On sequence space representations of Hörmander–Beurling spaces

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Abstract

It is shown that $B^{\text{loc}}_{p',1/\tilde{k}}(\Omega)$ is isomorphic to $(B^c_{p,k}(\Omega))'_b$ (Ω open set in $\mathbb{R}^n$, $1 \leq p < \infty$, k Beurling–Björck weight) extending a Hörmander's result (the proof we give is valid in the vector–valued case, too). As a consequence, and using Vogt's representation theorems and weighted L_p–spaces of entire analytic functions, a number of results on sequence space representations of Hörmander–Beurling are given.

Key words: Beurling ultradistributions, Hörmander spaces, Hörmander–Beurling spaces
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1 Introduction and notation

In [13, Chapter XV] Hörmander studies the behaviour of the Fourier–Laplace transform in the space $B^c_{2,k}(\Omega) = \operatorname{ind}_{\overrightarrow{K \Subset \Omega}} [B_{2,k} \cap \mathcal{E}'(K)]$ when Ω is an open convex set in $\mathbb{R}^n$ and k is a temperate weight function on $\mathbb{R}^n$, and then proves a theorem on the representation of solutions of the equation $P(D)u = 0$ by integrals of exponential solutions ($P(D)$ is a constant coefficient partial

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differential operator). For this he obtains an appropiate collection of seminorms defining the inductive limit topology of $B^c_{2,k}(\Omega)$, proves the isomorphism $(B^c_{2,k}(\Omega))'_b \simeq B^{\mathrm{loc}}_{2,1/\tilde{k}}(\Omega)$ and shows that every continuous seminorm in $B^c_{2,k}(\Omega)$ is bounded by a seminorm of the form $u \to \left(\int |\hat{u}(\zeta)|^2 e^{-2\phi(\zeta)} d\lambda(\zeta)\right)^{1/2}$ where $\hat{u}$ is the Fourier–Laplace transform of u and ϕ is plurisubharmonic (see [13, Section 15.2]). In this paper we extend the former isomorphism to Beurling–Björck weights [1] and as a consequence (and using Vogt's representation theorems [33] and weighted L_p–spaces of entire analytic functions [25,30]) a number of results on sequence space representations of Hörmander spaces in the sense of Beurling–Björck [1] (=Hörmander–Beurling spaces) are given. This research pursues the study on Hörmander–Beurling spaces carried out in [1,6,12,13,29,33] and [24,25,27,28,32] (see also [14]).

The organization of the paper is as follows. Section 2 contains some basic facts about scalar and vector–valued Beurling ultradistributions and the definitions of the spaces which are considered in the paper. In Section 3 we show that $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ is isomorphic to $(B^c_{p,k}(\Omega, E))'_b$ when $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \leq p < \infty$ and E is a Banach space whose dual E' possesses the Radon–Nikodým property (see Theorem 3.2), and we propose the following question: Are the spaces $BV^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ and $(B^c_{p,k}(\Omega, E))'_b$ isomorphic (E is any Banach space)? (Problem 3.4). In Section 4, by using the previous isomorphism, some representation theorems of Vogt [33, Theorems 5.2, 6.2] and the solution to Problem 4.11 in [24] given by Cembranos and Mendoza in [3], we partially answer the Problem 4.10 in [24] (see Theorem 4.4). We also show that, in general, $B^{\mathrm{loc}}_{\infty,k}(\Omega, E)$ is not isomorphic to either $B^{\mathrm{loc}}_{\infty,k}(\Omega)\widehat{\otimes}_\varepsilon E$ or $B^{\mathrm{loc}}_{\infty,k}(\Omega)\widehat{\otimes}_\pi E$. Next it is shown that $B^c_{p,k}(\Omega, l_q)$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$) is isomorphic to $\bigoplus_{j=0}^\infty G_j$ (resp. $\prod_{j=0}^\infty H_j$) where G_0 (resp. H_0) is isomorphic to $l_p(l_q)$ and G_j (resp. H_j) is isomorphic to a complemented subspace of $l_p(l_q)$ for $j = 1, 2, \ldots$. Then we describe the structure of the complemented normed subspaces of $B^{\mathrm{loc}}_{p,k}(\Omega)$, $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ and $\prod_{j=1}^m B^{\mathrm{loc}}_{p_j,k_j}(\Omega_j, l_p)$. We also give a new proof (based on our representation theorem $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$) of a well known result on linear partial differential operators.

Notation. The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $L_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. The dual of E is denoted by E' and is given the strong topology so that $E' = L_b(E, \mathbb{C})$. $E\widehat{\otimes}_\varepsilon F$ (resp. $E\widehat{\otimes}_\pi F$) is the completion of the injective (resp. projective) tensor product of E and F. If E and F are (topologically) isomorphic we put $E \simeq F$. If E is isomorphic to a complemented subspace of F we write $E < F$. We put $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous (we replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F). If $(E_n)_{n=1}^\infty$ is a sequence of locally convex spaces, $\prod_{n=1}^\infty E_n$ ($E^{\mathbb{N}}$ if $E_n = E$ for all n) is the topological product of the spaces E_n; $\bigoplus_{n=1}^\infty E_n$ ($E^{(\mathbb{N})}$ if $E_n = E$ for all n) is the locally convex direct sum of the spaces E_n.

Let $1 \le p \le \infty$, $k : \mathbb{R}^n \to (0, \infty)$ a Lebesgue measurable function, and E a Fréchet space. Then $L_p(E)$ is the set of all (equivalence classes of) Bochner measurable functions $f : \mathbb{R}^n \to E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|^p dx\right)^{1/p}$ is finite (with the usual modification when $p = \infty$) for all $\|\cdot\| \in \mathrm{cs}(E)$ (see, e.g. [8]). $L_{p,k}(E)$ denotes the set of all Bochner measurable functions $f : \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for all $f \in L_{p,k}(E)$ and for all $\|\cdot\| \in \mathrm{cs}(E)$, $L_{p,k}(E)$ becomes a Fréchet space isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\xi x} dx$. If f is a function on $\mathbb{R}^n$ then $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$. The letter C will always denote a positive constant, not necessarily the same at each occurrence.

Finally we recall the definition of A_p^* functions. A positive, locally integrable function ω on $\mathbb{R}^n$ is in A_p^* provided, for $1 < p < \infty$,

$$\sup_R \left(\frac{1}{|R|}\int_R \omega dx\right)\left(\frac{1}{|R|}\int_R \omega^{-p'/p} dx\right)^{p/p'} < \infty$$

where R runs over all bounded n–dimensional intervals. The basic properties of these functions can be found in [7, Chapter IV]

2 Spaces of vector–valued (Beurling) ultradistributions

In this section we collect some basic facts about vector–valued (Beurling) ultradistributions and we recall the definitions of the vector–valued Hörmander–Beurling spaces. Comprehensive treatments of the theory of (scalar or vector–valued) ultradistributions can be found in [1], [10] and [15], [16], [17]. Our notations are based on [1] and [30, pp. 14–19].

Let $\mathcal{M}$ (or $\mathcal{M}_n$) be the set of all functions ω on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties:

(i) $\sigma(0) = 0$,
(ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2} dt < \infty$ (Beurling's condition),
(iii) there exist a real number a and a positive number b such that

$$\sigma(t) \ge a + b\log(1+t) \qquad \text{for all } t \ge 0 \ .$$

The assumption (ii) is essentially the Denjoy–Carleman non–quasianalyticity condition (see [1, Sect. 1.5]). The two most prominent examples of functions

$\omega \in \mathcal{M}$ are given by $\omega(x) = \log(1+|x|)^d$, $d > 0$, and $\omega(x) = |x|^\beta$, $0 < \beta < 1$.

If $\omega \in \mathcal{M}$ and E is a Fréchet space, we denote by $D_\omega(E)$ the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda = \int_{\mathbb{R}^n} \|\hat{f}(\xi)\| \, e^{\lambda\omega(\xi)} d\xi < \infty$ for all $\lambda > 0$ and for all $\|\cdot\| \in \mathrm{cs}(E)$. For each compact subset K of $\mathbb{R}^n$, $D_\omega(K, E) = \{f \in D_\omega(E) : \operatorname{supp} f \subset K\}$, equipped with the topology induced by the family of seminorms $\{\|\cdot\|_\lambda : \|\cdot\| \in \mathrm{cs}(E), \lambda > 0\}$, is a Fréchet space and $D_\omega(E) = \operatorname{ind}_{\overrightarrow{K}} D_\omega(K, E)$ becomes a strict (LF)–space. If Ω is any open set in $\mathbb{R}^n$, $D_\omega(\Omega, E)$ is the subspace of $D_\omega(E)$ consisting of all functions f with $\operatorname{supp} f \subset \Omega$. $D_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology: $D_\omega(\Omega, E) = \operatorname{ind}_{\overrightarrow{K \subset \Omega}} D_\omega(K, E)$. Let $S_\omega(E)$ be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\| < \infty$ and $\sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha \hat{f}(x)\| < \infty$ for all multi–indices α, all positive numbers λ and all $\|\cdot\| \in \mathrm{cs}(E)$. $S_\omega(E)$ with the topology induced by the above family of seminorms is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $S_\omega(E)$. If $E = \mathbb{C}$ then $D_\omega(E)$ and $S_\omega(E)$ coincide with the spaces D_ω and S_ω (see [1]). Let us recall that, by Beurling's condition, the space D_ω is non–trivial and the usual procedure of the resolution of unity can be established with D_ω–functions (see [1, Theorem 1.3.7]). Furthermore, $D_\omega \overset{d}{\hookrightarrow} D$ (see [1, Theorem 1.3.18]) and D_ω is nuclear ([33, Corollary 7.5]). On the other hand, $D_\omega = D \cap S_\omega$, $D_\omega \overset{d}{\hookrightarrow} S_\omega \overset{d}{\hookrightarrow} S$ (see [1, Proposition 1.8.6, Theorem 1.8.7]) and S_ω is nuclear also (see [10, p. 320]). If $\mathcal{E}_\omega$ is the set of multipliers on D_ω, i.e., the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\varphi f \in D_\omega$ for all $\varphi \in D_\omega$, then $\mathcal{E}_\omega$ with the topology generated by the seminorms $\{f \to \|\varphi f\|_\lambda = \int_{\mathbb{R}^n} |\widehat{\varphi f}(\xi)| e^{\lambda\omega(\xi)} \, d\xi : \lambda > 0, \varphi \in D_\omega\}$ becomes a nuclear Fréchet space (see [33, Corollary 7.5]) and $D_\omega \overset{d}{\hookrightarrow} \mathcal{E}_\omega$. Using the above results and [17, Theorem 1.12] we can identify $S_\omega(E)$ with $S_\omega \widehat{\otimes}_\varepsilon E$. However, though $D_\omega \otimes E$ is dense in $D_\omega(E)$, in general $D_\omega(E)$ is not isomorphic to $D_\omega \widehat{\otimes}_\varepsilon E$ (cf., e.g. [9, Chapter II, p. 83]). A continuous linear operator from D_ω into E is said to be a (Beurling) ultradistribution with values in E. We write $D'_\omega(E)$ for the space of all E–valued (Beurling) ultradistributions endowed with the bounded convergence topology, thus $D'_\omega(E) = L_b(D_\omega, E)$. $D'_\omega(\Omega, E) = L_b(D_\omega(\Omega), E)$ is the space of all (Beurling) ultradistributions on Ω with values in E. A continuous linear operator from S_ω into E is said to be an E–valued tempered ultradistribution. $S'_\omega(E)$ is the space of all E–valued tempered ultradistributions equipped with the bounded convergence topology, i.e., $S'_\omega(E) = L_b(S_\omega, E)$. The Fourier transformation $\mathcal{F}$ is an automorphism of $S'_\omega(E)$.

If $\omega \in \mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a positive constant N such that $k(x+y) \le e^{N\omega(x)} k(y)$ for all x and y in $\mathbb{R}^n$ [1, Definition 2.1.1] (when $\omega(x) = \log(1+|x|)$ the functions k of the corresponding class $\mathcal{K}_\omega$ are called temperate weight functions, see [13, Definition 10.1.1]). If k, k_1, $k_2 \in \mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1 k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [1, Theorem 2.1.3]). If $u \in L_1^{\mathrm{loc}}$ and $\int_{\mathbb{R}^n} \varphi(x)u(x)\,dx = 0$ for all $\varphi \in D_\omega$, then $u = 0$ a.e. (see [1]). This result, the Hahn–Banach theorem and [5, Chapter II, Corollary 7] prove that if $k \in \mathcal{K}_\omega$, $p \in [1,\infty]$ and E is a Fréchet space, we can identify $f \in L_{p,k}(E)$ with the E–valued tempered ultradistribution $\varphi \to \langle \varphi, f\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)\,dx$, $\varphi \in S_\omega$, and $L_{p,k}(E) \hookrightarrow S'_\omega(E)$. If $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $p \in [1,\infty]$ and E is a Fréchet space, we denote by $B_{p,k}(E)$ the set of all E–valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\langle \varphi, \hat{T}\rangle = \int_{\mathbb{R}^n} \varphi(x)f(x)\,dx$, $\varphi \in S_\omega$. $B_{p,k}(E)$ with the seminorms $\{\|T\|_{p,k} = \left((2\pi)^{-n}\int_{\mathbb{R}^n} \|k(x)\hat{T}(x)\|^p dx\right)^{1/p} : \|\cdot\| \in \mathrm{cs}(E)\}$ (usual modification if $p = \infty$), becomes a Fréchet space isomorphic to $L_{p,k}(E)$. Spaces $B_{p,k}(E)$ are called Hörmander–Beurling spaces with values in E (see [12], [13], [33] for the scalar case and [25], [27], [32] for the vector–valued case). We denote by $B^{\mathrm{loc}}_{p,k}(\Omega, E)$ (see [12], [13], [33] and [24], [25], [27]) the space of all E–valued ultradistributions $T \in D'_\omega(\Omega, E)$ such that, for every $\varphi \in D_\omega(\Omega)$, the map $\varphi T : S_\omega \to E$ defined by $\langle u, \varphi T\rangle = \langle u\varphi, T\rangle$, $u \in S_\omega$, belongs to $B_{p,k}(E)$. The space $B^{\mathrm{loc}}_{p,k}(\Omega, E)$ is a Fréchet space with the topology generated by the seminorms $\{\|\cdot\|_{p,k,\varphi} : \varphi \in D_\omega(\Omega),\ \|\cdot\| \in \mathrm{cs}(E)\}$, where $\|T\|_{p,k,\varphi} = \|\varphi T\|_{p,k}$ for $T \in B^{\mathrm{loc}}_{p,k}(\Omega, E)$, and $B^{\mathrm{loc}}_{p,k}(\Omega, E) \hookrightarrow D'_\omega(\Omega, E)$. We shall also use the spaces $B^c_{p,k}(\Omega, E)$ which generalize the scalar spaces $B^c_{p,k}(\Omega)$ considered by Hörmander in [13], by Vogt in [33] and by Björck in [1]. If ω, k, p, Ω and E are as above, then $B^c_{p,k}(\Omega, E) = \bigcup_{j=1}^{\infty}[B_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)]$ (here (K_j) is any fundamental sequence of compact subsets of Ω and $\mathcal{E}'_\omega(K_j, E)$ denotes the set of all $T \in D'_\omega(E)$ such that $\operatorname{supp} T \subset K_j$). Since for every compact $K \subset \Omega$, $B_{p,k}(E) \cap \mathcal{E}'_\omega(K, E)$ is a Fréchet space with the topology induced by $B_{p,k}(E)$, it follows that $B^c_{p,k}(\Omega, E)$ becomes a strict (LF)–space (strict (LB)–space if E is a Banach space): $B^c_{p,k}(\Omega, E) = \operatorname{ind}_{\underset{j}{\to}}[B_{p,k}(E) \cap \mathcal{E}'_\omega(K_j, E)]$. These spaces are studied in [24], [25] and [27].

3 The dual of $B^c_{p,k}(\Omega, E)$

In [13, Chapter XV] Hörmander studies the behaviour of the Fourier–Laplace transform in the space $B^c_{2,k}(\Omega) = \operatorname{ind}_{\underset{K}{\to}}[B_{2,k} \cap \mathcal{E}'(K)]$ when Ω is an open convex set in $\mathbb{R}^n$ and k is a temperate weight function on $\mathbb{R}^n$. For this he discusses the inductive limit topology in $B^c_{2,k}(\Omega)$, proves the isomorphism $\left(B^c_{2,k}(\Omega)\right)'_b \simeq B^{\mathrm{loc}}_{2,1/\check{k}}(\Omega)$ [13, Section 15.2] and shows that every continuous seminorm in $B^c_{2,k}(\Omega)$ is bounded by a seminorm of the form

$$u \longrightarrow \left(\int |\hat{u}(\zeta)|^2 e^{-2\phi(\zeta)} d\lambda(\zeta)\right)^{1/2}$$

where $\hat{u}$ is the Fourier–Laplace transform of u and ϕ is plurisubharmonic. In this section we extend the former isomorphism to Hörmander spaces in the sense of Beurling–Björck [1] and prove that $(B^c_{p,k}(\Omega, E))'_b \simeq B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega, E')$ when $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p < \infty$ and E is a Banach space. A number of applications of this duality will be given in the next section.

Let us recall that a $D_\omega(\Omega)$–partition of unity in Ω (=open set in $\mathbb{R}^n$) is a sequence (θ_j) in $D_\omega(\Omega)$ such that: i) $\theta_j \ge 0$ for $j = 1, 2, \dots$, ii) $\sum_j \theta_j \equiv 1$ in Ω, iii) For every compact set $K \subset \Omega$ there exist a positive integer m and a bounded open set W such that $K \subset W \subset \bar{W} \subset \Omega$ and $\sum_{j=1}^m \theta_j \equiv 1$ in W.

Lemma 3.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p \le \infty$, and E a Banach space. Let (θ_j) be a $D_\omega(\Omega)$–partition of unity in Ω. Then the inductive limit topology on $B^c_{p,k}(\Omega, E)$ is generated by the seminorms*

$$\|T\|_{(C_j)} = \sum_{j=1}^{\infty} C_j \|\theta_j T\|_{p,k} \ , \qquad T \in B^c_{p,k}(\Omega, E) \ ,$$

varying (C_j) in $\mathbb{R}^{\mathbb{N}}_+$.

Proof. See Proposition 3.10 of [27]. □

In the next result we will need the spaces $l_1(C_j, E)$ and $l_\infty(C_j, E)$: If (C_j) is a sequence in $\mathbb{R}^{\mathbb{N}}_+$ and E is a Banach space then $l_1(C_j, E)$ (resp. $l_\infty(C_j, E)$) denotes the set of all sequences $(x_j) \in E^{\mathbb{N}}$ such that $\|(x_j)\|_1 = \sum_{j=1}^{\infty} C_j \|x_j\|_E < \infty$ (resp. $\|(x_j)\|_\infty = \sup_j C_j \|x_j\|_E < \infty$). With the norm $\|\cdot\|_1$ (resp. $\|\cdot\|_\infty$) $l_1(C_j, E)$ (resp. $l_\infty(C_j, E)$) becomes a Banach space.

Theorem 3.2. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p < \infty$, and let E be a Banach space whose dual E' possesses the Radon–Nikodým property. Then $B^{loc}_{p',1/\tilde{k}}(\Omega, E')$ is isomorphic to $(B^c_{p,k}(\Omega, E))'_b$.*

Proof. Choose a fixed $D_\omega(\Omega)$–partition of unity (θ_j) in Ω and let L be an element in $(B^c_{p,k}(\Omega, E))'$. By Lemma 3.1 we can find a sequence (C_j) in $\mathbb{R}^{\mathbb{N}}_+$ such that

$$|L(T)| \le \sum_{j=1}^{\infty} C_j \|\theta_j T\|_{p,k} \ , \qquad T \in B^c_{p,k}(\Omega, E) \ .$$

Then the linear mapping

$$\begin{array}{rcl} Z : B^c_{p,k}(\Omega, E) & \longrightarrow & l_1(C_j, B_{p,k}(E)) \\ T & \rightarrow & (\theta_j T) \end{array}$$

is continuous. Furthermore, since each T can be written in the form $T = \sum_{j=1}^m \theta_j T$ (m varying with T), we conclude that Z is injective. Now we consider the linear form $L \circ Z^{-1}$. Since $|L \circ Z^{-1}((\theta_j T))| \leq \|(\theta_j T)\|_1$, the Hahn–Banach theorem shows that there exists a linear form $(L \circ Z^{-1})^{-} \in (l_1(C_j, B_{p,k}(E)))'$ of norm at most 1 which extends $L \circ Z^{-1}$. Then, by the isometric isomorphism

$$A : l_\infty(\frac{1}{C_j}, B_{p',1/k}(E')) \longrightarrow (l_1(C_j, B_{p,k}(E)))'$$

defined by $< (T_j), A((S_j)) >= (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} < \hat{T}_j(x), \hat{S}_j(x) > dx$, we can find $(S_j) \in l_\infty(\frac{1}{C_j}, B_{p',1/k}(E'))$ such that $A((S_j)) = (L \circ Z^{-1})^{-}$, and so

$$L \circ Z^{-1}((\theta_j T)) = L(T) = (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} < \widehat{\theta_j T}(x), \hat{S}_j(x) > dx$$

for each $T \in B^c_{p,k}(\Omega, E)$. Next we shall prove that the linear mapping

$$\begin{array}{rcl} \Phi : (B^c_{p,k}(\Omega, E))'_b & \longrightarrow & B^{\rm loc}_{p',1/\tilde{k}}(\Omega, E') \\ L & \rightarrow & \sum_{j=1}^\infty \theta_j \tilde{S}_j \end{array}$$

(the series $\sum_{j=1}^\infty \theta_j \tilde{S}_j$ converges in $B^{\rm loc}_{p',1/\tilde{k}}(\Omega, E')$ since this space is a Fréchet space and $\sum_{j=1}^\infty \|\theta_j \tilde{S}_j\|_{p',1/\tilde{k},\varphi} = \sum_{j=1}^\infty \|(\theta_j \varphi)\tilde{S}_j\|_{p',1/\tilde{k}} < \infty$ for each $\varphi \in D_\omega(\Omega)$ in virtue of the properties of the sequence (θ_j)) is an isomorphism. Let us see that Φ is well defined. Let $(L \circ Z^{-1})^{=}$ another extension of $L \circ Z^{-1}$ to $l_1(C_j, B_{p,k}(E))$ and let $(S^1_j) \in l_\infty(\frac{1}{C_j}, B_{p',1/k}(E'))$ the sequence which represents this extension. Let us check that $\sum_{j=1}^\infty \theta_j \tilde{S}_j = \sum_{j=1}^\infty \theta_j \tilde{S}^1_j$. By Fourier's inversion formula, the properties of the Bochner integral and the embedding $B^{\rm loc}_{p',1/\tilde{k}}(\Omega, E') \hookrightarrow D'_\omega(\Omega, E')$ (see Section 2) we have for all $\varphi \in D_\omega(\Omega)$ and all $e \in E$

$$< \varphi, \sum_{j=1}^\infty \theta_j \tilde{S}_j >= \sum_{j=1}^\infty < \varphi, \theta_j \tilde{S}_j >= \sum_{j=1}^\infty < \varphi\theta_j, \tilde{S}_j >= (2\pi)^{-n} \sum_{j=1}^\infty < \widehat{\varphi\theta_j}, \hat{S}_j >$$

and

$$\begin{aligned} (2\pi)^{-n} < e, \sum_{j=1}^\infty < \widehat{\varphi\theta_j}, \hat{S}_j >> &= (2\pi)^{-n} \sum_{j=1}^\infty < e, < \widehat{\varphi\theta_j}, \hat{S}_j >> \\ &= (2\pi)^{-n} \sum_{j=1}^\infty < e, \int_{\mathbb{R}^n} \widehat{\theta_j \varphi}(x) \hat{S}_j(x)\, dx > \\ &= (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} < (\theta_j(\varphi \otimes e))^\wedge(x), \hat{S}_j(x) > dx \\ &= L(\varphi \otimes e)\ . \end{aligned}$$

Repeating the argument with $\sum_{j=1}^\infty \theta_j \tilde{S}^1_j$ we conclude that $\sum_{j=1}^\infty \theta_j \tilde{S}_j = \sum_{j=1}^\infty \theta_j \tilde{S}^1_j$. Now let $(C'_j) \in \mathbb{R}^{\mathbb{N}}_+$ another sequence such that $|L(T)| \leq \sum_{j=1}^\infty C'_j \|\theta_j T\|_{p,k}$ for

$T \in B^c_{p,k}(\Omega, E)$. Let Z' be the corresponding operator, let $(L \circ Z'^{-1})^-$ be an extension of $L \circ Z'^{-1}$ to $l_1(C'_j, B_{p,k}(E))$ and let $(S'_j) \in l_\infty(\frac{1}{C'_j}, B_{p',1/k}(E'))$ the sequence which represents this extension, then $L(T) = (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} < \widehat{\theta_j T}(x), \hat{S}'_j(x) > dx$, $T \in B^c_{p,k}(\Omega, E)$, and also $< e, < \varphi, \sum_{j=1}^\infty \theta_j \tilde{S}'_j >>= L(\varphi \otimes e)$ for $\varphi \in D_\omega(\Omega)$ and $e \in E$. Then Φ is well defined. If $\Phi(L) = 0$ then $< e, < \varphi, \Phi(L) >>= 0 = L(\varphi \otimes e)$ for all $\varphi \in D_\omega(\Omega)$ and all $e \in E$, thus $L = 0$ on $D_\omega(\Omega) \otimes E$. Since this space is dense in $D_\omega(\Omega, E)$ (see Section 2) and $D_\omega(\Omega, E) \overset{d}{\hookrightarrow} B^c_{p,k}(\Omega, E)$ (see Proposition 3.6 of [27]), it follows that $L = 0$. Consequently, Φ is one–to–one. Furthermore, Φ is surjective: Let (χ_j) a sequence in $D_\omega(\Omega)$ such that $\chi_j = 1$ in a compact neighborhood of $\operatorname{supp}\theta_j$. Let S be an element of $B^{\text{loc}}_{p',1/\tilde{k}}(\Omega, E')$. Then we have (convergence in $B^{\text{loc}}_{p',1/\tilde{k}}(\Omega, E')$) $S = \sum_{j=1}^\infty \theta_j S = \sum_{j=1}^\infty (\theta_j \chi_j) S = \sum_{j=1}^\infty \theta_j (\chi_j S) = \sum_{j=1}^\infty \theta_j \tilde{X}_j$ where $X_j = \widetilde{\chi_j S}$. Now we define the functional

$$L(T) = (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} < \widehat{\theta_j T}(x), \hat{X}_j(x) > dx \ , \qquad T \in B^c_{p,k}(\Omega, E) \ .$$

Since

$$\begin{aligned} |L(T)| &\leq (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} \|\widehat{\theta_j T}(x)\|_E \, k(x) \, \|\hat{X}_j(x)\|_{E'} \frac{1}{k(x)} \, dx \\ &\leq \sum_{j=1}^\infty \|\theta_j T\|_{p,k} \|X_j\|_{p',1/k} \end{aligned}$$

for all $T \in B^c_{p,k}(\Omega, E)$, it follows that $L \in (B^c_{p,k}(\Omega, E))'$. Then $\Phi(L) = S$ and Φ is surjective.

Now we prove that Φ^{-1} is continuous: Let A be a bounded set in $B^c_{p,k}(\Omega, E)$. Since this space is a strict (LB)–space, there is a compact set M in Ω such that A is contained and bounded in the step $B_{p,k}(E) \cap \mathcal{E}'_\omega(M, E)$ (see [18, (4) p. 223]). Take a sequence (χ_j) in $D_\omega(\Omega)$ such that $\chi_j = 1$ in a compact neighborhood of $\operatorname{supp}\theta_j$, $j = 1, 2, \ldots$, and let m be such that $\theta_j = 0$ in M for all $j > m$. Then, taking into account Proposition 3.4 of [27] and that every $S \in B^{\text{loc}}_{p',1/\tilde{k}}(\Omega, E')$ can be written in the form $S = \sum_{j=1}^\infty \theta_j \tilde{X}_j$ with $X_j = \widetilde{\chi_j S}$, we get

$$\begin{aligned} \sup_{T \in A} |\Phi^{-1}(S)(T)| &= \sup_{T \in A} \left| (2\pi)^{-n} \sum_{j=1}^\infty \int_{\mathbb{R}^n} < \widehat{\theta_j T}(x), \hat{X}_j(x) > dx \right| \\ &\leq \sup_{T \in A} \sum_{j=1}^m \|\theta_j T\|_{p,k} \|X_j\|_{p',1/k} \\ &\leq \sup_{T \in A} \sum_{j=1}^m \|\theta_j\|_{1,M_k} \|T\|_{p,k} \|S\|_{p',1/\tilde{k},\chi_j} \end{aligned}$$

$$\leq C\sum_{j=1}^{m}\|\theta_j\|_{1,M_k}\|S\|_{p',1/\tilde{k},\chi_j}$$

for all $S\in B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega,E')$ (C is a constant >0). Hence it follows the continuity of Φ^{-1}. Then Φ becomes an isomorphism since $B^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega,E')$ and $(B^{c}_{p,k}(\Omega,E))'_b$ are Fréchet spaces ($B^{c}_{p,k}(\Omega,E)$ is a (DF)–space by [18, (4) p. 402] and so its strong dual is a Fréchet space (see [18, (1) p. 397])). The proof is complete. □

Remark 3.3. When $k(x)$ is a temperate weight function, $p=2$ and $E=\mathbb{C}$, our theorem yields the isomorphism which appears in [13, p. 279].

In [32] the spaces $BV_{p,k}(E)$ are introduced (by using the natural embedding of the space $V_p(k^p dx,E)$ of the finitely additive E–valued measures of bounded p–variation into the space $S'_\omega(E)$) and the isometric isomorphism $BV_{p',1/k}(E')\simeq(B_{p,k}(E))'$ is shown (E is any Banach space and $1\leq p<\infty$). In view of this result and our Theorem 3.2 we can define the space

$$BV^{\mathrm{loc}}_{p,k}(\Omega,E)=\{T\in D'_\omega(\Omega,E):\varphi T\in BV_{p,k}(E)\text{ for all }\varphi\in D_\omega(\Omega)\}$$

(equipped with the topology generated by the family of seminorms $\{T\rightarrow\|(2\pi)^{-n/p}\widehat{\varphi T}\|_{V_p(k^p dx,E)}:\varphi\in D_\omega(\Omega)\}$ when $p<\infty$ (resp. $\{T\rightarrow\|\widehat{\varphi T}\|_{V_\infty(\frac{1}{k}dx,E)}:\varphi\in D_\omega(\Omega)\}$ if $p=\infty$)) and propose the following question.

Problem 3.4. Let Ω be an open set in $\mathbb{R}^n$, $\omega\in\mathcal{M}$, $k\in\mathcal{K}_\omega$, $1\leq p<\infty$ and let E be a Banach space. Are the spaces $BV^{\mathrm{loc}}_{p',1/\tilde{k}}(\Omega,E')$ and $(B^{c}_{p,k}(\Omega,E))'_b$ isomorphic?

4 On sequence space representations of spaces of ultradistributions

In this section we give a number of results on sequence space representations of spaces of distributions and ultradistributions. Based on these and using the solution to Problem 4.11 in [24] given by Cembranos and Mendoza in [3], we partially answer the Problem 4.10 in [24]. We also give a new proof of a well known result: The short sequence

$$0\longrightarrow N(P(D))\longrightarrow B^{\mathrm{loc}}_{p,k}(\Omega)\overset{P(D)}{\longrightarrow}B^{\mathrm{loc}}_{p,k/P'}(\Omega)\longrightarrow 0$$

does not split ($P(D)$ is an elliptic operator with constant coefficients and $P'(\xi)=\left(\sum_\alpha|\partial^\alpha P(\xi)|^2\right)^{1/2}$). (The proof we give is based on the isomorphism $B^{\mathrm{loc}}_{p,k}(\Omega)\simeq l_p^{\mathbb{N}}$.)

We shall omit the proof of the following simple result.

Lemma 4.1. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p \le \infty$, and let $(E_j)_{j=1}^{\infty}$ be a sequence of Banach spaces. Then the space $B_{p,k}^{loc}(\Omega, \prod_{j=1}^{\infty} E_j)$ is isomorphic to $\prod_{j=1}^{\infty} B_{p,k}^{loc}(\Omega, E_j)$.*

Theorem 4.2. *Let Ω be an open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, and let E be a Banach space. Then: (1) $B_{1,k}^{c}(\Omega, E)$ is isomorphic to $(l_1(E))^{(\mathbb{N})}$, (2) $B_{1,k}^{loc}(\Omega, E)$ is isomorphic to $(l_1(E))^{\mathbb{N}}$, (3) If E is a dual space and has the Radon–Nikodým property then $B_{\infty,k}^{loc}(\Omega, E)$ is isomorphic to $(l_\infty(E))^{\mathbb{N}}$.*

Proof. (1) and (2) The proof given in [33] is also valid in the vector–valued case and for weights $k \in \mathcal{K}_\omega$. (3) Suppose $E \simeq F'$ and recall that if $(E_j)_{j=1}^{\infty}$ is a sequence of Banach spaces then the space $(\bigoplus_{j=1}^{\infty} E_j)'_b$ is isomorphic to $\prod_{j=1}^{\infty} E'_j$ (see [18, p. 287]). Then, taking into account Theorem 3.2 and (1), we get

$$B_{\infty,k}^{\mathrm{loc}}(\Omega, E) \simeq \left(B_{1,1/\tilde{k}}^{c}(\Omega, F)\right)'_b \simeq \left((l_1(F))^{(\mathbb{N})}\right)'_b \simeq (l_\infty(E))^{\mathbb{N}} .$$

□

Theorem 4.3. *$l_\infty(l_1)$ and $l_1(l_\infty)$ are not isomorphic.*

Proof. See [3, Theorem 1]. □

Next we answer the Problem 4.10 in [24] when $q = \infty$.

Theorem 4.4. *If Ω_1 is an open set in $\mathbb{R}^{n_1}$, $\omega_1 \in \mathcal{M}_{n_1}$ and $k_1 \in \mathcal{K}_{\omega_1}$ (resp. Ω_2 open set in $\mathbb{R}^{n_2}$, $\omega_2 \in \mathcal{M}_{n_2}$, $k_2 \in \mathcal{K}_{\omega_2}$), then the spaces $B_{1,k_1}^{loc}(\Omega_1, B_{\infty,k_2}^{loc}(\Omega_2))$ and $B_{\infty,k_2}^{loc}(\Omega_2, B_{1,k_1}^{loc}(\Omega_1))$ are not isomorphic.*

Proof. By using the previous results we have the isomorphisms

$$\begin{aligned} B_{1,k_1}^{\mathrm{loc}}(\Omega_1, B_{\infty,k_2}^{\mathrm{loc}}(\Omega_2)) &\simeq B_{1,k_1}^{\mathrm{loc}}(\Omega_1, l_\infty^{\mathbb{N}}) \simeq \left(B_{1,k_1}^{\mathrm{loc}}(\Omega_1, l_\infty)\right)^{\mathbb{N}} \\ &\simeq \left((l_1(l_\infty))^{\mathbb{N}}\right)^{\mathbb{N}} \simeq (l_1(l_\infty))^{\mathbb{N}} \end{aligned}$$

and

$$\begin{aligned} B_{\infty,k_2}^{\mathrm{loc}}(\Omega_2, B_{1,k_1}^{\mathrm{loc}}(\Omega_1)) &\simeq B_{\infty,k_2}^{\mathrm{loc}}(\Omega_2, l_1^{\mathbb{N}}) \simeq \left(B_{\infty,k_2}^{\mathrm{loc}}(\Omega_2, l_1)\right)^{\mathbb{N}} \\ &\simeq \left((l_\infty(l_1))^{\mathbb{N}}\right)^{\mathbb{N}} \simeq (l_\infty(l_1))^{\mathbb{N}} . \end{aligned}$$

Suppose now that our iterated spaces are isomorphic. Then $(l_1(l_\infty))^{\mathbb{N}}$ and $(l_\infty(l_1))^{\mathbb{N}}$ are also isomorphic. Hence it follows (by [4]) that there exist positive integers α, β such that $l_1(l_\infty) < (l_\infty(l_1))^\alpha \simeq l_\infty(l_1)$ and $l_\infty(l_1) < (l_1(l_\infty))^\beta \simeq l_1(l_\infty)$. Then, using Pelczynski's decomposition method, we conclude that

$l_1(l_\infty) \simeq l_\infty(l_1)$. This contradicts Theorem 4.3. In consequence, $B^{\rm loc}_{1,k_1}(\Omega_1, B^{\rm loc}_{\infty,k_2}(\Omega_2))$ and $B^{\rm loc}_{\infty,k_2}(\Omega_2, B^{\rm loc}_{1,k_1}(\Omega_1))$ are not isomorphic. □

Remark 4.5. 1. We must point out that the space $B^{\rm loc}_{\infty,k_2}(\Omega_2, B^{\rm loc}_{1,k_1}(\Omega_1))$ even contains no complemented subspace isomorphic to $B^{\rm loc}_{1,k_1}(\Omega_1, B^{\rm loc}_{\infty,k_2}(\Omega_2))$ (see the proof of Theorem 4.4 and use the final remarks of [3]).

2. Note also that, in general, $B^{\rm loc}_{\infty,k}(\Omega, E)$ is not isomorphic to either $B^{\rm loc}_{\infty,k}(\Omega)\widehat{\otimes}_\varepsilon E$ or $B^{\rm loc}_{\infty,k}\widehat{\otimes}_\pi E$: In fact, let $1 \le p < \infty$ and assume that $B^{\rm loc}_{\infty,k}(\Omega, l_p)$ is isomorphic to $B^{\rm loc}_{\infty,k}(\Omega)\widehat{\otimes}_\varepsilon l_p$. Then, by virtue of [19, (5) p. 282], [19, (2) p. 287], Theorem 4.2 and a result of Cembranos and Freniche [2, Theorem 3.2.1], we get

$$(l_\infty(l_p))^{\mathbb{N}} \simeq l_\infty^{\mathbb{N}}\widehat{\otimes}_\varepsilon l_p \simeq (l_\infty\widehat{\otimes}_\varepsilon l_p)^{\mathbb{N}} \simeq (C(\beta\mathbb{N})\widehat{\otimes}_\varepsilon l_p)^{\mathbb{N}} \simeq (C(\beta\mathbb{N}, l_p))^{\mathbb{N}} > c_0^{\mathbb{N}} .$$

Hence it follows, arguing as in Theorem 4.4, that $l_\infty(l_p)$ contains a complemented copy of c_0. Then, by a result of Leung and Räbiger [2, Theorem 5.1.1], l_p also contains a complemented copy of c_0. This contradiction shows that $B^{\rm loc}_{\infty,k}(\Omega, l_p)$ and $B^{\rm loc}_{\infty,k}\widehat{\otimes}_\varepsilon l_p$ are not isomorphic. On the other hand, since by Theorem 4.2 and [19, (5) p. 194] we have

$$\begin{aligned} B^{\rm loc}_{\infty,k}(\Omega, l_1) &\simeq (l_\infty(l_1))^{\mathbb{N}} \\ B^{\rm loc}_{\infty,k}(\Omega)\widehat{\otimes}_\pi l_1 &\simeq l_\infty^{\mathbb{N}}\widehat{\otimes}_\pi l_1 \simeq (l_\infty\widehat{\otimes}_\pi l_1)^{\mathbb{N}} \simeq (l_1(l_\infty))^{\mathbb{N}} , \end{aligned}$$

it follows that the spaces $B^{\rm loc}_{\infty,k}(\Omega, l_1)$ and $B^{\rm loc}_{\infty,k}(\Omega)\widehat{\otimes}_\pi l_1$ are not isomorphic.

In the next theorem the following elementary fact will be used: "Let $F = \operatorname{ind}_{\vec{j}} F_j$ be the strict inductive limit of a properly increasing sequence $F_1 \subset F_2 \subset \cdots$ of Banach spaces. Assume that every F_j is a complemented subspace of F_{j+1} and that G_j is a topological complement of F_j in F_{j+1}. Then, the mapping $F_1 \oplus G_1 \oplus G_2 \oplus \cdots \to F : (f_1, g_1, g_2, \dots) \to f_1 + g_1 + g_2 + \cdots$ is an isomorphism". We will also need the weighted L_p–spaces of vector–valued entire analytic functions $L^K_{p,k}(E)$ and the operators $S_K(f) = \mathcal{F}^{-1}(\chi_K \hat{f})$ (see [25]).

Theorem 4.6. *Let Ω be an open set in $\mathbb{R}^n$. Assume $1 < p, q < \infty$ and let k be a temperate weight function on $\mathbb{R}^n$ with $k^p \in A^*_p$. Then the space $B^c_{p,k}(\Omega, l_q)$ (resp. $B^{loc}_{p,k}(\Omega, l_q)$) is isomorphic to $\bigoplus_{j=0}^\infty G_j$ (resp. $\prod_{j=0}^\infty H_j$) where G_0 (resp. H_0) is isomorphic to $l_p(l_q)$ and G_j (resp. H_j) is isomorphic to a complemented subspace of $l_p(l_q)$ for $j = 1, 2, \dots$.*

Proof. Let (K_j) be a covering of Ω consisting of compact sets such that $K_j \subset \mathring{K}_{j+1}$, $K_j = \overline{\mathring{K}_j}$ and $\mathring{K}_j$ has the segment property (we may also assume, without loss of generality, that each K_j is a finite union of n–dimensional compact intervals). Then $B^c_{p,k}(\Omega, l_q) = \operatorname{ind}_{\underset{j}{\to}}[B_{p,k}(l_q) \cap \mathcal{E}'(K_j, l_q)]$. In this inductive limit, the step $B_{p,k}(l_q) \cap \mathcal{E}'(K_j, l_q)$ is isomorphic (via the Fourier transform) to $L^{-K_j}_{p,k}(l_q)$ and this space is isomorphic, by Corollaries 4.2 and 5.1 of [25], to $l_p(l_q)$. Furthermore, $L^{-K_j}_{p,k}(l_q)$ is a complemented subspace of $L^{-K_{j+1}}_{p,k}(l_q)$: $L^{-K_j}_{p,k}(l_q) \oplus [\ker S_{-K_j} \cap L^{-K_{j+1}}_{p,k}(l_q)] = L^{-K_{j+1}}_{p,k}(l_q)$. Thus, the space $G_j = \ker S_{-K_j} \cap L^{-K_{j+1}}_{p,k}(l_q)$ is isomorphic to an infinite–dimensional complemented subspace of $l_p(l_q)$. Then, by using the former result, we obtain

$$B^c_{p,k}(\Omega, l_q) \simeq L^{-K_1}_{p,k}(l_q) \oplus G_1 \oplus G_2 \oplus \cdots \simeq l_p(l_q) \oplus G_1 \oplus G_2 \oplus \cdots .$$

Next, since $1/\tilde{k}$ is a temperate weight function on $\mathbb{R}^n$ such that $1/\tilde{k}^{p'} \in A^*_{p'}$, we see that $B^c_{p',1/\tilde{k}}(\Omega, l_{q'}) \simeq \bigoplus_{j=0}^{\infty} B_j$ where $B_0 \simeq l_{p'}(l_{q'})$ and $B_j < l_{p'}(l_{q'})$ for $j = 1, 2, \ldots$. Therefore, by Theorem 3.2, we get

$$B^{\mathrm{loc}}_{p,k}(\Omega, l_q) \simeq \left(B^c_{p',1/\tilde{k}}(\Omega, l_{q'})\right)'_b \simeq \Big(\bigoplus_{j=0}^{\infty} B_j\Big)'_b \simeq \prod_{j=0}^{\infty} B'_j = \prod_{j=0}^{\infty} H_j$$

(here $H_j = B'_j$) where $H_0 \simeq l_p(l_q)$ and $H_j < l_p(l_q)$ for $j = 1, 2, \ldots$, and the proof is complete. □

Remark 4.7. 1. Let Ω, p and k as in Theorem 4.6. In [25, Corollary 5.3] the space $B^c_{p,k}(\Omega, E)$ is showed to be isomorphic to $l_p^{(\mathbb{N})}$ if $\dim E < \infty$ or $E = l_p$, and to $(l_p(l_2))^{(\mathbb{N})}$ if $E = l_2$. By duality (Theorem 3.2) it follows that $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$, $B^{\mathrm{loc}}_{p,k}(\Omega, l_p) \simeq l_p^{\mathbb{N}}$ and $B^{\mathrm{loc}}_{p,k}(\Omega, l_2) \simeq (l_p(l_2))^{\mathbb{N}}$.

2. Note that, in general, $B^{\mathrm{loc}}_{p,k}(\Omega, E)$ is not isomorphic to either $B^{\mathrm{loc}}_{p,k}(\Omega)\widehat{\otimes}_\varepsilon E$ or $B^{\mathrm{loc}}_{p,k}(\Omega)\widehat{\otimes}_\pi E$: In fact, let Ω, p, q and k as in Theorem 4.6 and assume that $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ is isomorphic to $B^{\mathrm{loc}}_{p,k}(\Omega)\widehat{\otimes}_\varepsilon l_q$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega)\widehat{\otimes}_\pi l_q$). Then, by Theorem 4.6, the previous note, [19, (5), p. 282] and [19, (5), p. 194], we get

$$\prod_{j=0}^{\infty} H_j \simeq l_p^{\mathbb{N}}\widehat{\otimes}_\varepsilon l_q \simeq \left(l_p\widehat{\otimes}_\varepsilon l_q\right)^{\mathbb{N}} \qquad \left(\text{resp. } \prod_{j=0}^{\infty} H_j \simeq \left(l_p\widehat{\otimes}_\pi l_q\right)^{\mathbb{N}}\right)$$

where $H_0 \simeq l_p(l_q)$ and $H_j < l_p(l_q)$ for $j = 1, 2, \ldots$. Hence it follows, reasoning as in Theorem 4.4, that $l_p(l_q) \simeq l_p\widehat{\otimes}_\varepsilon l_q$ (resp. $l_p\widehat{\otimes}_\pi l_q$) but this is false when $p' \leq q$ (resp. $p \leq q'$) by a result of Holub [11, Proposition 3.7] (resp. [11, Proposition 3.6]). In consequence, the spaces $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ and $B^{\mathrm{loc}}_{p,k}(\Omega)\widehat{\otimes}_\varepsilon l_q$ (resp. $B^{\mathrm{loc}}_{p,k}(\Omega)\widehat{\otimes}_\pi l_q$) are not isomorphic when $p' \leq q$ (resp. $p \leq q'$).

3. By using the previous results we can describe the structure of the complemented (normed) subspaces of $B^{\mathrm{loc}}_{p,k}(\Omega)$, $B^{\mathrm{loc}}_{p,k}(\Omega, l_q)$ and $\prod_{i=1}^m B^{\mathrm{loc}}_{p_i,k_i}(\Omega_i, l_{p_i})$: (i) Let X be an infinite–dimensional complemented (normed) subspace of $B^{\mathrm{loc}}_{p,k}(\Omega)$ (Ω open set in $\mathbb{R}^n$, $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$ and $p \in \{1, \infty\}$ or k temperate weight function on $\mathbb{R}^n$ such that $k^p \in A^*_p$ and $p \in (1, \infty)$). Then $B^{\mathrm{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$ and thus X becomes a complemented subspace of l_p. This implies, since l_p

is prime [20, Theorems 2.a.3, 2.a.7], that $X \simeq l_p$. (ii) Let X be an infinite–dimensional complemented (normed) subspace of $B^{\text{loc}}_{p,k}(\Omega, l_q)$ (Ω open set in $\mathbb{R}^n$, $p, q \in (1, \infty)$ and k temperate weight function on $\mathbb{R}^n$ with $k^p \in A^*_p$). Then, since $B^{\text{loc}}_{p,k}(\Omega, l_q) < (l_p(l_q))^{\mathbb{N}}$ in virtue of Theorem 4.6, X becomes a complemented subspace of $l_p(l_q)$. This implies, in the case $q = 2$, that X is isomorphic to either l_2, l_p, $l_2 \oplus l_p$ or $l_p(l_2)$ by a result of Odell [26]. (iii) Let X be an infinite–dimensional complemented (normed) subspace of $\prod_{i=1}^m B^{\text{loc}}_{p_i,k_k}(\Omega_i, l_{p_i})$ (Ω_i open set in $\mathbb{R}^n$, $1 < p_1 < \cdots < p_m < \infty$, k_i temperate weight function on $\mathbb{R}^n$ with $k_i^{p_i} \in A^*_{p_i}$, $i = 1, \ldots, m$). Then, since

$$\prod_{i=1}^m B^{\text{loc}}_{p_i,k_i}(\Omega_i, l_{p_i}) \simeq \prod_{i=1}^m l_{p_i}^{\mathbb{N}} \simeq (l_{p_1} \oplus \cdots \oplus l_{p_m})^{\mathbb{N}} ,$$

we have that $X < l_{p_1} \oplus \cdots \oplus l_{p_m}$ and so there exist $1 \le i_1 < \cdots < i_k \le m$ such that $X \simeq l_{p_{i_1}} \oplus \cdots \oplus l_{p_{i_k}}$ in virtue of [20, Theorem 2.c.14].

4. We omit the proof of the following result:

$$B^{\text{loc}}_{p_1,k_1}(\Omega_1, l_{q_1}) \simeq B^{\text{loc}}_{p_2,k_2}(\Omega_2, l_{q_2}) \Longleftrightarrow p_1 = p_2 \text{ and } q_1 = q_2$$

(Ω_i open set in $\mathbb{R}^n$, $p_i, q_i \in (1, \infty)$, k_i temperate weight function on $\mathbb{R}^n$ with $k_i^{p_i} \in A^*_{p_i}$, $i = 1, 2$).

We conclude this section by showing a result on linear partial differential operators (the result is well known, see e.g. [21], [22], [31] and [34]). The proof we give is based on our representation theorem $B^{\text{loc}}_{p,k}(\Omega) \simeq l_p^{\mathbb{N}}$.

Theorem 4.8. *Let Ω be an open set in $\mathbb{R}^n$ ($n \ge 2$), $1 < p < \infty$, k a temperate weight function on $\mathbb{R}^n$ such that $k^p \in A^*_p$ and $P(D)$ an elliptic operator with constant coefficients. Then the short sequence*

$$0 \longrightarrow N(P(D)) \longrightarrow B^{loc}_{p,k}(\Omega) \xrightarrow{P(D)} B^{loc}_{p,k/P'}(\Omega) \longrightarrow 0$$

is exact and does not split, i.e., the operator $P(D)$ has no continuous linear right inverse (here $N(P(D))$ is the kernel of $P(D)$).

Proof. $P(D)$ is well defined by [13, Theorem 10.1.11] and the short sequence is exact in virtue of [13, Corollary 10.8.2] and [13, Theorem 10.6.7]. The closed subspace $N(P(D))$ of $B^{\text{loc}}_{p,k}(\Omega)$ coincides, algebraic and topologically, with the subspace $N(\Omega) = \{f \in \mathcal{E}(\Omega) : P(D)f = 0\}$ of $\mathcal{E}(\Omega)$ (by [12, Theorem 1.11.10], [12, Theorem 1.11.11] and the closed graph theorem) and thus it is a nuclear Fréchet space. Note also that, for every connected component O of Ω, the space $N(O)$ equipped with the topology induced by $\mathcal{E}(O)$, is a nuclear Fréchet space with continuous norms (since all $f \in N(O)$ is real analytic in O, see e.g. [1, Corollary 4.1.4]) isomorphic to a complemented subspace of $N(P(D))$). Now assume that the short sequence splits. Then $N(P(D))$ is a complemented subspace of $B^{\text{loc}}_{p,k}(\Omega)$. Since this space is isomorphic to $l_p^{\mathbb{N}}$ by Remark 4.7/1, it follows that, for any connected component O of Ω, the space $N(O)$ becomes isomorphic to an infinite–dimensional ($n \geq 2$) complemented subspace of $l_p^{\mathbb{N}}$. This implies, taking into account a result of Metafune and Moscatelli [23, Theorem 1.2], that $N(O)$ is isomorphic to either l_p, $l_p \times \omega$, ω or $l_p^{\mathbb{N}}$. This contradiction completes the proof. □

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