Some embedding theorems for Hörmander-Beurling spaces

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Dedicated to Professor Manuel Valdivia on the occasion of his 80th birthday

Abstract

In this paper we prove a number of results on sequence space representations and embedding theorems of Hörmander-Beurling spaces. As a consequence and using sharp results of Meise, Taylor and Vogt, a result of Kaballo on short sequences and hypoelliptic operators is extended to $\omega$-hypoelliptic differential operators and to the vector-valued setting.

Key words: Beurling ultradistributions, Hörmander spaces, Hörmander-Beurling spaces, $\omega$–hypoelliptic differential operators.

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1 Introduction and notations

It is well known that the Hörmander spaces $B_{p,k}$, $B_{p,k}^{\text{loc}}(\Omega)$ and $B_{p,k}^{c}(\Omega)$ play a crucial role in the theory of linear partial differential operators (see [2,15,16]). Our research pursues the study on Hörmander spaces and Hörmander spaces in the sense of Beurling and Björck [2] (=Hörmander-Beurling spaces) carried out in [2,8,14–16,19,40,45] and [5,29–31,36,37,44] (see also [18]). In this paper we prove a number of results on sequence space representations and

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embedding theorems of Hörmander-Beurling spaces (extending corresponding results of [29–31]) and as a consequence, and using results of Meise, Taylor and Vogt [24], a result of Kaballo [19] on short sequences and hypoelliptic differential operators is extended to ω–hypoelliptic differential operators and to the vector-valued setting.

The organization of the paper is as follows. Section 2 contains some basic facts about scalar and vector-valued Beurling ultradistributions and the definitions of the spaces which are considered in the paper. In Section 3 we generalize to UMD spaces the Theorem 4.6 of [31], we prove an embedding (and sequence spaces representation) theorem for vector-valued Hörmander-Beurling spaces, we give a result of Rosenthal type [38] (every weakly compact subset of $\mathcal{B}_{\infty,k}^\text{loc}(\Omega, E)$ is separable when $E$ is a closed subspace of $l_\infty^N$) (see Remark 3.1.1), we prove an embedding theorem when $E$ is non-separable Fréchet space and we pose the following question: Is $\mathcal{B}_{\infty,k}^\text{loc}(\Omega, l_\infty)$ isomorphic to a complemented subspace of $l_\infty^N$? (see Remark 3.1.3). In Section 4 we show that, in general, the topology induced by $\mathcal{B}_{p,k}^\text{loc}(\Omega, E)$ on $\mathcal{B}_{p,k}^\text{loc}(\Omega) \otimes E$ is strictly finer than the $\varepsilon$ topology and strictly coarser than the $\pi$ topology (our example extends to $1 < p < \infty$, by using a different technique, the example studied in [31, Remark 4.7.2]) and we pose another question: Are the spaces $\mathcal{B}_{\infty,k}^\text{loc}(\Omega, l_\infty)$ and $\mathcal{B}_{\infty,k}^\text{loc}(\Omega) \hat{\otimes}_\varepsilon l_\infty$ isomorphic? We also give a sequence space representation theorem when $E$ is a nuclear Fréchet space (for example it is shown that if $E \simeq s$ or $s^N$ then $\mathcal{B}_{\infty,k}^\text{loc}(\Omega, E)$ is isomorphic to $(\mathcal{D}_{L^n})^N$). Then, using results of Meise, Taylor and Vogt [24], we extend a result of Kaballo [19] to ω–hypoelliptic differential operators.

**Notations.** The linear spaces we use are defined over $\mathbb{C}$. Let $E$ and $F$ be locally convex spaces. Then $\mathcal{L}_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The (topological) dual of $E$ is denoted by $E'$ and is given the strong topology so that $E' = \mathcal{L}_b(E, \mathbb{C})$. $E \hat{\otimes}_\varepsilon F$ (resp. $E \hat{\otimes}_\pi F$) is the completion of the injective (resp. projective) tensor product of $E$ and $F$. If $E$ and $F$ are (topologically) isomorphic we put $E \simeq F$. If $E$ is isomorphic to a subspace (resp. complemented subspace) of $F$ we write $E \subset F$ (resp. $E < F$). We put $E \hookrightarrow F$ if $E$ is a linear subspace of $F$ and the canonical injection is continuous (we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if $E$ is also dense in $F$). If $(E_n)_{n=1}^\infty$ is a sequence of locally convex spaces, $\prod_{n=1}^\infty E_n$ ($E^N$ if $E_n = E$ for all $n$) is the topological product of the spaces $E_n$, $\bigoplus_{n=1}^\infty E_n$ ($E^{(N)}$ if $E_n = E$ for all $n$) is the locally convex direct sum of the spaces $E_n$. The Fréchet space defined by the projective sequence of Fréchet spaces $E_n$ and linking maps $A_n$ will be denoted by $\text{proj}(E_n, A_n)$ (or $\text{proj}E_n$, for short). This projective limit is said to be reduced if $\text{Im}P_j = E_j$ for $j = 1, 2, ..., \text{being } P_j : \text{proj}(E_n, A_n) \to E_j : (e_n) \mapsto e_j$. If the $E_n$ are Banach spaces and the maps $A_n$ are surjective then $\text{proj}(E_n, A_n)$ is said to be a quojection (see e.g. [28]).

Let $1 \leq p \leq \infty$, $k : \mathbb{R}^n \to (0, \infty)$ a Lebesgue measurable function, and $E$ a Fréchet space. Then $L_p(E)$ is the set of all (equivalence classes of) Bochner
measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|^p dx\right)^{1/p}$ is finite (with the usual modification when $p = \infty$) for all $\|\cdot\| \in \text{cs}(E)$ (see, e.g., [10]). $L_{p,k}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for all $f \in L_{p,k}(E)$ and for all $\|\cdot\| \in \text{cs}(E)$, $L_{p,k}(E)$ becomes a Fréchet space isomorphic to $L_p(E)$. When $E$ is the field $\mathbb{C}$, we simply write $L_p$ and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of $f$, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-i\xi x} dx$. If $f$ is a function on $\mathbb{R}^n$ then $\hat{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$.

Finally we recall the definition of $A^*_p$ functions. A positive, locally integrable function $\omega$ on $\mathbb{R}^n$ is in $A^*_p$ provided, for $1 < p < \infty$,

$$\sup_R \left(\frac{1}{|R|} \int_R \omega dx\right) \left(\frac{1}{|R|} \int_R \omega^{-\nu/p'} dx\right)^{p'/p} < \infty,$$

where $R$ runs over all bounded $n$-dimensional intervals. The basic properties of these functions can be found in [9].

2 Spaces of Beurling ultradistributions. Hörmander-Beurling spaces

In this section we collect some basic facts about vector-valued (Beurling) ultradistributions and we recall the definitions of the vector-valued Hörmander-Beurling spaces. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [2,13,20,21]. Our notations are based on [2] and [41].

Let $\mathcal{M}$ (or $\mathcal{M}_n$) be the set of all functions $\omega$ on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty]$ with the following properties:

(i) $\sigma(0) = 0$,
(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^\beta} dt < \infty \quad \text{(Beurling’s condition)},$
(iii) there exist a real number $a$ and a positive number $b$ such that

$$\sigma(t) \geq a + b \log(1 + t) \quad \text{for all} \quad t \geq 0.$$

The assumption (ii) is essentially the Denjoy-Carleman non-quasianalyticity condition (see [2]). The two most prominent examples of functions $\omega \in \mathcal{M}$ are given by $\omega(x) = \log(1 + |x|)^d$, $d > 0$, and $\omega(x) = |x|^\beta$, $0 < \beta < 1$.

If $\omega \in \mathcal{M}$ and $E$ is a Fréchet space, we denote by $\mathcal{D}_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|f(\xi)\|e^{\lambda\omega(\xi)} d\xi < \infty$, for all $\lambda > 0$ and for all $\|\cdot\| \in \text{cs}(E)$. For each compact subset $K$ of $\mathbb{R}^n$, $\mathcal{D}_\omega(K, E) = \{f \in \mathcal{D}_\omega(E) : \text{supp} f \subset K\}$, equipped with the topology induced by the family of seminorms $\{\|\cdot\|_\lambda : \|\cdot\| \in \text{cs}(E), \lambda > 0\}$, is a Fréchet space and $\mathcal{D}_\omega(E) = \operatorname{ind}_K \mathcal{D}_\omega(K, E)$ becomes a strict (LF)-space. If $\Omega$ is any open set in $\mathbb{R}^n$, $\mathcal{D}_\omega(\Omega, E)$ is the subspace of $\mathcal{D}_\omega(E)$ consisting of all functions $f$ with $\text{supp} f \subset \Omega$. $\mathcal{D}_\omega(\Omega, E)$ is endowed with the corresponding inductive limit
topology: \( D_\omega(\Omega, E) = \bigcap_K \mathcal{D}_\omega(K, E) \). Let \( S_\omega(E) \) be the set of all functions \( f \in L_1(E) \) such that both \( f \) and \( \hat{f} \) are infinitely differentiable functions on \( \mathbb{R}^n \) with \( \sup_{x \in \mathbb{R}^n} e^{\lambda \omega(x)} \left\| \partial^\alpha f(x) \right\| < \infty \) and \( \sup_{x \in \mathbb{R}^n} e^{\lambda \omega(x)} \left\| \partial^\alpha \hat{f}(x) \right\| < \infty \) for all multi-indices \( \alpha \) and all positive numbers \( \lambda \) and all \( \left\| \cdot \right\| \in \cos(E) \). \( S_\omega(E) \) with the topology induced by the above family of seminorms is a Fréchet space and the Fourier transformation \( \mathcal{F} \) is an automorphism of \( S_\omega(E) \). If \( E = \mathbb{C} \) then \( \mathcal{D}_\omega(E) \) and \( S_\omega(E) \) coincide with the spaces \( \mathcal{D}_\omega \) and \( S_\omega \) (see [2]). Let us recall that, by Beurling’s condition, the space \( \mathcal{D}_\omega \) is non-trivial and the usual procedure of the resolution of unity can be established with \( \mathcal{D}_\omega \)-functions (see [2]). Furthermore \( \mathcal{D}_\omega \overset{d}{\rightarrow} \mathcal{D} \) (see [2]) and \( \mathcal{D}_\omega \) is nuclear [45]. On the other hand, \( \mathcal{D}_\omega = \mathcal{D} \cap S_\omega \), \( \mathcal{D}_\omega \overset{d}{\rightarrow} S_\omega \overset{d}{\rightarrow} S \) (see [2]) and \( S_\omega \) is nuclear too (see [13]). If \( \mathcal{E}_\omega \) is the set of multipliers on \( \mathcal{D}_\omega \), i.e., the set of all functions \( \varphi : \mathbb{R}^n \rightarrow \mathbb{C} \) such that \( \varphi f \in \mathcal{D}_\omega \), for all \( \varphi \in \mathcal{E}_\omega \), then \( \mathcal{E}_\omega \) with the topology generated by the seminorms \( \{ \varphi \rightarrow \left\| \varphi f \right\|_\lambda = \int_{\mathbb{R}^n} |\varphi f(\xi)|e^{\lambda \omega(\xi)}d\xi : \lambda > 0, \varphi \in \mathcal{D}_\omega \} \) becomes a nuclear Fréchet space (see [45]) and \( \mathcal{D}_\omega \overset{d}{\rightarrow} \mathcal{E}_\omega \). Using the above results and [21, Theorem 1.12] we can identify \( S_\omega(E) \) with \( S_\omega \hat{\otimes} E \). However, though \( \mathcal{D}_\omega \otimes E \) is dense in \( \mathcal{D}_\omega(E) \), in general \( \mathcal{D}_\omega(E) \) is not isomorphic to \( \mathcal{D}_\omega \hat{\otimes} E \) (cf., e.g., [12]). A continuous linear operator from \( \mathcal{D}_\omega \) into \( E \) is said to be a (Beurling) ultradistribution with values in \( E \). We write \( \mathcal{D}_\omega'(E) \) for the space of all \( E \)-valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus \( \mathcal{D}_\omega'(E) = \mathcal{L}_b(\mathcal{D}_\omega, E) \). \( \mathcal{D}_\omega'(\Omega, E) = \mathcal{L}_b(\mathcal{D}_\omega(\Omega), E) \) is the space of all (Beurling) ultradistributions on \( \Omega \) with values in \( E \). A continuous linear operator from \( \mathcal{S}_\omega \) into \( E \) is said to be an \( E \)-valued tempered ultradistribution. \( \mathcal{S}_\omega'(E) \) is the space of all \( E \)-valued tempered ultradistributions equipped with the bounded convergence topology, i.e., \( \mathcal{S}_\omega'(E) = \mathcal{L}_b(\mathcal{S}_\omega, E) \). The Fourier transformation \( \mathcal{F} \) is an automorphism of \( \mathcal{S}_\omega'(E) \).

If \( \omega \in \mathcal{M} \), then \( \mathcal{K}_\omega \) is the set of all positive functions \( k \) on \( \mathbb{R}^n \) for which there exists a positive constant \( N \) such that \( k(x + y) \leq e^{N \omega(x)}k(y) \) for all \( x \) and \( y \) in \( \mathbb{R}^n \), cf. [2] (when \( \omega(x) = \log(1 + |x|) \) the functions \( k \) of the corresponding class \( \mathcal{K}_\omega \) are called temperate weight functions, see [16]). If \( k, k_1, k_2 \in \mathcal{K}_\omega \) and \( s \) is a real number then \( \log k \) is uniformly continuous, \( k^s \in \mathcal{K}_\omega \), \( k_1k_2 \in \mathcal{K}_\omega \) and \( M_k(x) = \sup_{y \in \mathbb{R}^n} \frac{k(x + y)}{k(y)} \in \mathcal{K}_\omega \) (see [2]). If \( u \in L_1^\infty \) and \( \int_{\mathbb{R}^n} \varphi(x)u(x)dx = 0 \) for all \( \varphi \in \mathcal{D}_\omega \), then \( u = 0 \) a.e. (see [2]). This result, the Hahn-Banach theorem and [7, Chapter II, Corollary 7] prove that if \( k \in \mathcal{K}_\omega \), \( p \in [1, \infty) \) and \( E \) is a Fréchet space, we can identify \( f \in L_{p,k}(E) \) with the \( E \)-valued tempered ultradistribution \( \varphi \rightarrow \langle \varphi, f \rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)dx, \varphi \in S_\omega \), and \( L_{p,k}(E) \hookrightarrow S_\omega'(E) \). If \( \omega \in \mathcal{M}, k \in \mathcal{K}_\omega, p \in [1, \infty] \) and \( E \) is a Fréchet space, we denote by \( \mathcal{B}_{p,k}(E) \) the set of all \( E \)-valued tempered ultradistributions \( T \) for which there exists a function \( f \in L_{p,k}(E) \) such that \( \langle \varphi, T \rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)dx, \varphi \in S_\omega \). \( \mathcal{B}_{p,k}(E) \) with the seminorms \( \{ \left\| T \right\|_{p,k} = ((2\pi)^{-n} \int_{\mathbb{R}^n} \left\| k(x)\hat{T}(\xi) \right\|^p dx)^{1/p} : \left\| \cdot \right\| \in \cos(E) \} \) (usual modification if \( p = \infty \)), becomes a Fréchet space isomorphic to \( L_{p,k}(E) \). Spaces \( \mathcal{B}_{p,k}(E) \) are called Hörmander-Beurling spaces with values in \( E \) (see [2] for the scalar case and [44] for the vector-valued case). We denote by
of Fréchet spaces $E$ (see [30]) the space of all $E$-valued ultradistributions $T \in \mathcal{D}'(\Omega, E)$ such that, for every $\varphi \in \mathcal{D}_\omega(\Omega)$, the map $\varphi T : \mathcal{S}_\omega \to E$ defined by $\langle u, \varphi T \rangle = \langle u\varphi, T \rangle$, $u \in \mathcal{S}_\omega$, belongs to $\mathcal{B}_{p,k}(E)$. The space $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ is a Fréchet space with the topology generated by the seminorms $\{\| \cdot \|_{p,k,\varphi} : \varphi \in \mathcal{D}_\omega(\Omega), \| \cdot \| \in \text{cs}(E)\}$, where $\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k}$ for $T \in \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$, and $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E) \hookrightarrow \mathcal{D}'(\Omega, E)$. We shall also use the spaces $\mathcal{B}^c_{p,k}(\Omega, E)$ which generalize the scalar spaces $\mathcal{B}^c_{p,k}(\Omega)$ considered by Hörmander in [16], by Vogt in [45] and by Björck in [2]. If $\omega$, $k$, $p$, $\Omega$ and $E$ are as above, then $\mathcal{B}^c_{p,k}(\Omega, E) = \bigcup_{j=1}^\infty [\mathcal{B}_{p,k}(E) \cap \mathcal{E}_{\omega}^j(K_j, E)]$ (here $(K_j)$ is any fundamental sequence of compact subsets of $\Omega$ and $\mathcal{E}_{\omega}^j(K_j, E)$ denotes the set of all $T \in \mathcal{D}_\omega(E)$ such that $\text{supp} T \subset K_j$). Since for every compact $K \subset \Omega$, $\mathcal{B}_{p,k}(E) \cap \mathcal{E}_{\omega}^j(K_j, E)$ is a Fréchet space with the topology induced by $\mathcal{B}_{p,k}(E)$, it follows that $\mathcal{B}^c_{p,k}(\Omega, E)$ becomes a strict (LF)-space (strict (LB)-space if $E$ is a Banach space): $\mathcal{B}^c_{p,k}(\Omega, E) = \varinjlim_j [\mathcal{B}_{p,k}(E) \cap \mathcal{E}_{\omega}^j(K_j, E)]$. These spaces are studied in [36] and [31].

3 An embedding theorem

In this section we generalize to UMD spaces the Theorem 4.6 of [31], we prove an embedding theorem for vector-valued Hörmander-Beurling spaces (Theorem 3.1, see also Remark 3.1.2) and we give a result of Rosenthal type [38] (every weakly compact subset of $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ is separable when $E$ is a closed subspace of $l_1^N$; see Remark 3.1.1).

We shall need the following technical result.

Lemma 3.1 Let $\Omega$ be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $1 \leq p \leq \infty$. Let $E = \text{proj}(E_j, A_j)$ be the reduced projective limit of the projective sequence of Fréchet spaces $E_j$ and linking maps $A_j$. Then the map

$$P : \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E) \longrightarrow \text{proj} \left( \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E_j), \mathcal{A}_j \right) : T \mapsto \left( P_j \circ T \right)^\infty_1$$

is an isomorphism ($\mathcal{A}_j$ is the map $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E_{j+1}) \to \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E_j) : T \mapsto A_j \circ T$) and this projective limit is reduced if $p < \infty$. If $E = \prod_{j=1}^\infty E_j$ then the space $\mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$ is isomorphic to $\prod_{j=1}^\infty \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E_j)$.

Proof. Although the proof of the lemma is straightforward, for the sake of completeness we give here the proof of the surjectivity of $P$: Let $(T_j)^\infty_1$ be any element in $\text{proj} \left( \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E_j), \mathcal{A}_j \right)$. For each $\varphi \in \mathcal{D}_\omega(\Omega)$ and each $j \geq 1$, we have $A_j \left( \langle \varphi, T_{j+1} \rangle \right) = \langle \varphi, A_j \circ T_{j+1} \rangle = \langle \varphi, T_j \rangle$ and so $\left( \langle \varphi, T_j \rangle \right)^\infty_1 \in \text{proj}(E_j, A_j)$. Let $T : \mathcal{D}_\omega \to E$ be defined by $\langle \varphi, T \rangle := \left( \langle \varphi, T_j \rangle \right)^\infty_1$ for $\varphi \in \mathcal{D}_\omega(\Omega)$. Let us prove that $T \in \mathcal{B}^{\text{loc}}_{p,k}(\Omega, E)$, i.e., that for every $\varphi \in \mathcal{D}_\omega(\Omega)$ there is an $f \in L_{p,k}(E)$ such that $\langle \theta, \varphi T \rangle = \int_{\mathbb{R}^n} \theta(x)f(x)dx$ for all $\theta \in \mathcal{S}_\omega$. Given such a $\varphi$ let $f_j \in L_{p,k}(E_j)$, $j = 1, 2, \ldots$, such that $\langle \theta, \varphi T_j \rangle = \int_{\mathbb{R}^n} \theta(x)f_j(x)dx$ for all $\theta \in \mathcal{S}_\omega$. Then, for
every \( \theta \in S_{\omega} \), we have
\[
\int_{\mathbb{R}^n} \theta(x) A_j \circ f_{j+1}(x) \, dx = A_j \left( \int_{\mathbb{R}^n} \theta(x) f_j(x) \, dx \right) = A_j \left( \left( \theta, (\varphi T_{j+1})^\wedge \right) \right) = \left( \theta, A_j \circ (\varphi T_{j+1})^\wedge \right) = \left( \theta, (\varphi T_j)^\wedge \right) = \int_{\mathbb{R}^n} \theta(x) f_j(x) \, dx.
\]
that is, \( \int_{\mathbb{R}^n} \theta(x) \left[ A_j \circ f_{j+1}(x) - f_j(x) \right] \, dx = 0 \). Hence it follows (see Section 2) that \( A_j \circ f_{j+1}(x) = f_j(x) \) for almost all \( x \in \mathbb{R}^n \). Then, modifying the functions \( f_j \) in a nullset if necessary, we get \( \left( f_j(x) \right)_1 \in \text{proj}(E_j, A_j) \) for all \( x \in \mathbb{R}^n \). It is easy to verify that the function \( f(x) = \left( f_j(x) \right)_1 \) is Bochner measurable. In fact, if \( \phi \in E' \) we can find \( N \geq 1 \) and \( (e'_1, \ldots, e'_N) \in E'_1 \times \cdots \times E'_N \) (see, e.g. [25]) such that \( \langle (e_j)^\infty_1, \phi \rangle = \sum_{j=1}^N \langle e_j, e'_j \rangle, (e_j)^\infty_1 \in E \). Thus \( \phi \circ f = \sum_{j=1}^N e'_j \circ f_j \) is measurable. Moreover, if \( N_j \) is a nullset such that \( f_j | (\mathbb{R}^n \setminus N_j) \) is separable, then \( f(\mathbb{R}^n \cup N_j) \) is also separable. Hence by the Pettis’ measurability theorem (in Fréchet spaces, see e.g. [10]) it follows that \( f \) is Bochner measurable. Then, by using the properties of the \( f_j \), \( j = 1, 2, \ldots \), we conclude that \( f \in L_{p,k}(E) \). Finally, since \( \int_{\mathbb{R}^n} \theta(x) f_j(x) \, dx = \left( \int_{\mathbb{R}^n} \theta(x) f_j(x) \, dx \right)_1 = \left( \theta, (\varphi T_{j+1})^\wedge \right) = \left( \theta, (\varphi T_j)^\wedge \right) = \theta(\varphi, T) = \langle \theta, \varphi T \rangle \) for all \( \theta \in S_{\omega} \), it follows that \( T \in B_{p,k}^{\text{loc}}(\Omega, E) \). Thus \( P \) is surjective. ■

The next lemma generalizes to UMD spaces the Theorem 4.6 of [31]. We will reason as we did in [31] but we will use Theorem 4.2 of [29] instead of Corollary 4.2 of [29]. For convenience of the reader we will give a complete proof. The following elementary fact will be used: “Let \( F = \text{ind} F_j \) be the strict inductive limit of a properly increasing sequence \( F_1 \subset F_2 \subset \cdots \) of Banach spaces. Assume that every \( F_j \) is a complemented subspace of \( F_{j+1} \) and that \( G_j \) is a topological complement of \( F_j \) in \( F_{j+1} \). Then the mapping \( F_1 \oplus G_1 \oplus G_2 \oplus \cdots \to F : (f_1, g_1, g_2, \ldots) \to f_1 + g_1 + g_2 + \cdots \) is an isomorphism”. We will also need the weighted \( L_p \)-spaces of vector-valued entire analytic functions \( L_{p,k}(E) \) and the operators \( S_K(f) = F^{-1}(\chi_K \hat{f}) \) (see [29] and [41]).

**Lemma 3.2** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( p \in (1, \infty) \) and \( k \) a temperate weight function on \( \mathbb{R}^n \) with \( k^p \in A_{\omega}^* \). Let \( E \) be a Banach space with the UMD-property. Then the space \( B_{p,k}^{\text{loc}}(\Omega, E) \) is isomorphic to \( \prod_{j=0}^\infty H_j \) where \( H_0 \) is isomorphic to \( \ell_p(E) \) and \( H_j \) is isomorphic to a complemented subspace of \( \ell_p(E) \) for \( j = 1, 2, \ldots \).

**Proof.** Let \( (K_j) \) be a covering of \( \Omega \) consisting of compact sets such that \( K_j \subset \overline{K}_{j+1}, K_j = \overline{K}_j \) and \( \overline{K}_j \) has the segment property (we may also assume, without loss of generality, that each \( K_j \) is a finite union of \( n \)-dimensional compact intervals). Then \( B_{p,k}^{\text{loc}}(\Omega, E) = \text{ind} \left[ B_{p,k}(E) \cap E'(K_j, E) \right] \). In this inductive limit, the step \( B_{p,k}(E) \cap E'(K_j, E) \) is isomorphic (via Fourier transform) to \( L_{p,k}^{-K_j}(E) \) and this space is isomorphic, by Theorem 4.2 and Corollary 5.1 of [29], to \( \ell_p(E) \). Furthermore, \( L_{p,k}^{-K_j}(E) \) is a complemented subspace of \( L_{p,k}^{-K_{j+1}}(E) : L_{p,k}^{-K_{j+1}}(E) = L_{p,k}^{-K_j}(E) \oplus \left[ \text{ker} S_{-K_j} \cap L_{p,k}^{-K_{j+1}}(E) \right] \). Thus, the space \( G_j = \text{ker} S_{-K_j} \cap L_{p,k}^{-K_{j+1}}(E) \) is isomorphic to an infinite-dimensional
complemented subspace of \( l_p(E) \). Then, by using the former result, we obtain \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \approx \mathcal{B}_{p,k}^\text{K}(E) \oplus G_1 \oplus G_2 \oplus \cdots \approx l_p(E) \oplus G_1 \oplus G_2 \oplus \cdots \). Next, since \( 1/\hat{k} \) is a temperate weight function on \( \mathbb{R}^n \) such that \( 1/\hat{k}^{j} \in A_p^{\ast} \) and \( E' \in UMD \) (see [39]), we see that \( \mathcal{B}_{p,1/k}^\text{loc}(\Omega, E') \approx \bigoplus_{j=0}^\infty B_j \) where \( B_0 \approx l_p(E') \) and \( B_j < l_p(E') \) for \( j = 1, 2, \ldots \). Therefore, by Theorem 3.2 of [31] (see [16] also), we get \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \approx \left( \mathcal{B}_{p,1/k}^\text{loc}(\Omega, E') \right)' \approx \left( \bigoplus_{j=0}^\infty B_j \right)' \approx \prod_{j=0}^\infty H_j \) (here \( H_j = B_j' \)) where \( H_0 \approx l_p(E) \) and \( H_j < l_p(E) \) for \( j = 1, 2, \ldots \), and the proof is complete.

**Remark.** One can improve Lemma 3.2 by using [45]. Indeed, using the arguments of [45] it can be shown that \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \approx (\mathcal{B}_{p,k}(E) \cap E'(Q, E))^N \) where \( Q = [0, 1]^n \). Then, reasoning as in the lemma, we obtain the isomorphism \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \approx (l_p(E))^N \).

We now present the main result of this section, an embedding (and sequence space representation) theorem for vector-valued Hörmander-Beurling spaces (see also Remark 3.1). We also pose a related question (Remark 3.1.3): Is \( \mathcal{B}_{\infty,k}^\text{loc}(\Omega, l_\infty) \) isomorphic to a complemented subspace of \( l_\infty^N \)? We will use the Fréchet spaces \( l_p^+ = \bigcap_{p>q} l_p \) and \( l_p^- = \bigcap_{p<q} l_p([0, 1]) \) (these spaces have an interest in the structure theory of Fréchet spaces and are primary and have all nuclear \( \Lambda(\alpha) \)-spaces as complemented subspaces, see [27] and [3]).

**Theorem 3.1** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( \omega \in \mathcal{M} \), \( k \in \mathcal{K}_\omega \) and \( 1 \leq p, q \leq \infty \), and let \( E \) be a Fréchet space.

1. If \( p < \infty \) and \( E \) is separable then \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \) is isomorphic to a subspace of \( \left( C([0, 1]) \right)^N \) and this space does not contain any complemented copy of \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \).

2. If \( E \) is separable and infinite-dimensional and \( E \not\cong \mathbb{C}^N \) then \( \mathcal{B}_{\infty,k}^\text{loc}(\Omega, E) \) is isomorphic to a subspace of \( l_\infty^N \) but this space does not contain any complemented copy of \( \mathcal{B}_{\infty,k}^\text{loc}(\Omega, E) \).

3. Suppose \( E \subset F^N \) (resp. \( F^N \)) where \( F \) is a Banach space. Then \( l_1^N \subset \mathcal{B}_{1,k}^\text{loc}(\Omega, E) \subset \left( l_1(F) \right)^N \) (resp. \( \left( l_1(F) \right)^N \)). If \( F \) is a dual space and has the Radon-Nikodým property, then \( l_\infty^N \subset \mathcal{B}_{\infty,k}^\text{loc}(\Omega, E) \subset \left( l_\infty(F) \right)^N \) (resp. \( \left( l_\infty(F) \right)^N \)). If \( F \) has the UMD-property then \( l_p^N \subset \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \subset \left( l_p(F) \right)^N \) (resp. \( \left( l_p(F) \right)^N \)) provided that \( 1 < p < \infty \) and \( k \) is a temperate weight with \( k^p \in A_p^{\ast} \); in particular, \( \mathcal{B}_{p,k}^\text{loc}(\Omega, l_1^N) \) is isomorphic to \( l_1^N \).

4. Suppose \( 1 < p < \infty \) and that \( k \) is a temperate weight with \( k^p \in A_p^{\ast} \), and let \( E = l_q^+ \) with \( q < \infty \) (resp. \( l_q^-([0, 1]) \)) with \( 1 < q \). Let \( (q_j)_1^\infty \) be any sequence such that \( q_j \not\in q \) (resp. \( q_j \not\in q \)). Then \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \) is isomorphic to a subspace of \( G : = \left( \prod_{j=1}^\infty l_p(q_j) \right)^N \) (resp. \( H : = \left( \prod_{j=1}^\infty l_p(q_j([0, 1])) \right)^N \)) but \( G \) (resp. \( H \)) does not contain any complemented copy of \( \mathcal{B}_{p,k}^\text{loc}(\Omega, E) \).

5. Let \( p, k, q \) and \( (q_j)_1^\infty \) be as in 4. Let \( X \) be a Banach subspace of \( \mathcal{B}_{p,k}^\text{loc}(\Omega, l_q^+) \)
Suppose now that $F((L_{q_1} \oplus \ldots \oplus L_{q_m}))$. Then $X$ is isomorphic to a subspace of $l_p(l_{q_1} \oplus \ldots \oplus l_{q_m})$ (resp. $l_p(L_{q_1}([0,1]) \oplus \ldots \oplus L_{q_m}([0,1]))$ for some integer $m$.

**Proof.** 1. The first claim is a consequence from the fact that every separable Fréchet space is isomorphic to a subspace of $(C([0,1]))^N$ (see e.g. [1, p.51]). Now suppose that $(C([0,1]))^N$ contains a complemented copy of $B_{p,k}^\text{loc}(\Omega, E)$.

Then $(C([0,1]))^N$ also contains a complemented copy of $B_{p,k}^\text{loc}(\Omega)$ since this space is clearly isomorphic to a complemented subspace of $B_{p,k}^\text{loc}(\Omega, E)$. Hence it follows, if $p = 1$, that $(C([0,1]))^N$ contains a complemented copy of $l_1^N$ (the proof given in [45] of the isomorphism $B_{p,k}^\text{loc}(\Omega) \simeq l_1^N$ is also valid for weights $k \in K_\omega$). Then $l_1$ becomes isomorphic to a complemented subspace of $C([0,1])$ (see e.g. [6]) which contradicts Corollary 2 in [33]. In case $p > 1$ we can apply Proposition 3.7 in [26] and obtain the isomorphism $B_{p,k}^\text{loc}(\Omega) \simeq C^N$. This contradicts the fact that $B_{p,k}^\text{loc}(\Omega)$ is a non-Montel Fréchet space (see [15, Theorem 2.3.9] and [16]). Consequently, $(C([0,1]))^N$ does not contain any complemented copy of $B_{p,k}^\text{loc}(\Omega, E)$.

2. We know that $E \subset l_\infty^N ([1, p.51]),$ that $L_\infty \simeq l_\infty ([23])$ and that $L_\infty(L_\infty) \subset \left(L_1(L_1)\right)^N \simeq l_1^N$ (but $L_\infty(L_\infty) \neq l_\infty$, see [4]). Hence and from Lemma 3.1 it follows that $B_{\infty,k}^\text{loc}(\Omega, E) \subset B_{\infty,k}^\text{loc}(\Omega, l_\infty^N) \simeq \left(\left(B_{\infty,k}^\text{loc}(\Omega, L_\infty)\right)^N\right) \subset \left(\left(L_\infty(L_\infty)\right)^N\right) \simeq \left(l_\infty^N\right) \subset l_1^N$.

3. By Lemma 3.1 and by [45] and [31, Theorem 4.2(2)], we have $l_1^N \simeq B_{1,k}^\text{loc}(\Omega) \subset B_{1,k}^\text{loc}(\Omega, F^N) \simeq \left(l_1(F)^N\right) \subset \left(l_\infty(F)^N\right) \simeq \left(l_1^N\right)$.

If $F$ is a dual space and has the Radon-Nikodým property then $l_\infty^N \simeq B_{\infty,k}^\text{loc}(\Omega) \subset B_{\infty,k}^\text{loc}(\Omega, F^N) \simeq \left(l_\infty(F)^N\right) \simeq \left(l_1^N\right)$ by virtue of Lemma 3.1 and [31, Theorem 4.2(3)].

Suppose now that $F$ has the UMD-property, $1 < p < \infty$ and $k^p \in A_p^\ast$. By using [31, Remark 4.7(1)] (see also [14]), Lemma 3.1 and Lemma 3.2, we get $l_p^N \simeq B_{p,k}^\text{loc}(\Omega) \subset B_{p,k}^\text{loc}(\Omega, E) \subset (\text{resp. } <) B_{p,k}^\text{loc}(\Omega, F^N) \simeq \left(\left(l_p(F)^N\right)\right) \simeq \left(l_p^N\right)$. Hence and from [42, (1)p.331] it follows that $B_{p,k}^\text{loc}(\Omega, l_p^N) \simeq l_p^N$ (see also [31, Remark 4.7(1)] or [14]).

4. Since the proofs of both claims are similar, we shall only proceed with the proof of the second one.

Put $E = L_q - ([0,1])$ and let $(q_j)$ be a sequence such that $q_j < q$. Then, tak-
ing into account Lemma 3.1 and Lemma 3.2 (the spaces \( L_{q_j}([0, 1]) \) have the UMD-property, see e.g. [39]), we have
\[
\mathcal{B}_{p,k}^{\text{loc}}(\Omega, E) \subset \mathcal{B}_{p,k}^{\text{loc}}\left(\Omega, \prod_{j=1}^{\infty} L_{q_j}([0, 1])\right) \simeq \prod_{j=1}^{\infty} \mathcal{B}_{p,k}^{\text{loc}}(\Omega, L_{q_j}([0, 1])) \simeq \left( \prod_{j=1}^{\infty} l_p(L_{q_j}([0, 1])) \right)^N.
\]

H. Furthermore, since all complemented subspace of a quojection is a quojection (see [28]), \( H \) is a quojection (actually \( H \cong \prod_{r=1}^{\infty} X_r \) where each \( X_r \) coincides with some \( l_p(L_{q_j}([0, 1])) \) where each \( X_r \) (resp. \( X_r \)) coincides with some \( l_p(l_q) \) (resp. \( l_p(L_{q_j}([0, 1])) \)).

5. Let \( X \) be a Banach subspace of \( \mathcal{B}_{p,k}^{\text{loc}}(\Omega, L_{q^+}) \) (resp. \( \mathcal{B}_{p,k}^{\text{loc}}(\Omega, L_{q^-}([0, 1])) \)). By using 4 we see that \( X \) is isomorphic to a subspace of \( \prod_{r=1}^{\infty} Y_r \) (resp. \( \prod_{r=1}^{\infty} X_r \)) where each \( Y_r \) (resp. \( X_r \)) coincides with some \( l_p(l_q) \) (resp. \( l_p(L_{q_j}([0, 1])) \)), thus ([6]) \( X \) becomes isomorphic to a subspace of \( l_p(l_{q_1} \oplus \cdots \oplus l_{q_m}) \) (resp. \( l_p(L_{q_j}([0, 1]) \oplus \cdots \oplus L_{q_m}([0, 1]))) \) for some integer \( m \).

**Remark 3.1**

1. In [38] Rosenthal showed that if \((\Omega, \Sigma, \mu)\) is a finite measure space then every weakly compact subset of \( L_\infty(\mu) \) is norm separable. By using this result it is easy to show that if \( E \subset \ell^1_\infty \) then every weakly compact subset of \( \mathcal{B}_{\infty,k}^{\text{loc}}(\Omega, E) \) (and hence every WCG subspace of \( \mathcal{B}_{\infty,k}^{\text{loc}}(\Omega, E) \)) is separable. In fact, let \( K \) be a weakly compact subset of \( \mathcal{B}_{\infty,k}^{\text{loc}}(\Omega, E) \). Then \( K \) becomes a weakly compact subset of \( \left( L_\infty([0, 1]) \right)^\infty \) (see the proof of Theorem 3.1(2) and recall that \( l_\infty \simeq L_\infty([0, 1]) \)). Now the weak topology
\[
\sigma\left(\left( L_\infty([0, 1]) \right)^\infty, \left( (L_\infty([0, 1]))^\infty \right)^\prime \right)
\]
is the product of the weak topologies (see, e.g. [17, p.167]). Consequently the projection of \( K \) on every factor \( L_\infty([0, 1]) \) is weakly compact and, by the Rosenthal’s result, is norm separable. Hence it follows that \( K \) is separable in \( \left( L_\infty([0, 1]) \right)^\infty \) and so is separable in \( \mathcal{B}_{\infty,k}^{\text{loc}}(\Omega, E) \).

2. Evidently it is possible to replace \( C([0, 1]) \) by \( l_\infty \) in Theorem 3.1(1). In the non-separable case we have the following extension: “Let \( p < \infty \) be. Let \( E \) be a non-separable Fréchet space and let \( I \) be a set such that \( \text{card} I = \text{dens} E \). Then \( \mathcal{B}_{p,k}^{\text{loc}}(\Omega, E) \subset (l_\infty(I))^\infty \) and this space does not contain any complemented copy of \( \mathcal{B}_{p,k}^{\text{loc}}(\Omega, E) \).” In fact, let \( (E_j)_{j=1}^{\infty} \) be a sequence of Banach spaces, with \( \text{dens} E_j \leq \text{dens} E \) for all \( j \), such that \( E \) is isomorphic to a subspace of \( \prod_{j=1}^{\infty} E_j \) (see, e.g. [1, p.34]). Since \( \text{dens} L_p(E_j) \leq \text{card} I \), we get \( L_p(E_j) \subset l_\infty(I) \) ([1, p.50]) and

\[
\mathcal{B}_{p,k}^{\text{loc}}(\Omega, E) \subset \mathcal{B}_{p,k}^{\text{loc}}(\Omega, \prod_{j=1}^{\infty} E_j) \simeq \prod_{j=1}^{\infty} \mathcal{B}_{p,k}^{\text{loc}}(\Omega, E_j) \subset \prod_{j=1}^{\infty} (l_p(E_j))^\infty \subset \prod_{j=1}^{\infty} (l_\infty(I))^\infty \simeq (l_\infty(I))^\infty.
\]

Finally, since \( l_\infty(I) = C(\beta I) \) (\( \beta I \) is the Stone-Čech compactification of \( I \) re-
garded in its discrete topology) and \( \beta I \) is extremally disconnected, we apply [26, Proposition 3.12].

3. We finish this note by posing the following question: Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( \omega \in \mathcal{M} \) and \( k \in \mathcal{K}_\omega \). Is \( B_{\infty,k}^{\text{loc}}(\Omega, l_\infty) \) isomorphic to a complemented subspace of \( l_\infty^N \)? (If the answer to this question were yes, \( B_{\infty,k}^{\text{loc}}(\Omega, l_\infty) \) would be isomorphic to \( l_\infty^N \) since \( B_{\infty,k}^{\text{loc}}(\Omega) \simeq l_\infty^N < B_{\infty,k}^{\text{loc}}(\Omega, l_\infty) \) implies \( B_{\infty,k}^{\text{loc}}(\Omega, l_\infty) \simeq l_\infty^N \) in virtue of [42, (1) p.331]).

4 On sequence space representations of Hörmander-Beurling spaces and applications

In this section a number of results on sequence space representations of vector-valued Hörmander-Beurling spaces are given (Theorem 4.1; see also Lemma 3.2, [30] and [31]). As a consequence, and using sharp results of Meise, Taylor and Vogt [24], a result of Kaballo (see [19]) on short sequences and hypoelliptic differential operators is extended to \( \omega \)-hypoelliptic differential operators and to the vector-valued setting.

**Lemma 4.1** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( \omega \in \mathcal{M} \), \( k \in \mathcal{K}_\omega \) and \( 1 \leq p < \infty \). Let \( E \) be a Fréchet space. Then the topology induced by \( B_{p,k}^{\text{loc}}(\Omega, E) \) on \( B_{p,k}^{\text{loc}}(\Omega) \otimes E \) is intercalated between the \( \varepsilon \) and \( \pi \) topologies.

**Proof.** Taking into account the corresponding fundamental systems of seminorms the proof is immediate since, for every \( \varphi \in D_\omega(\Omega) \) and every \( \| \cdot \| \in \text{cs}(E) \), we have

\[
\| T \|_{p,k,\varphi} \leq \inf \left\{ \sum_{i=1}^{m} \| u_j \|_{p,k,\varphi} \| e_j \| : T = \sum_{i=1}^{m} u_j \otimes e_j \right\}
\]

for all \( T \in B_{p,k}^{\text{loc}}(\Omega) \otimes E \), and, for every neighborhood \( U \) of 0 in \( B_{p,k}^{\text{loc}}(\Omega) \) and every \( \| \cdot \| \in \text{cs}(E) \), we have

\[
\sup_{(\xi,e') \in U^{\text{\prime} \otimes E}} \left| \sum_{j=1}^{m} \langle u_j, \xi \rangle \langle e_j, e' \rangle \right| \leq \max_{1 \leq i \leq r} \| T \|_{p,k,\varphi,i}
\]

(here \( \varphi_1, \ldots, \varphi_r \in D_\omega(\Omega) \) generate \( U \) and \( V = \{ e \in E : \| e \| \leq 1 \} \)) for all \( T = \sum_{i=1}^{m} u_j \otimes e_j \in B_{p,k}^{\text{loc}}(\Omega) \otimes E \). \( \blacksquare \)

**Remark 4.1** 1. Note that, in general, the topology induced by \( B_{p,k}^{\text{loc}}(\Omega, E) \) on \( B_{p,k}^{\text{loc}}(\Omega) \otimes E \) is strictly finer than the \( \varepsilon \) topology and strictly coarser than the \( \pi \) topology: In fact let \( 1 < p < \infty \), let \( k \) a temperate weight function on \( \mathbb{R}^n \) with \( k^p \in A_p \) and assume that \( B_{p,k}^{\text{loc}}(\Omega, l_p) \) contains a complemented copy of \( B_{p,k}^{\text{loc}}(\Omega) \otimes_{\text{a}} l_p \). Then, by [31, Remark 4.7(1)] (see also Theorem 3.1(3)) and [22, (5) p.282], we get \( B_{p,k}^{\text{loc}}(\Omega) \otimes_{\text{a}} l_p \simeq l_p \otimes_{\text{a}} l_p \simeq (l_p \otimes_{\text{a}} l_p)^{\text{N}} < B_{p,k}^{\text{loc}}(\Omega, l_p) \simeq l_p^{\text{N}} \). Hence and from [6] it follows that \( l_p \otimes_{\text{a}} l_p < l_p \), that is to say (since \( l_p \) is prime [23, Theorem 2.4.3]), that \( l_p \otimes_{\text{a}} l_p \simeq l_p \). But this is false since \( l_p \otimes_{\text{a}} l_p \) fails to have the
uniform approximation property (UAP, for short; see [34, p.350]) whereas \( l_p \in \text{UAP} \) by [35]. Therefore, \( B_{p,k}^{\text{loc}}(\Omega) \otimes l_p \) can not be isomorphic to a complemented subspace of \( B_{p,k}^{\text{loc}}(\Omega,l_p) \). In particular, since \( B_{p,k}^{\text{loc}}(\Omega) \otimes l_p \) is dense in \( B_{p,k}^{\text{loc}}(\Omega,l_p) \), the \( \varepsilon \) topology is strictly coarser than the topology induced by \( B_{p,k}^{\text{loc}}(\Omega,l_p) \). (A different proof, for the case \( 2 \leq p < \infty \), is given in [31, Remark 4.7(2)]). In a similar way it can be shown that the topology induced by \( B_{p,k}^{\text{loc}}(\Omega,l_p) \) on \( B_{1,k}^{\text{loc}}(\Omega) \otimes l_1 \) is strictly coarser than the \( \pi \) topology (recall that \( l_p \otimes l_p \notin \text{UAP} \) [34, p.350]).

2. If \( p = 1 \) and \( k \) is any weight in \( K_\omega \) one can argue as in 1 (by using [31, Theorem 4.2(3)] and the well known fact that \( l_1 \otimes l_1 \) is not isomorphic to \( l_1 \) [7, Chapter VIII]) and show that the topology induced by \( B_{1,k}^{\text{loc}}(\Omega,l_1) \) on \( B_{1,k}^{\text{loc}}(\Omega) \otimes l_1 \) is strictly finer than the \( \varepsilon \) topology.

3. The assertions in the above notes continue to hold when one replaces \( l_p \) by \( l_p^N \) in 1 and \( l_1 \) by \( l_1^N \) in 2.

4. Notice also that if the answer to the posed question in Remark 3.1.3 were affirmative, then \( B_{\infty,k}^{\text{loc}}(\Omega) \hat{\otimes} l_\infty \) would not be isomorphic to \( B_{\infty,k}^{\text{loc}}(\Omega,l_\infty) \) for any \( k \in K_\omega \). In fact, if these spaces were isomorphic then, by [31, Theorem 4.2(3)], [22, (5) p.282], [22, (2) p.287] and a result of G. Freniche [4, Theorem 3.2.1], we would have \( l_1^N \simeq (l_\infty \hat{\otimes} l_\infty)^N \simeq \bigotimes_{\varepsilon_\lambda} l_\infty \simeq \bigotimes_{\varepsilon_\lambda} l_\infty^N \simeq (C(\beta N, l_\infty)) \). Therefore \( c_0 \) would become a complemented subspace of \( l_\infty \) which contradicts a classical result of Phillips (see e.g. [4, Corollary 1.3.2]).

**Theorem 4.1** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( \omega \in \mathcal{M} \), \( k \in K_\omega \) and \( 1 \leq p < \infty \).

Let \( E \) be a nuclear Fréchet space. Then

(a) \( B_{p,k}^{\text{loc}}(\Omega,E) = B_{p,k}^{\text{loc}}(\Omega) \hat{\otimes} E \)

(b) if \( p = 1 \), or, \( 1 < p < \infty \) and \( k \) is a temperate weight with \( k^p \in A_p^* \), then \( B_{p,k}^{\text{loc}}(\Omega,E) \simeq \left( l_p(E) \right)^N \)

(c) if \( p = 1 \), or, \( 1 < p < \infty \) and \( k \) is a temperate weight with \( k^p \in A_p^* \), and \( E \simeq s \) or \( s^N \), then \( B_{p,k}^{\text{loc}}(\Omega,E) \simeq (\mathcal{D}_{L_p})^N \)

(d) if \( E \) is infinite dimensional and \( E \not\simeq \mathbb{C}^N \), then \( B_{p,k}^{\text{loc}}(\Omega,E) \) is isomorphic to a (non complemented) subspace of \( (L_p([0,1]))^N \)

(e) if \( E \) is a power series space of finite type, then \( B_{p,k}^{\text{loc}}(\Omega,E) \) is isomorphic to a complemented subspace of \( B_{p,k}^{\text{loc}}(\Omega, l_q) \) (resp. \( B_{p,k}^{\text{loc}}(\Omega, L_q([0,1])) \)) for any \( q \in [1, \infty[ \) (resp. \( q \in ]1, \infty[ \))

(f) if \( X \) is a Banach subspace of \( B_{p,k}^{\text{loc}}(\Omega,E) \), then \( X \) is isomorphic to a subspace of \( L_p([0,1]) \)

(g) if \( p = 1 \), or, \( 1 < p < \infty \) and \( k \) is a temperate weight with \( k^p \in A_p^* \), and \( X \) is a Banach subspace of \( B_{p,k}^{\text{loc}}(\Omega,E) \), then \( X \) is isomorphic to a subspace of \( l_p \)

(h) if \( 1 < p_1, p_2 < \infty \), and \( k_1, k_2 \) are temperate weights such that \( k_1^{p_1} \in A_{p_1}^* \), \( k_2^{p_2} \in A_{p_2}^* \), then \( B_{p_1,k_1}^{\text{loc}}(\Omega,E) \simeq B_{p_2,k_2}^{\text{loc}}(\Omega,E) \) if and only if \( p_1 = p_2 \)

(i) \( B_{p,k}^{\text{loc}}(\Omega,E) \) is quasinormable, and if \( p > 1 \) every quotient of \( B_{p,k}^{\text{loc}}(\Omega,E) \) by a closed subspace is reflexive
(j) every exact sequence \( 0 \rightarrow B_{p,k}^{\text{loc}}(\Omega) \rightarrow G \rightarrow E \rightarrow 0 \) where \( G \) is a Fréchet space, \( 1 < p < \infty \) and \( k \) is a temperate weight with \( k^p \in A_p^* \), splits.

Proof. (a) This is an immediate consequence of Lemma 4.1, the nuclearity of \( E \), the denseness of \( D_p(\Omega) \otimes E \) in \( B_{p,k}^{\text{loc}}(\Omega, E) \) (use [36, Proposition 3.4]) and the completeness of \( B_{p,k}^{\text{loc}}(\Omega, E) \).

(b) By using (a), [31, Theorem 4.2], [31, Remark 4.7(1)], [22, (5) p.282], [22, (5) p.198] and [22, (5) p.291], we get
\[
B_{p,k}^{\text{loc}}(\Omega, E) = B_{p,k}^{\text{loc}}(\Omega) \hat{\otimes}_\varepsilon E \simeq l^N_p \hat{\otimes}_\varepsilon E \simeq \left( l^N_p \hat{\otimes}_\varepsilon E \right)^N \simeq \left( l_p(E) \right)^N.
\]

(c) By Valdivia [43] and Vogt [45], we know that \( D_{L^p} \) is isomorphic to \( l_p \hat{\otimes}_\varepsilon s \). Hence and from (b) and [22, (5) p.282] it follows that \( B_{p,k}^{\text{loc}}(\Omega, s^N) \simeq \left( l_p \hat{\otimes}_\varepsilon s^N \right)^N \simeq \left( (l_p \hat{\otimes}_\varepsilon s)^N \right) \simeq \left( l_p \hat{\otimes}_\varepsilon s \right)^N \simeq (D_{L^p})^N \).

(d) The space \( E \) is isomorphic to a subspace of \( \left( L_p([0,1]) \right)^N \) (see e.g. [17, p.483]). Hence and from Lemma 3.1 it follows that
\[
B_{p,k}^{\text{loc}}(\Omega, E) \subset B_{p,k}^{\text{loc}}(\Omega, \left( L_p([0,1]) \right)^N) \simeq \left( B_{p,k}^{\text{loc}}(\Omega, L_p([0,1])) \right)^N \subset \left( \left( L_p(L_p([0,1])) \right)^N \right) \simeq \left( \left( L_p([0,1]) \right)^N \right) \simeq \left( L_p([0,1]) \right)^N.
\]

Now we prove that \( B_{p,k}^{\text{loc}}(\Omega, E) \) can not be isomorphic to a complemented subspace of \( \left( L_p([0,1]) \right)^N \). If this were not the case, \( E \) would also be isomorphic to a complemented subspace of \( \left( L_p([0,1]) \right)^N \). Then \( E \) would become a quojection (see e.g. [26]) and thus \( E \simeq \mathbb{C}^N \) (see again [26]), a contradiction.

(e) We know that all nuclear \( \Lambda_1(\alpha) \)–spaces are complemented subspaces of \( l_q \) when \( 1 \leq q < \infty \) [27] and of \( L_q([0,1]) \) when \( 1 < q \leq \infty \) [3]. Thus, if \( E = \Lambda_1(\alpha) \), we have \( B_{p,k}^{\text{loc}}(\Omega, \Lambda_1(\alpha)) < B_{p,k}^{\text{loc}}(\Omega, l_q^+) \) (resp. < \( B_{p,k}^{\text{loc}}(\Omega, L_q^-([0,1])) \)).

(f) By (d) \( X \) is isomorphic to a subspace of \( \left( L_p([0,1]) \right)^N \) and thus (see [6]) isomorphic to a subspace of \( L_p([0,1]) \).

(g) Since \( E \) is isomorphic to a subspace of \( l^N_p \) [17, p.483], we may apply Theorem 3.1(3) and conclude that \( X \) is also isomorphic to a subspace of \( l^N_p \). Thus [6] \( X \) becomes isomorphic to a subspace of \( l_p \).

(h) \( \Rightarrow \) From [31, Remark 4.7(1)], the hypothesis and (g) it follows that \( l_{p_1} \subset l_{p_2} \) (and \( l_{p_2} \subset l_{p_1} \)). As is well known this implies \( p_1 = p_2 \). \( \iff \) It suffices to apply (b).

(i) Taking into account (b) and recalling that the product of a family of quasi-normable spaces is quasi-normable [11, p.107] and that the tensor product \( \hat{\otimes}_\varepsilon \) of a Banach space and a nuclear space is also quasi-normable [12, Ch. II, Proposition 13 p.76], we see that \( B_{p,k}^{\text{loc}}(\Omega, E) \) becomes a quasi-normable space. Finally, since \( B_{p,k}^{\text{loc}}(\Omega, E) \subset \left( L_p([0,1]) \right)^N \) (see the proof of (d)), we conclude the proof by virtue of [11, Corollary p.101].

(j) Since the Fréchet space \( B_{p,k}^{\text{loc}}(\Omega) \) is a quojection (we know that this space is isomorphic to \( l^N_p \), see [31] or [14]) it suffices to apply [46, Theorems 5.2 and
Remark 4.2 1. Concerning Theorem 4.1 (c) let us recall that a large number of standard spaces of test functions are isomorphic to $s$ or $s^N$. For example, $S(\mathbb{R}^n) \simeq s$ \cite{25}, $\mathcal{D}(K) \simeq s$ ($K$ is a compact set in $\mathbb{R}^n$ such that $\bar{K} \neq \emptyset$; see \cite{25} and \cite{46}), $C^\infty(\Omega) \simeq s^N$ ($\Omega$ is an open set in $\mathbb{R}^n$; see \cite{25} and \cite{46}), $C^\infty(V) \simeq s$ ($V$ is an $n$–dimensional compact $C^\infty$–differentiable manifold; see \cite{25}), $C^\infty(W) \simeq s^N$ ($W$ is an $n$–dimensional $C^\infty$–differentiable manifold not compact and countable at infinity; see \cite{25}).

2. It is well known (see \cite{25}) that the space $A(\mathbb{C}^d)$ of all entire analytic functions cannot be isomorphic to either $s$ or $s^N$ but it is isomorphic to a complemented subspace of $s$. However, if $p$ and $k$ are as in Theorem 4.1 (c), $\mathcal{B}_{p,k}^{\text{loc}}(\Omega, A(\mathbb{C}^d))$ and $(\mathcal{D}_{L^p})^N$ are isomorphic. In fact, we know that

$$\mathcal{B}_{p,k}^{\text{loc}}(\Omega, A(\mathbb{C}^d)) \simeq \mathcal{B}_{p,k}^{\text{loc}}(\Omega) \otimes \mathcal{A}(\mathbb{C}^d) \simeq \mathcal{I}_{p,k}^{\mathbb{C}^d} \mathcal{A}(\mathbb{C}^d) \simeq (\mathcal{I}_{p,k}^{\mathbb{C}^d} \mathcal{A}(\mathbb{C}^d))^N$$

and that $A(\mathbb{C}^d) \simeq \Lambda_\infty(\alpha)$ with $\alpha_n = n^{1/\alpha}$. But, by \cite[1.1 Proposition]{47} (the proof given there works for any $p \geq 1$) we have $\mathcal{I}_{p,k}^{\mathbb{C}^d} \mathcal{A}(\mathbb{C}^d) \simeq \mathcal{I}_{p,k}^{\mathbb{C}^d} s$, therefore $\mathcal{B}_{p,k}^{\text{loc}}(\Omega, A(\mathbb{C}^d)) \simeq (\mathcal{D}_{L^p})^N$.

In \cite{19} Kaballo showed that the short sequence $0 \rightarrow N(P(D)) \rightarrow \mathcal{B}_{p,k,P}^{\text{loc}}(\Omega) \rightarrow 0$ is an $(\epsilon L)$–triple when the differential operator $P(D)$ is hypoelliptic and it does not split when $P(D)$ is elliptic (recall that a short exact sequence of locally convex spaces $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ is called an $(\epsilon L)$–triple, if for every Banach space $X$ the mapping $q \otimes \text{id} : F \otimes X \rightarrow G \otimes X$ is surjective). In the next theorem this result is extended to $\omega$-hypoelliptic differential operators and to the vector-valued setting. The extension is essentially a consequence of results of Meise, Taylor and Vogt \cite[Theorem 2.10, Corollary 2.16]{24} (see also Vogt \cite{46}) and Theorem 4.1. We will consider weights in the class $\mathcal{M}^*$ ($\omega \in \mathcal{M}^*$ if $\omega(x) = \sigma(|x|) \in \mathcal{M}$ and $\sigma$ is as in \cite[Definition 1.1]{24}). For example, the weight $\omega(x) = |x|^\beta$ belongs to $\mathcal{M}^*$ when $0 < \beta < 1$. On the other hand, if $P(x) = \sum_{|a| \leq m} a_a x^a$ is a complex polynomial in $n$ variables then $P'(x)$ denotes the function $x \rightarrow \left(\sum_{|a| \geq 0} |\partial^a P(x)|^2\right)^{1/2}$. An open set $\Omega \subset \mathbb{R}^n$ is called $P$–convex ($P$–convex for supports in \cite[Definition 10.6.1]{16}) if to every compact set $K \subset \Omega$ there exists another compact set $K' \subset \Omega$ such that $\phi \in \mathcal{D}(\Omega)$ and supp $P(-D)\phi \subset K$ implies supp $\phi \subset K'$. Finally we refer the reader to \cite{2,15,16} for the theory of linear partial differential operators.

**Theorem 4.2** Let $P(D)$ be a linear partial differential operator with constant coefficients in $\mathbb{R}^n$ ($n \geq 2$), $\Omega$ an open subset of $\mathbb{R}^n$, $\omega \in \mathcal{M}^*$, $k \in K_\omega$ and $1 \leq p < \infty$.

(1) If $P(D)$ is $\omega$–hypoelliptic and $\Omega$ is $P$–convex, then the short sequence

$$0 \rightarrow N(P(D)) \rightarrow \mathcal{B}_{p,k,P}^{\text{loc}}(\Omega) \xrightarrow{P(D)} \mathcal{B}_{p,k}^{\text{loc}}(\Omega) \rightarrow 0$$
is exact, it does not split and it is an \((\epsilon L)\)-triple (here \(N(D)\) is the kernel of \(P(D)\)). The dual sequence

\[
0 \rightarrow \left( B_{p,k}(\Omega) \right)^{\prime} \overset{P(D)}{\rightarrow} \left( B_{p,k;P'}(\Omega) \right)^{\prime} \rightarrow \left( N(P(D)) \right)^{\prime} \rightarrow 0
\]

is topologically exact and it does not split either.

(2) If \(P(D)\) is \(\omega\)-hypoelliptic, \(\Omega\) is \(\bar{P}\)-convex and \(1 < p < \infty\), there exist a short sequence

\[
0 \rightarrow B_{p,k}(\Omega) \rightarrow B_{p,k;P'}(\Omega) \rightarrow \left( N(P(-D)) \right)^{\prime} \rightarrow 0
\]

which is topologically exact and it does not split.

(3) If \(P(D)\) is \(\omega\)-hypoelliptic, \(\Omega\) is \(P\)-convex and \(E\) is a nuclear Fréchet space, the short sequence

\[
0 \rightarrow N(P(E)(D)) \rightarrow B_{p,k;P'}(\Omega, E) \overset{P_{E}(D)}{\rightarrow} B_{p,k}(\Omega, E) \rightarrow 0
\]

is exact and an \((\epsilon L)\)-triple (here \(P_{E}(D)\) : \(\mathcal{D}'(\Omega, E) \rightarrow \mathcal{D}'(\omega, \varphi)\) is defined by \(\langle \varphi, P(E)(D)T \rangle = \langle P(-D)\varphi, T \rangle\) for all \(\varphi \in \mathcal{D}(\omega, \varphi)\) and all \(T \in \mathcal{D}'(\omega, \varphi))\).

**Proof.** 1. It follows from the hypothesis and [2, Theorem 3.3.3] that \(P(D)\) is a continuous linear operator of \(B_{p,k;P'}(\Omega)\) (resp. \(E_{\omega}(\Omega)\)) onto \(B_{p,k}(\Omega)\) (resp. \(E_{\omega}(\Omega)\)). Furthermore \(N(P(D))\) coincides, algebraically and topologically, with the subspace \(\{ f \in E_{\omega}(\Omega) : P(D)f = 0 \}\) of \(E_{\omega}(\Omega)\) in virtue of [2, Theorem 4.1.1], the embedding \(E_{\omega}(\Omega) \hookrightarrow B_{p,k;P'}(\Omega)\) [2, Theorem 2.3.5] and the closed graph theorem; thus \(N(P(D))\) is a nuclear Fréchet space (\(E_{\omega}(\Omega)\) is nuclear by [45]). It is then clear that the diagram

\[
\begin{array}{ccc}
0 & \rightarrow & N(P(D)) \\
\uparrow & & \uparrow \\
0 & \rightarrow & \mathcal{E}_{\omega}(\Omega)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & N(P(D)) \\
\uparrow \quad & \quad \uparrow & \\
0 & \rightarrow & \mathcal{E}_{\omega}(\Omega)
\end{array}
\]

is commutative. Since, by the Meise-Taylor-Vogt theorem [24, Theorem 2.10, Corollary 2.16], the second row of this diagram does not split, it follows that the first row does not split either (see [32]). The first row is an \((\epsilon L)\)-triple by the nuclearity of \(N(P(D))\) and [19, Theorem 2.9]. Next consider the dual diagram
0 \longrightarrow (\mathcal{B}^\text{loc}_{p,k}(\Omega))' \xrightarrow{\iota P(D)} (\mathcal{B}^\text{loc}_{p,k,P'}(\Omega))' \longrightarrow (N(P(D)))' \longrightarrow 0

0 \longrightarrow \mathcal{E}_\omega'(\Omega) \xrightarrow{\iota P(D)} \mathcal{E}_\omega'(\Omega) \longrightarrow (N(P(D)))' \longrightarrow 0

This diagram is also commutative and since $N(P(D))$ is quasinormable (see e.g. [25, Corollary 28.5]) its rows are topologically exact sequences (use [25, Proposition 26.18]). Its second row does not split because the second row of the previous diagram does not split either and the space $\mathcal{E}_\omega(\Omega)$ is reflexive (see [32]). Hence it follows that the first row does not split either.

2. Since $P(D) = P(-D)$ and $\Omega$ is $P-$convex, it follows from 1 that the short sequence $0 \longrightarrow \left(\mathcal{B}^\text{loc}_{p',1/k}(\Omega)\right)' \xrightarrow{\iota P(D)} \left(\mathcal{B}^\text{loc}_{p',-k/P}p(\Omega)\right)' \longrightarrow \left(N(P(-D))\right)' \longrightarrow 0$ is topologically exact and it does not split. Using the isomorphisms [31, Theorem 3.2] $\left(\mathcal{B}^\text{loc}_{p',1/k}(\Omega)\right)' \approx \mathcal{B}^\text{loc}_{p,k}(\Omega)$, $\left(\mathcal{B}^\text{loc}_{p',-k/P}p(\Omega)\right)' \approx \mathcal{B}^\text{loc}_{p,k/P}(\Omega)$ one easily concludes the proof.

3. According to 1 we have the exact sequence $0 \longrightarrow N(P(D)) \longrightarrow \mathcal{B}^\text{loc}_{p,k,P'}(\Omega) \overset{P(D)}{\longrightarrow} \mathcal{B}^\text{loc}_{p,k}(\Omega) \longrightarrow 0$ then also $0 \longrightarrow N(P(D)) \overset{\otimes}{\longrightarrow} E \longrightarrow \mathcal{B}^\text{loc}_{p,k,P'}(\Omega) \otimes_{\varepsilon} E \longrightarrow \mathcal{B}^\text{loc}_{p,k}(\Omega) \otimes_{\varepsilon} E \longrightarrow 0$ is exact (the second arrow is injective by [22, Proposition 5 p.277] and $P(D) \otimes_{\varepsilon} \text{id}$ is surjective by the nuclearity of $E$ and [22, Proposition 7 p.189]). On the other hand from [22, Proposition 7 p.189] and [22, Proposition 7 p.174] it follows that $N(P_E(D)) = N(P(D) \otimes_{\varepsilon} \text{id}) = N(P(D)) \otimes E \mathcal{B}^\text{loc}_{p,k,P'}(\Omega) \otimes_{\varepsilon} E = N(P(D)) \otimes_{\varepsilon} E$. Furthermore, by virtue of Theorem 4.1(a), we have $\mathcal{B}^\text{loc}_{p,k,P'}(\Omega) \otimes_{\varepsilon} E = \mathcal{B}^\text{loc}_{p,k,P'}(\Omega, E)$ and $\mathcal{B}^\text{loc}_{p,k}(\Omega) \otimes_{\varepsilon} E = \mathcal{B}^\text{loc}_{p,k}(\Omega, E)$. Therefore we have the exact sequence $0 \longrightarrow N(P_E(D)) \longrightarrow \mathcal{B}^\text{loc}_{p,k,P'}(\Omega, E) \overset{P_E(D)}{\longrightarrow} \mathcal{B}^\text{loc}_{p,k}(\Omega, E) \longrightarrow 0$. Finally the nuclearity of $N(P_E(D))$ and Theorem 2.9 in [19] show that this sequence is also an $(\varepsilon L)-$triple. 

Remark. For results on the splitting of partial differential operators between $\mathcal{B}^\text{loc}_{p,k}$-spaces in the temperate case see also [14].

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References


