{8.19}$$

A strict-feedback structure is now evident in (8.19), where the spatial approximation has lead to a interconnection of first-order systems with time constant equal to the inverse of the discretization step. We find a backstepping transformation $(X, u_j) \mapsto (X, w_j)$ such that the system in the transformed variables takes the form

$$\dot{\zeta}(t) = \begin{bmatrix} A+BK & B & 0 \\ 0 & -\frac{1}{h} & \frac{1}{h} \\ 0 & 0 & -\frac{1}{h} \end{bmatrix} \zeta(t), \tag{8.20}$$

where $\zeta(t) = [X(t)^T, w_0(t), w_1(t)]^T$. Defining $w_0(t) = u_0(t) - KX(t)$, differentiating, using the first and second rows in (8.19), imposing the second row in (8.20) and solving for $w_1(t)$, leads to

$$w_1(t) = u_1(t) - K(X + h\dot{X}(t)). \tag{8.21}$$

Differentiating (8.21), using the first and third row in (8.19), imposing the third row in (8.20) and solving for $U(t)$, leads to

$$U(t) = K\left(X(t) + 2h\dot{X}(t) + h^2\ddot{X}(t)\right) = K\left(X(t) + D\dot{X}(t) + \frac{D^2}{4}\ddot{X}(t)\right). \tag{8.22}$$

The term between brackets in (8.22) can be regarded as a pseudo-prediction of the state D units of time ahead, while the one in (8.14) is an exact prediction. However, the similarity between both controllers is evident. The same happens between (8.21) and the backsteeping transformation (8.13), where both terms in brackets are predictions over a fraction of the total delay. However, the infinite backstepping transformation seems the appropriate way to follow, provided that (8.6) is infinite-dimensional, in the first place.

8.1.4 PDE observer for systems with measurement delay

The following developments were introduced in (Krstic et al. 2008). They can also be found in the monograph (Krstic 2009c). Consider the system

$$\dot{X}(t) = AX(t) + BU(t), \tag{8.23}$$

$$Y(t) = CX(t-D), \tag{8.24}$$

with $D \geq 0$ a constant measurement delay. The output equation (8.24) can be represented by the following PDE system

$$u_t(x,t) = u_x(x,t), \tag{8.25}$$

$$u(D,t) = CX(t), \tag{8.26}$$

$$Y(t) = u(0,t). \tag{8.27}$$

The following result introduces a novel observer with an ODE-PDE cascade structure that estimates both the state of the ODE and the so-called sensor state, i.e., the PDE state.

Theorem 8.1. *(Krstic et al. 2008) The observer*

$$\dot{\hat{X}}(t) = A\hat{X}(t) + e^{AD}L\left(Y(t) - \hat{u}(0,t)\right), \tag{8.28}$$

$$\hat{u}_t(x,t) = \hat{u}_x(x,t) + Ce^{Ax}L\left(Y(t) - \hat{u}(0,t)\right), \tag{8.29}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{8.30}$$

where L is chosen such that $A - LC$ is Hurwitz, guarantees that the observer error is exponentially stable in the sense of the norm

$$|X(t) - \hat{X}(t)|^2 + \int_0^D |u(x,t) - \hat{u}(x,t)|^2\,\mathrm{d}x.$$

It was mentioned before that the control derived by means of the backstepping approach was equivalent to those originally derived in (Manitius et al. 1979; Kwon et al. 1980; Artstein 1982). However, the observer (8.28)-(8.30) is substantially different from the classical delay-compensating observer results in (Watanabe et al. 1981b; Klamka 1982). Those were based on estimating the delayed state and then advancing it D seconds ahead, as follows

$$\dot{\Xi}(t) = A\Xi(t) + BU(t-D) + L(Y(t) - C\Xi(t)), \tag{8.31}$$

$$\hat{X}(t) = e^{AD}\Xi(t) + \int_{t-D}^{t} e^{A(t-\theta)}BU(\theta)\,\mathrm{d}\theta. \tag{8.32}$$

A reduced-order version of (8.31)-(8.32) is easily obtained as

$$\begin{aligned}\dot{\hat{X}}(t) &= A\hat{X}(t) + BU(t)\\ &\quad + e^{AD}L\left(Y(t) - Ce^{-AD}\hat{X}(t) + C\int_{t-D}^{t} e^{A(t-D-\theta)}BU(\theta)\,\mathrm{d}\theta\right),\end{aligned} \tag{8.33}$$

which is the original form reported in (Watanabe et al. 1981b; Klamka 1982).

Remark 8.1. *The observer (8.28)-(8.30) does not contain distributed integral terms, in contrast to (8.31)-(8.32) and (8.33). Instead, the infinite-dimensionality is represented by a partial differential equation. This is a crucial fact that is recurrently exploited by the contributions reported in the following chapters.*

8.2 ODEs with diffusive dynamics

The problem of compensating actuator or sensor dynamics dominated by diffusion was addressed in (Krstic 2009b). A system with diffusive actuator dynamics can be modeled by the following ODE-PDE cascade

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{8.34}$$
$$u_t(x,t) = u_{xx}(x,t), \tag{8.35}$$
$$u_x(0,t) = 0, \tag{8.36}$$
$$u(D,t) = U(t), \tag{8.37}$$

where $X \in \mathrm{R}^n$ is the ODE state, U is the control input and u is the PDE state. While the ODE with input delay (8.5)-(8.7) has a transfer function representation

$$X(s) = (sI - A)^{-1}Be^{-sD}U(s),$$

the ODE-PDE cascade (8.34)-(8.37) is represented by

$$X(s) = (sI - A)^{-1}B\frac{1}{\cosh(D\sqrt{s})}U(s).$$

In the rest of this section, an infinite-dimensional controller for compensation of actuator diffusive dynamics is introduced. The controller is obtained as a direct application of the backstepping techniques developed in (Smyshlyaev et al. 2004; Smyshlyaev et al. 2005) for parabolic systems. An observer is also introduced for systems with diffusive sensor dynamics.

8.2.1 *Backstepping controller design*

The backstepping procedure with the same candidate transformation (8.8) above can be applied to the diffusive case. The reader is referred to (Krstic 2009b) or (Krstic 2009c) for details. The resulting compensating controller has the form

$$U(t) = K\begin{bmatrix} I & 0 \end{bmatrix}\left\{M(D)\begin{bmatrix} I \\ 0 \end{bmatrix}X(t) + \int_0^D \left(\int_0^{D-y} M(\xi)\,\mathrm{d}\xi\right)\begin{bmatrix} I \\ 0 \end{bmatrix}Bu(y,t)\,\mathrm{d}y\right\} \tag{8.38}$$

where

$$M(x) = e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}x}.$$

Remark 8.2. *It was recently shown in (Sanz et al. 2018c), one of the contributions of this thesis, that the control law above can be alternatively written as (see proof of Lemma 9.4)*

$$U(t) = K\left\{\cosh(D\sqrt{A})X(t) + \int_0^D \left(\int_0^{D-y} \cosh(\xi\sqrt{A})\,\mathrm{d}\xi\right) Bu(y,t)\,\mathrm{d}y\right\}.$$

8.2.2 *PDE observer for systems with diffusive sensor dynamics*

The observer introduced in the following theorem is the counterpart of the one in Theorem 8.1 for systems with diffusive sensor dynamics.

Theorem 8.2. *(Krstic 2009b) The observer*

$$\dot{\hat{X}}(t) = A\hat{X}(t) + M(D)L\left(Y(t) - \hat{u}(0,t)\right), \tag{8.39}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + CM(x)L\left(Y(t) - \hat{u}(0,t)\right), \tag{8.40}$$

$$\hat{u}_x(0,t) = 0, \tag{8.41}$$

$$\hat{u}(D,t) = C\hat{X}(t), \tag{8.42}$$

where L is chosen such that $A - LC$ is Hurwitz, guarantees that the observer error is exponentially stable in the sense of the norm

$$|X(t) - \hat{X}(t)|^2 + \int_0^D |u(x,t) - \hat{u}(x,t)|^2\,\mathrm{d}x.$$

Remark 8.3. *Note that, again, the observer resulting from the application of the backstepping approach does not contain distributed integral terms. This fact is exploited in Chapter 9 to derive an observer-based controller, different from* (8.38)*, which avoids the integral terms.*

8.3 Contributions in this part

The contributions reported in the following chapters are summarized here.

Chapter 9 is based on both (Sanz et al. 2018a), accepted to be presented at the *American Control Conference 2018* and its journal version (Sanz et al. 2018c), provisionally accepted for its publication in *Transactions of Automatic Control*. This work is also focused on avoiding integral terms in the control law (see Section 2.3.1). Observer-based controllers are proposed for systems with delay or diffusive dynamics. In contrast to other similar approaches developed for input-delayed systems, this novel method allows stabilization of exponentially unstable plants with arbitrarily large input-delay.

Chapter 10 is based on (Sanz et al. 2018b), submitted to *Automatica* and currently under review. In this work, the infinite-dimensional observer (8.28)-(8.30) is extended to systems with time-varying matrices and/or time-varying delays. The observer is further exploited to develop an exponentially stabilizing controller.

Chapter 9

Observer-based compensation of infinite-dimensional actuator dynamics

This chapter deals with robust observer-based output feedback stabilization of systems whose actuator dynamics can be described in terms of partial differential equations. More specifically, delay dynamics (first-order hyperbolic PDE) and diffusive dynamics (parabolic PDE) are considered. The proposed controllers have a PDE observer-based structure. The main novelty is that stabilization for an arbitrarily large delay or diffusion domain length is achieved, while distributed integral terms in the control law are avoided. The exponential stability of the closed-loop in both cases is proved using Lyapunov-Krasovskii functionals, even in the presence of small uncertainties in the time delay or the diffusion coefficient.

9.1 Introduction

Traditional predictor-based controllers, as developed in (Manitius et al. 1979; Kwon et al. 1980; Artstein 1982), use infinite-dimensional feedback laws, whose discretization may cause problems in their practical implementation, as discussed in the literature (Mondié et al. 2003; Zhong 2004). More recently, the application of backstepping techniques developed for first-order hyperbolic PDEs has also led to controllers with distributed integrals, when applied to time delay systems (Krstic et al. 2008). Modeling the delay phenomenon as a transport PDE has been shown to provide a solid framework with ample tools for analysis and design (Krstic 2009c). In this context, input-delay systems are just a particular case of a broader class of systems with infinite-dimensional actuator dynamics, which have attracted attention recently, and whose stabilizing controllers also involve distributed (sometimes double) integrals of the actuator state (Krstic 2009b; Krstic 2009a).

Stabilization of input-delayed systems without distributed terms has been pursued in different directions. A successful approach consists of ignoring the distributed terms in the traditional predictor leading to a static feedback control law. Following this idea, the truncated predictor feedback, introduced in (Zhou et al. 2012), achieves stabilization of marginally stable systems. The pseudo-predictor feedback further extends this technique to exponentially unstable systems, although arbitrarily large delays can only be handled for polynomially unstable systems (Zhou 2014).

Another approach is based on designing observers to estimate the predicted state, rather than explicitly computing it. Let us briefly develop this idea. Given the LTI system $\dot{\mathcal{X}}(t) = A\mathcal{X}(t) + B\mathcal{U}(t-D)$ with input delay $D \geq 0$ and output measurement $\mathcal{Y}(t) = C\mathcal{X}(t)$, the predicted state, $\mathcal{P}(t) = \mathcal{X}(t+D)$, can be shown to satisfy the equation $\dot{\mathcal{P}}(t) = A\mathcal{P}(t) + B\mathcal{U}(t)$, while the output can be expressed as $\mathcal{Y}(t) = C\mathcal{P}(t-D)$. Therefore, the stabilization problem can be solved by generating an asymptotically convergent estimate, $\hat{\mathcal{P}}(t)$, through the delayed measurement $\mathcal{Y}(t)$ and then applying the control law $\mathcal{U}(t) = K\hat{\mathcal{P}}(t)$, such that $A + BK$ is Hurwitz. The problem of state observation via a delayed output measurement has been recurrently approached in the literature without the need of integral terms (Germani et al. 2002; Ahmed-Ali et al. 2009; Ahmed-Ali et al. 2012). In these works, a chain of sequential observers is used, in which each of the components estimates a prediction of the stated over an interval, whose length equals a fraction of the delay, achieving asymptotic stability for arbitrarily large delays as the number of sequential predictors goes to infinity. However, it has been only recently that this fact has gained increasing attention among researchers to deal with input-delayed systems. This idea was first devised in (Besancon et al. 2007) and further extended with an LMI-based design methodology in (Najafi et al. 2013). In the past few years, this technique has been extended to time-varying delays (Léchappé et al. 2016), input and output delays (Zhou et al. 2017) and nonlinear time-varying systems (Mazenc et al. 2017b).

9.1.1 Problem statement

This chapter deals with two classes of systems whose actuator dynamics can be described in terms of PDEs. The first type of systems considered are those described by

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{9.1}$$
$$u_t(x,t) = u_x(x,t), \tag{9.2}$$
$$u(D,t) = U(t), \tag{9.3}$$
$$Y(t) = CX(t), \tag{9.4}$$

where $D \geq 0$ is the spatial domain length and A, B, C are matrices with appropriate dimensions. As discussed in see Section 8.1.2, it is a well-known result that, if the whole state is available, the stabilization of (9.1)-(9.3) can be achieved by the predictive feedback control law

$$U(t) = KP(t),$$
$$P(t) = e^{AD}X(t) + \int_0^D e^{A(D-y)}Bu(y,t)\,\mathrm{d}y, \tag{9.5}$$

where the vector K is such that $A + BK$ is Hurwitz.

The second type of systems treated here are those described by

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{9.6}$$
$$u_t(x,t) = u_{xx}(x,t), \tag{9.7}$$
$$u_x(0,t) = 0, \tag{9.8}$$
$$u(D,t) = U(t), \tag{9.9}$$
$$Y(t) = CX(t). \tag{9.10}$$

where $D \geq 0$ is the spatial domain length. In this case, the actuator dynamics (9.7) is governed by a parabolic PDE, the so-called heat equation. Therefore, the control action undergoes a diffusive process before reaching the ODE. Since the spatial domain length is arbitrary, the diffusion coefficient is taken to be unity without loss of generality. As described in Section 8.2.1, if the whole state is accessible, a stabilizing control law for (9.6)-(9.9) is given by

$$\begin{aligned} U(t) &= K\Pi(t), \\ \Pi(t) &= M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y, \end{aligned} \tag{9.11}$$

where the vector K is again to be chosen such that $A+BK$ is Hurwitz, and

$$m(s) = \int_0^s M(\xi)\,\mathrm{d}\xi, \tag{9.12}$$

$$M(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & A \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{9.13}$$

While $P(t)$ in (9.5) is the predicted state D units of time ahead, i.e., $P(t) = X(t+D)$, the meaning of $\Pi(t)$ in (9.11) is more difficult to interpret in the time domain. Note that, in the Laplace domain, the PDE system (9.2)-(9.3) is represented by $u(0,s) = e^{-sD}U(s)$, and thus $P(s) = e^{sD}X(s)$ cancels out the effect of the actuator dynamics. Analogously, noting that the PDE system (9.7)-(9.9) has a transfer function representation $u(0,s) = U(s)/\cosh(\sqrt{s}D)$, as pointed out in (Krstic 2009b), it can be drawn that $\Pi(s) = \cosh(\sqrt{s}D)X(s)$, for the actuator dynamics to be counteracted. We shall refer to $\Pi(t)$ as the “anti-diffused” state.

9.1.2 Motivation

An obvious limitation of the control laws (9.5) and (9.11) is that they require full state measurement. To overcome this issue, the control laws can be alternatively computed using estimates $\hat{X}(t)$ and $\hat{u}(x,t)$, generated by a suitable observer. However, another handicap lies in the fact that these control laws are actually integral equations, since the control action appears explicitly on the LHS and under an integral sign on the RHS. Therefore, the discretization of the integral term for its implementation can lead to instability (Zhong 2004).

In what follows, an output-based control strategy is introduced, which achieves exponential stabilization while avoiding the distributed integral terms. The key idea behind the proposed control laws is to design observers to estimate the predicted state $P(t)$ for a system with delay actuator dynamics, or the “anti-diffused” state $\Pi(t)$, for a system with diffusive actuator dynamics. Robustness under uncertainties in the delay size or the diffusion coefficient are also considered.

9.2 Delay dynamics

The observer-based controller for the case of delay actuator dynamics is introduced in this section. The case of perfectly known delay is presented next, while robustness is addressed below in 9.2.2

9.2.1 Nominal case

Let us define

$$v(x,t) = Ce^{Ax}X(t) + C\int_0^x e^{A(x-y)}Bu(y,t)\,dy. \tag{9.14}$$

Computing the time derivative of (9.5) and the spatial and temporal derivatives of (9.14), using (9.1)-(9.4), one arrives at the following ODE-PDE cascade system

$$\dot{P}(t) = AP(t) + BU(t), \tag{9.15}$$

$$v_t(x,t) = v_x(x,t), \tag{9.16}$$

$$v(D,t) = CP(t), \tag{9.17}$$

$$Y(t) = v(0,t), \tag{9.18}$$

where an integration by parts in the variable y and the fact that A and e^{Ax} commute for all x was used in (9.15)-(9.16), while (9.17)-(9.18) follow simply by evaluating (9.14) at $x = D$ and $x = 0$, respectively. The original input-delay system (9.1)-(9.4) has been then mapped through the transformation $(X,u) \mapsto (P,v)$ into the virtual system (9.15)-(9.18), in which the delay is affecting the output.

Theorem 9.1. *Given matrices K and L such that $A+BK$ and $A-LC$ are Hurwitz, the closed-loop system composed of* (9.1)-(9.4) *and*

$$\dot{\hat{P}}(t) = A\hat{P}(t) + BU(t) + e^{AD}L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.19}$$

$$\hat{v}_t(x,t) = \hat{v}_x(x,t) + Ce^{Ax}L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.20}$$

$$\hat{v}(D,t) = C\hat{P}(t), \tag{9.21}$$

$$U(t) = K\hat{P}(t), \tag{9.22}$$

is exponentially stable in the sense of the norm

$$\Big(|X(t)|^2 + \|u(t)\|^2 + |\hat{P}(t)|^2 + \|\hat{v}(t)\|^2\Big)^{1/2}.$$

Proof. Let us introduce the mapping $(P, v) \rightarrow (\tilde{P}, \tilde{v})$, by defining the error variables

$$\tilde{P}(t) \triangleq P(t) - \hat{P}(t), \tag{9.23}$$

$$\tilde{v}(x,t) \triangleq v(x,t) - \hat{v}(x,t). \tag{9.24}$$

Differentiating (9.23)-(9.24) and using (9.19)-(9.21), the observer error system is obtained as

$$\dot{\tilde{P}}(t) = A\tilde{P}(t) - e^{AD}L\tilde{v}(0,t), \tag{9.25}$$

$$\tilde{v}_t(x,t) = \tilde{v}_x(x,t) - Ce^{Ax}L\tilde{v}(0,t), \tag{9.26}$$

$$\tilde{v}(D,t) = C\tilde{P}(t). \tag{9.27}$$

On the other hand, using (9.22) and (9.23), the system (9.1)-(9.3) can be written as

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{9.28}$$

$$u_t(x,t) = u_x(x,t), \tag{9.29}$$

$$u(D,t) = KP(t) - K\tilde{P}(t), \tag{9.30}$$

Now, let us introduce the mappings $(X, u) \mapsto (X, w)$ and $(\tilde{P}, \tilde{v}) \mapsto (\tilde{P}, \tilde{w})$, defined by the backstepping transformations

$$w(x,t) = u(x,t) - Ke^{Ax}X(t) - \int_0^x Ke^{A(x-y)}Bu(y,t)\,\mathrm{d}y, \tag{9.31}$$

$$\tilde{w}(x,t) = \tilde{v}(x,t) - Ce^{A(x-D)}\tilde{P}(t), \tag{9.32}$$

respectively, which transform (9.25)-(9.27) and (9.28)-(9.30) into

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{9.33}$$

$$w_t(x,t) = w_x(x,t), \tag{9.34}$$

$$w(D,t) = -K\tilde{P}(t), \tag{9.35}$$

$$\dot{\tilde{P}}(t) = (A - e^{AD}LCe^{-AD})\tilde{P}(t) - e^{AD}L\tilde{w}(0,t), \tag{9.36}$$

$$\tilde{w}_t(x,t) = \tilde{w}_x(x,t), \tag{9.37}$$

$$\tilde{w}(D,t) = 0, \tag{9.38}$$

where (9.34) followed from an integration by parts and (9.37) used the fact that A and e^{Ax} commute for all x. The overall transformation $(X, u, \hat{P}, \hat{v}) \mapsto (X, w, \tilde{P}, \hat{w})$

can be written as

$$w(x,t) = u(x,t) - Ke^{Ax}X(t) - \int_0^x Ke^{A(x-y)}Bu(y,t)\,\mathrm{d}y, \tag{9.39}$$

$$\tilde{P}(t) = e^{AD}X(t) + \int_0^D e^{A(D-y)}Bu(y,t)\,\mathrm{d}y - \hat{P}(t), \tag{9.40}$$

$$\tilde{w}(x,t) = Ce^{A(x-D)}\hat{P}(t) - \hat{v}(x,t) - C\int_x^D e^{A(x-y)}Bu(y,t)\,\mathrm{d}y. \tag{9.41}$$

while its inverse is given by

$$u(x,t) = w(x,t) + Ke^{(A+BK)x}X(t) + \int_0^x Ke^{(A+BK)(x-y)}Bw(y,t)\,\mathrm{d}y, \tag{9.42}$$

$$\begin{aligned}\hat{P}(t) &= e^{AD}X(t) - \tilde{P}(t) + \int_0^D e^{A(D-y)}Bw(y,t)\,\mathrm{d}y \\ &\quad + \left(\int_0^D e^{A(D-y)}BKe^{(A+BK)y}\,\mathrm{d}y\right)X(t) \\ &\quad + \int_0^D \left(\int_z^D e^{A(D-y)}BKe^{(A+BK)(y-z)}\,\mathrm{d}y\right)Bw(z,t)\,\mathrm{d}z,\end{aligned} \tag{9.43}$$

$$\begin{aligned}\hat{v}(x,t) &= Ce^{Ax}X(t) - Ce^{A(x-D)}\tilde{P}(t) - \tilde{w}(x,t) + C\int_0^x e^{A(x-y)}Bw(y,t)\,\mathrm{d}y \\ &\quad + C\left(\int_0^x e^{A(x-y)}BKe^{(A+BK)y}\mathrm{d}y\right)X(t) \\ &\quad + C\int_0^x \left(\int_z^x e^{A(x-y)}BKe^{(A+BK)(y-z)}\,\mathrm{d}y\right)Bw(z,t)\,\mathrm{d}z.\end{aligned} \tag{9.44}$$

In order to assess stability, let us choose the Lyapunov-Krasovskii functional

$$\begin{aligned}V(t) &= X(t)^T S_1 X(t) + \frac{a}{2}\int_0^D (1+x)w(x,t)^2\,\mathrm{d}x \\ &\quad + b\tilde{P}(t)^T T^T S_2 T\tilde{P}(t) + \frac{c}{2}\int_0^D (1+x)\tilde{w}(x,t)^2\,\mathrm{d}x\end{aligned} \tag{9.45}$$

where the constants $a, b, c > 0$ are specified in the subsequent analysis, $T = e^{-AD}$ is defined for the sake of brevity, and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$ are the solutions to the Lyapunov equations

$$S_1(A+BK) + (A+BK)^T S_1 = -Q_1, \tag{9.46}$$

$$S_2(A-LC) + (A-LC)^T S_2 = -Q_2, \tag{9.47}$$

for some symmetric positive definite matrices Q_1 and Q_2, respectively. Using integration by parts and the fact that T and A commute, the time derivative of (9.45) along the trajectories of (9.33)-(9.38) can be written as

$$\begin{aligned}\dot{V} = & -X^T Q_1 X + 2X^T S_1 B w(0,t) \\ & + \frac{a}{2}(1+D)w(D,t)^2 - \frac{a}{2}w(0,t)^2 - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x \\ & - b\tilde{P}^T T^T Q_2 T\tilde{P} - 2b\tilde{P}^T T^T S_2 L\tilde{w}(0) \\ & - \frac{c}{2}\tilde{w}(0,t)^2 - \frac{c}{2}\int_0^D \tilde{w}(x,t)^2\,\mathrm{d}x, \end{aligned} \tag{9.48}$$

which can be bounded by

$$\begin{aligned}\dot{V} \le & -\frac{\underline{\lambda}(Q_1)}{2}|X|^2 + \frac{2|S_1 B|^2}{\underline{\lambda}(Q_1)}w(0,t)^2 \\ & + \frac{a}{2}(1+D)w(D,t)^2 - \frac{a}{2}w(0,t)^2 - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x \\ & - b\frac{\underline{\lambda}(Q_2)}{2}|T\tilde{P}|^2 + \frac{2b|S_2 L|^2}{\underline{\lambda}(Q_2)}\tilde{w}(0,t)^2 \\ & - \frac{c}{2}\tilde{w}(0,t)^2 - \frac{c}{2}\int_0^D \tilde{w}(x,t)^2\,\mathrm{d}x \\ \le & -\frac{\underline{\lambda}(Q_1)}{2}|X|^2 + \left(\frac{2|S_1 B|^2}{\underline{\lambda}(Q_1)} - \frac{a}{2}\right)w(0,t)^2 \\ & + \left(\frac{a}{2}(1+D)|KT^{-1}|^2 - b\frac{\underline{\lambda}(Q_2)}{2}\right)|T\tilde{P}|^2 \\ & - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x - \frac{c}{2}\int_0^D \tilde{w}(x,t)^2\,\mathrm{d}x \\ & + \left(\frac{2b|S_2 L|^2}{\underline{\lambda}(Q_2)} - \frac{c}{2}\right)\tilde{w}(0,t)^2. \end{aligned} \tag{9.49}$$

Choosing

$$a = \frac{4|S_1 B|^2}{\underline{\lambda}(Q_1)},\ b = \frac{2a(1+D)|KT^{-1}|^2}{\underline{\lambda}(Q_2)},\ c = \frac{4b|S_2 L|^2}{\underline{\lambda}(Q_2)},$$

leads to

$$\dot{V} \le -\frac{\underline{\lambda}(Q_1)}{2}|X|^2 - \frac{a}{2}\int_0^D w(x,t)^2\,\mathrm{d}x$$

$$- b\frac{\underline{\lambda}(Q_2)}{4}|T\tilde{P}|^2 - \frac{c}{2}\int_0^D \tilde{w}(x,t)^2\,\mathrm{d}x.$$

Next, observe that

$$\begin{aligned}\dot{V} \le & -\frac{\underline{\lambda}(Q_1)}{2\overline{\lambda}(S_1)}X(t)^T S_1 X(t) \\ & -\frac{1}{1+D}\frac{a}{2}\int_0^D (1+x)w(x,t)^2\,\mathrm{d}x \\ & -\frac{\underline{\lambda}(Q_2)}{4\overline{\lambda}(S_2)}b\tilde{P}(t)^T T^T S_2 T\tilde{P}(t) \\ & -\frac{1}{1+D}\frac{c}{2}\int_0^D (1+x)\tilde{w}(x,t)^2\,\mathrm{d}x. \end{aligned} \tag{9.50}$$

Therefore, from (9.45) and (9.50) it follows that

$$\dot{V}(t) \le -\mu V(t), \tag{9.51}$$

where

$$\mu = \min\left\{\frac{\underline{\lambda}(Q_1)}{2\overline{\lambda}(S_1)}, \frac{\underline{\lambda}(Q_2)}{4\overline{\lambda}(S_2)}, \frac{1}{1+D}\right\}.$$

Now, from (9.45), one can find that

$$\psi_1 \Xi(t) \le V(t) \le \psi_2 \Xi(t), \tag{9.52}$$

where

$$\Xi(t) = |X(t)|^2 + \|w(t)\|^2 + |\tilde{P}(t)|^2 + \|\tilde{w}(t)\|^2,$$

and

$$\begin{aligned}\psi_1 &= \min\left\{\underline{\lambda}(S_1), \frac{a}{2}, b\underline{\lambda}(T^T S_2 T), \frac{c}{2}\right\}, \\ \psi_2 &= \max\left\{\overline{\lambda}(S_1), \frac{a(1+D)}{2}, b\overline{\lambda}(T^T S_2 T), \frac{c(1+D)}{2}\right\}.\end{aligned}$$

Hence, from (9.51)-(9.52) and the comparison principle, the following exponential stability estimate is obtained for the transformed system

$$\Xi(t) \le \frac{\psi_2}{\psi_1}\Xi(0)e^{-\mu t},\ \forall t \ge 0. \tag{9.53}$$

Now, it is necessary to show the exponential stability of the original system, that is, in the sense of the norm

$$\Upsilon(t) = |X(t)|^2 + \|u(t)\|^2 + |\hat{P}(t)|^2 + \|\hat{v}(t)\|^2.$$

Using (9.39)-(9.41) and (9.42)-(9.44), one can show there exist constants α_i, β_i such that

$$\Xi(t) \leq \alpha_1|X|^2 + \alpha_2\|u\|^2 + \alpha_3|\hat{P}|^2 + \alpha_4\|\hat{v}\|^2,$$
$$\Upsilon(t) \leq \beta_1|X|^2 + \beta_2\|w\|^2 + \beta_3|\tilde{P}|^2 + \beta_4\|\tilde{w}\|^2,$$

for all $t \geq 0$, from which it follows that

$$\phi_1\Upsilon(t) \leq \Xi(t) \leq \phi_2\Upsilon(t), \tag{9.54}$$

being $\phi_1 = 1/\max\beta_i$ and $\phi_2 = \max\alpha_i$. Therefore, one gets the exponential stability estimate

$$\Upsilon(t) \leq \frac{\psi_2}{\psi_1}\frac{\phi_2}{\phi_1}\Upsilon(0)e^{-\mu t}, \; \forall t \geq 0,$$

thus completing the proof. □

9.2.2 *Robustness to delay mismatch*

Let us consider now that the input delay size is not accurately known. Such scenario can be modeled by the following ODE-PDE cascade

$$\dot{X}(t) = AX(t) + Bu(\Delta D, t), \tag{9.55}$$
$$u_t(x,t) = u_x(x,t), \tag{9.56}$$
$$u(D_0, t) = U(t), \tag{9.57}$$
$$Y(t) = CX(t), \tag{9.58}$$

where $X \in \mathbb{R}^n$ is the ODE state, A, B and C are matrices with appropriate dimensions and $u \in \mathcal{C}^1$ is the PDE state, whose spatial domain is given by

$$x \in [\underline{x}, D_0], \qquad \underline{x} = \min\{0, \Delta D\}.$$

The system (9.55)-(9.57) is equivalent to an LTI system with input delay of $D = D_0 - \Delta D$ units of time, where $D_0 \geq 0$ is the assumed plant delay and ΔD is a bounded delay mismatch. If the whole state is available and ΔD is known, the global asymptotic stabilization to zero of (9.55)-(9.57) can be achieved by the predictive feedback control law $U(t) = KP(t)$, where

$$P(t) = e^{AD}X(t) + \int_{\Delta D}^{D_0} e^{A(D_0-y)}Bu(y,t)\,\mathrm{d}y, \tag{9.59}$$

and the vector K is such that $A + BK$ is Hurwitz. Let us also define

$$v(x,t) = Ce^{A(x-\Delta D)}X(t) + C\int_{\Delta D}^{x} e^{A(x-y)}Bu(y,t)\,\mathrm{d}y. \tag{9.60}$$

Computing the time derivative of (9.59) and the spatial and temporal derivatives of (9.60), and using (9.55)-(9.58), one arrives at the following ODE-PDE cascade system

$$\dot{P}(t) = AP(t) + BU(t), \tag{9.61}$$

$$v_t(x,t) = v_x(x,t), \tag{9.62}$$

$$v(D_0,t) = CP(t), \tag{9.63}$$

$$Y(t) = v(\Delta D,t), \tag{9.64}$$

where an integration by parts in the variable y and the fact that A and e^{Ax} commute for all x was used in (9.61)-(9.62), and (9.63)-(9.64) follow simply by evaluating (9.60) at $x = D_0$ and $x = \Delta D$, respectively. The original input-delay system (9.55)-(9.58) has been then mapped into the virtual system (9.61)-(9.64), in which the delay is affecting the output. Let us introduce the error variables

$$\tilde{P}(t) \triangleq P(t) - \hat{P}(t), \tag{9.65}$$

$$\tilde{v}(x,t) \triangleq v(x,t) - \hat{v}(x,t). \tag{9.66}$$

Differentiating (9.65)-(9.66), using (9.93)-(9.95) and (9.61)-(9.64), and adding and subtracting $v(0,t)$, the observer error system can be written as

$$\dot{\tilde{P}}(t) = A\tilde{P}(t) - e^{AD_0}L\tilde{v}(0,t) - e^{AD_0}L\mathcal{I}(t), \tag{9.67}$$

$$\tilde{v}_t(x,t) = \tilde{v}_x(x,t) - Ce^{Ax}L\tilde{v}(0,t) - Ce^{Ax}L\mathcal{I}(t), \tag{9.68}$$

$$\tilde{v}(D_0,t) = C\tilde{P}(t). \tag{9.69}$$

where $\mathcal{I}(t) = v(\Delta D,t) - v(0,t) = \int_0^{\Delta D} v_x(x,t)\,\mathrm{d}x$, which follows from the Newton-Leibniz formula. Now, let us introduce the mappings $(X,u) \mapsto (X,w)$ and

$(\tilde{P}, \tilde{v}) \mapsto (\tilde{P}, \tilde{w})$, defined by the backstepping transformations

$$w(x,t) = u(x,t) - Ke^{A(x-\Delta D)}X(t) - \int_{\Delta D}^{x} Ke^{A(x-y)}Bu(y,t)\,\mathrm{d}y, \tag{9.70}$$

$$\tilde{w}(x,t) = \tilde{v}(x,t) - Ce^{A(x-D_0)}\tilde{P}(t). \tag{9.71}$$

Using (9.96) and the transformations (9.70)-(9.71), the systems (9.55)-(9.57), (9.67)-(9.69) are mapped into

$$\dot{X}(t) = (A+BK)X(t) + Bw(\Delta D, t), \tag{9.72}$$

$$w_t(x,t) = w_x(x,t), \tag{9.73}$$

$$w(D_0,t) = -K\tilde{P}(t), \tag{9.74}$$

$$\begin{aligned}\dot{\tilde{P}}(t) &= (A - e^{AD_0}LCe^{-AD_0})\tilde{P}(t) - e^{AD_0}L\tilde{w}(0,t) \\ &\quad - e^{AD_0}L\mathcal{I}(t),\end{aligned} \tag{9.75}$$

$$\tilde{w}_t(x,t) = \tilde{w}_x(x,t), \tag{9.76}$$

$$\tilde{w}(D_0,t) = 0, \tag{9.77}$$

respectively, where (9.73) followed from an integration by parts, (9.75) used (9.71) with $x = 0$, and (9.76) used the fact that A and e^{Ax} commute for all x. Also, using (9.96) and (9.65), the system (9.61)-(9.63) can be written as

$$\dot{P}(t) = (A+BK)P(t) - BK\tilde{P}(t), \tag{9.78}$$

$$v_t(x,t) = v_x(x,t), \tag{9.79}$$

$$v(D_0,t) = CP(t). \tag{9.80}$$

Remark 9.1. *In the nominal case, i.e., $\Delta D = 0$, the coupling term $\mathcal{I}(t)$ would vanish and thus the stability of (9.72)-(9.77) could be proved without taking (9.78)-(9.80) into account, as it was done before in Section 9.2.1. Since this is not the case, the three subsystems are analyzed altogether in the proof of the Theorem 9.2 below. The following lemma is introduced first for the sake of clarity.*

Lemma 9.1. *The overall transformation $(X, u, \hat{P}, \hat{v}) \mapsto (X, w, \tilde{P}, \tilde{w}, P, v)$ can be written as*

$$w(x,t) = u(x,t) - Ke^{A(x-\Delta D)}X(t) - \int_{\Delta D}^{x} Ke^{A(x-y)}Bu(y,t)\,\mathrm{d}y, \tag{9.81}$$

$$\tilde{P}(t) = e^{AD}X(t) + \int_{\Delta D}^{D_0} e^{A(D_0-y)}Bu(y,t)\,\mathrm{d}y - \hat{P}(t), \tag{9.82}$$

$$\tilde{w}(x,t) = Ce^{A(x-D_0)}\hat{P}(t) - \hat{v}(x,t) - C\int_{x}^{D_0} e^{A(x-y)}Bu(y,t)\,\mathrm{d}y. \tag{9.83}$$

$$P(t) = e^{AD}X(t) + \int_{\Delta D}^{D_0} e^{A(D_0-y)}Bu(y,t)\,dy \tag{9.84}$$

$$v(x,t) = Ce^{A(x-\Delta D)}X(t) + C\int_{\Delta D}^{x} e^{A(x-y)}Bu(y,t)\,dy. \tag{9.85}$$

while its inverse transformation is given by

$$u(x,t) = w(x,t) + Ke^{(A+BK)(x-\Delta D)}X(t) + \int_{\Delta D}^{x} Ke^{(A+BK)(x-y)}Bw(y,t)\,dy, \tag{9.86}$$

$$\hat{P}(t) = P(t) - \tilde{P}(t), \tag{9.87}$$

$$\hat{v}(x,t) = v(x,t) - \tilde{w}(x,t) - Ce^{A(x-D_0)}\tilde{P}(t). \tag{9.88}$$

Proof. Only the fact that (9.86) is the inverse of (9.81) is proved here since the other expressions follow from straightforward manipulations. The transformation (9.81) can be compactly written as

$$w(x,t) = u(x,t) - f(x-\Delta D)X(t) - (g \star u)(x,t) \tag{9.89}$$

where $f(x) = Ke^{Ax}$, $g(x) = Ke^{Ax}B$ and $\star$ denotes the convolution operator in the x variable, i.e., $(g \star u)(x,t) = \int_{-\infty}^{\infty} g(x-y)u(y,t)\,dy$. Note that the limits of the integral can be truncated assuming that $g : [0,\infty)$ and provided that $u : [\Delta D, D_0] \times [0,\infty)$. Taking the Laplace transform of (9.89) yields

$$w(\sigma,t) = \Gamma u(\sigma,t) - K(\sigma I - A)^{-1}e^{-\Delta D\sigma}X(t) \tag{9.90}$$

where σ is the Laplace argument and $\Gamma = I - K(\sigma I - A)^{-1}B$. Solving (9.90) for $u(\sigma,t)$ yields

$$u(\sigma,t) = \Gamma^{-1}w(\sigma,t) + \Gamma^{-1}K(\sigma I - A)^{1}e^{-\Delta D\sigma}X(t) \tag{9.91}$$

where $\Gamma^{-1} = I + K(\sigma I - A - BK)^{-1}B$, which follows by the Woodbury identity. Adding and subtracting $K(\sigma I - A - BK)^{-1}$ to $\Gamma^{-1}K(\sigma I - A)^{1}$ and using the identity $(\sigma I - A - BK)^{-1}\left(I - BK(\sigma I - A)^{-1}\right) = (\sigma I - A)^{-1}$ leads to

$$\Gamma^{-1}K(\sigma I - A)^{1} = K(\sigma I - A - BK)^{-1}. \tag{9.92}$$

Finally, plugging (9.92) into (9.91) and taking the inverse Laplace transform yields (9.86), which completes the proof. □

Theorem 9.2. *Consider the closed-loop system composed of* (9.55)-(9.58) *and*

$$\dot{\hat{P}}(t) = A\hat{P}(t) + BU(t) + e^{AD_0}L\left(Y(t) - \hat{v}(0,t)\right), \tag{9.93}$$

$$\hat{v}_t(x,t) = \hat{v}_x(x,t) + Ce^{Ax}L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.94}$$

$$\hat{v}(D_0,t) = C\hat{P}(t), \tag{9.95}$$

$$U(t) = K\hat{P}(t), \tag{9.96}$$

where K and L are such that $A + BK$ and $A - LC$ are Hurwitz. Then, there exists a $\delta > 0$ such that for all $|\Delta D| \leq \delta$, i.e., for all $D \in [D_0 - \delta, D_0 + \delta]$, the zero solution of the $(X, u, \hat{P}, \hat{v})$-system is exponentially stable, that is, there exist positive constants R and ρ such that for all initial conditions

$$(X_0, u_0, \hat{P}_0, \hat{v}_0) \in \mathbb{R}^n \times L_2(\underline{x}, D_0) \times \mathbb{R}^n \times H_1(\underline{x}, D_0),$$

it holds that $\Upsilon(t) \leq R\Upsilon(0)e^{-\rho t}$, where

$$\Upsilon(t) = |X(t)|^2 + \|u(t)\|^2_{L_2[\underline{x},D_0]} + |\hat{P}(t)|^2 + \|\hat{v}(t)\|^2_{H_1[\underline{x},D_0]}.$$

Proof. In order to assess stability, let us choose the Lyapunov-Krasovskii functional

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{9.97}$$

where

$$\begin{aligned}
V_1(t) &= X(t)^T S_1 X(t) + \frac{a_1}{2}\int_{\Delta D}^{D_0} e^x w(x,t)^2\,\mathrm{d}x + \frac{a_1}{4}\int_{\underline{x}}^{\Delta D} e^x w(x,t)^2\,\mathrm{d}x,\\
V_2(t) &= b_0 P(t)^T S_1 P(t) + \frac{b_1}{2}\int_{\underline{x}}^{D_0} e^x v(x,t)^2\,\mathrm{d}x + \frac{b_2}{2}\int_{\underline{x}}^{D_0} e^x v_x^2(x,t)\,\mathrm{d}x,\\
V_3(t) &= c_0 \tilde{P}(t)^T T^T S_2 T \tilde{P}(t) + \frac{c_1}{2}\int_0^{D_0} e^x \tilde{w}(x,t)^2\,\mathrm{d}x\\
&\quad + \frac{c_1}{4}\int_{\underline{x}}^0 e^x \tilde{w}^2(x,t)\,\mathrm{d}x + \int_{\underline{x}}^{D_0} e^x \tilde{w}_x^2(x,t)\,\mathrm{d}x,
\end{aligned}$$

the constants $a_i, b_i, c_i > 0$ are specified in the subsequent analysis, $T = e^{-AD_0}$ is defined for the sake of brevity, and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$ are the solutions to the Lyapunov equations

$$S_1(A + BK) + (A + BK)^T S_1 = -Q_1, \tag{9.98}$$

$$S_2(A - LC) + (A - LC)^T S_2 = -Q_2, \tag{9.99}$$

for some symmetric positive definite matrices Q_1 and Q_2, respectively. Using integration by parts[1], the time derivative of $V_1(t)$ along the trajectories of (9.72)-(9.77) is given by

$$\begin{aligned}\dot{V}_1(t) &= -X^T Q_1 X + 2X^T S_1 Bw(\Delta D, t)\\&\quad + \frac{a_1}{2} e^{D_0} w(D_0, t)^2 - \frac{a_1}{2} e^{\Delta D} w(\Delta D, t)^2\\&\quad - \frac{a_1}{2}\int_{\Delta D}^{D_0} e^x w(x,t)^2 \,\mathrm{d}x + \frac{a_1}{4} e^{\Delta D} w(\Delta D, t)^2\\&\quad - \frac{a_1}{4} e^{\underline{x}} w(\underline{x}, t)^2 - \frac{a_1}{4}\int_{\underline{x}}^{\Delta D} e^x w^2(x,t)\,\mathrm{d}x\\&\leq -\frac{\underline{\lambda}(Q_1)}{2}|X|^2 + \left(\frac{2|S_1 B|^2}{\underline{\lambda}(Q_1)} - \frac{a_1}{4} e^{\Delta D}\right) w(\Delta D, t)^2\\&\quad + \frac{a_1}{2} e^{D_0} |K|^2 |\tilde{P}|^2 - \frac{a_1}{2}\int_{\Delta D}^{D_0} e^x w(x,t)^2\,\mathrm{d}x\\&\quad - \frac{a_1}{4}\int_{\underline{x}}^{\Delta D} e^x w(x,t)^2\,\mathrm{d}x \qquad (9.100)\end{aligned}$$

where (9.98) was used and Young's inequality was employed to upper bound the second term. Proceeding in a very similar fashion, the derivative of $V_2(t)$ along the trajectories of (9.78)-(9.80) is obtained as

$$\begin{aligned}\dot{V}_2 &= -b_0 P^T Q_1 P - 2b_0 P^T S_1 BK\tilde{P}\\&\quad + \frac{b_1}{2} e^{D_0} v(D_0, t)^2 - \frac{b_1}{2} e^{\underline{x}} v^2(\underline{x}, t) - \frac{b_1}{2}\int_{\underline{x}}^{D_0} e^x v(x,t)^2 \mathrm{d}x\\&\quad + \frac{b_2}{2} e^{D_0} v_t(D_0, t)^2 - \frac{b_2}{2} e^{\underline{x}} v_t(\underline{x}, t)^2 - \frac{b_2}{2}\int_{\underline{x}}^{D_0} e^x v(x,t)^2 \mathrm{d}x\\&\leq \left(-\frac{b_0 \underline{\lambda}(Q_1)}{2} + \frac{b_1}{2} e^{D_0} |C|^2 + \frac{b_2}{2} e^{D_0} \kappa_1\right)|P|^2\\&\quad + \left(\frac{2b_0 |S_1 BK|^2}{\underline{\lambda}(Q_1)} + \frac{b_2}{2} e^{D_0} \kappa_2\right)|\tilde{P}|^2\\&\quad - \frac{b_1}{2}\int_{\underline{x}}^{D_0} e^x v(x,t)^2 \mathrm{d}x - \frac{b_2}{2}\int_{\underline{x}}^{D_0} e^x v_x(x,t)^2 \mathrm{d}x, \qquad (9.101)\end{aligned}$$

[1] By the differentiation under the integral sign rule one has that $\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b e^x w(x,t)^2\,\mathrm{d}x = \int_a^b 2e^x w(x,t) w_x(x,t)\,\mathrm{d}x$, where (9.76) was used. Then, applying integration by parts leads to $\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b e^x w(x,t)^2\,\mathrm{d}x = e^b w(b,t)^2 - e^a w(a,t)^2 - \int_a^b e^x w(x,t)^2\,\mathrm{d}x$.

where the bound

$$v_t(D_0,t)^2 \leq \kappa_1|P|^2 + \kappa_2|\tilde{P}|^2,$$

with $\kappa_1 = 2|C(A+BK)|^2$ and $\kappa_2 = 2|CBK|^2$ was employed, which follows by differentiating (9.64), plugging (9.61) in, squaring both sides and then using Young's inequality. Similarly, the time derivative of $V_3(t)$ along the trajectories of (9.72)-(9.77) can be written as

$$\begin{aligned}
\dot{V}_3 =& -c_0\tilde{P}^T T^T Q_2 T\tilde{P} - 2c_0\tilde{P}^T T^T S_2 L\tilde{w}(0,t) - 2c_0\tilde{P}^T T^T S_2 L\mathcal{I}(t)\\
&- \frac{c_1}{2}\tilde{w}(0,t)^2 - \frac{c_1}{2}\int_0^{D_0} e^x\tilde{w}(x,t)^2\,\mathrm{d}x\\
&+ \frac{c_1}{4}\tilde{w}(0,t)^2 - \frac{c_1}{4}e^{\underline{x}}\tilde{w}(\underline{x},t)^2 - \frac{c_1}{4}\int_{\underline{x}}^0 e^x\tilde{w}(x,t)^2\,\mathrm{d}x\\
&- e^{\underline{x}}\tilde{w}_x(\underline{x},t)^2 - \int_{\underline{x}}^{D_0} e^x\tilde{w}_x^2(x,t)\,\mathrm{d}x\\
\leq& -\frac{c_0\underline{\lambda}(T^TQ_2T)}{4}|\tilde{P}|^2 + \frac{4c_0|T^TS_2L|^2}{\underline{\lambda}(T^TQ_2T)}\mathcal{I}(t)^2\\
&+ \left(\frac{2c_0|T^TS_2L|^2}{\underline{\lambda}(T^TQ_2T)} - \frac{c_1}{4}\right)\tilde{w}(0,t)^2 - \int_{\underline{x}}^{D_0} e^x\tilde{w}_x^2(x,t)\,\mathrm{d}x\\
&- \frac{c_1}{2}\int_0^{D_0} e^x\tilde{w}(x,t)^2\,\mathrm{d}x - \frac{c_1}{4}\int_{\underline{x}}^0 e^x\tilde{w}(x,t)^2\,\mathrm{d}x
\end{aligned} \tag{9.102}$$

where the fact that T and A commute was taken into account, (9.99) was used, and Young's inequality was employed to upper bound the second term ($2a^Tb \leq |a|^2/2 + 2|b|^2$) and the third one ($2a^Tb \leq |a|^2/4 + 4|b|^2$). Gathering (9.100), (9.101) and (9.102), and choosing

$$b_0 = \frac{c_0\underline{\lambda}(T^TQ_2T)\underline{\lambda}(Q_1)}{16|S_1BK|^2},\ c_0 = \frac{8a_1e^{D_0}|K|^2}{\underline{\lambda}(T^TQ_2T)},$$
$$a_1 = \frac{8|S_1B|^2}{\underline{\lambda}(Q_1)e^{\Delta D}},\ b_1 = \frac{b_0\underline{\lambda}(Q_1)}{2e^{D_0}|C|^2},\ c_1 = \frac{8c_0|T^TS_2L|^2}{\underline{\lambda}(T^TQ_2T)},$$

the derivative of (9.97) is given by

$$\begin{aligned}
\dot{V}(t) \leq& -\frac{\underline{\lambda}(Q_1)}{2}|X|^2 - \frac{a_1}{2}\int_{\Delta D}^{D_0} e^x w(x,t)^2\,\mathrm{d}x\\
&- \frac{a_1}{4}\int_{\underline{x}}^{\Delta D} e^x w(x,t)^2\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&- \left(\frac{b_0 \underline{\lambda}(Q_1)}{4} - \frac{b_2}{2} e^{D_0} \kappa_1 \right) |P|^2 \\
&- \frac{b_1}{2} \int_{\underline{x}}^{D_0} e^x v(x,t)^2 \mathrm{d}x \\
&- \left(\frac{c_0 \underline{\lambda}(T^T Q_2 T)}{16} - \frac{b_2}{2} e^{D_0} \kappa_2 \right) |\tilde{P}|^2 \\
&- \frac{c_1}{2} \int_0^{D_0} e^x \tilde{w}(x,t)^2 \, \mathrm{d}x - \frac{c_1}{4} \int_{\underline{x}}^0 e^x \tilde{w}(x,t)^2 \, \mathrm{d}x \\
&+ \left(\frac{4c_0 |T^T S_2 L|^2}{\underline{\lambda}(T^T Q_2 T)} \delta e^\delta - \frac{b_2}{2} \right) \int_{\underline{x}}^{\delta} e^x v_x^2(x,t) \, \mathrm{d}x \\
&- \frac{b_2}{2} \int_\delta^{D_0} e^x v_x^2(x,t) \, \mathrm{d}x - \int_{\underline{x}}^{D_0} e^x \tilde{w}_x^2(x,t) \, \mathrm{d}x
\end{aligned} \tag{9.103}
$$

in which the following bound was used

$$
\begin{aligned}
\mathcal{I}(t)^2 &\leq |\Delta D| \int_{\min\{0,\Delta D\}}^{\max\{0,\Delta D\}} v_x^2(x,t) \, \mathrm{d}x \\
&\leq \delta \int_{\underline{x}}^{\delta} v_x^2(x,t) \, \mathrm{d}x \leq \delta e^\delta \int_{\underline{x}}^{\delta} e^x v_x^2(x,t) \, \mathrm{d}x,
\end{aligned}
$$

where the first inequality follows from Jensen's, the second holds because the integral of a positive function is an increasing function of its upper limit and $|\Delta D| \leq \delta$, and the third one follows from the fact that $e^\delta e^x \geq 1, \forall x \in [\underline{x}, \delta]$. Next, choosing

$$
b_2 < \frac{1}{4e^{D_0}} \min \left\{ \frac{b_0 \underline{\lambda}(Q_1)}{\kappa_1}, \frac{c_0 \underline{\lambda}(T^T Q_2 T)}{4\kappa_2} \right\},
$$

and selecting δ such that

$$
\delta e^\delta < \frac{b_2 \underline{\lambda}(T^T Q_2 T)}{8c_0 |T^T S_2 L|^2},
$$

it follows from (9.103) and (9.97) that

$$
\dot{V}(t) \leq -\mu V(t), \tag{9.104}
$$

where

$$
\mu = \min \left\{ \frac{\underline{\lambda}(Q_1)}{8\overline{\lambda}(S_1)}, \frac{\underline{\lambda}(T^T Q_2 T)}{32\overline{\lambda}(S_2)}, \left(\frac{8c_0 |T^T S_2 L|^2}{b_2 \underline{\lambda}(T^T Q_2 T)} \delta e^\delta - 1 \right) \right\}.
$$

From (9.97), one can find that

$$\psi_1 \Xi(t) \leq V(t) \leq \psi_2 \Xi(t), \tag{9.105}$$

where

$$\Xi(t) = |X|^2 + \|w\|^2 + |P|^2 + \|v\|^2_{H_1} + |\tilde{P}|^2 + \|\tilde{w}\|^2_{H_1},$$

and

$$\psi_1 = \max\left\{\underline{\lambda}(S_1), b_0\underline{\lambda}(S_1), c_0\underline{\lambda}(T^T S_2 T), \frac{a_1 e^{\underline{x}}}{4}, \frac{b_1 e^{\underline{x}}}{2}, \frac{b_2 e^{\underline{x}}}{2}, \frac{c_1 e^{\underline{x}}}{4}, e^{\underline{x}}\right\},$$
$$\psi_2 = \max\left\{\overline{\lambda}(S_1), b_0\overline{\lambda}(S_1), c_0\overline{\lambda}(T^T S_2 T), \frac{a_1 e^{D_0}}{2}, \frac{b_1 e^{D_0}}{2}, \frac{b_2 e^{D_0}}{2}, \frac{c_1 e^{D_0}}{2}, e^{D_0}\right\}.$$

Integrating (9.104) and then using (9.105), the following exponential stability estimate is obtained for the transformed system

$$\Xi(t) \leq \frac{\psi_2}{\psi_1}\Xi(0)e^{-\mu t}, \forall t \geq 0. \tag{9.106}$$

Now, it is necessary to show the exponential stability of the original system, that is, in the sense of the norm

$$\Upsilon(t) = |X|^2 + \|u\|^2 + |\hat{P}|^2 + \|\hat{v}\|^2_{H_1}.$$

Using (9.81)-(9.83) and (9.86)-(9.88), one can show there exist constants α_i and β_i in $[1, \infty)$ such that

$$\begin{aligned}
\Xi(t) &\leq \alpha_1|X|^2 + \alpha_2\|u\|^2 + \alpha_3|\hat{P}|^2 + \alpha_4\|\hat{v}\|^2_{H_1}, \\
\Upsilon(t) &\leq \beta_1|X|^2 + \beta_2\|w\|^2 + \beta_3|P|^2 + \beta_4\|v\|^2_{H_1} \\
&\quad + \beta_5|\tilde{P}|^2 + \beta_6\|\tilde{w}\|^2_{H_1},
\end{aligned}$$

for all $t \geq 0$, from which it follows that

$$\phi_1\Upsilon(t) \leq \Xi(t) \leq \phi_2\Upsilon(t), \tag{9.107}$$

being $\phi_1 = 1/\max\beta_i$ and $\phi_2 = \max\alpha_i$. Therefore, one gets the exponential stability estimate

$$\Upsilon(t) \leq \frac{\psi_2}{\psi_1}\frac{\phi_2}{\phi_1}\Upsilon(0)e^{-\mu t}, \forall t \geq 0,$$

thus completing the proof. □

9.3 Diffusive dynamics

The methodology developed in this section is analogous to the one introduced above for systems with delay actuator dynamics. The nominal case is presented first, for the sake of clarity. Then, an uncertainty in the diffusion coefficient is considered below in Section 9.3.2, which needs of a stability proof substantially more complicated.

9.3.1 Nominal case

Let us define

$$v(x,t) = CM(x)X(t) + C\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y. \tag{9.108}$$

Lemma 9.2. *The original system* (9.6)-(9.10) *is mapped through* (9.11) *and* (9.108) *into the virtual system*

$$\dot{\Pi}(t) = A\Pi(t) + BU(t), \tag{9.109}$$
$$v_t(x,t) = v_{xx}(x,t), \tag{9.110}$$
$$v(D,t) = C\Pi(t), \tag{9.111}$$
$$v_x(0,t) = 0, \tag{9.112}$$
$$Y(t) = v(0,t). \tag{9.113}$$

Proof. Let us introduce

$$q(x,y) = \int_0^{x-y} M(\xi)\,\mathrm{d}\xi, \tag{9.114}$$

to rewrite (9.11), (9.108) in a more compact notation

$$\Pi(t) = M(D)X(t) + \int_0^D q(D,y)Bu(y,t)\,\mathrm{d}y, \tag{9.115}$$
$$v(x,t) = CM(x)X(t) + C\int_0^x q(x,y)Bu(y,t)\,\mathrm{d}y. \tag{9.116}$$

Note that (9.114) satisfies the following relations

$$q_{xx}(x,y) = q_{yy}(x,y), \tag{9.117}$$
$$q(x,x) = 0, \tag{9.118}$$
$$q_y(x,y) = -M(x-y). \tag{9.119}$$

Differentiating (9.115), using (9.6)-(9.7) and integrating twice by parts, yields

$$\begin{aligned}\dot{\Pi}(t) = {} & M(D)[AX(t) + Bu(0,t)] - q(D,0)Bu_y(0,t) \\ & - q_y(D,D)Bu(D,t) + q_y(D,0)Bu(0,t) \\ & + \int_0^D q_{yy}(D,y)Bu(y,t)\,\mathrm{d}y. \end{aligned} \tag{9.120}$$

Using (9.8) and (9.119) one can simplify (9.120) to

$$\begin{aligned}\dot{\Pi}(t) = {} & AX(t) + Bu(D,t) \\ & + \int_0^D q_{yy}(D,y)Bu(y,t)\,\mathrm{d}y. \end{aligned} \tag{9.121}$$

On the other hand, direct computations on (9.13) show that

$$M(0) = I, \tag{9.122}$$

$$M'(0) = 0, \tag{9.123}$$

$$M''(\xi) = AM(\xi). \tag{9.124}$$

Integrating (9.124) from 0 to $x - y$ on both sides and using (9.123) yields

$$M'(x-y) = A\int_0^{x-y} M(\xi)\,\mathrm{d}\xi,$$

, and thus by (9.114) it follows that

$$M'(x-y) = Aq(x,y). \tag{9.125}$$

Differentiating (9.119) and using (9.125) leads to

$$q_{yy}(x,y) = M'(x-y) = Aq(x,y). \tag{9.126}$$

Plugging (9.126) into (9.121) and using (9.115) yields (9.109). On the other hand, (9.110) can be obtained by computing the first-in-time and second-in-space derivatives of (9.116), subtracting them, and using (9.117)-(9.119), (9.122)-(9.124). Finally, (9.112)-(9.113) follow simply by evaluating (9.108) at $x = D$ and $x = 0$, respectively. □

Theorem 9.3. *Given matrices K and L such that $A + BK$ and $A - LC$ are Hurwitz, the closed-loop system composed of* (9.6)-(9.9) *and*

$$\dot{\hat{\Pi}}(t) = A\hat{\Pi}(t) + BU(t) + M(D)L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.127}$$

$$\hat{v}_t(x,t) = \hat{v}_{xx}(x,t) + CM(x)L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.128}$$
$$\hat{v}_x(0,t) = 0, \tag{9.129}$$
$$\hat{v}(D,t) = C\hat{\Pi}(t), \tag{9.130}$$
$$U(t) = K\hat{\Pi}(t), \tag{9.131}$$

is exponentially stable in the sense of the norm

$$\left(|X(t)|^2 + \|u(t)\|^2 + |\hat{\Pi}(t)|^2 + \|\hat{v}(t)\|^2\right)^{1/2}.$$

Proof. Let us introduce another transformation, $(\Pi, v) \to (\tilde{\Pi}, \tilde{v})$, by defining the error variables

$$\tilde{\Pi}(t) \triangleq \Pi(t) - \hat{\Pi}(t), \tag{9.132}$$
$$\tilde{v}(x,t) \triangleq v(x,t) - \hat{v}(x,t). \tag{9.133}$$

Differentiating (9.132)-(9.133) and using (9.127)-(9.130), the observer error system is obtained as

$$\dot{\tilde{\Pi}}(t) = A\tilde{\Pi}(t) - M(D)L\tilde{v}(0,t), \tag{9.134}$$
$$\tilde{v}_t(x,t) = \tilde{v}_{xx}(x,t) - CM(x)L\tilde{v}(0,t), \tag{9.135}$$
$$\tilde{v}_x(0,t) = 0, \tag{9.136}$$
$$\tilde{v}(D,t) = C\tilde{\Pi}(t). \tag{9.137}$$

On the other hand, using (9.131) and (9.132), the system (9.6)-(9.9) can be written as

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{9.138}$$
$$u_t(x,t) = u_{xx}(x,t), \tag{9.139}$$
$$u_x(0,t) = 0, \tag{9.140}$$
$$u(D,t) = K\Pi(t) - K\tilde{\Pi}(t). \tag{9.141}$$

Let us introduce the mappings $(X,u) \mapsto (X,w)$ and $(\tilde{\Pi},\tilde{v}) \mapsto (\tilde{\Pi},\tilde{w})$, defined by the backstepping transformations

$$w(x,t) = u(x,t) - KM(x)X(t) - K\int_0^x m(x-y)Bu(y,t)\,dy, \tag{9.142}$$
$$\tilde{w}(x,t) = \tilde{v}(x,t) - CM(x)M(D)^{-1}\tilde{\Pi}(t), \tag{9.143}$$

respectively, which transform (9.134)-(9.137) and (9.138)-(9.141) into

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{9.144}$$
$$w_t(x,t) = w_{xx}(x,t), \tag{9.145}$$
$$w_x(0,t) = 0, \tag{9.146}$$
$$w(D,t) = -K\tilde{\Pi}(t), \tag{9.147}$$
$$\dot{\tilde{\Pi}}(t) = (A - M(D)LCM(D)^{-1})\tilde{\Pi}(t) - M(D)L\tilde{w}(0,t), \tag{9.148}$$
$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t), \tag{9.149}$$
$$\tilde{w}_x(0,t) = 0, \tag{9.150}$$
$$\tilde{w}(D,t) = 0, \tag{9.151}$$

where (9.145) followed after two integrations by parts, (9.123) was used to obtain (9.146) and (9.150), while (9.149) used (9.124) and the fact that A and $M(x)$ commute for all x. As described in Appendix C, the overall transformation $(X, u, \hat{\Pi}, \hat{v}) \mapsto (X, w, \tilde{\Pi}, \tilde{w})$ can be written as

$$w(x,t) = u(x,t) - KM(x)X(t) - K\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y, \tag{9.152}$$
$$\tilde{\Pi}(t) = M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y - \hat{\Pi}(t), \tag{9.153}$$
$$\tilde{w}(x,t) = CM(x)M(D)^{-1}\hat{\Pi}(t) - \hat{v}(x,t) + C\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y - CM(x)M(D)^{-1}\int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y, \tag{9.154}$$

while the inverse of (9.152)-(9.154) is given by

$$u(x,t) = w(x,t) + KN(x)X(t) + K\int_0^x n(x-y)Bw(y,t)\,\mathrm{d}y, \tag{9.155}$$
$$\hat{\Pi}(t) = M(D)X(t) - \tilde{\Pi}(t) + \int_0^D m(D-y)Bw(y,t)\,\mathrm{d}y$$

$$
\begin{aligned}
&+ \left(\int_0^D m(D-y)BKN(y)\,\mathrm{d}y \right) X(t) \\
&+ \int_0^D \left(\int_z^D m(D-y)BKn(y-z)\,\mathrm{d}y \right) \\
&\times Bw(z,t)\,\mathrm{d}z,
\end{aligned} \tag{9.156}
$$

$$
\begin{aligned}
\hat{v}(x,t) = {} & CM(x)X(t) - CM(x)M(D)^{-1}\tilde{\Pi}(t) - \tilde{w}(x,t) \\
&+ C\int_0^x m(x-y)Bw(y,t)\,\mathrm{d}y \\
&+ C\left(\int_0^x m(x-y)BKn(x-y)\,\mathrm{d}y \right) X(t) \\
&+ C\int_0^x \left(\int_z^x m(x-y)BKn(y-z)\,\mathrm{d}y \right) \\
&\times Bw(z,t)\,\mathrm{d}z
\end{aligned} \tag{9.157}
$$

In order to assess stability, let us choose the Lyapunov-Krasovskii functional

$$
\begin{aligned}
V(t) = {} & X^T S_1 X + \frac{a}{2}\|w\|^2 \\
&+ b\tilde{\Pi}^T M^{-T} S_2 M^{-1}\tilde{\Pi} + \frac{c}{2}\|\tilde{w}\|^2
\end{aligned} \tag{9.158}
$$

where $M = M(D)$ for the sake of brevity and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$ are the solutions to the Lyapunov equations

$$
S_1(A+BK) + (A+BK)^T S_1 = -Q_1, \tag{9.159}
$$

$$
S_2(A-LC) + (A-LC)^T S_2 = -Q_2. \tag{9.160}
$$

for some $Q_1 = Q_1^T$, $Q_2 = Q_2^T$. Using integration by parts, the time derivative of (9.158) along the trajectories of (9.144)-(9.151) is computed as

$$
\begin{aligned}
\dot{V}(t) = {} & -X^T Q_1 X + 2X^T S_1 Bw(0,t) \\
&+ aw(D)w_x(D) - a\|w_x\|^2 - c\|\tilde{w}_x\|^2 \\
&- b\tilde{\Pi}^T M^{-T} Q_2 M^{-1}\tilde{\Pi} - 2b\tilde{\Pi}^T M^{-T} S_2 L\tilde{w}(0),
\end{aligned}
$$

which can be bounded by

$$
\dot{V}(t) \leq -\frac{\lambda_{\min}(Q_1)}{2}|X|^2 + \frac{2|S_1 B|^2}{\lambda_{\min}(Q_1)} w(0,t)^2
$$

$$
\begin{aligned}
&+2aw(D,t)^2+\frac{a}{2}w_x(D)^2-a\|w_x\|^2-c\|\tilde{w}_x\|^2 \\
&-b\frac{\lambda_{\min}(Q_2)}{2}|M^{-1}\tilde{\Pi}|^2+\frac{2b|S_2L|^2}{\lambda_{\min}(Q_2)}\tilde{w}(0,t)^2.
\end{aligned} \tag{9.161}
$$

To proceed, note that the following bound holds

$$
\begin{aligned}
w(0,t)^2 &\le w(D,t)^2+2\|w\|\|w_x\| \\
&\le w(D,t)^2+\|w\|^2+\|w_x\|^2 \\
&\le 2w(D)^2+(4D^2+1)\|w_x\|^2,
\end{aligned} \tag{9.162}
$$

where the first inequality is Agmon's, the second follows from Young's and the last one follows by applying Poincaré's inequality, that is, $\|w\|^2 \le w(D,t)^2 + 4D^2\|w_x\|^2$. Using (9.162), the inequality (9.161) can be further bounded as

$$
\begin{aligned}
\dot{V}(t) \le &-\frac{\lambda_{\min}(Q_1)}{2}|X|^2+\frac{2(4D^2+1)|S_1B|^2}{\lambda_{\min}(Q_1)}\|w_x\|^2 \\
&+\left(2a+\frac{4|S_1B|^2}{\lambda_{\min}(Q_1)}\right)w(D,t)^2-\frac{a}{2}\|w_x\|^2 \\
&-c\|\tilde{w}_x\|^2-b\frac{\lambda_{\min}(Q_2)}{2}|M^{-1}\tilde{\Pi}|^2 \\
&+\frac{2b|S_2L|^2}{\lambda_{\min}(Q_2)}\tilde{w}(0,t)^2
\end{aligned} \tag{9.163}
$$

Let us choose

$$
a=\frac{8(4D^2+1)|S_1B|^2}{\lambda_{\min}(Q_1)}, \tag{9.164}
$$

and use the fact that $-4D^2\|w_x\|^2 \le -\|w\|^2 + w(D,t)^2$, to write

$$
\begin{aligned}
\dot{V}(t) \le &-\frac{\lambda_{\min}(Q_1)}{2}|X|^2-\frac{a}{16D^2}\|w\|^2+\frac{a}{16D^2}w(D,t)^2 \\
&+\left(2a+\frac{4|S_1B|^2}{\lambda_{\min}(Q_1)}\right)w(D,t)^2-c\|\tilde{w}_x\|^2 \\
&-b\frac{\lambda_{\min}(Q_2)}{2}|M^{-1}\tilde{\Pi}|^2+\frac{2b|S_2L|^2}{\lambda_{\min}(Q_2)}\tilde{w}(0,t)^2.
\end{aligned} \tag{9.165}
$$

Using (9.147) and the fact that $\tilde{w}(0,t)^2 \leq 4D\|\tilde{w}_x\|^2$, which follows also from Agmon's and Poincaré's inequalities provided that $\tilde{w}(D,t) = 0$, (9.165) can be further bounded by

$$\begin{aligned}\dot{V}(t) \leq & -\frac{\lambda_{\min}(Q_1)}{2}|X|^2 - \frac{a}{16D^2}\|w\|^2 \\ & + \left(\chi - b\frac{\lambda_{\min}(Q_2)}{2}\right)|M^{-1}\tilde{\Pi}|^2 \\ & + \left(\frac{8Db|S_2L|^2}{\lambda_{\min}(Q_2)} - c\right)\|\tilde{w}_x\|\end{aligned} \tag{9.166}$$

where $\chi = |KM|^2\left(\left(2+\frac{1}{16D^2}\right)a + \frac{4|S_1B|^2}{\lambda_{\min}(Q_1)}\right)$. Choosing

$$b = \frac{4\chi}{\lambda_{\min}(Q_2)}, \quad c = \frac{16Db|S_2L|^2}{\lambda_{\min}(Q_2)}, \tag{9.167}$$

leads to

$$\begin{aligned}\dot{V}(t) \leq & -\frac{\lambda_{\min}(Q_1)}{2}|X|^2 - \frac{a}{16D^2}\|w\|^2 \\ & - \frac{\lambda_{\min}(Q_2)}{4}b|M^{-1}\Pi|^2 - \frac{c}{2}\|\tilde{w}_x\|^2.\end{aligned} \tag{9.168}$$

Next, we observe that

$$\begin{aligned}\dot{V}(t) \leq & -\frac{\lambda_{\min}(Q_1)}{2\lambda_{\max}(S_1)}X^TS_1X - \frac{1}{8D^2}\frac{a}{2}\|w\|^2 \\ & - \frac{\lambda_{\min}(Q_2)}{4\lambda_{\max}(S_2)}\Pi^TM^{-T}S_2M^{-1}\Pi - \frac{1}{2D}\frac{c}{2}\|\tilde{w}\|^2.\end{aligned}$$

where Poincaré's inequality has been used. Therefore, it follows that

$$\dot{V}(t) \leq -\mu V(t), \tag{9.169}$$

where

$$\mu = \min\left\{\frac{\lambda_{\min}(Q_1)}{2\lambda_{\max}(S_1)}, \frac{\lambda_{\min}(Q_2)}{4\lambda_{\max}(S_2)}, \frac{1}{8D^2}, \frac{1}{2D}\right\}.$$

Now, from (9.158), one can find that

$$\psi_1\Xi(t) \leq V(t) \leq \psi_2\Xi(t), \tag{9.170}$$

where

$$\Xi(t) = |X(t)|^2 + \|w(t)\|^2 + |\tilde{\Pi}(t)|^2 + \|\tilde{w}(t)\|^2,$$

and

$$\psi_1 = \min\left\{\lambda_{\min}(S_1), \frac{a}{2}, b|M^{-1}|^2\lambda_{\min}(S_2), \frac{c}{2}\right\},$$
$$\psi_2 = \max\left\{\lambda_{\max}(S_1), \frac{a}{2}, b|M^{-1}|^2\lambda_{\max}(S_2), \frac{c}{2}\right\}.$$

Hence, the following exponential stability estimate is obtained for the transformed system

$$\Xi(t) \leq \frac{\psi_2}{\psi_1}\Xi(0)e^{-\mu t}, \; \forall t \geq 0. \tag{9.171}$$

Now, an estimate is derived in terms of

$$\Upsilon(t) = |X(t)|^2 + \|u(t)\|^2 + |\hat{\Pi}(t)|^2 + \|\hat{v}(t)\|^2.$$

Using (9.152)-(9.154) and (9.155)-(9.157), one can show there exist constants α_i, β_i such that

$$\Xi(t) \leq \alpha_1|X|^2 + \alpha_2\|u\|^2 + \alpha_3|\hat{\Pi}|^2 + \alpha_4\|\hat{v}\|^2,$$
$$\Upsilon(t) \leq \beta_1|X|^2 + \beta_2\|w\|^2 + \beta_3\|\tilde{\Pi}\|^2 + \beta_4\|\tilde{w}\|^2,$$

for all $t \geq 0$, from which it follows that

$$\phi_1\Upsilon(t) \leq \Xi(t) \leq \phi_2\Upsilon(t), \tag{9.172}$$

being $\phi_1 = 1/\max\beta_i$, $\phi_2 = \max\alpha_i$. Therefore, from (9.171)-(9.172), one gets the exponential stability estimate

$$\Upsilon(t) \leq \frac{\psi_2}{\psi_1}\frac{\phi_2}{\phi_1}\Upsilon(0)e^{-\mu t}, \; \forall t \geq 0,$$

completing the proof. □

9.3.2 Robustness to diffusion coefficient mismatch

In this section, an uncertainty in the diffusion coefficient is considered. Such an scenario can be represented by the following ODE-PDE cascade

$$\dot{X}(t) = AX(t) + Bu(0,t), \tag{9.173}$$
$$u_t(x,t) = \epsilon u_{xx}(x,t), \tag{9.174}$$
$$u_x(0,t) = 0, \tag{9.175}$$
$$u(D,t) = U(t), \tag{9.176}$$
$$Y(t) = CX(t). \tag{9.177}$$

where $D \geq 0$ is the spatial domain length, $\epsilon = \epsilon_0 + \Delta\epsilon$ is the diffusion coefficient, in which $\epsilon_0 \neq 0$ is known and $\Delta\epsilon > -\epsilon_0$ is a small additive uncertainty. If the whole state is available and ϵ is known, a stabilizing control law for (9.6)-(9.9) is given by $U(t) = K\Pi(t)$, where

$$\Pi(t) = M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y, \tag{9.178}$$

the vector K is again to be chosen such that $A + BK$ is Hurwitz, and

$$m(s) = \frac{1}{\epsilon}\int_0^s M(\xi)\,\mathrm{d}\xi, \tag{9.179}$$
$$M(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & \frac{A}{\epsilon} \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{9.180}$$

This result is a slightly modified version of Theorem 1 in Krstic 2009b. Let us also define

$$v(x,t) = CM(x)X(t) + C\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y. \tag{9.181}$$

Lemma 9.3. *The original system (9.173)-(9.177) is mapped through (9.178) and (9.181) into the virtual system*

$$\dot{\Pi}(t) = A\Pi(t) + BU(t), \tag{9.182}$$
$$v_t(x,t) = \epsilon v_{xx}(x,t), \tag{9.183}$$
$$v(D,t) = C\Pi(t), \tag{9.184}$$
$$v_x(0,t) = 0, \tag{9.185}$$
$$Y(t) = v(0,t), \tag{9.186}$$

Proof. Let us introduce

$$q(x,y) = \frac{1}{\epsilon}\int_0^{x-y} M(\xi)B\,\mathrm{d}\xi, \tag{9.187}$$

to rewrite (9.178), (9.181) in a more compact notation

$$\Pi(t) = M(D)X(t) + \int_0^D q(D,y)u(y,t)\,\mathrm{d}y, \tag{9.188}$$

$$v(x,t) = CM(x)X(t) + C\int_0^x q(x,y)u(y,t)\,\mathrm{d}y. \tag{9.189}$$

Note that (9.187) satisfies the following relations

$$q_{xx}(x,y) = q_{yy}(x,y), \tag{9.190}$$

$$q(x,x) = 0, \tag{9.191}$$

$$\epsilon q_y(x,y) = -M(x-y)B. \tag{9.192}$$

On the other hand, direct computations on (9.180) show that

$$M(0) = I, \tag{9.193}$$

$$M'(0) = 0, \tag{9.194}$$

$$M''(\xi) = \epsilon^{-1}AM(\xi). \tag{9.195}$$

Differentiating (9.188), using (9.173)-(9.174) and integrating twice by parts, yields

$$\begin{aligned}\dot{\Pi}(t) = M(D)[AX(t) + Bu(0,t)] + \Big(&q_y(D,0)u(0,t)\\ &- q_y(D,D)u(D,t) - q(D,0)u_x(0,t)\\ &+ \int_0^D q_{yy}(D,y)u(y,t)\,\mathrm{d}y\Big)\epsilon.\end{aligned} \tag{9.196}$$

Using (9.175), (9.192) and (9.193), one can simplify (9.196) to

$$\begin{aligned}\dot{\Pi}(t) = M(D)AX(t) + Bu(D,t)&\\ + \int_0^D q_{yy}(D,y)Bu(y,t)\,\mathrm{d}y.&\end{aligned} \tag{9.197}$$

Now, integrating (9.195) from 0 to $x-y$ on both sides and using (9.194) yields

$$M'(x-y) = \epsilon^{-1}A\int_0^{x-y} M(\xi)\,\mathrm{d}\xi = Aq(x,y). \tag{9.198}$$

where the last equality follows from (9.187). Differentiating (9.192) and using (9.198) leads to

$$\epsilon q_{yy}(x,y) = Aq(x,y). \tag{9.199}$$

Plugging (9.199) evaluated at $x = D$ into (9.197) and using (9.188) yields (9.182). On the other hand, (9.183) can be obtained by computing the first-in-time and second-in-space derivatives of (9.189), subtracting them, and using (9.190)-(9.192), (9.193)-(9.195). Finally, (9.185)-(9.186) follow simply by evaluating (9.181) at $x = D$ and $x = 0$, respectively. □

Let us define the error variables as

$$\tilde{\Pi}(t) \triangleq \Pi(t) - \hat{\Pi}(t), \tag{9.200}$$

$$\tilde{v}(x,t) \triangleq v(x,t) - \hat{v}(x,t). \tag{9.201}$$

Differentiating (9.200)-(9.201) and using (9.232)-(9.235), the observer error system is obtained as

$$\dot{\tilde{\Pi}}(t) = A\tilde{\Pi}(t) - M_0(D)L\tilde{v}(0,t), \tag{9.202}$$

$$\tilde{v}_t(x,t) = \epsilon_0\tilde{v}_{xx}(x,t) - CM_0(x)L\tilde{v}(0,t) + \Delta\epsilon v_{xx}(x,t), \tag{9.203}$$

$$\tilde{v}_x(0,t) = 0, \tag{9.204}$$

$$\tilde{v}(D,t) = C\tilde{\Pi}(t). \tag{9.205}$$

Now, let us introduce the backstepping transformations

$$\begin{aligned} w(x,t) = u(x,t) &- KM(x)X(t) \\ &- K\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y, \end{aligned} \tag{9.206}$$

$$\tilde{w}(x,t) = \tilde{v}(x,t) - CM_0(x)M_0(D)^{-1}\tilde{\Pi}(t), \tag{9.207}$$

Using (9.236) and the transformations (9.206)-(9.207) the systems (9.173)-(9.176) and (9.202)-(9.205) are mapped into

$$\dot{X}(t) = (A + BK)X(t) + Bw(0,t), \tag{9.208}$$
$$w_t(x,t) = \epsilon w_{xx}(x,t), \tag{9.209}$$
$$w_x(0,t) = 0, \tag{9.210}$$
$$w(D,t) = -K\tilde{\Pi}(t), \tag{9.211}$$
$$\begin{aligned}\dot{\tilde{\Pi}}(t) = {} & (A - M_0(D)LCM_0(D)^{-1})\tilde{\Pi}(t) \\ & - M_0(D)L\tilde{w}(0,t),\end{aligned} \tag{9.212}$$
$$\tilde{w}_t(x,t) = \epsilon_0\tilde{w}_{xx}(x,t) + \Delta\epsilon v_{xx}, \tag{9.213}$$
$$\tilde{w}_x(0,t) = 0, \tag{9.214}$$
$$\tilde{w}(D,t) = 0, \tag{9.215}$$

where (9.209) followed after two integrations by parts, (9.194) was used to obtain (9.210) and (9.214), while (9.213) used that $M_0''(x) = \epsilon_0^{-1}AM_0(x)$ and the fact that A and $M_0(x)$ commute for all x. Also, using (9.236) and (9.200), the system (9.182)-(9.185) can be written as

$$\dot{\Pi}(t) = (A + BK)\Pi(t) - BK\tilde{\Pi}(t), \tag{9.216}$$
$$v_t(x,t) = \epsilon v_{xx}(x,t), \tag{9.217}$$
$$v_x(0,t) = 0, \tag{9.218}$$
$$v(D,t) = C\Pi(t). \tag{9.219}$$

Lemma 9.4. *The overall transformation* $(X, u, \hat{\Pi}, \hat{v}) \mapsto (X, w, \tilde{\Pi}, \tilde{w}, \Pi, v)$ *can be written as*

$$\begin{aligned} w(x,t) = {} & u(x,t) - KM(x)X(t) \\ & - K\int_0^x m(x-y)Bu(y,t)\,dy, \end{aligned} \tag{9.220}$$
$$\begin{aligned} \tilde{\Pi}(t) = {} & M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,dy \\ & - \hat{\Pi}(t), \end{aligned} \tag{9.221}$$
$$\begin{aligned} \tilde{w}(x,t) = {} & CM_0(x)M_0(D)^{-1}\hat{\Pi}(t) - \hat{v}(x,t) \\ & + CM(D)X(t) + C\int_0^x m(x-y)Bu(y,t)\,dy \\ & - CM_0(x)M_0(D)^{-1}\Big(M(D)X(t) \end{aligned}$$

$$+\int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y\Big), \tag{9.222}$$

$$\Pi(t) = M(D)X(t) + \int_0^D m(D-y)Bu(y,t)\,\mathrm{d}y, \tag{9.223}$$

$$v(x,t) = CM(x)X(t) + C\int_0^x m(x-y)Bu(y,t)\,\mathrm{d}y, \tag{9.224}$$

while its inverse transformation is given by

$$u(x,t) = w(x,t) + KN(x)X(t) + K\int_0^x n(x-y)Bw(y,t)\,\mathrm{d}y, \tag{9.225}$$

$$\hat{\Pi}(t) = \Pi(t) - \tilde{\Pi}(t), \tag{9.226}$$

$$\hat{v}(x,t) = v(x,t) - CM_0(x)M(D)^{-1}\tilde{\Pi}(t) - \tilde{w}(x,t). \tag{9.227}$$

where

$$n(s) = \frac{1}{\epsilon}\int_0^s N(\xi)\,\mathrm{d}\xi,$$

$$N(\xi) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & \frac{A+BK}{\epsilon} \\ I & 0 \end{bmatrix}\xi} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Proof. Again, only the proof of (9.225) is given as the other expressions are easily obtained. Let us define $R = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$. Because of the anti-diagonal structure of R, it is verified that $\Phi^T R^{2j+1}\Phi = 0$ and $\Phi^T R^{2j}\Phi = (A/\epsilon)^j$, for all $j \in \{0, 1, \ldots, \infty\}$. Therefore, using the Taylor expansion of the matrix exponential we have that $\Phi^T e^{Rx}\Phi = \sum_{n=0}^{\infty}(\Phi^T R^n \Phi)x^n/n! = \sum_{n=0}^{\infty}(A/\epsilon)^n x^{2n}/(2n)! = \sum_{n=0}^{\infty}(x\sqrt{A/\epsilon})^{2n}/(2n)! = \cosh(x\sqrt{A/\epsilon})$. Then, one can compactly rewrite (9.220) as

$$w(x,t) = u(x,t) - f(x)X(t) - (g \star u)(x,t) \tag{9.228}$$

where $f(x) = K\cosh(x\sqrt{A\epsilon})$, $g(x) = \epsilon^{-1}K\int_0^x f(\xi)\,\mathrm{d}\xi B$ and $\star$ denotes the convolution operator in the x variable. Taking the Laplace transform of (9.228) yields

$$w(\sigma,t) = \Gamma u(\sigma,t) - K\sigma(\sigma^2 I - A/\epsilon)^{-1}X(t) \tag{9.229}$$

where σ is the Laplace argument and $\Gamma = I - \epsilon^{-1}K(\sigma^2 I - A/\epsilon)^{-1}B$. Solving (9.229) for $u(\sigma, t)$ yields

$$u(\sigma, t) = \Gamma^{-1}w(\sigma, t) + \Gamma^{-1}K\sigma(\sigma^2 - A/\epsilon)^{-1}X(t) \tag{9.230}$$

where $\Gamma^{-1} = I + \epsilon^{-1}K(\sigma^2 I - (A+BK)/\epsilon)^{-1}B$, which follows by the Woodbury identity. Now, adding and subtracting $K\sigma(\sigma^2 I - (A+BK)/\epsilon)^{-1}$ to $\Gamma^{-1}K\sigma(\sigma^2 - A/\epsilon)^{-1}$ and using the identity

$$(\sigma^2 I - (A+BK)/\epsilon)^{-1}\left[I - \epsilon^{-1}BK(\sigma^2 I - A\epsilon)^{-1}\right] = (\sigma^2 I - A\epsilon)^{-1}$$

leads to

$$\Gamma^{-1}K\sigma(\sigma^2 - A/\epsilon)^{-1} = K\sigma(\sigma^2 I - (A+BK)/\epsilon)^{-1}. \tag{9.231}$$

Finally, plugging (9.231) into (9.230) and taking the inverse Laplace transform yields (9.225), which completes the proof. □

Theorem 9.4. *Consider the closed-loop system composed of* (9.173)-(9.177) *and*

$$\dot{\hat{\Pi}}(t) = A\hat{\Pi}(t) + BU(t) + M_0(D)L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.232}$$

$$\hat{v}_t(x,t) = \epsilon_0\hat{v}_{xx}(x,t) + CM_0(x)L\big(Y(t) - \hat{v}(0,t)\big), \tag{9.233}$$

$$\hat{v}_x(0,t) = 0, \tag{9.234}$$

$$\hat{v}(D,t) = C\hat{\Pi}(t), \tag{9.235}$$

$$U(t) = K\hat{\Pi}(t), \tag{9.236}$$

where

$$M_0(x) = \begin{bmatrix} I & 0 \end{bmatrix} e^{\begin{bmatrix} 0 & \frac{A}{\epsilon_0} \\ I & 0 \end{bmatrix} x} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

the matrices K and L are such that $A + BK$ and $A - LC$ are Hurwitz and $\epsilon_0 > 0$. Then, there exists a $\delta > 0$ such that for all $|\Delta\epsilon| \le \delta$, i.e., for all $\epsilon \in [\epsilon_0 - \delta, \epsilon_0 + \delta]$, the zero solution of the $(X, u, \hat{\Pi}, \hat{v})$-system is exponentially stable, that is, there exist positive constants R and ρ such that for all initial conditions

$$(X_0, u_0, \hat{\Pi}_0, \hat{v}_0) \in \mathbb{R}^n \times H_1(0,D) \times \mathbb{R}^n \times H_1(0,D),$$

it holds that $\Upsilon(t) \le R\Upsilon(0)e^{-\rho t}$, where

$$\Upsilon(t) = |X(t)|^2 + \|u(t)\|^2_{H_1[0,D]} + |\hat{P}(t)|^2 + \|\hat{v}(t)\|^2_{H_1[0,D]}.$$

Proof. In order to assess stability, let us choose the Lyapunov-Krasovskii functional

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{9.237}$$

where

$$\begin{aligned} V_1(t) &= a_0 X^T S_1 X + \frac{a_1}{2\epsilon}\|w\|^2 + \frac{a_2}{2\epsilon}\|w_x\|^2 \\ V_2(t) &= b_0 \Pi^T S_1 \Pi + \frac{b_1}{2\epsilon}\|\nu\|^2 + \frac{b_2}{2\epsilon}\|\nu_x\|^2 \\ V_3(t) &= \tilde{\Pi}^T M_0^{-T} S_2 M_0^{-1} \tilde{\Pi} + \frac{c_1}{2\epsilon_0}\left(\|\tilde{w}\|^2 + \|\tilde{w}_x\|^2\right), \end{aligned}$$

where $M_0 = M_0(D)$ for the sake of brevity and $S_1 = S_1^T > 0$, $S_2 = S_2^T > 0$ are the solutions to the Lyapunov equations

$$S_1(A + BK) + (A + BK)^T S_1 = -Q_1, \tag{9.238}$$

$$S_2(A - LC) + (A - LC)^T S_2 = -Q_2. \tag{9.239}$$

for some symmetric positive definite matrices Q_1 and Q_2. Using integration by parts[2], the time derivative of $V_1(t)$ along the trajectories of (9.208)-(9.211) can be written as

$$\begin{aligned} \dot{V}_1(t) &= -a_0 X^T Q_1 X + 2a_0 X^T S_1 B w(0,t) \\ &\quad + a_1 w(D,t) w_x(D,t) - a_1\|w_x\|^2 \\ &\quad + a_2 w_x(D,t) w_{xx}(D,t) - a_2\|w_{xx}\|^2 \\ &\leq -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 + \frac{2a_0|S_1 B|^2}{\underline{\lambda}(Q_1)} w(0,t)^2 \\ &\quad + \frac{D a_1^2}{a_2} w(D,t)^2 + \frac{a_2}{4D} w_x(D,t)^2 \\ &\quad + \frac{a_2}{4D} w_x(D,t)^2 + D a_2 w_{xx}(D,t)^2 \\ &\quad - a_1\|w_x\|^2 - a_2\|w_{xx}\|^2 \end{aligned} \tag{9.240}$$

where Young's inequality was used conveniently used multiple times. To proceed, some inequalities are derived next. By the fundamental theorem of cal-

[2] By the differentiation under the integral sign rule one has that $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|w(t)\|^2 = \int_0^D w(x,t) w_{xx}(x,t)\,\mathrm{d}x$, where (9.213) was used. Then, applying integration by parts leads to $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|w(t)\|^2 = w(D,t)w_x(D,t) - w(0,t)w_x(0,t) - \|w_x(t)\|^2$.

culus and Jensen's inequality, $\left(w(D,t)-w(0,t)\right)^2 = \left(\int_0^D w_x\,\mathrm{d}x\right)^2 \leq D\|w_x\|^2$, and then expanding the squared difference and employing Young's inequality to upper bound the cross term leads to

$$w(0,t)^2 \leq 2w(D,t)^2 + 2D\|w_x\|^2, \tag{9.241}$$

Proceeding in a similar way with w_x and $\tilde{w}$, and using (9.210) and (9.215), respectively, yields

$$w_x(D,t)^2 \leq D\|w_{xx}\|^2, \tag{9.242}$$

$$\tilde{w}(0,t)^2 \leq D\|\tilde{w}_x\|^2. \tag{9.243}$$

Integrating $\|w\|^2$ by parts and using Young's inequality conveniently, leads to $\|w\|^2 \leq 2Dw(D,t)^2 + 4D^2\|w_x\|^2$, from which we get

$$-\|w_x\|^2 \leq \frac{1}{2D}w(D,t)^2 - \frac{1}{4D^2}\|w\|^2, \tag{9.244}$$

follows. Using the same procedure with $\|w_x\|^2$ and taking (9.210) into account yields

$$-\|w_{xx}\|^2 \leq -\frac{1}{4D^2}\|w_x\|^2. \tag{9.245}$$

Using (9.241)-(9.242) into (9.240) and selecting

$$a_1 = \frac{8Da_0|S_1B|^2}{\underline{\lambda}(Q_1)},$$

yields

$$\begin{aligned}\dot{V}_1(t) \leq & -\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2 + Da_2w_{xx}(D,t)^2 \\ & + \left(\frac{4a_0|S_1B|^2}{\underline{\lambda}(Q_1)} + \frac{Da_1^2}{a_2}\right)w(D,t)^2 \\ & - \frac{a_1}{2}\|w_x\|^2 - \frac{a_2}{2}\|w_{xx}\|^2\end{aligned} \tag{9.246}$$

Now, using (9.244)-(9.245) into (9.246), one can write

$$\dot{V}_1(t) \leq -\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2 + Da_2w_{xx}(D,t)^2$$

$$+\left(\frac{4a_0|S_1B|^2}{\underline{\lambda}(Q_1)}+\frac{Da_1^2}{a_2}+\frac{a_1}{4D}\right)w(D,t)^2$$
$$-\frac{a_1}{8D^2}\|w\|^2-\frac{a_2}{8D^2}\|w_x\|^2 \tag{9.247}$$

Furthermore, using (9.209), (9.211) and (9.212),

$$w_{xx}(D,t)^2 \leq \kappa_1|\tilde{\Pi}|^2+\kappa_2\tilde{w}(0,t)^2 \tag{9.248}$$

where $\kappa_1 = 2\epsilon^{-2}|K(A-M_0LCM_0^{-1})|^2$ and $\kappa_2 = 2\epsilon^{-2}|KM_0L|^2$. Using (9.211), (9.243) and (9.248) into (9.247) yields

$$\dot{V}_1(t) \leq -\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2+D^2a_2\kappa_2\|\tilde{w}_x\|^2$$
$$+\left(\left(\frac{4a_0|S_1B|^2}{\underline{\lambda}(Q_1)}+\frac{Da_1^2}{a_2}+\frac{a_1}{4D}\right)|K|^2\right.$$
$$\left.+Da_2\kappa_1\right)|\tilde{\Pi}|^2-\frac{a_1}{8D^2}\|w\|^2-\frac{a_2}{8D^2}\|w_x\|^2 \tag{9.249}$$

Similarly as before, using integration by parts and Young's inequality, the time derivative of $V_2(t)$ along the trajectories of (9.216)-(9.218) can be bounded by

$$\begin{aligned}\dot{V}_2(t) =& -b_0\Pi^TQ_1\Pi-2b_0\Pi^TS_1BK\tilde{\Pi}\\ &+b_1v(D,t)v_x(D,t)-b_1\|v_x\|^2\\ &+b_2v_x(D,t)v_{xx}(D,t)-b_2\|v_{xx}\|^2\\ \leq& -\frac{b_0\underline{\lambda}(Q_1)}{2}|\Pi|^2+\frac{2b_0|S_1BK|^2}{\underline{\lambda}(Q_1)}|\tilde{\Pi}|^2\\ &+\frac{Db_1^2}{b_2}v(D,t)^2+\frac{b_2}{4D}v_x(D,t)^2\\ &+\frac{b_2}{4D}v_x(D,t)^2+Db_2v_{xx}(D,t)^2\\ &-b_1\|v_x\|^2+b_2\|v_{xx}\|^2\end{aligned}$$

$$
\begin{aligned}
&\leq -\left(\frac{b_0\underline{\lambda}(Q_1)}{2}+\frac{Db_1^2|C|^2}{b_2}+\frac{b_1|C|^2}{2D}+Db_2\kappa_3\right)|\Pi|^2\\
&\quad+\left(\frac{2b_0|S_1BK|^2}{\underline{\lambda}(Q_1)}+Db_2\kappa_4\right)|\tilde{\Pi}|^2\\
&\quad-\frac{b_1}{4D^2}\|\nu\|^2-\frac{b_2}{2}\|\nu_{xx}\|^2
\end{aligned}
\tag{9.250}
$$

where the inequalities

$$
\nu_x(D,t)^2 \leq D\|\nu_{xx}\|^2 \tag{9.251}
$$

$$
-\|\nu_x\|^2 \leq \frac{1}{2D}\nu(D,t)^2-\frac{1}{4D^2}\|\nu\|^2, \tag{9.252}
$$

$$
\nu_{xx}(D,t)^2 \leq \kappa_3|\Pi|^2+\kappa_4|\tilde{\Pi}|^2, \tag{9.253}
$$

with $\kappa_3 = 2\epsilon^{-2}|C(A+BK)|^2$ and $\kappa_4 = 2\epsilon^{-2}|CBK|^2$ were used. Note that (9.251) and (9.252) follow by the same procedures used to derive (9.242) and (9.244), respectively, whereas (9.253) follows from (9.216)-(9.219). Choosing

$$
b_1 = \min\left\{\sqrt{\frac{b_0\underline{\lambda}(Q_1)b_2}{8D|C|^2}}, \frac{b_0\underline{\lambda}(Q_1)D}{4|C|^2}\right\}, \tag{9.254}
$$

in (9.250) yields

$$
\begin{aligned}
\dot{V}_2(t) &\leq -\left(\frac{b_0\underline{\lambda}(Q_1)}{4}+Db_2\kappa_3\right)|\Pi|^2\\
&\quad+\left(\frac{2b_0|S_1BK|^2}{\underline{\lambda}(Q_1)}+Db_2\kappa_4\right)|\tilde{\Pi}|^2\\
&\quad-\frac{b_1}{4D^2}\|\nu\|^2-\frac{b_2}{2}\|\nu_{xx}\|^2.
\end{aligned}
\tag{9.255}
$$

Again, integrating by parts, using Young's inequality and (9.243), the derivative of $V_3(t)$ along the trajectories of (9.212)-(9.215) can be bounded by

$$
\begin{aligned}
\dot{V}_3(t) &= -\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})|\tilde{\Pi}|^2-2M_0^{-T}S_2L\tilde{w}(0,t)\\
&\quad-c_1\|\tilde{w}_x\|^2+c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D\tilde{w}(x,t)\nu_{xx}(x,t)\,\mathrm{d}x\\
&\quad-c_1\|\tilde{w}_{xx}\|^2+c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D\tilde{w}_x(x,t)\nu_{xx}(x,t)\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq -\frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{2}|\tilde{\Pi}|^2 \\
&\quad + \left(\frac{8D|M_0^{-T}S_2L|^2}{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})} - c_1\right)\|\tilde{w}_x\|^2 \\
&\quad + c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}(x,t)\nu_{xx}(x,t)\,\mathrm{d}x \\
&\quad - c_1\|\tilde{w}_{xx}\|^2 + c_1\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}_x(x,t)\nu_{xx}(x,t)\,\mathrm{d}x
\end{aligned} \tag{9.256}
$$

Integrating $\|\tilde{w}\|^2$ and $\|\tilde{w}_x\|^2$ by parts, using Young's inequality and taking (9.214)-(9.215) into account, one can show that

$$
\|\tilde{w}\|^2 \leq 4D^2\|\tilde{w}_x\|^2, \qquad \|\tilde{w}_x\|^2 \leq 4D^2\|\tilde{w}_{xx}\|^2. \tag{9.257}
$$

Using Cauchy-Schwartz, Young and (9.257), the following bounds are derived

$$
\begin{aligned}
\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}\nu_{xx}\,\mathrm{d}x &\leq \frac{1}{4}\|\tilde{w}_x\|^2 + 4D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\|\nu_{xx}\|^2 \\
\frac{\Delta\epsilon}{\epsilon_0}\int_0^D \tilde{w}_x\nu_{xx}\,\mathrm{d}x &\leq \frac{1}{2}\|\tilde{w}_{xx}\|^2 + 2D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2\|\nu_{xx}\|^2,
\end{aligned}
$$

which plugged into (9.256) and after choosing

$$
c_1 = \frac{16D|M_0^{-T}S_2L|^2}{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})},
$$

yield

$$
\begin{aligned}
\dot{V}_3(t) &\leq -\frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{2}|\tilde{\Pi}|^2 \\
&\quad - \frac{c_1}{4}\|\tilde{w}_x\|^2 - \frac{c_1}{2}\|\tilde{w}_{xx}\|^2 \\
&\quad + 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\|\nu_{xx}\|^2
\end{aligned} \tag{9.258}
$$

Gathering (9.249), (9.255) and (9.258), and selecting

$$
a_1 = \min\left\{\sqrt{\frac{a_2\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{24D|K|^2}}, \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})D}{6|K|^2}\right\}
$$

$$a_0 = \frac{\underline{\lambda}(M_0^{-T} Q_2 M_0^{-1}) \underline{\lambda}(Q_1)}{72|S_1 B|^2 |K|^2},$$
$$b_0 = \frac{\underline{\lambda}(M_0^{-T} Q_2 M_0^{-1}) \underline{\lambda}(Q_1)}{24|S_1 B K|^2},$$

leads to

$$\begin{aligned}
\dot{V}(t) = & -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 - \left(\frac{b_0 \underline{\lambda}(Q_1)}{4} - D b_2 \kappa_3\right)|\Pi|^2 \\
& - \left(\frac{\underline{\lambda}(M_0^{-T} Q_2 M_0^{-1})}{4} - D b_2 \kappa_4 - D a_2 \kappa_1\right)|\tilde{\Pi}|^2, \\
& - \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2 \\
& - \frac{b_1}{4D^2}\|v\|^2 - \left(\frac{b_2}{2} - 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\right)\|v_{xx}\|^2 \\
& - \left(\frac{c_1}{4} - D^2 a_2 \kappa_2\right)\|\tilde{w}_x\|^2 - \frac{c_1}{2}\|\tilde{w}_{xx}\|^2
\end{aligned} \tag{9.259}$$

Now, choosing

$$a_2 = \frac{1}{8D}\min\left\{\frac{c_1}{D\kappa_2}, \frac{\underline{\lambda}(M_0^{-T} Q_2 M_0^{-1})}{2\kappa_1}\right\},$$
$$b_2 = \frac{1}{8D}\min\left\{\frac{b_0 \underline{\lambda}(Q_1)}{\kappa_3}, \frac{\underline{\lambda}(M_0^{-T} Q_2 M_0^{-1})}{2\kappa_4}\right\},$$

into (9.259) yields

$$\begin{aligned}
\dot{V}(t) = & -\frac{a_0 \underline{\lambda}(Q_1)}{2}|X|^2 - \frac{b_0 \underline{\lambda}(Q_1)}{8}|\Pi|^2 \\
& - \frac{\underline{\lambda}(M_0^{-T} Q_2 M_0^{-1})}{8}|\tilde{\Pi}|^2 \\
& - \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2 \\
& - \frac{b_1}{4D^2}\|v\|^2 - \left(\frac{b_2}{2} - 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\right)\|v_{xx}\|^2 \\
& - \frac{c_1}{8}\|\tilde{w}_x\|^2 - \frac{c_1}{2}\|\tilde{w}_{xx}\|^2.
\end{aligned} \tag{9.260}$$

Integrating $\|\nu_x\|^2$ by parts, using Young's inequality and taking (9.214) into account, one gets $\|\nu_x\|^2 \leq 4D^2\|\nu_{xx}\|^2$, which can be used, along with (9.257), to further bound (9.260) as

$$\begin{aligned}\dot{V}(t) = &-\frac{a_0\underline{\lambda}(Q_1)}{2}|X|^2 - \frac{b_0\underline{\lambda}(Q_1)}{8}|\Pi|^2 \\ &- \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{8}|\tilde{\Pi}|^2 \\ &- \frac{a_1}{8D^2}\|w\|^2 - \frac{a_2}{8D^2}\|w_x\|^2 \\ &- \frac{b_1}{4D^2}\|\nu\|^2 - \left(\frac{b_2}{2} - 6D^2\left(\frac{\Delta\epsilon}{\epsilon_0}\right)^2 c_1\right)\frac{1}{4D^2}\|\nu_x\|^2 \\ &- \frac{c_1}{32D^2}\|\tilde{w}\|^2 - \frac{c_1}{8D^2}\|\tilde{w}_x\|^2. \end{aligned} \tag{9.261}$$

Assuming $|\Delta\epsilon| \leq \delta$ and selecting

$$\delta < \frac{\epsilon_0}{2D}\sqrt{\frac{b_2}{3c_1}},$$

it follows from (9.237) and (9.3.2) that

$$\dot{V}(t) \leq \mu V(t)$$

where

$$\begin{aligned}\mu = \min\Bigg\{&\frac{\underline{\lambda}(Q_1)}{8\overline{\lambda}(S_1)}, \frac{\underline{\lambda}(M_0^{-T}Q_2M_0^{-1})}{8\overline{\lambda}(M_0^{-T}S_2M_0^{-1})}, \\ &\left(\frac{1}{2} - \frac{6D^2(\Delta\epsilon/\epsilon_0)^2c_1}{b_2}\right)\frac{\epsilon}{2D^2}, \frac{\epsilon_0}{16D^2}\Bigg\}.\end{aligned}$$

Now, from (9.237), one can find that

$$\psi_1\Xi(t) \leq V(t) \leq \psi_2\Xi(t), \tag{9.262}$$

where

$$\Xi(t) = |X|^2 + \|w\|_{H_1}^2 + |\tilde{\Pi}|^2 + \|\tilde{w}\|_{H_1}^2 + |\Pi|^2 + \|\nu\|_{H_1}^2,$$

and

$$\begin{aligned}\psi_1 = \min \Big\{ & a_0 \underline{\lambda}(S_1), b_0 \underline{\lambda}(S_1), \underline{\lambda}(M_0^{-T} S_2 M_0^{-1}), \\ & \frac{1}{2\epsilon} \min\{a_1, a_2, b_1, b_2\}, \frac{c_1}{2\epsilon_0} \Big\}, \\ \psi_2 = \max \Big\{ & a_0 \overline{\lambda}(S_1), b_0 \overline{\lambda}(S_1), \overline{\lambda}(M_0^{-T} S_2 M_0^{-1}), \\ & \frac{1}{2\epsilon} \max\{a_1, a_2, b_1, b_2\}, \frac{c_1}{2\epsilon_0} \Big\}.\end{aligned}$$

Hence, the following exponential stability estimate is obtained for the transformed system

$$\Xi(t) \leq \frac{\psi_2}{\psi_1} \Xi(0) e^{-\mu t}, \; \forall t \geq 0. \tag{9.263}$$

Now, an estimate is derived in terms of

$$\Upsilon(t) = |X|^2 + \|u\|_{H_1}^2 + |\hat{\Pi}|^2 + \|\hat{v}\|_{H_1}^2.$$

Using (9.220)-(9.224) and (9.225)-(9.227), one can show there exist constants α_i, β_i such that

$$\begin{aligned}\Xi(t) &\leq \alpha_1 |X|^2 + \alpha_2 \|u\|_{H_1}^2 + \alpha_3 |\hat{\Pi}|^2 + \alpha_4 \|\hat{v}\|_{H_1}^2, \\ \Upsilon(t) &\leq \beta_1 |X|^2 + \beta_2 \|w\|_{H_1}^2 + \beta_3 \|\tilde{\Pi}\|^2 + \beta_4 \|\tilde{w}\|_{H_1}^2 \\ &\quad + \beta_5 |\Pi|^2 + \beta_6 \|v\|_{H_1}^2,\end{aligned}$$

for all $t \geq 0$, from which it follows that

$$\phi_1 \Upsilon(t) \leq \Xi(t) \leq \phi_2 \Upsilon(t), \tag{9.264}$$

being $\phi_1 = 1/\max \beta_i$, $\phi_2 = \max \alpha_i$. Therefore, from (9.263)-(9.264), one gets the exponential stability estimate

$$\Upsilon(t) \leq \frac{\psi_2}{\psi_1} \frac{\phi_2}{\phi_1} \Upsilon(0) e^{-\mu t}, \; \forall t \geq 0,$$

completing the proof. □

9.4 Simulations

The proposed control strategies are illustrated in this section using a second-order system defined by

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

which has an exponentially unstable mode (the poles of the system are located at $s = \pm 1$). The feedback gain matrices are chosen as $K = L^T = [-2, -2]$, which guarantee $A + BK$ and $A - LC$ Hurwitz, being all their poles located at $s = -1$.

9.4.1 *Delay case*

First, we consider the case modeled by (9.55)-(9.58), in which the input is affected by a delay $D = 1$. The control law (9.93)-(9.96) is implemented using an upwind scheme (first order accurate both in time and space) for the PDE discretization. Simulation results are shown in Fig. 1 for the nominal case, that is, with $D_0 = 1$. Note that the system (solid blue) runs in an open-loop fashion until the control action reaches the system at $t = D$. The observer estimates $\hat{P}_1$ and $\hat{P}_2$, which are actually D units of time ahead predictions, are shown delayed by D units of time (dashed red), to match the actual state (blue). One can also see that the value of $\hat{v}$ at the spatial location $x = 0$ contains an actual estimation of the output (dashed black), as expected. The bottom plot shows the control law (blue) and the actual signal that reaches the ODE (black), which is simply delayed by D units of time.

Robustness is also illustrated in Fig. 2, where a $+5\%$ additive disturbance in the time delay is considered, that is, $D = 1.05$. One can see that the asymptotic stability is preserved in spite of the uncertainty.

9.4.2 *Diffusion case*

Now, we consider the case modeled by (9.173)-(9.177), in which the input undergoes a diffusive process through a domain of length $D = 1$ with a diffusive coefficient $\epsilon = 1$. The control law (9.232)-(9.236) is implemented using a first-order-in-time and second-order-in-space discretization for the PDE. Simulation results are shown in Fig. 3 for the nominal case, that is $\epsilon_0 = \epsilon = 1$. The system states are depicted at the top and central plots (blue). Recall that $\hat{\Pi}_1$ and $\hat{\Pi}_2$, are actually the "anti-diffused" state estimates, as discussed in Remark 1. Then, we plot the observer estimates after undergoing a diffusion process through a domain of length $D = 1$ and with $\epsilon = 1$ (dashed red), to see that they match

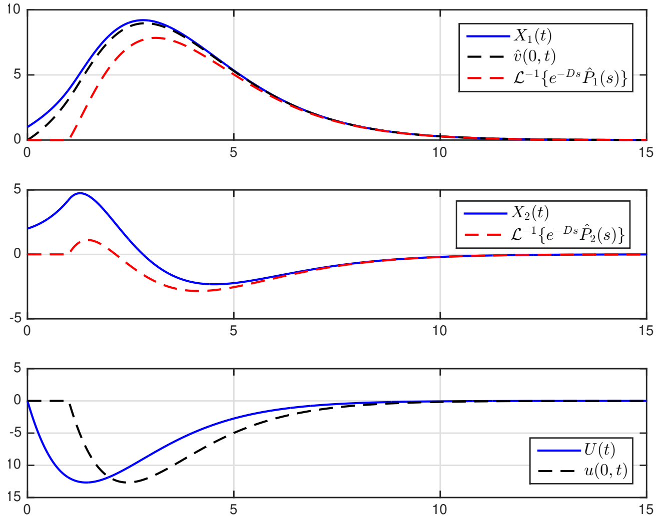


Figure 9.1: Nominal simulation of a plant with delay actuator dynamics, where $X(0) = [1,\ 2]^T$, $\hat{P}(0) = [0,\ 0]^T$ and $\hat{v}(x,0) = 0$, $\forall x \in [0,1)$

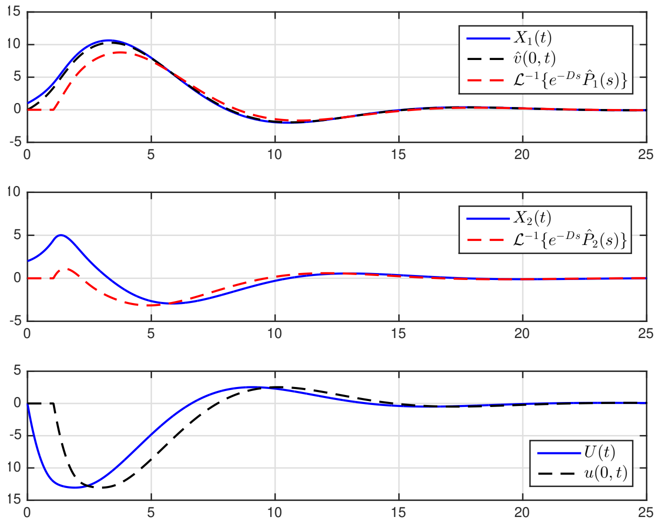


Figure 9.2: Robust simulation of a plant with delay actuator dynamics, where $X(0) = [1,\ 2]^T$, $\hat{P}(0) = [0,\ 0]^T$ and $\hat{v}(x,0) = 0$, $\forall x \in [0,1)$

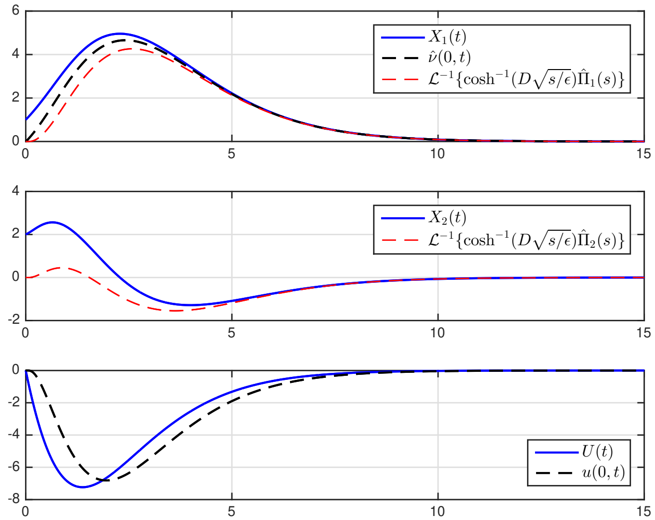


Figure 9.3: Nominal simulation of a plant with diffusive actuator dynamics, where $X(0) = [1,\, 2]^T$, $\hat{\Pi}(0) = [0,\, 0]^T$ and $\hat{v}(x, 0) = 0$, $\forall x \in [0, 1)$

the actual state (blue). One can also see that the value of $\hat{v}$ at the spatial location $x = 0$ contains an actual estimation of the output (dashed black), as expected. The bottom plot shows the control law (blue) and the actual signal that reaches the ODE (black).

Robustness is also illustrated for this case, performing one more simulation in which $\epsilon = 2$ while we keep $\epsilon_0 = 1$. The results are shown in Fig. 4, where it can be seen that small oscillations appear but stability is preserved.

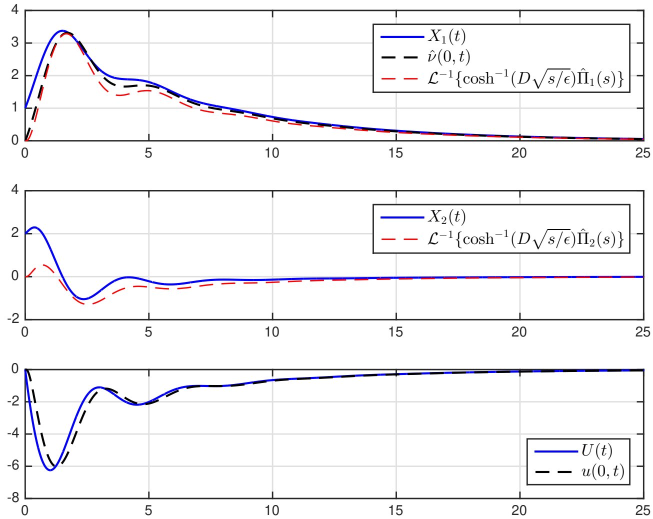


Figure 9.4: Robust simulation of a plant with diffusive actuator dynamics, where $X(0) = [1,\, 2]^T$, $\hat{\Pi}(0) = [0,\, 0]^T$ and $\hat{\nu}(x, 0) = 0$, $\forall x \in [0, 1)$

Chapter 10

LTV systems with time-varying measurement delay

This chapter deals with output-feedback stabilization of linear time-varying (LTV) systems with time-varying measurement delay. An important result is the extension of the PDE-ODE observer reported in the previous chapter to LTV systems with time-varying delays. The exponential stability of the estimation error is established for arbitrarily large time-varying delays. Then, an observer-based controller is also introduced and the closed-loop stability is proved. The separation principle holds and the design of both the observer and controller gains can be done as if there was no delay. Some simulations illustrate the feasibility of this approach.

10.1 Introduction

The state estimation problem in the presence of delayed measurements is ubiquitous in engineering applications and it has attracted attention over the past decades (Watanabe et al. 1981b; Klamka 1982; Souza et al. 2001; Germani et al. 2002; Fridman et al. 2003; Krstic et al. 2008). The problem of stabilizing an LTV system is also of interest in practice, as it arises in trajectory tracking of nonlinear systems (Rugh et al. 2000). See (Zhou 2016) for recent results on stability of LTV systems. Only a few works have been reported in which LTV systems with measurement delay are considered (Pila et al. 1999; Basin et al. 2010; Song et al. 2017)

As shown in (Krstic et al. 2008), time-delay systems can be modeled as the interconnection of an ODE (plant dynamics) and a first-order hyperbolic PDE (delay dynamics). In this framework, the approach consists of finding an invertible backstepping transformation that maps the ODE-PDE cascade into an exponentially stable target system (Smyshlyaev et al. 2004; Smyshlyaev et al. 2005). This methodology has been shown to be very useful when dealing with time-delay systems, providing rigorous tools for analysis and design, which have been exploited in many ways (Krstic 2009c; Krstic 2010b; Bresch-Pietri et al. 2014; Ahmed-Ali et al. 2018; Sanz et al. 2018c). A tutorial on compensating infinite-dimensional actuator and sensor dynamics is given in (Krstic 2010a). When applied to input-delay systems, the backstepping approach has led to predictor-based controllers, equivalent to those originally derived in (Manitius et al. 1979; Kwon et al. 1980; Artstein 1982). However, when applied to systems with measurement delays, the resulting observer is substantially different (see Remarks 4-7 in (Krstic et al. 2008) for a discussion) from those originally proposed in (Watanabe et al. 1981b; Klamka 1982), which involved distributed integral terms.

In the context of systems with measurement delays, a common approach consists of using a conventional observer structure and then estimating upper bounds on the admissible delay, for which powerful techniques are available (Souza et al. 2001; Fridman et al. 2003; Zhang et al. 2008); see also the monograph (Fridman 2014). A different strategy consists of proposing new observers that specifically handle the delay. Early results were based on estimating a delayed state and then projecting it ahead via a state predictor (Watanabe et al. 1981b; Klamka 1982). This approach can be also applied when the delay is time-varying, see Section 6.3 in (Krstic 2009c). In (Pila et al. 1999), an H_∞ filter is derived for LTV systems with constant measurement delay. However, these approaches need the online computation of a distributed integral over the past control actions, whose approximation may lead to instability (Van Assche et al. 1999; Zhong 2004). A new approach for state observation with long constant delays was introduced in (Germani et al. 2002), where a chain of observers was employed. The idea is that each of the observers in the chain is in charge of predicting the system state over a fraction of the total delay. This novel idea has been also extended to time-varying delays (Cacace et al. 2010; Cacace et al. 2013; Cacace et al. 2014), achieving prescribed exponential stability, if the delay size is below a suitable bound. A different technique was used in (Zhou et al. 2013), based on a truncated predictor that neglects the integral term. Stabilization of systems with input/output delays whose eigenvalues are located on the closed left-half plane was achieved. A similar approach, although from a different perspective, is proposed in (Cacace et al. 2015). As mentioned before, a novel observer with an ODE-PDE cascade structure was introduced in (Krstic et al. 2008), which handled arbitrarily large constant delays. Among the aforementioned works, only (Pila et al. 1999) considers LTV systems.

In this chapter, a generalization of the observer derived in (Krstic et al. 2008) is proposed, with a twofold contribution. One one hand, time-varying delays are considered and, on the other hand, the observer is also adapted to deal with LTV systems. The open-loop plant is allowed to contain exponentially unstable modes and arbitrarily large time-varying delays can be handled. To the best of the author's knowledge, this is a departure from previous works, even for LTI systems. Furthermore, the proposed observer is used to derive an exponentially stabilizing output-feedback controller for LTI/LTV systems with time-varying measurement delay.

10.2 Time-varying delays in the PDE framework

Time-varying delays can be also modeled by means of PDEs. Let $\phi(t)$ be a continuously differentiable function that incorporates a measurement delay. One can express the function $\phi(t)$ in a more standard form $\phi(t) = t - D(t)$, where $D(t)$ is a time-varying delay. Consider now the system

$$\dot{X}(t) = AX(t) + BU(\phi(t)), \tag{10.1}$$

which we would like to express in the PDE framework. Let us introduce the following choice for the PDE state

$$u(x,t) = U(\phi(t + x(\phi^{-1}(t) - t))). \tag{10.2}$$

It is readily verified that $u(0,t) = U(\phi(t))$ and $u(1,t) = U(t)$. Therefore, the system (10.1) can be represented as

$$\dot{X}(t) = AX(t) + BU(\phi(t)), \tag{10.3}$$
$$u_t(x,t) = \pi(x,t)u_x(x,t), \tag{10.4}$$
$$u(1,t) = U(t). \tag{10.5}$$

where $\pi(x,t)$ is the the speed of propagation, given by

$$\pi(x,t) = \frac{1 + x\left(\frac{\mathrm{d}(\phi^{-1}(t))}{\mathrm{d}t} - 1\right)}{\phi^{-1}(t) - t}.$$

Note that the latter follows from the quotient between the temporal and spatial derivatives of (10.2). The following assumptions are needed to obtain a meaningful stability analysis.

Assumption 10.1. *The delay function ϕ satisfies*

$$t - \phi(t) > 0, \quad \forall t \geq 0.$$

Note that Assumption 10.1 can be alternatively stated as $\phi^{-1}(t) - t > 0$. In both cases, it implies that the time-varying delay is strictly greater than zero. This is needed because, if $D(t) = 0$ for any t, then the propagation speed is infinite (see the denominator of $\pi(x,t)$ above) and the transport PDE representation does not make sense.

Assumption 10.2. *The delay function ϕ is continuously differentiable and satisfies*

$$\dot{\phi}(t) > 0, \quad \forall t \geq 0.$$

This assumption implies that $\phi(t)$ is strictly-increasing, which guarantees the existence of its inverse.

10.2.1 Predictor-based controller for time-varying input delay

The time-varying predictor controller takes the form

$$U(t) = K\left(e^{A(\phi^{-1}(t)-t)}X(t) + \int_t^{\phi^{-1}(t)} e^{A(\phi^{-1}(t)-s)}BU(\phi(s))\,\mathrm{d}s\right) \tag{10.6}$$

It was proved in (Krstic 2010b) that this controller guarantees the exponential stability of the resulting closed-loop in the sense of the norm $|X(t)|^2 + \|u(t)\|^2$. Although the proof is not presented here, it is pointed out that many technical aspects are borrowed from it in the proof of the proposed observers introduced further below.

10.2.2 *Predictor-based observer for time-varying measurement delay*

As discussed in Section 8.1.4, one of the approaches to deal with measurement delays is to design an observer for the delayed state and then project it ahead in an open-loop manner. The following observer-predictor, introduced in (Krstic 2010b),

$$\dot{\Xi}(t) = \dot{\phi}(t)[A\Xi(t) + BU(\phi(t)) + L(Y(t) - C\Xi(t)], \tag{10.7}$$

$$\hat{X}(t) = e^{A(t-\phi(t))}\Xi(t) + \int_{\phi(t)}^{t} e^{A(t-s)}BU(s)\,\mathrm{d}s, \tag{10.8}$$

is the counterpart of (8.31)-(8.32) for time-varying delays

10.2.3 *A new PDE observer for time-varying measurement delay*

The observer introduced next is one of the contributions of this thesis. Inspired by (10.7)-(10.8), the following observer is proposed

$$\dot{\hat{X}}(t) = A\hat{X}(t) + BU(t) + \dot{\phi}(t)e^{A(t-\phi(t))}L\big(Y(t) - \hat{u}(0,t)\big), \tag{10.9}$$

$$\hat{u}_t(x,t) = \pi(x,t)\hat{u}_x(x,t) + C\dot{\phi}(t)e^{Ax(t-\phi(t))}L\big(Y(t) - \hat{u}(0,t)\big), \tag{10.10}$$

$$\hat{u}(1,t) = C\hat{X}(t), \tag{10.11}$$

which is the extension of (8.28)-(8.30) for the case of time-varying delays. The proof can be found below in Corollary 10.1, although the observer is presented here for completeness. The novel observer (10.9)-(10.11) has the advantage of avoiding distributed integral terms over the past control actions, in contrast to (10.7)-(10.8).

10.3 Linear time-varying systems

Consider the following LTV system with measurement delay

$$\dot{X}(t) = A(t)X(t) + B(t)U(t), \tag{10.12}$$
$$Y(t) = C(\phi(t))X(\phi(t)), \tag{10.13}$$
$$X(\theta) = X_0(\theta), \quad \forall\theta \in [\phi(t_0), t_0], \tag{10.14}$$

where $X(t) \in \mathbb{R}^n$ is the state, $U(t) \in \mathbb{R}^m$ is the control action, the system matrices $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ and $C : \mathbb{R} \to \mathbb{R}^{q\times n}$ are piecewise-continuous uniformly-bounded functions, $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a continuously differentiable function that incorporates the measurement delay, $X_0 : [\phi(t_0), t_0] \to \mathbb{R}^n$ denotes the initial condition and $t_0 \geq 0$ is the initial time.

Assumption 10.3. *The delay function ϕ is known, continuously differentiable and there exist $\underline{D}, \overline{D}, c_1, c_2 > 0$ such that*

$$\underline{D} < t - \phi(t) \leq \overline{D},$$

and

$$c_1 < \dot{\phi}(t) \leq c_2,$$

hold for all $t \geq 0$.

Assumption 10.4. *There exists a time-varying matrix function $L : \mathbb{R} \to \mathbb{R}^{n\times q}$ such that the origin of $\dot{\mathcal{X}}(t) = \big(A(t) - L(t)C(t)\big)\mathcal{X}(t)$ is uniformly exponentially stable for all $t \geq \phi(t_0)$. In other words, there exist positive definite, bounded, symmetric, time-varying matrices $W_1(t), Q_1(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $\dot{W}_1(t) + W_1(t)F_1(t) + F_1^T W_1(t) \leq -Q_1(t)$, $\forall t \geq \phi(t_0)$, where $F_1(t) = A(t) - L(t)C(t)$.*

The function $\phi(t)$ can be expressed as $\phi(t) = t - D(t)$, where $D : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is the measurement delay. Since both c_2 and $\overline{D}$ are allowed to be arbitrarily large, the main limitations imposed by Assumption 1 are $\dot{d}(t) < 1$ and $D(t) > \underline{D}$. The former is a common assumption when dealing with time-varying delays (Zhou et al. 2013), while the latter has to do with the PDE representation of the delay, in which the propagation speed becomes infinity if the delay is zero (Krstic 2010b). On the other hand, Assumption 2 is needed in the subsequenty analysis to guarantee the stabilization of the LTV system.

Recall that any solution of the homogeneous equation $\dot{X}(t) = A(t)X(t)$ can be written as $X(t) = \Phi(t, t_0)X(t_0)$, where

$$\Phi(t, t_0) = P(t)P^{-1}(t_0), \tag{10.15}$$

is the so-called state transition matrix, and $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ is the solution of the matrix initial value problem $\dot{P}(t) = A(t)P(t)$, $P(0) = I$. The methodology here developed is based on representing the time-varying measurement delay as a transport phenomenon, which is modeled by a first-order hyperbolic PDE. Similarly to (Krstic 2010b), let us introduce the following state for the transport equation

$$u(x,t) = C(\varphi(x,t))X(\varphi(x,t)), \quad \forall x \in [0,1] \tag{10.16}$$

where

$$\varphi(x,t) = \phi(t) + x(t - \phi(t)), \tag{10.17}$$

leading to the boundary values $u(0,t) = Y(t)$ and $u(1,t) = C(t)X(t)$. The system (10.12)-(10.13) can be then represented by the following ODE-PDE cascade

$$\dot{X}(t) = A(t)X(t) + B(t)U(t), \tag{10.18}$$

$$u_t(x,t) = \pi(x,t)u_x(x,t), \tag{10.19}$$

$$u(1,0) = C(t)X(t), \tag{10.20}$$

$$Y(t) = u(0,t), \tag{10.21}$$

with

$$\pi(x,t) = \frac{\dot{\phi}(t) + x(1 - \dot{\phi}(t))}{t - \phi(t)}. \tag{10.22}$$

Remark 10.1. Note that the PDE observer state is nothing but $u(x,t) = Y(\phi^{-1}(\varphi(x,t)))$, which can be seen from (10.13) and (10.16). Then, the PDE observer state contains a prediction of the output over the interval $[t, \phi^{-1}(t)]$. As an example, let us consider $\phi(t) = t - D$, which represents a constant delay $D > 0$, leading to $u(x,t) = Y(t + xD)$, $\forall x \in [0,1]$, that is, a prediction of $Y(t)$ over the time window $[0, D]$.

10.3.1 PDE observer for LTV systems with time-varying delay

The main result is stated in the following theorem. Note that, for the sake of clarity, in what follows $\varphi(x,t)$ and $\phi(t)$ are often denoted simply by φ and ϕ, respectively.

Theorem 10.1. *The observer*

$$\dot{\hat{X}}(t) = A(t)\hat{X}(t) + B(t)U(t) + \dot{\phi}(t)\Phi(t,\phi)L(\phi)\big(Y(t) - \hat{u}(0,t)\big), \tag{10.23}$$

$$\hat{u}_t(x,t) = \pi(x,t)\hat{u}_x(x,t) + \dot{\phi}(t)C(\varphi)\Phi(\varphi,\phi)L(\phi)\big(Y(t) - \hat{u}(0,t)\big), \tag{10.24}$$

$$\hat{u}(1,0) = C(t)\hat{X}(t). \tag{10.25}$$

is uniformly exponentially convergent, in the sense that, there exist $M, \mu > 0$*, such that* $\Upsilon(t) \leq M\Upsilon(t_0)e^{-\mu(t-t_0)}$*,* $\forall t \geq t_0 \geq 0$*, where*

$$\Upsilon(t) = |X(t) - \hat{X}(t)|^2 + \int_0^1 |u(x,t) - \hat{u}(x,t)|^2 \, dx.$$

Proof. Let us define the errors

$$\tilde{X}(t) = X(t) - \hat{X}(t) \tag{10.26}$$

$$\tilde{u}(x,t) = u(x,t) - \hat{u}(x,t). \tag{10.27}$$

Differentiating (10.26)-(10.27) and using (10.18)-(10.21) and (10.23)-(10.25), the error system is obtained as

$$\dot{\tilde{X}}(t) = A(t)\tilde{X}(t) - \dot{\phi}(t)\Phi(t,\phi)L(\phi)\tilde{u}(0,t), \tag{10.28}$$

$$\tilde{u}_t(x,t) = \pi(x,t)\tilde{u}_x(x,t) - \dot{\phi}(t)C(\varphi)\Phi(\varphi,\phi)L(\phi)\tilde{u}(0,t), \tag{10.29}$$

$$\tilde{u}(1,t) = C(t)\tilde{X}(t). \tag{10.30}$$

To proceed, let us recall the following properties of the state transition matrix

$$\Phi(t,t) = I, \tag{10.31}$$

$$\Phi_t(t,\tau) = A(t)\Phi(t,\tau), \tag{10.32}$$

$$\Phi_\tau(t,\tau) = -\Phi(t,\tau)A(\tau), \tag{10.33}$$

$$\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0). \tag{10.34}$$

Using (10.32)-(10.33), the partial derivatives of (10.15) can be written as

$$\Phi_x(\varphi,t) = \varphi_x(x,t)A(\varphi)\Phi(\varphi,t), \tag{10.35}$$

$$\Phi_t(\varphi,t) = \varphi_t(x,t)A(\varphi)\Phi(\varphi,t) - \Phi(\varphi,t)A(t), \tag{10.36}$$

where the chain differentiation rule was employed. Furthermore, evaluating (10.36) at $x = 0$ and using that $\varphi(0, t) = \phi(t)$ yields

$$\Phi_t(\phi, t) = \dot{\phi}(t) A(\phi) \Phi(\phi, t) - \Phi(\phi, t) A(t). \tag{10.37}$$

Let us consider the transformation

$$\tilde{Z}(t) = \Phi(\phi(t), t) \tilde{X}(t), \tag{10.38}$$
$$\tilde{w}(x, t) = \tilde{u}(x, t) - C(\varphi(x, t)) \Phi(\varphi(x, t), t) \tilde{X}(t), \tag{10.39}$$

which, as it will be shown next, maps (10.28)-(10.30) into the exponentially stable target system

$$\dot{\tilde{Z}}(t) = \dot{\phi}(t) F_1(\phi) \tilde{Z}(t) - \dot{\phi}(t) L(\phi) C(\phi) \tilde{w}(0, t), \tag{10.40}$$
$$\tilde{w}_t(x, t) = \pi(x, t) \tilde{w}_x(x, t), \tag{10.41}$$
$$\tilde{w}(1, t) = 0, \tag{10.42}$$

where $F_1(t)$ was defined in Assumption 2. The time derivative of (10.38) is given by

$$\begin{aligned} \dot{\tilde{Z}}(t) &= \Phi_t(\phi, t) \tilde{X}(t) + \Phi(\phi, t) \dot{\tilde{X}}(t) \\ &= \dot{\phi}(t) A(\phi) \tilde{Z}(t) - \dot{\phi}(t) L(\phi) \tilde{u}(0, t) \end{aligned} \tag{10.43}$$

where (10.28), (10.31), (10.34) and (10.37) were employed in the second row. Evaluating (10.39) at $x = 0$ and using (10.38) yields

$$\tilde{u}(0, t) = \tilde{w}(0, t) + C(\phi) \tilde{Z}(t),$$

which can be plugged into (10.43), leading to (10.40). On the other hand, computing the spatial and temporal derivatives of (10.39) yields

$$\begin{aligned} \tilde{w}_t(x, t) = {} & \pi(x, t) \tilde{u}_x(x, t) + \dot{\phi}(t) \Phi(\varphi, \phi) L(\phi) \tilde{u}(0, t) \\ & - \varphi_t C'(\varphi) \Phi(\varphi, t) \tilde{X}(t) - C(\varphi) \Phi_t(\varphi, t) \tilde{X}(t) \\ & - C(\varphi) \Phi(\varphi, t) [A(t) \tilde{X}(t) \\ & - \dot{\phi}(t) \Phi(t, \phi) L(\phi) \tilde{u}(0, t)], \\ \tilde{w}_x(x, t) = {} & \tilde{u}_x(x, t) - \varphi_x C'(\phi) \Phi(\varphi, t) \tilde{X}(t) \\ & - C(\phi) \Phi(\varphi, t) \tilde{X}(t), \end{aligned}$$

and thus one can see that (10.41) holds if

$$\pi(x,t)\varphi_x(x,t) - \varphi_t(x,t) = 0,$$
$$\pi(x,t)\Phi_x(\varphi,t) - \Phi_t(\varphi,t) - \Phi(\varphi,t)A(t) = 0,$$
$$\Phi(\varphi,\phi) - \Phi(\varphi,t)\Phi(t,\phi) = 0.$$

The first identity is readily verified from (10.17) and (10.22). Plugging (10.35)-(10.36) into the left side of the second identity yields

$$\big(\pi(x,t)\varphi_x(x,t) - \varphi_t(x,t)\big)A(\varphi)\Phi(\varphi,t),$$

which is zero provided that so is the term in brackets. The last identity follows from the property (10.34). Finally, evaluating (10.39) at $x = 1$ and using (10.30), one can see that (10.42) holds if $\Phi(\varphi(1,t),t) = I$, which follows from the fact that $\varphi(1,t) = t$ and the property (10.31).

Then, the (invertible) transformation (10.38)-(10.39) maps the error system (10.28)-(10.30) into the target system (10.40)-(10.42), which is next proved to be exponentially stable. Let us consider the functional

$$V_1(t) = \int_0^1 e^{bx}|\tilde{w}(x,t)|^2\,\mathrm{d}x. \tag{10.44}$$

The time-derivative of (10.44) along the trajectories of (10.29)-(10.30) is given by

$$\begin{aligned}\dot{V}_1(t) &= \int_0^1 2e^{bx}\pi(x,t)\tilde{w}(x,t)^T\tilde{w}_x(x,t)\,\mathrm{d}x\\ &= -\pi(0,t)|\tilde{w}(0,t)|^2 - \int_0^1 \sigma(x,t)e^{bx}|\tilde{w}(x,t)|^2\,\mathrm{d}x,\end{aligned} \tag{10.45}$$

where integration by parts was applied and $\sigma(x,t) = b\pi(x,t) + \pi_x(x,t)$. Observe that

$$\pi(0,t) = \frac{\dot{\phi}(t)}{t-\phi(t)}, \tag{10.46}$$

and

$$\sigma(x,t) = \frac{1 + (b-1)\dot{\phi}(t) - bx(\dot{\phi}(t)-1)}{t-\phi(t)}, \tag{10.47}$$

which follow from (10.22). From (10.46), one has that

$$\pi(0,t) \geq \frac{c_1}{\overline{D}}. \tag{10.48}$$

On the other hand, since (10.47) is a linear function of x, its minimum occurs either at $x = 0$ or $x = 1$, and thus

$$\sigma(x,t) \geq \min\{\frac{1+(b-1)\dot{\phi}(t)}{t-\phi(t)}, \frac{1-\dot{\phi}(t)+b}{t-\phi(t)}\}. \tag{10.49}$$

Choosing $b \geq \max\{1, c_2\}$ in (10.49), yields

$$\sigma(x,t) \geq \frac{\beta^*}{\overline{D}}, \tag{10.50}$$

with $\beta^* = \min\{1+(b-1)c_1, 1-c_2+b\} > 0$, and thus plugging (10.48) and (10.50) into (10.45) leads to

$$\dot{V}_1(t) \leq -\frac{c_1}{\overline{D}}|\tilde{w}(0,t)|^2 - \frac{\beta^*}{\overline{D}}\int_0^1 e^{bx}|\tilde{w}(x,t)|^2\,\mathrm{d}x. \tag{10.51}$$

Now, let us look at the Lyapunov function

$$V_2(t) = \tilde{Z}(t)^T W_1(\phi(t))\tilde{Z}(t), \tag{10.52}$$

whose derivative along the trajectories of (10.40) can be bounded, using Young's inequality, by

$$\begin{aligned}
\dot{V}_2(t) &= \dot{\phi}(t)\tilde{Z}^T\left(W_1(\phi)F_1(\phi) + F_1^T(\phi)W_1(\phi)\right. \\
&\quad \left. + \dot{W}_1(\phi)\right)\tilde{Z} + 2\dot{\phi}(t)\tilde{Z}^T W_1(\phi)L(\phi)C(\phi)\tilde{w}(0,t) \\
&\leq -c_1 q_1|\tilde{Z}|^2 + 2c_2\tau_1|\tilde{Z}||\tilde{w}(0,t)| \\
&\leq -\frac{c_1 q_1}{2}|\tilde{Z}|^2 + \frac{2c_2^2\tau_1^2}{c_1 q_1}|\tilde{w}(0,t)|^2, \quad \forall t \geq t_0,
\end{aligned} \tag{10.53}$$

where $\tau_1 = \|W_1(t)L(t)C(t)\|_\infty$, Assumptions 1-2 were used and $q_1 > 0$ is defined such that $Q(t) \geq q_1 I$, $\forall t \geq \phi(t_0)$. Let us choose now

$$V_0(t) = \frac{2\overline{D}c_2^2\tau_1^2}{c_1^2 q_1}V_1(t) + V_2(t), \tag{10.54}$$

whose derivative can be bounded using (10.51) and (10.53) by

$$\dot{V}_0(t) \leq -\frac{c_1 q_1}{2}|\tilde{Z}|^2 - \frac{2\beta^* c_2^2\tau_1^2}{c_1^2 q_1}\int_0^1 e^{bx}|\tilde{w}(x,t)|^2\,\mathrm{d}x, \tag{10.55}$$

for all $t \geq t_0$. From (10.54)-(10.55), one has that $\dot{V}_0(t) \leq -\mu V_0(t)$, $\forall t \geq t_0$ and thus, by the comparison principle,

$$V_0(t) \leq e^{-\mu(t-t_0)} V_0(t_0), \tag{10.56}$$

where $\mu = \min\left\{\frac{c_1 q_1}{2w_2}, \frac{\beta^*}{\overline{D}}\right\}$, and $w_1, w_2 > 0$ are defined such that $w_1 I \leq W_1(t) \leq w_2 I$. Let us denote $\Xi(t) = |\tilde{Z}(t)|^2 + \int_0^1 \tilde{w}^2(x,t)\,dx$. The following relation holds

$$\psi_1 \Xi(t) \leq V_0(t) \leq \psi_2 \Xi(t), \tag{10.57}$$

where

$$\psi_1 = \min\{w_1, \frac{2\overline{D}c_2^2\tau_1^2}{c_1^2 q_1}\},$$
$$\psi_2 = \max\{w_2, \frac{2\overline{D}c_2^2\tau_1^2 e^b}{c_1^2 q_1}\}.$$

Therefore, using (10.56) and (10.57) yields

$$\Xi(t) \leq \frac{\psi_2}{\psi_1} e^{-\mu(t-t_0)} \Xi(t_0), \quad \forall t \geq t_0. \tag{10.58}$$

Let us denote $\Upsilon(t) = |X(t)|^2 + \int_0^1 \tilde{u}(x,t)\,dx$ and write the inverse of (10.38)-(10.39) as

$$\tilde{X}(t) = \Phi(t, \phi(t))\tilde{Z}(t), \tag{10.59}$$
$$\tilde{u}(x,t) = \tilde{w}(x,t) + C(\varphi(x,t))\Phi(\varphi(x,t), \phi(t))\tilde{Z}(t). \tag{10.60}$$

From (10.38)-(10.39) and (10.59)-(10.60), one can show that there exist $\rho_1, \rho_2 > 0$ such that

$$\rho_1 \Upsilon(t) \leq \Xi(t) \leq \rho_2 \Upsilon(t), \tag{10.61}$$

and thus the exponential stability estimate

$$\Upsilon(t) \leq \frac{\rho_2}{\rho_1}\frac{\psi_2}{\psi_1} e^{-\mu(t-t_0)} \Upsilon(t_0), \quad \forall t \geq t_0,$$

follows from (10.58) and (10.61), which completes the proof.

□

Remark 10.2. Following Remark 3, one can define $\hat{Y}(\phi^{-1}(\varphi(x,t)) = \hat{u}(x,t)$, $\tilde{Y}(t) = Y(t) - \hat{Y}(t)$ and use the change of variable $s = \phi^{-1}(\varphi(x,t))$, to rewrite

$\Upsilon(t)$ as

$$\Upsilon(t) = |\tilde{X}(t)|^2 + \frac{1}{D(t)} \int_t^{\phi^{-1}(t)} |\tilde{Y}(s)|^2 \dot{\phi}(s)\, ds.$$

The latter points out the fact that the proposed observer produces an exponentially convergent prediction of the output over the time window $[t, \phi^{-1}(t)]$, $\forall t \geq t_0$.

The particular case of LTI systems is highlighted in the following corollary. To the best of the author's knowledge, exponentially stable observers for LTI systems with arbitrarily large time-varying delays have not been yet proposed in the literature and thus this result is also novel.

Corollary 10.1. *For an LTI system, that is, with $A(t) = A$, $B(t) = B$ and $C(t) = C$, the observer (10.9)-(10.11) where L is chosen such that $A - LC$ is Hurwitz, guarantees that $\hat{X}$, $\hat{u}$ exponentially converge to X, u, in the sense that, there exist $M, \mu > 0$, such that $\Upsilon(t) \leq M\Upsilon(0)e^{-\mu t}$, $\forall t \geq 0$.*

Proof. The state transition matrix of an LTI system is given by $\Phi(t, t_0) = e^{A(t-t_0)}$. Then, one has that $\Phi(t, \phi) = e^{A(t-\phi(t))}$, $\Phi(\varphi, \phi) = e^{Ax(t-\phi(t))}$ and thus the corollary follows from (10.23)-(10.25). □ □

10.3.2 Observer-based controller

The following assumption is needed, which is the counterpart of Assumption 2, for control purposes.

Assumption 10.5. *There exists a time-varying matrix function $K : \mathbb{R} \to \mathbb{R}^{m\times n}$ such that the origin of $\dot{\mathcal{X}}(t) = \big(A(t) + B(t)K(t)\big)\mathcal{X}(t)$ is uniformly exponentially stable for all $t \geq t_0$. In other words, there exist positive definite, bounded, symmetric, time-varying matrices $W_2(t), Q_2(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $\dot{W}_2(t) + W_2(t)F_2(t) + F_2(t)^T W_2(t) \leq -Q_2(t), \forall t \geq t_0$, where $F_2(t) = A(t) + B(t)K(t)$.*

Theorem 10.2. *The closed-loop composed of the system (10.12)-(10.13), the observer (10.23)-(10.25) and the control law*

$$U(t) = K(t)\hat{X}(t), \tag{10.62}$$

is uniformly exponentially stable in the sense that, there exist $R, \rho > 0$ such that for all initial conditions $(X_0(\cdot), \hat{X}_0, \hat{u}_0(\cdot, t_0)) \in L_2(\phi(t_0), t_0) \times \mathbb{R}^n \times L_2(0, 1)$, the following *holds* $\Gamma(t) \leq R\Gamma(t_0)e^{-\rho(t-t_0)}$, *where*

$$\Gamma(t) = |X(t)|^2 + \frac{1}{D(t)} \int_{\phi(t)}^{t} |C(\theta)X(\theta)|^2 \, d\theta + |\hat{X}(t)|^2 + \int_0^1 \hat{u}(x,t)^2 \, dx.$$

Proof. Let us plug (10.62) into (10.12) and use (10.26) and (10.59) to obtain

$$\dot{X}(t) = F_2(t)X(t) - B(t)K(t)\Phi(t,\phi(t))\tilde{Z}(t). \tag{10.63}$$

Let us consider $L_1(t) = \gamma \int_0^1 e^{bx}|u(x,t)|^2 \, dx$, with $\gamma > 0$. Using similar arguments to those in the proof of Theorem 4, the derivative of $L_1(t)$ along the trajectories of (10.18)-(10.20) can be bounded by

$$\dot{L}_1(t) \leq \gamma e^b \underline{D} \|C\|_\infty^2 |X|^2 - \frac{\gamma\beta^*}{\overline{D}} \int_0^1 e^{bx}|u(x,t)|^2 \, dx \tag{10.64}$$

Now, let us consider also $L_2(t) = \kappa X(t)^T W_2(t) X(t)$ with $\kappa > 0$, whose derivative along (10.63) can be bounded by

$$\begin{aligned}\dot{L}_2(t) &\leq -\kappa q_2 |X|^2 + 2\kappa |X^T W_2 B K \Phi(t,\phi)\tilde{Z}| \\ &\leq -\frac{\kappa q_2}{2}|X|^2 + \frac{2\kappa\tau_2^2}{q_2}|\tilde{Z}|^2, \; \forall t \geq t_0,\end{aligned} \tag{10.65}$$

where $\tau_2 = \|W_2(t)B(t)K(t)\Phi(t,\phi(t))\|_\infty$ and Assumption (10.5) was used, which implies the existence of a $q_2 > 0$ such that $Q_2(t) \geq q_2 I$, $\forall t \geq t_0$. Defining $L_0(t) = L_1(t) + L_2(t)$, using (10.64)-(10.65) and choosing $\gamma = \kappa q_2/(4e^b \underline{D}\|C\|_\infty^2)$, yields

$$\begin{aligned}\dot{L}_0(t) \leq & -\frac{\kappa q_2}{4}|X|^2 + \frac{2\kappa\tau_2^2}{q_2}|\tilde{Z}|^2 \\ & - \frac{\gamma\beta^*}{\overline{D}} \int_0^1 e^{bx}|u(x,t)|^2 \, dx, \; \forall t \geq t_0.\end{aligned} \tag{10.66}$$

Now, let us choose the a composite Lyapunov function $\mathcal{V}(t) = L_0(t) + V_0(t)$, whose derivative can be bounded, using (10.55) and (10.66), and selecting $\kappa = c_1 q_1 q_2/(8\tau_2^2)$, by

$$\dot{\mathcal{V}}(t) \leq -\frac{\kappa q_2}{4}|X|^2 - \frac{c_1 q_1}{4}|\tilde{Z}|^2 - \frac{\gamma\beta^*}{\overline{D}} \int_0^1 e^{bx}|u(x,t)|^2 \, dx$$

$$-\frac{2\beta^* c_2^2 \tau_1^2}{c_1^2 q_1}\int_0^1 e^{bx}\tilde{w}^2(x,t)\,dx.$$

Then $\dot{\mathcal{V}}(t) \leq -\rho\mathcal{V}(t)$, $\forall t \geq t_0$ and an exponential stability estimate $\mathcal{V}(t) \leq e^{-\rho(t-t_0)}\mathcal{V}(t_0)$, $\forall t \geq t_0$, in terms of $(X, \tilde{Z}, u, \tilde{w})$, is established. Proceeding as in the proof of Theorem 2, this estimate can be related to the original variables $(X, \tilde{X}, u, \tilde{u})$ using the inverse transformation (10.59)-(10.60) and then to $(X, \hat{X}, u, \hat{u})$ using (10.26)-(10.27), leading to $\Gamma(t) \leq R\Gamma(t_0)e^{-\rho(t-t_0)}$, $\forall t \geq 0$, with $\Gamma(t) = |X(t)|^2 + \int_0^1 |u(x,t)|^2\,dx + |\hat{X}(t)|^2 + \int_0^1 |\hat{u}(x,t)|^2\,dx$. This is is omitted here for brevity. Finally, introducing (10.16) into $\Gamma(t)$ and performing the change of variable $\theta = \varphi(x,t)$ completes the proof. □ □

10.4 Simulations

Let us consider (10.12)-(10.13) with $A(t) = (2+t^2)/(1+t^2)$, $\phi(t) = t - D(t)$ and $D(t) = 1 - 0.5\sin t$. Note that $A : [0,\infty) \to [2,1)$, and thus the open-loop system is potentially unstable. The control of such a system is challenging, even for constant delay and constant coefficient (Middleton et al. 2007). Since $\dot{\phi}(t) = 1 + 0.5\cos t$, Assumption 1 is satisfied with $\underline{D} = 0.5$, $\overline{D} = 1.5$, $c_1 = 0.5$ and $c_2 = 1.5$. It is readily verified that $P(t) = e^{t+\mathrm{atan}\,t}$ is a fundamental matrix and thus $\Phi(t,t_0) = e^{t-t_0+\mathrm{atan}\,t-\mathrm{atan}\,t_0}$. Choosing $W_1(t) = W_2(t) = 1/2$, Assumptions 2 and 7 are fulfilled with any $L > \|A\|_\infty$ and any $K < \|A\|_\infty$, respectively, and thus the gains are selected as $L = -K = 2$. The initial time is set to $t_0 = 0$ and the initial condition is arbitrary chosen as $X_0(\theta) = \{0, \forall\theta \in [\phi(0), 0); X_0(0) = 1\}$. The control law (10.62) with the observer (10.23)-(10.25) is implemented, where

$$\pi(x,t) = \frac{1 + (1-x)0.5\cos t}{1 - 0.5\sin t}.$$

A first-order approximation both in time (Δt) and space (Δx) is used for the discretization. The so-called Courant-Friedrichs-Lewy (CFL) condition, $\Delta x > \Delta t \sup_{x,t} |\pi(x,t)|$, must hold in order to guarantee numerical stability. The time step is arbitrarily selected as $\Delta t = 0.005\ s$ and thus $\Delta x = 0.015$, provided that the bound $\sup_{x,t} |\pi(x,t)| \leq 3$ is easily obtained from $\pi(x,t)$ above.

The simulation results are shown in Fig. 1. One can see how the system runs in open-loop over a period of time, during which there is no measurable information about the system (see dotted line). The exact time can be computed by solving $\phi(t) = 0$ for $t \geq 0$, which yields $t \approx 1.5$. After that, one can see how the estimated state (dashed) converges to the actual one (full), which in turn, is

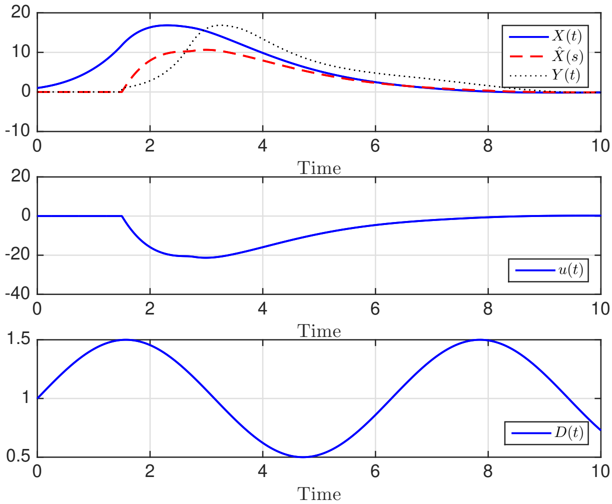


Figure 10.1: Simulation results. Actual state, estimated state and output (top); control action (center); time-varying delay (bottom)

driven exponentially to zero. The control action and the time-varying delay are also depicted in the central and bottom plots, respectively.

Conclusions

In this thesis, the problem of robustly stabilizing systems with input/output delays has been investigated. The contributions herein reported have been focused on two objectives: i.) improving disturbance rejection performance, and ii.) overcoming the implementation issues of conventional predictive controllers. The first goal has been achieved by combining disturbance observers with predictor-based controllers. The second goal has been pursued by exploiting the so-called sequential predictor approach, which has gained popularity over the past years. Provided that this strategy uses a predictor in observer form, it is well-suited to be combined with disturbance observers, leading to a solution that achieves the two objectives stated above. A summary of the different strategies reported in this thesis can be found in Table 10.1.

The performance improvement of a predictive controller equipped with a disturbance observer over other strategies has been proved in Chapter 5. This is an important result of this thesis, as it guarantees from a mathematical standpoint that the increment in complexity of the controller is paid back in terms of performance. In practice, not only the achievable performance has to be taken into account, but also the simplicity when tuning the controller. In Chapters 4, 5 and 7, it was shown both in simulations and experiments, how the reference and disturbance responses are decoupled. The tuning of the primary controller can be performed using conventional design techniques, while the observer is adjusted to reach a trade-off between disturbance rejection performance and robustness. This is a highly celebrated feature for control engineers. In this context, the generalization of the Smith Predictor introduced in Chapter 4 should be emphasized, which is well-suited for its applicability in industry.

The potential of LMI-based design was also illustrated in Chapter 6. This is a remarkable result in which asymptotic trajectory tracking and rejection of disturbances with known dynamics is achieved, while guaranteeing a prescribed level of attenuation of unmodeled disturbances. This is the result of applying existing LMI-based H_∞ design techniques. Furthermore, recall that this approach

Table 10.1: Summary of the different strategies reported in this thesis. System: Linear time-invariant (LTI), Linear Time-Varying (LTV), Nonlinear (NL); Disturbance: Matched (M), Mismatched (MM); Delay: Input (I), Output (O), Constant (C), Time-varying (TV); Delay compensation: Smith Predictor (SP), Finite Spectrum Assignment (FSA), Sequential predictors (SSPs), Sequential observers (SSOs), Partial differential equation (PDE); Control: State feedback (SF), Output feedback (OF); Disturbance compensation: General purpose input/output estimator (GPIO), Extended state observer (ESO), Uncertainty and disturbance estimator (UDE)

	System	Dist.	Delay	Delay comp.	Control	Dist. comp.
Ch. 4	LTI	M	I/C	SP-like	OF	Any
Ch. 5	LTI	M	I/C	FSA	SF	GPIO
Ch. 6	LTI	MM	I/C	SSPs	OF	ESO
Ch. 7	NL	M	I & O / C	FSA	SF	UDE
Ch. 9	LTI	-	I/C	SSPs PDE	OF	-
Ch. 10	LTV	-	O/TV	SSOs PDE	OF	-

makes use of sequential predictors, whose implementation is straightforward, thus achieving the two objectives aforementioned. This result may be less attractive for practitioners, however, due to the need of implementing complex LMIs for tuning.

A slightly different line of work has consisted of pushing forward the sequential predictor approach to deal with arbitrarily large delays. This has been achieved by exploiting the modeling of the delay phenomenon by means of partial differential equations in Part III of this thesis. In Chapter 9, an observer-based controller has been proposed to compensate arbitrarily large input delays, in which the observer is infinite-dimensional. Furthermore, this technique has been extended also to systems with diffusive actuator dynamics. Its extension to other types of PDEs, like the wave dynamics, seems also feasible.
Chapter 10 is focused on output delays. In this context, the proposed strategy resembles the sequential observer approach. An infinite-dimensional observer is proposed that achieves exponential stabilization of the estimation error for systems with time-varying matrices and/or time-varying delays.
A reasonable continuation of this work is to use the results in Chapter 12 as the basis to propose an observer-based controller for time-varying systems with time-varying input delay. Although preliminary studies show some technical difficulties, this is a feasible line of research in the future.

The results of this thesis can be extended in several directions, some of which have been already pointed out. Among the limitations of the proposed controllers, there is one that should deserve special attention in future research, namely, the robustness to delay mismatch. Bounds on admissible delay uncertainty can often be computed, as in Chapter 5, which is a crucial design requirement when tuning the controller. However, for unstable systems, these bounds are, in any case, small. Delay-adaptive strategies have been proposed in the literature. It would be interesting to study whether they can be used to increase the tolerance of the proposed controllers to uncertain delays.

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