Vector measures on δ–rings and representation theorems for Banach lattices with the σ-Fatou property

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Chapter 1

Introduction

1.1 The aim of this work.

The representation problem for Banach lattices using vector measures consists on determining which Banach lattices can be identified as spaces $L^1(\nu)$ and $L^1_w(\nu)$ of integrable functions and weakly integrable functions with respect to a vector measure ν, respectively. It is known that every order continuous Banach lattice with a weak unit can be identified with a space $L^1(\nu)$, where ν is defined in this case on a σ–algebra (see [6, Theorem 8]). If the existence of a weak order unit is not assumed (this is the case for instance of $l^\infty(\Gamma)$ with Γ uncountable), it is still possible to represent it but using in this case a vector measure on a δ–ring (see [9, Theorem 4]). In the case that the space has the σ–Fatou property and it has a weak unit belonging to E_a the order continuous part of the lattice, it can be identified with the space $L^1_w(\nu)$ where ν is also defined on a σ–algebra (see [7, Theorem 2.5]). A similar result is possible if we forget about the weak unit and consider vector measures defined on a δ-ring, as it happens in the case when E is order continuous. Indeed, if E has the Fatou property and E_a is order dense in E, then E is order isometric to $L^1_w(\nu)$ (see [9, Theorem 8]).

Even if there exists a weak unit in the space there are two different possibilities to represent the space which are essentially different. Namely,

1) *The first one involves vector measures defined on σ–algebras. In this*

case the existence of a weak unit in the space is necessary and the representation theorem when applied to the case of Banach function spaces over σ–finite (but not finite) measure spaces as Banach lattice is given by a multiplication operator different of the identity map.

2) *In the second one, no weak unit is needed, the vector measure ν is defined on a δ–ring and we must consider the extension of the integration theory to vector measures defined on δ–rings due to Lewis [12] and Masani and Niemi [15, 16], and use the associated spaces $L^1(\nu)$ studied in [8]. Now, if we consider $X(\mu)$ a Banach function space over a σ-finite measure space and we forget about the existence of a weak unit in the space, the order isometry $T : X(\mu) \to L^1(\nu)$ can be given, as in the general case, by the identity map, i.e. both spaces can be directly identified having the same elements.*

In this work we are interested in provide additional information on these representations that improves the knowledge on the behaviour of general Banach lattices and their integral representations. More precisely, we are interested in developing the representation theorem for a particular class of Banach lattices using the approach explained in 2). In particular, we deal with Banach lattices with the σ-Fatou property that are "*locally order continuous*", i.e. there is a class of projections in the space such that the range of all of them lies in the order continuous part of the lattice and each element can be expressed as a sum of the components given by these projections. The canonical example of these spaces is $\sum_{\ell^\infty(I)} L^1(\mu_i)$, where each μ_i is a probability measure, and I is a non necessarily countable index set. Therefore, our results can be applied also when there is no weak unit in the lattice E. In Chapter 2 we present the canonical example that motivates our results, it is also a concrete example which shows, precisely, the differences explained in the approaches above. Chapter 4 is devoted to prove our main result.

Another line of research appears when we try to solve this problem. The properties that satisfy a vector measure ν defined on a δ–ring are directly related to the lattice properties of the spaces $L^1(\nu)$ of integrable functions with respect to it. It will be also the aim of this work to study the effect of certain properties of ν on the lattice properties of the space $L^1_w(\nu)$ of weakly integrable functions with respect to ν and Chapter 3 is devoted to develop our results in this context.

In fact, it is well-known that the space $L^1(\nu)$ of integrable functions with respect to a vector measure ν on a σ-algebra is always order continuous. For such a measure, the space $L^1_w(\nu)$ of weakly integrable functions always has the Fatou property, the σ-order continuous part $(L^1_w(\nu))_a$ of $L^1_w(\nu)$ coincides with $L^1(\nu)$ and the "Fatou completion" of $L^1(\nu)$ –the smaller Fatou Banach function space containing $L^1(\nu)$– is $L^1_w(\nu)$. We show that, when the decomposition properties regarding σ-finiteness of ν of the δ-ring $\mathcal{R}$ where the measure is defined become weaker, those basic facts that hold for the case of σ-algebras (i.e. finite or σ-finite vector measures) are not true any more. We also provide characterizations of these lattice properties for $L^1(\nu)$ and $L^1_w(\nu)$ in terms of the decomposition properties of $\mathcal{R}$ with respect to ν.

We will consider three decomposition properties for ν, local σ-finiteness, weak local σ-finiteness and $\mathcal{R}$-decomposability. Each of them characterize a lattice property of $L^1_w(\nu)$: super order density of $L^1(\nu)$ in $L^1_w(\nu)$, the equality $(L^1_w(\nu))_a = L^1(\nu)$ and a strong version of the Fatou property that implies weak local σ-finiteness, respectively.

1.2 Preliminaries and notation.

We shall deal with real Banach spaces. The topological dual of a Banach space X is denoted by X^* and the unit ball of X by B_X.

A Banach lattice is a Banach space E with norm $\|\cdot\|$, endowed with an order relation $\leq$ which is compatible with the algebraic structure of E such that for every pair x, y of elements of E, there exists the supremum and the infimum of x and y and satisfying that if $x, y \in E$ with $|x| \leq |y|$, then $\|x\| \leq \|y\|$, where $|x| = \sup\{x, -x\}$. An *ideal* of E is a closed subspace F of E such that $y \in F$ whenever $y \in E$ with $|y| \leq |x|$ for some $x \in F$. An ideal F in E is said to be *order dense* if for every $0 \leq x \in E$ there exists an upwards directed system $0 \leq x_\tau \uparrow x$ such that $(x_\tau)_\tau \subset F$. Similarly, an ideal F in E is said to be *super order dense* if for every $0 \leq x \in E$ there exists a sequence $0 \leq x_n \uparrow x$ such that $(x_n)_{n\geq 1} \subset F$. A *weak unit* of E is an element $0 \leq e \in E$ such that $\inf\{x, e\} = 0$ implies $x = 0$. We also use the notation $x \wedge y$ for $\inf\{x, y\}$.

A Banach lattice is *σ–order continuous* if order bounded increasing sequences are norm convergent. If this property holds for arbitrary downwards directed systems of elements, it is said to be *order continuous*. We call *order continuous part* E_{an} of E to the largest order continuous ideal in E. It can be described as $E_{an} = \{x \in E : |x| \geq x_\tau \downarrow 0 \text{ implies } \|x_\tau\| \downarrow 0\}$ (see [19, Theorem 102.8]). Similarly, the *σ-order continuous part* E_a of E is the largest σ-order continuous ideal in E, which can be described as $E_a = \{x \in E : |x| \geq x_n \downarrow 0 \text{ implies } \|x_n\| \downarrow 0\}$ (see [19, Theorem 102.8]).

E is said to be *Dedekind σ–complete* if every non-empty finite or countable subset which is bounded from above has a supremum and it is *Dedekind complete* if this is the case for every non-empty subset of E which is bounded from above. E is said to have *the weak Fatou property for monotone sequences* if it follows from $0 \leq x_n \uparrow (n \geq 1)$ and $\sup_{n\geq 1} \|x_n\| < \infty$ that $x := \sup_{n\geq 1} x_n$ exists in E. In this case, E is a Dedekind σ–complete Banach lattice [19, Theorem 113.1]. If, in addition, $\|x\| = \sup_{n\geq 1} \|x_n\|$, then E is said to have the *Fatou property for monotone sequences* (or simply *σ–Fatou property*). In a similar way, the space E is said to have the *weak Fatou property for directed sets* if it follows from $0 \leq x_\tau \uparrow$ and $\sup_\tau \|x_\tau\| < \infty$ that $x := \sup_\tau x_\tau$ exists in E. As above, E is now Dedekind complete ([19, Theorem 107.5]). Moreover, if $\|x\| = \sup_\tau \|x_\tau\|$, we will say that E has the *Fatou property for directed sets*.

An operator $T\colon E \to F$ between Banach lattices is said to be an *order isometry* if it is a linear isometry which is also an order isomorphism, that is, T is linear, one to one, onto, $\|Tx\|_F = \|x\|_E$ for all $x \in E$ and $T(\inf\{x, y\}) = \inf\{Tx, Ty\}$ for all $x, y \in E$. In this case, we say that E and F are *order isometrics*.

A *Banach function space* over a measure space $(\Omega, \Sigma, \lambda)$ is a Banach space E of (equivalence classes of) measurable functions which are integrable with respect to λ on sets of finite measure, contains all characteristic functions of sets of finite measure, and satisfies $f \in E$ with $\|f\| \leq \|g\|$ if $|f| \leq |g|$ with $g \in E$. Our main reference for Banach lattices and Banach function spaces is [13]; we also use [1, 3, 17, 18].

Throughout this work $\mathcal{R}$ will be a *δ–ring* of subsets of Ω, that is a ring of sets closed under countable intersections, and $\mathcal{R}^{loc}$ the σ–algebra of subsets A of Ω which are locally in $\mathcal{R}$, i.e. such that $A \cap B \in \mathcal{R}$ for every $B \in \mathcal{R}$. Measurability of the functions will be defined with respect to $\mathcal{R}^{loc}$ and the space of measurable real functions on $(\Omega, \mathcal{R}^{loc})$ will be denoted by $\mathcal{M}(\mathcal{R}^{loc})$. We will also use the notation $\mathcal{S}(\mathcal{R}^{loc})$ for the set consisting of simple functions based on the σ–algebra $\mathcal{R}^{loc}$ and $\mathcal{S}(\mathcal{R})$ for the set of simple functions which are supported in $\mathcal{R}$.

A vector measure $\nu\colon \mathcal{R} \to X$ is a set function such that $\sum_{n\geq 1} \nu(A_n)$ converges to $\nu(\cup_{n\geq 1} A_n)$ in X for every sequence $(A_n)_{n\geq 1}$ of pairwise disjoint sets in $\mathcal{R}$ with $\cup_{n\geq 1} A_n \in \mathcal{R}$.

The *semivariation* $\|\nu\|$ of ν in $\mathcal{R}^{loc}$ is given by $\|\nu\|(A) = \sup\{|x^*\nu|(A) : x^* \in B_{X^*}\}$, $A \in \mathcal{R}^{loc}$, where $|x^*\nu|$ is the variation of the measure $x^*\nu\colon \mathcal{R} \to \mathbb{R}$, that is $|x^*\nu|$ is the countably additive measure $|x^*\nu| : \mathcal{R}^{loc} \to [0, \infty]$ defined by $|x^*\nu|(A) = \sup\{\sum |x^*\nu(A_i)| \ : \ (A_i) \,\text{finite disjoint sequence in}\ \mathcal{R} \cap 2^A\}$. The semivariation is always finite on $\mathcal{R}$, and

$$\frac{\|\nu\|(A)}{2} \leq \sup\left\{\|\nu(B)\| \,:\, B \in \mathcal{R} \cap 2^A\right\} \leq \|\nu\|(A), \quad A \in \mathcal{R}^{loc},$$

see [12, Section 2], [16, Lemma 3.4 and Corollary 3.5]. A set $B \in \mathcal{R}^{loc}$ is *ν–null* if $\|\nu\|(B) = 0$, and a property holds *ν–almost everywhere* (ν–a.e. for short) if it holds except on a ν–null set.

Following Lewis [12] and Masani and Niemi [15], [16] we define now what an integrable functions with respect to a vector measure on a δ–ring is. If $\nu\colon \mathcal{R} \to X$ is a vector measure, we write $L^1_w(\nu)$ for the space of functions in $\mathcal{M}(\mathcal{R}^{loc})$ which are integrable with respect to the scalar measure $x^*\nu$ for all $x^* \in X^*$. In $L^1_w(\nu)$, functions which are equal ν–a.e. are identified;

$L^1_w(\nu)$ is a Banach space when the norm

$$\|f\|_\nu = \sup\Big\{ \int_\Omega |f| d\,|x^*\nu| \,:\, x^* \in B_{X^*} \Big\},$$

is considered.

Moreover, it is a Banach lattice having the σ–Fatou property for the ν–a.e. order and an it is an ideal of measurable functions, that is, if $|f| \leq |g|$ ν–a.e. with $f \in \mathcal{M}(\mathcal{R}^{loc})$ and $g \in L^1_w(\nu)$, then $f \in L^1_w(\nu)$.

Although $L^1_w(\nu)$ is not in general a Banach function space over any $x^*\nu$, it still satisfies that convergence in norm of a sequence implies ν–a.e. convergence of some subsequence (see [16, Lemma 3.13] or [19, 100.6]).

A function $f \in L^1_w(\nu)$ is *integrable with respect to* ν if for each $A \in \mathcal{R}^{loc}$ there is a (unique) vector denoted by $\int_A f d\nu \in X$, such that $x^*\Big(\int_A f d\nu\Big) = \int_A f dx^*\nu$ for all $x^* \in X^*$. Sometimes we write $\int f d\nu$ for $\int_\Omega f d\nu$. We denote by $L^1(\nu)$ the space of integrable functions with respect to ν.

Notice that for simple functions $\varphi = \sum_{i=1}^n a_i \chi_{A_i} \in \mathcal{S}(\mathcal{R})$ we have that $\varphi \in L^1(\nu)$ with $\int_A \varphi \, d\nu = \sum_{i=1}^n a_i \nu(A_i \cap A)$, $A \in \mathcal{R}^{loc}$. Furthermore, $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$ (see [12, Theorem 3.5]).

The space $L^1(\nu)$ is a Banach lattice for the order structure of $L^1_w(\nu)$; in fact, it is an ideal of measurable functions and also within $L^1_w(\nu)$; see [16, Theorem.4.10]. By [12, Theorem 3.3], $L^1(\nu)$ is also order continuous and may not have weak unit (see [8, Example 2.2] for a concrete example).

The integration operator $f \in L^1(\nu) \to \int f d\nu \in X$ is linear and continuous with $\| \int f d\nu\| \leq \|f\|_\nu$.

A vector measure $\nu \colon \mathcal{R} \to E$ with values in a Banach lattice E is *positive* if $\nu(A) \geq 0$ for all $A \in \mathcal{R}$. In this case, the integration operator $I_\nu \colon L^1(\nu) \to E$ is positive (i.e. $I_\nu(f) \geq 0$ whenever $0 \leq f \in L^1(\nu)$) and it can be checked that $\|f\|_\nu = \|I_\nu(|f|)\|$ for all $f \in L^1(\nu)$.

Each vector measure ν defined on a σ–algebra satisfies that $\chi_\Omega \in L^1(\nu)$ and so $\|\nu\|(\Omega) = \|\chi_\Omega\|_\nu < \infty$, that is, ν is bounded. It is relevant for this paper that this does not hold in general for vector measures defined on δ–rings ([8, Example 2.1]).

As we have already pointed out, the properties of a vector measure ν defined on a δ-ring $\mathcal{R}$ influence the space $L^1(\nu)$. Let us recall the consequences on the lattice properties of $L^1(\nu)$ that produce the strong additivity and the σ–finiteness of ν.

A measure ν is *strongly additive* if $(\nu(A_n))_{n\geq 1}$ converges to zero whenever $(A_n)_{n\geq 1}$ is a sequence of disjoint subsets of $\mathcal{R}$. It is known that ν is strongly additive if and only if $\chi_\Omega \in L^1(\nu)$ —in this case, χ_Ω is a weak unit for $L^1(\nu)$— and if and only if all bounded measurable functions belong to $L^1(\nu)$. Under this requirement ν is bounded and $L^1(\nu)$ coincides with the space $L^1(\widehat{\nu})$, where $\widehat{\nu} : \mathcal{R}^{loc} \to X$ is a vector measure that extends ν. $L^1(\nu)$ is then a Banach function space over $(\Omega, \mathcal{R}^{loc}, |x_0^*\nu|)$, where $|x_0^*\nu|$ is a bounded control measure for ν, $x_0^* \in B_{X^*}$, [8, Theorem 2.6].

A measure ν is said to be *σ–finite* with respect to $\mathcal{R}$ (just σ–finite for short) if there is a sequence $(A_n)_{n\geq 1}$ in $\mathcal{R}$ and a ν–null set $N \in \mathcal{R}^{loc}$ such that $\Omega = (\cup_{n\geq 1} A_n) \cup N$.

Every vector measure defined on a σ–algebra is strongly additive and range bounded. In the case of vector measures defined on a δ–ring, each strongly additive measure is σ–finite (see [4, Lemma 1.1]) but the opposite is not true (see [8, Example 2.1]).

In the general case of not strongly additive measures, there is no relation between σ–finiteness and boundedness of the measure (see [8, Example 2.2]). However —as for general Banach function spaces— σ–finiteness of a measure ν is equivalent to the existence of a weak unit in the space $L^1(\nu)$ and to the existence of a bounded local control measure for ν (see [8, Theorem 3.3]). Recall that a countable additive vector measure $\lambda : \mathcal{R} \to [0, \infty]$ is a *local control measure for* ν if it satisfies that $\lim_{A\subset B, \lambda(A)\to 0} \|\nu(A)\| = 0$ for all $B \in \mathcal{R}$ and that every ν–null set in $\mathcal{R}^{loc}$ is also λ–null.

We cannot assure in this case that $L^1(\nu)$ is a Banach function space with respect to any measure space $(\Omega, \mathcal{R}^{loc}, \lambda)$, being λ a local control measure for ν, but if ν is σ–finite, then $L^1(\nu)$ is an order continuous Banach lattice with weak unit, and so $L^1(\nu)$ is order isometric to $L^1(\widehat{\nu})$ for some vector measure $\widehat{\nu}$ defined on a σ–algebra of sets (see [6, Theorem 8]). A more concrete description can be done. If the measure ν is σ–finite and g is a weak unit for $L^1(\nu)$, then $L^1(\nu)$ is order isomorphic and isometric to $L^1(\nu_g)$, where $\nu_g : \mathcal{R}^{loc} \to X$ is the vector measure given by $\nu_g(A) = \int_A g\, d\nu$ (see [8, Theorem 3.5]).

When no restrictions are imposed to the vector measure ν on a δ–ring, the space $L^1(\nu)$ is still an order continuous Banach lattice and so it can be represented as an unconditional direct sum of a family of disjoint ideals, each of them with weak unit. Moreover by [6, Theorem 8] each one of these ideals is the space L^1 of a vector measure on a σ–algebra.

Again, a concrete representation of these spaces can be given: $L^1(\nu)$ can be written as an unconditional direct sum of disjoint ideals, each of them being order isometric to $L^1(\nu_A)$, where each ν_A is the vector measure ν restricted to a σ–algebra as $A \cap \mathcal{R}$ for some $A \in \mathcal{R}$.

More precisely, in the proof of Theorem 3.1 in [4], it is shown that there exists a maximal family $\{A_\alpha : \alpha \in \Delta\}$ of non ν-null sets in $\mathcal{R}$ with $A_\alpha \cap A_\beta$ ν-null for $\alpha \neq \beta$. Let ν_α be the restriction of ν to the σ-algebra $A_\alpha \cap \mathcal{R} = \{B \in \mathcal{R} : B \subset A_\alpha\}$ and consider the bounded linear projections $P_\alpha : L^1(\nu) \to L^1(\nu)$ given by $P_\alpha(f) = f\chi_{A_\alpha}$. Then, if $f \in L^1(\nu)$, there exists a countable subset I in Δ such that $f = \sum_{\alpha \in I} f\chi_{A_\alpha}$ ν-a.e. and the sum converges unconditionally in $L^1(\nu)$. Thus f is uniquely represented as an unconditional direct sum of elements of the family of disjoint closed ideals $(P_\alpha(L^1(\nu)))_{\alpha\in\Delta}$ where every space $P_\alpha(L^1(\nu)$ is order isometric to $L^1(\nu_\alpha)$ ([8, Theorem 3.6]).

Chapter 2

Motivation

2.1 Representation theorems.

As it was already mentioned in the Introduction, vector measures can be used for representing Banach lattices as spaces $L^1(\nu)$ and $L^1_w(\nu)$ of integrable functions and weakly integrable functions, respectively. Let us explain briefly the details of the corresponding representation theorems.

• For the case of an order continuous Banach lattice E with a weak unit, by a classical representation theorem, there exists a Banach function space $X(\mu)$ over a positive finite measure space (Ω, Σ, μ) and an isometric order isomorphism $\Phi : X(\mu) \to E$ ([13, Theorem 1.b.14]; see also [19, Theorem 120.10]). As shown in the proof of the theorem, there exists a vector measure $\eta : \Sigma \to X(\mu)^+$ defined by $\eta(A) := \chi_A, A \in \Sigma$ such that the integration map $I_\eta : f \mapsto \int f \, d\eta$ is an order isometry of $L^1(\eta)$ onto $X(\mu)$. Then, for the E^+-valued vector measure $\nu := \Phi \circ \eta$, the map $I_\nu = \Phi \circ I_\eta$ is an order isometry of $L^1(\nu)$ onto E which maps the weak unit χ_Ω of $L^1(\nu)$ onto a weak unit of E.

• When E is an order continuous Banach lattice, the key for constructing the vector measure is a classical result which ensures that E can be then decomposed into an unconditionally direct sum of a family of mutually disjoints ideals $\{E_\alpha\}_{\alpha \in \Delta}$, each E_α having a weak unit. That is, every $e \in E$ has a unique representation $e = \sum_{\alpha \in \Delta} e_\alpha$ with $e_\alpha \in E_\alpha$, only countably many

$e_\alpha \neq 0$ and the series converging unconditionally ([13, Proposition 1.a.9]).

Each E_α is an order continuous Banach lattice with a weak unit, then, by the representation theorem described above, there exist a σ-algebra Σ_α of parts of an abstract set Ω_α and a positive vector measure $\nu_\alpha\colon \Sigma_\alpha \to E_\alpha$ such that the integration operator $I_{\nu_\alpha}\colon L^1(\nu_\alpha) \to E_\alpha$ is an order isometry.

Consider now the set $\Omega = \cup_{\alpha\in\Delta}\big(\{\alpha\} \times \Omega_\alpha\big)$, that is

$$\Omega = \big\{(\alpha, \omega) : \alpha \in \Delta \text{ and } \omega \in \Omega_\alpha\big\}.$$

If we denote $\cup_{\alpha\in\Delta}(\{\alpha\} \times A_\alpha) = \big\{(\alpha, \omega) : \alpha \in \Delta \text{ and } \omega \in A_\alpha\big\}$, where $A_\alpha \subset \Omega_\alpha$ for all $\alpha \in \Delta$, and for every $\Gamma \subset \Delta$ we write $\cup_{\alpha\in\Gamma}(\{\alpha\} \times A_\alpha) = \cup_{\alpha\in\Delta}(\{\alpha\} \times A_\alpha)$ whenever $A_\alpha = \emptyset$ for all $\alpha \in \Delta\backslash\Gamma$, then the family $\mathcal{R}$ of sets $\cup_{\alpha\in\Delta}(\{\alpha\} \times A_\alpha)$ satisfying that $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \Delta$ and there exists a finite set $I \subset \Delta$ such that A_α is ν_α-null for all $\alpha \in \Delta\backslash I$, is a δ-ring of parts of Ω.

Moreover, $\mathcal{R}^{loc} = \big\{ \cup_{\alpha\in\Delta} (\{\alpha\} \times A_\alpha) : A_\alpha \in \Sigma_\alpha \text{ for all } \alpha \in \Delta\big\}$.

Let $\nu\colon \mathcal{R} \to E$ be the vector measure defined by $\nu\big(\cup_{\alpha\in\Delta} (\{\alpha\} \times A_\alpha)\big) = \sum_{\alpha\in\Delta} \nu_\alpha(A_\alpha)$, then the space $L^1(\nu)$ is order isometric to E. Even more, the integration operator $I_\nu\colon L^1(\nu) \to E$ is an order isometry.

- If E is a Banach lattice satisfying the σ–Fatou property with a weak unit belonging to the σ–order continuous part E_a of E, then there exists a vector measure ν defined on a σ–algebra such that E is order isometric to $L^1_w(\nu)$. In the proof it is noted that in this case E_a is also order continuous. Indeed, E_a is an ideal of E which is Dedekind σ–complete as it is σ–Fatou ([19, Theorem 113.1]). Then, E_a is also Dedekind σ–complete and, as it is σ–order continuous, it follows that it is order continuous ([13, Proposition 1.a.8]). The proof of the representation of E as an $L^1_w(\nu)$ consists in taking a vector measure ν such that $L^1(\nu)$ is order isometric to E_a via the integration operator I_ν, and extending I_ν to $L^1_w(\nu)$. The result is that this extension is an order isometry from $L^1_w(\nu)$ onto E.

- Finally, let E be a Banach lattice with the Fatou property and such that the order continuous part E_{an} of E is order dense in E. In this case,

E has the σ–Fatou property and then $E_{an} = E_a$, as we have already note. Then, we can take the vector measure ν associated to E_a as in Section 3 of [9], and so $I_\nu\colon L^1(\nu) \to E_a$ is an order isometry. The way as the vector measure is constructed is crucial. In fact, $L^1_w(\nu)$ has actually the Fatou property, where it is an open question if it is so in the general case. Then it is possible to extend I_ν to the space $L^1_w(\nu)$ in a way that the extension is an order isometry between $L^1_w(\nu)$ and E.

When our framework are order continuous Banach function spaces, the existence of a weak unit in it, is equivalent to the space to be defined over a σ–finite measure space. Let us remark how the order isometry which we obtain onto the corresponding L^1 space works.

Let $X(\mu)$ be an order continuous Banach function space based on a positive, finite measure space (Ω, Σ, μ). Note that the constant function χ_Ω is then a weak unit in $X(\mu)$. In these conditions $X(\mu)$ is then order isometric to an space $L^1(\nu)$ of integrable functions over a vector measure on a σ–algebra. In fact, let sim Σ denote the vector space of all Σ–simple functions. The $X(\mu)$–valued set function $\nu : A \mapsto \chi_A$, $A \in \Sigma$ is a positive vector measure and $I_\nu : L^1(\nu) \to X(\mu)$ is an order isomorphism onto its range such that $I_\nu(\varphi) = \varphi$, for every $\varphi \in$ sim Σ. Since sim Σ is dense in both $L^1(\nu)$ and $X(\nu)$, it follows that $I_\nu(f) = f$ for every $f \in L^1(\nu)$; in other words, $L^1(\nu) = X(\mu)$ and $I_\nu = id_{X(\mu)}$. Moreover, the norms in $L^1(\nu)$ and $X(\mu)$ are equal.

Let $X(\mu)$ be an order continuous Banach function space over a σ-finite (but not finite) measure space (Ω, Σ, μ). Note that the function χ_Ω is not a weak unit of $X(\mu)$ (it is not even in $X(\mu)$), and so, the representation theorem using vector measures on σ–algebras when applied to this case cannot done the identity map since the order isometry that we obtain of $L^1(\nu)$ and $X(\mu)$ carries the weak unit χ_Ω of $L^1(\nu)$ onto a weak unit of $X(\mu)$ which necessarily fails to be the same. In this case, $X(\mu)$ is order isometric to another Banach function space over a positive, finite measure. More precisely, suppose $\Omega = \bigcup_{n=1}^{+\infty} A_n$ with $\mu(A_n) < +\infty$, $A_n \in \Sigma$ and let $g := \sum_1^\infty \frac{1}{2^n \mu(A_n)} \chi_{A_n}$, then g is a weak unit in $X(\mu)$. Define $\mu_g(A) := \int_A g d\mu$, $A \in \Sigma$ and consider

$X(\mu_g)$ the space of (classes of) measurable functions (note that μ and μ_g have the same null sets) with the norm $\|\cdot\|_{X(\mu_g)} = \|\cdot g\|_{X(\mu)}$. The space $X(\mu_g)$ is a Banach function space over the finite measure space (Ω, Σ, μ_g) and the multiplication operator $M_{g^{-1}} : X(\mu) \to X(\mu_g)$ is an order isometry. Applying now the previous result, there exists an $L^1(\nu)$ space of integrable functions with respect to a vector measure on a σ–algebra such that the identity map is an order isometry from $L^1(\nu)$ to $X(\mu_g)$. Consequently the multiplication operator $M_g : L^1(\mu) \to X(\mu)$ is now an order isometry.

This is not the case when we represent $X(\mu)$ as an L^1 space respect to a vector measure on a δ–ring. Both spaces can be now identified having the same elements. Indeed, suppose again $\Omega = \bigcup_{n=1}^{+\infty} A_n$ with $\mu(A_n) < +\infty$, $A_n \in \Sigma$, for every A_n there exists a countably additive measure $\nu_n : \Sigma_n \to A_n$ defined by $\nu_n(B) = \chi_B$, $B \in \Sigma_n$ (the σ–algebra of measurable sets in A_n) such that the identity map is an order isometry from $L^1(\nu_n)$ to A_n. Consider the δ–ring $\mathcal{R} = \{\cup_{n\in I} B_n \,:\, I \subset \mathbb{N} \text{ is finite }, B_n \in \Sigma_n\}$ and the vector measure $A = \cup_{n\in I} B_n \in \mathcal{R} \mapsto \nu(A) = \sum_{n\in I} \nu_n(B_n)$, then the integral operator is the identity map and a biyection that preserves order and norm between the spaces $L^1(\nu)$ and $X(\mu)$.

Consequently, the representation provided in the approaches above are different. In the next section we present a concrete example which shows, precisely, these differences.

2.2 The canonical example.

Let us consider the σ–finite measure space $([0, +\infty), \Sigma, \mu)$, with Σ the σ–algebra of Lebesgue measurable subsets and $\mu(A) := \sum_{n=1}^{\infty} \frac{1}{2^n} m(A \cup [n, n+1])$ where m is the Lebesgue measure on $[0, +\infty)$. Let us denote by $L^0([0, +\infty))$ the set consisting of real Lebesgue measurable functions on the interval $[0, +\infty)$ where μ–a.e. equal functions are identified. Finally let us denote by m_n the Lebesgue measure on the interval $[n, n+1]$ for $n \in \mathbb{N}$ (including 0). As usual we write $\int_n^{n+1} |f(x)|dx$ instead of $\int_{[n,n+1]} |f(x)|dm_n(x)$. In this section we shall deal with the space $E := \sum_{l^\infty} L^1(m_n)$, i.e.

$$E = \left\{ f \in L^0([0,+\infty)) : \, f \in L^1([n,n+1]), \, n \in \mathbb{N} \text{ and } \sup_{n\in\mathbb{N}} \int_n^{n+1} |f(x)| \, dx < \infty \right\}.$$

This function space is a Banach lattice when endowed with the μ–a.e. order and the norm

$$\|f\| := \sup_{n\geq 0} \int_n^{n+1} |f(x)| \, dx, \quad f \in E.$$

It is in fact a Banach function space on $([0,+\infty), \Sigma, \mu)$. This space is not σ–order continuous, since clearly $\chi_{[0,+\infty)} \in E$ but $\chi_{[0,+\infty)} \notin E_a$, where E_a is the σ–order continuous part of E, which is the space $E_a = \sum_{c_0} L^1(m_n)$, i.e.

$$\begin{aligned} E_a &= \left\{ f \in \sum_{l^\infty} L^1(m_n) : \lim_{n\to+\infty} \int_n^{n+1} |f(x)| \, dx = 0 \right\} \\ &= \left\{ f \in \sum_{l^\infty} L^1(m_n) : \big(\int_n^{n+1} |f(x)| \, dx \big)_{n\geq 0} \in c_0 \right\}. \end{aligned}$$

It is easy to see that E has the σ–Fatou property. Notice that the function $\chi_{[0,+\infty)}$ is obviously a weak unit for E that is not in E_a. However, the function $g := \sum_{n=0}^\infty \frac{1}{n+1} \chi_{[n,n+1]}$ is also a weak unit, but $g \in E_a$. This fact will be relevant in what follows.

Let us see which are the representations that can be obtained using the approaches 1) and 2) explained in the Introduction.

1) *First, since E is a σ–Fatou Banach lattice with weak unit g belonging to E_a, there is a vector measure ν defined on a σ–algebra and with values in E_a such that E is order isomorphic and isometric to $L^1_w(\nu)$. In this case, the set map m_g on the Lebesgue measurable sets with values on $E_a = \sum_{c_0} L^1(m_n)$ given by $m_g(A) = g\chi_A$ is a countably additive measure. The space of integrable functions $L^1(m_g)$ is*

$$\begin{aligned} L^1(m_g) &= \left\{ f \in L^0([0,+\infty)) : fg \in E_a \right\} \\ &= \left\{ f \in \sum_{l^\infty} L^1(m_n) : \lim_{n\to+\infty} \int_n^{n+1} |f(x)| g(x) \, dx = 0 \right\}. \end{aligned}$$

Consequently, the multiplication operator $M_g : L^1(m_g) \to E_a$ given by $M_g(f) := fg$ is an order isometry —but it does **not** *identify the functions since maps the function $\chi_{[0,+\infty)}$ into the function g, i.e. the elements are not the same—. Moreover, the same map M_g can be extended to be defined from $L^1_w(m_g)$ to E as a bijective isometry between both spaces.*

2) *On the other hand, we can consider the representation theorem for order continuous Banach lattices without considering the existence of weak unit in E in order to represent E_a as a space $L^1(\nu)$, where ν is a vector measure defined on a δ–ring and with values in E_a^+. Recall that $E_a = \sum_{c_0} L^1(m_n)$ and so E_a can be written as an unconditional sum of the Banach lattices $L^1(m_n)$ that are order continuous and have weak units $\chi_{[n,n+1]}$, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, if we define $\nu_n : \Sigma_n \to L^1(m_n)$ by $\nu_n(A) := \chi_A$, $A \in \Sigma_n$, the σ–algebra of measurable subsets of $[n, n+1]$, ν_n is a vector measure such that $L^1(\nu_n)$ is exactly (identifying functions) the space $L^1(m_n)$. If we consider now the δ–ring*

$$\mathcal{R} = \{A = \cup_{i \in I} A_i : I \subset \mathbb{N} \text{ is finite }, A_i \in \Sigma_i\}$$

and the measure

$$A = \cup_{i \in I} A_i \in \mathcal{R} \mapsto \nu(A) = \sum_{i \in I} \nu_i(A_i) \in E_a, \tag{2.2.1}$$

the identity is a lattice isomorphism and an isometry between the spaces $L^1(\nu)$ and E_a. The identity can also be seen as an isometric lattice isomorphism between $L^1_w(\nu)$ and E, since E satisfies the conditions of Theorems **??** *and 4.3 of this work.*

Notice that the same analysis that we have done for E can be done for a space $G := \sum_{l^p} L^1(m_n)$, $1 \le p < \infty$, and in this case the function $\chi_{[0,+\infty)}$ does not belong to G. However, it is order continuous and $L^1(\nu) = L^1_w(\nu)$ so there is a representation of G using a vector measure on a σ–algebra, but it does not identify functions directly. Notice also that the approach in 2) gives the same structure result if we consider a non countable sum $\sum_{l^p(I)} L^1(\mu_i)$, $1 \le p \le \infty$, for a family $\{\mu_i : i \in I\}$ of probability spaces. In this case, the approach in 1) cannot be done.

Chapter 3

The influence of the vector measure in the lattice structure of the spaces $L^1_w(\nu)$

3.1 Order continuity type properties.

In this section we introduce new requirements for the measure ν and we analyze the consequences on the spaces of weakly integrable functions with respect to ν.

We start first with a series of results on order continuity and related topics.

Definition 3.1 *We say that a vector measure ν is locally σ–finite with respect to a δ–ring $\mathcal{R}$ (locally σ-finite for short) if given a set $B \in \mathcal{R}^{loc}$ with $\|\nu\|(B) < \infty$, B can be written as $B = \big(\cup_{n\geq 1} A_n\big) \cup N$, with $A_n \in \mathcal{R}$ and $N \in \mathcal{R}^{loc}$ a ν–null set.*

If ν es σ–finite, then is clear that it is locally σ–finite. However, there are vector measures that are locally σ–finite that are not σ–finite. The following example shows this.

Example 3.2 *Let $\Gamma := [0,\infty)$ and consider the δ-ring $\mathcal{R} := \{A \subset \Gamma : A \text{ finite}\}$. We construct three examples of vector measures taking values in spaces with completely different topological and lattice properties by using the partition of $[0,\infty)$ given by the intervals $[n-1,n)$, $n \in \mathbb{N}$.*

(1) Let $\{e_n : n \in \mathbb{N}\}$ be the canonical basis of c_0. Let us define the set function $\eta : \mathcal{R} \to c_0$ by $\eta(A) := \sum_{n\geq 1} \frac{\mathrm{card}(A_n)}{2^n} e_n$, $A \in \mathcal{R}$ with $A = \cup_{n\geq 1} A_n$, where $A_n := A \cap [n-1,n)$. The function η is a vector measure on $\mathcal{R}$, and a set $N \in \mathcal{R}^{loc}$ is η-null if and only if $\eta(A) = 0, \forall A \in \mathcal{R} \cap 2^N$, and this happens if and only if $N = \emptyset$. Let us show that ν is locally σ-finite with respect to $\mathcal{R}$. If $B \in \mathcal{R}^{loc}$ is finite or countable, the conditions in the definition for the sets are clearly fulfilled. Consider an uncountable set $B \in \mathcal{R}^{loc}$. It is enough to see that $\|\eta\|(B) = +\infty$. Since B is uncountable, there is an interval $[n_0-1,n_0)$ such that $\mathrm{card}\big(B \cap [n_0-1,n_0)\big) = \infty$. Since the semivariation is a monotone function, we prove that $\|\eta\|(B \cap [n_0-1,n_0)) = \infty$. Equivalently, it can be seen that $\sup\{\|\eta(A)\|_{c_0} : A \in \mathcal{R} \cap 2^{B\cap[n_0-1,n_0)}\} = \infty$. Consider a finite set $A \subset B \cap [n_0-1,n_0)$ such that $\mathrm{card}(A) = n$; it holds that $\|\eta(A)\|_{c_0} = \frac{\mathrm{card}(A)}{2^{n_0}} = \frac{n}{2^{n_0}} \to \infty$ when $n \to \infty$, and so $\sup\{\|\eta(A)\|_{c_0} : A \in \mathcal{R} \cap 2^{B\cap[n_0-1,n_0)}\} = \infty$. Consequently, η is locally σ-finite with respect to $\mathcal{R}$. However, it is not σ-finite since $[0,+\infty)$ cannot be written as a countable union of finite sets.

(2) Consider now $\nu : \mathcal{R} \to l_1(\Gamma)$ defined by $\nu(A) := \sum_{m\geq 1} a_m \chi_{\{\gamma_m\}}$ where $A \in \mathcal{R}$ with $A = \{\gamma_1, \gamma_2, \ldots, \gamma_m\}$ and $a_m := 1/2^n$ if $\gamma_m \in [n-1,n)$. It is easy to see that ν is a vector measure on $\mathcal{R}$, and as in the example above a set $N \in \mathcal{R}^{loc}$ is ν-null if and only if $N = \emptyset$.

The measure ν is locally σ-finite with respect to $\mathcal{R}$, but it is not σ-finite. As in (1), to see that it is enough to prove that if $B \in \mathcal{R}^{loc}$ is uncountable then $\|\nu\|(B) = \infty$. Take such a set B and an interval $[n_0-1,n_0)$ satisfying that $\mathrm{card}\big(B \cap [n_0-1,n_0)\big) = \infty$. For every finite set $A \subset B \cap [n_0-1,n_0)$ we have that $\|\nu(A)\|_{l_1(\Gamma)} = \sum_{\gamma\in\Gamma} |\nu(A)(\gamma)| = \frac{\mathrm{card}(A)}{2^{n_0}} \to \infty$ when the size of A increases, and so $\|\nu\|(B) = \infty$. Therefore ν is locally σ-finite with respect to $\mathcal{R}$, but not σ-finite.

(3) Let $1 \leq p < \infty$. Consider the space $l_p(\Gamma)$ and the family of Lebesgue measure spaces that are considered to be disjoint $([0,1], \mathcal{B}, dt)_i$, $i \in \Gamma$. Take the sequence $(r_k)_k$ of the Rademacher functions in $[0,1]$ and consider the δ-

ring

$$\mathcal{R} = \{A = \cup_{i\in\Gamma_0} A_i : A_i \in \mathcal{B}\, \textit{each}\ A_i\ \textit{considered in}\ ([0,1],\mathcal{B},dt)_i,\ \ \Gamma_0\, \textit{finite}\}$$

and the vector measure $\nu : \mathcal{R} \to \bigoplus_{l_p(\Gamma)} c_0$ *given by*

$$\nu(A) = \nu(\bigcup_{i\in\Gamma_0} A_i) := \sum_{i\in\Gamma_0} \chi_{\{i\}}(\int_{A_i} r_k(t)\, dt)_k, \quad \Gamma_0 \in \mathcal{R}.$$

This vector measure is well defined as a consequence of the Riemann-Lebesgue Lemma, taking into account that the Rademacher functions define an orthonormal system in $L^2[0,1]$. *Clearly, it is not* σ*-finite. However, using the same arguments that in the examples above, it can be proved that it is locally* σ*-finite.*

In our first result we characterize the local σ–finiteness. We start with a lemma.

Lemma 3.3 *Let* $\mathcal{R}$ *be a* δ*–ring of subsets of* Ω*,* X *a Banach space and* $\nu : \mathcal{R} \to X$ *a locally* σ*–finite vector measure with respect to* $\mathcal{R}$*. Given* $\varphi \in \mathcal{S}(\mathcal{R}^{loc})$ *there is* $(\varphi_j)_{j\geq 1} \subset \mathcal{S}(\mathcal{R})$ *such that* $\phi_j \to \phi$*,* ν*–a.e.*

Proof. Let $\varphi = \sum_{i=1}^N a_i \chi_{B_i}$ with $a_i \in \mathbb{R}\setminus\{0\}$ and $A_i \in \mathcal{R}^{loc}$ for $1 \leq i \leq N$. Since for all $1 \leq i \leq N$ we have that $\|\nu\|(B_i) = \|\chi_{B_i}\|_\nu \leq |a_i|^{-1}\|\phi\|_\nu < \infty$, then by the local σ–finiteness of ν with respect to $\mathcal{R}$, we can find a sequence $(A_n^i) \subset \mathcal{R}$, that we can assume to be pairwise disjoint, and a ν–null set $N^i \in \mathcal{R}^{loc}$, such that $B_i = \big(\cup_{n\geq 1} A_n^i\big) \cup N^i$. Hence

$$\varphi = \sum_{i=1}^N a_i \chi_{B_i} = \sum_{i=1}^N a_i \chi_{\left(\cup_{n\geq 1} A_n^i\right)\cup N^i} = \sum_{i=1}^N a_i \chi_{\cup_{n\geq 1} A_n^i},$$

that can be written, just by reordering the sets, as

$$\varphi = \sum_{n=1}^{+\infty} b_n \chi_{A_n} = \lim_{j\to+\infty} \sum_{n=1}^{j} b_n \chi_{A_n},\ \ \nu\text{–a.e.}$$

$\square$

Theorem 3.4 *Let $\nu\colon \mathcal{R}\to X$ be a vector measure on a δ–ring of subsets of an abstract space Ω and with values in a Banach space X. The vector measure ν is locally σ–finite if and only if for every $0\le f\in L^1_w(\nu)$, there exists a sequence $(\varphi_n)\subset\mathcal{S}(\mathcal{R})$ such that $0\le\varphi_n\uparrow f$ ν–a.e. Consequently $L^1(\nu)$ is super order dense in $L^1_w(\nu)$.*

Proof. Suppose that ν is locally σ–finite and let $0\le f\in L^1_w(\nu)$. There exists a sequence $(\psi_n)\subset\mathcal{S}(\mathcal{R}^{loc})$ such that $0\le\psi_n\uparrow f$ ν–a.e. Then, there exists a ν–null set Z such that $\psi_n(\omega)\to f(\omega)$ for each $\omega\in\Omega\backslash Z$.

Fix n. We can write $\psi_n=\sum_{j=1}^{k_n}\alpha_j^n\chi_{A_j^n}$ with $(A_j^n)_j$ pairwise disjoint and $\alpha_j^n>0$. Then, taking $\beta_n=\min\{\alpha_1^n,...,\alpha_{k_n}^n\}$, it follows

$$\|\nu\|(Supp\ \psi_n)=\|\chi_{Supp\ \psi_n}\|_\nu\le\frac{1}{\beta_n}\|\psi_n\|_\nu\le\frac{1}{\beta_n}\|f\|_\nu<\infty.$$

So, there exist $(A_j^n)_j\subset\mathcal{R}$ and a ν–null set Z^n such that $Supp\ \psi_n=\big(\cup_j A_j^n\big)\cup Z^n$.

Define $\varphi_n=\psi_n\chi_{\cup_{i=1}^n\cup_{j=1}^n A_j^i}\in\mathcal{S}(\mathcal{R})$. Of course, $0\le\varphi_n\uparrow$ and $\varphi_n\le f$. Let us see that $\varphi_n\uparrow f$ ν–a.e. If $\omega\notin Supp\ f$, then $\varphi_n(\omega)=0\to f(\omega)=0$. Let $\omega\in Supp\ f\backslash\big(\cup_n Z^n\cup Z\big)$ and $\varepsilon>0$. Since $\omega\notin Z$, there exists n_ω such that

$$|f(\omega)-\psi_n(\omega)|<\varepsilon\quad\text{for all } n\ge n_\omega. \tag{3.1.1}$$

In other hand, since $\omega\in Supp\ f=\cup_k Supp\ \psi_k$, there exists k_ω such that $\omega\in Supp\ \psi_{k_\omega}=\big(\cup_j A_j^{k_\omega}\big)\cup Z^{k_\omega}$. As $\omega\notin Z^{k_\omega}$, there exists j_{k_ω} such that $\omega\in A^{k_\omega}_{j_{k_\omega}}\subset\cup_{i=1}^n\cup_{j=1}^n A_j^i$ for all $n\ge k_\omega, j_{k_\omega}$. From this and (3.1.2), for all $n\ge\tilde n_\omega=\max\{n_\omega,k_\omega,j_{k_\omega}\}$, it follows that

$$|f(\omega)-\varphi_n(\omega)|=|f(\omega)-\psi_n(\omega)|<\varepsilon,$$

that is, $0\le\varphi_n\uparrow f$ ν–a.e.

Conversely, suppose that for every $0\le f\in L^1_w(\nu)$, there exists a sequence $(\varphi_n)\subset\mathcal{S}(\mathcal{R})$ such that $0\le\varphi_n\uparrow f$ ν–a.e. Let $B\in\mathcal{R}^{loc}$ such that $\|B\|<\infty$. Then, $0\le\chi_B\in L^1_w(\nu)$. So, there exists a sequence $(\varphi_n)\subset\mathcal{S}(\mathcal{R})$ such that $0\le\varphi_n\uparrow\chi_B$ ν–a.e., that is, there exists a ν–null set Z such that $\varphi_n(\omega)\uparrow\chi_B(\omega)$ for each $\omega\in\Omega\backslash Z$. Then, $B=\big(\cup_n Supp\ \varphi_n\big)\cup\big(B\cap Z\big)$, where $Supp\ \varphi_n\in\mathcal{R}$ and $B\cap Z$ is ν–null. Finally, since $\mathcal{S}(\mathcal{R})\subseteq L^1_w(\nu)$, $L^1(\nu)$ is super order dense in $L^1_w(\nu)$. $\square$

Let us show an example of a space $L^1(\nu)$ of a locally σ–finite vector measure ν. Consider the vector measure given in Example 3.2(3). It is already known that the space $L^1(\eta)$ of the vector measure $\eta : \mathcal{B} \to c_0$ given by $\eta(A) := (\int_{A_i} r_k(t)\, dt)_k \in c_0$ coincides with $L^1[0,1]$ (see the example after Theorem 3 in [6]). An straightforward computation using this result shows that $L^1(\nu) = \bigoplus_{l_p(\Gamma)} L^1[0,1]$.

As we said in Preliminaries, for a vector measure ν defined on a δ–ring the space $L^1(\nu)$ can be written as an unconditional direct sum of spaces $L^1(\nu_A)$ where ν_A is the restriction of the measure ν to the σ–algebra $A \cap \mathcal{R}$ and $A \in \mathcal{R}$. This decomposition allows to see each function of $L^1(\nu)$ as a limit —in the norm and also in order— of the sequence of partial sums $(\sum_{i=1}^n f\chi_{A_i})_{n\geq 1}$, with $A_i \in \mathcal{R}$. As we shall show in what follows, a similar decomposition —but in order only— can be given for the space $L^1_w(\nu)$ when ν is a locally σ–finite vector measure with respect to $\mathcal{R}$.

Theorem 3.5 *Let $\mathcal{R}$ be a δ–ring of subsets of a set Ω and X a Banach space. If $\nu : \mathcal{R} \to X$ is a locally σ–finite vector measure with respect to $\mathcal{R}$ then the space $L^1_w(\nu)$ can then be written as a ν–a.e. pointwise direct sum of spaces $L^1_w(\nu_A)$, where for each $A \in \mathcal{R}$, ν_A is the vector measure ν restricted to a σ–algebra as $A \cap \mathcal{R}$. Concretely, each function $f \in L^1_w(\nu)$ can be written as a sum —in order— of a countable set of projections over these spaces $L^1_w(\nu_A)$. The converse is also true.*

Proof. Let $\{A_\alpha : \alpha \in \Delta\}$ be the maximal family of non ν–null sets in $\mathcal{R}$ such that $A_\alpha \cap A_\beta$ is ν–null for $\alpha \neq \beta$ and such that for each $B \in \mathcal{R}$, $B = \cup_{\alpha\in I}(B \cap A_\alpha) \cup N$ with I countable and $N \in \mathcal{R}$ a ν–null set explained in section 2. For each $\alpha \in \Delta$, consider again the σ–algebra of subsets of A_α, $\mathcal{R}_\alpha = A_\alpha \cap \mathcal{R} = \{B \in \mathcal{R} : B \subset A_\alpha\}$, let $\nu_\alpha : \mathcal{R}_\alpha \to X$ be the restriction of ν to the σ–algebra $\mathcal{R}_\alpha$ and consider the linear (bounded) projections $P_\alpha : L^1_w(\nu) \to L^1_w(\nu)$ given by $P_\alpha(f) = f\chi_{A_\alpha}$. Recall that the sequence $(P_\alpha(L^1_w(\nu)))_{\alpha\in\Delta}$ is a family of disjoint ideals of $L^1_w(\nu)$. Let $f \in L^1_w(\nu)$, we have to see that $f = \sum_{\alpha\in I} f\chi_{A_\alpha}$ ν–a.e. for a countable subset $I \subset \Delta$. Since by hypothesis ν is locally σ–finite with respect to $\mathcal{R}$, by Theorem 3.4, there is a sequence $(\varphi_n)_n \subset \mathcal{S}(\mathcal{R})$ that converges to f ν–a.e. On the other hand, for each simple function $\varphi \in \mathcal{S}(\mathcal{R})$, let us write $\varphi = \sum_{j=1}^k a_j \chi_{B_j}$ such that $(B_j)_j \subset \mathcal{R}$ are disjoint; we have that $\varphi = \sum_{j=1}^k a_j \chi_{\cup_{\alpha\in I_j}(B_j\cap A_\alpha)}$ ν–a.e., being $I_j \subset \Delta$ countable. Thus if we take $I = \cup_{j=1}^k I_j$, then $\varphi\chi_{A_\alpha} = 0$ ν–a.e. for all $\alpha \notin I$. In particular, for each n, consider $I_n \subset \Delta$ the countable set such that $\varphi_n\chi_{A_\alpha} = 0$ ν–a.e. for all $\alpha \notin I_n$. Then, taking into account that the support of f is contained in the union of the supports of the functions φ_n, we have that $f\chi_{A_\alpha} = 0$ ν–a.e. for all $\alpha \notin I$, where $I = \cup_{n\geq 1} I_n$ is countable and consequently, since $\varphi_n\chi_{A_\alpha} \uparrow f\chi_{A_\alpha}$, we obtain $f = \sum_{\alpha\in I} f\chi_{A_\alpha}$ ν–a.e.

For the converse, let $B \in \mathcal{R}^{loc}$ with $\|\nu\|(B) < \infty$. We have to prove that $B = (\cup_{n\geq 1} A_n) \cup N$, with $A_n \in \mathcal{R}$ and $N \in \mathcal{R}^{loc}$ ν–null. Since $\|\chi_B\|_{L^1_w(\nu)} = \|\nu\|(B) < \infty$, the characteristic function $\chi_B \in L^1_w(\nu)$, and then by hypothesis it can be written as $\chi_B = \sum_{\alpha\in I} \chi_B\chi_{A_\alpha}$ ν–a.e., where I is countable. Then, $\chi_B = \sum_{\alpha\in I} \chi_{B\cap A_\alpha} = \chi_{\cup_{\alpha\in I}(B\cap A_\alpha)}$ ν–a.e. and $B = \cup_{\alpha\in I}(B\cap A_\alpha) \cup N$ where $B\cap A_\alpha \in \mathcal{R}$ (by definition of $\mathcal{R}^{loc}$) and $N \in \mathcal{R}^{loc}$ is ν–null. □

In the case of spaces of integrable functions with respect to a vector measure defined on a σ–algebra, it is well known that $(L^1_w(\nu))_a = L^1(\nu)$ (see [7, p.192] or [18, p.144]). However it is an open question if it is the case when we work with vector measures defined on δ–rings. We present here an equivalent condition.

Definition 3.6 *We define a vector measure ν to be weakly locally σ-finite whit respect to $\mathcal{R}$ (weakly locally σ-finite for short) if for every $B \in \mathcal{R}^{loc}$ with $\|\nu\|(B) < \infty$ and satisfying that $\|\nu\|(B_n) \to 0$ whenever $(B_n) \subset B$ and $B_n \downarrow$ with $\cap_n B_n$ being ν-null, it follows that there exist $(A_n) \subset \mathcal{R}$ and a ν-null set N such that $B = \big(\cup_n A_n\big) \cup N$.*

Theorem 3.7 *The vector measure ν is weakly locally σ–finite if and only if $\big(L^1_w(\nu)\big)_a = L^1(\nu)$.*

Proof. Suppose that ν is weakly locally σ–finite. Note that $L^1(\nu) \subset \big(L^1_w(\nu)\big)_a$. Let us prove the converse containment. Let $0 \leq f \in \big(L^1_w(\nu)\big)_a$ and $(\psi_n) \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \psi_n \uparrow f$ ν–a.e. Then, there exists a ν–null set Z such that $\psi_n(\omega) \to f(\omega)$ for each $\omega \in \Omega\backslash Z$.

Fix n. We can write $\psi_n = \sum_{j=1}^{k_n} \alpha_j^n \chi_{A_j^n}$ with $(A_j^n)_j$ pairwise disjoint and $\alpha_j^n > 0$. Then, taking $\beta_n = \min\{\alpha_1^n, ..., \alpha_{k_n}^n\}$, it follows

$$\|\nu\|(Supp\ \psi_n) = \|\chi_{Supp\ \psi_n}\|_\nu \leq \frac{1}{\beta_n}\|\psi_n\|_\nu \leq \frac{1}{\beta_n}\|f\|_\nu < \infty.$$

Moreover, given $(B_k) \subset Supp\ \psi_n$ such that $B_k \downarrow$ with $\cap_k B_k$ being ν–null, we have that $\chi_{B_k} \downarrow 0$ ν–a.e. and $\chi_{B_k} \leq \chi_{Supp\ \psi_n} \leq \frac{1}{\beta_n} f \in \big(L_w^1(\nu)\big)_a$. So, $\|\nu\|(B_k) = \|\chi_{B_k}\|_\nu \to 0$. Therefore, there exist $(A_j^n)_j \subset \mathcal{R}$ and a ν–null set Z^n such that $Supp\ \psi_n = \big(\cup_j A_j^n\big) \cup Z^n$.

Define $\varphi_n = \psi_n \chi_{\cup_{i=1}^n \cup_{j=1}^n A_j^i} \in \mathcal{S}(\mathcal{R})$. Of course, $0 \leq \varphi_n \uparrow$ and $\varphi_n \leq f$. Let us see that $\varphi_n \uparrow f$ ν–a.e. If $\omega \notin Supp\ f$, then $\varphi_n(\omega) = 0 \to f(\omega) = 0$. Let $\omega \in Supp\ f \backslash \big(\cup_n Z^n \cup Z\big)$ and $\varepsilon > 0$. Since $\omega \notin Z$, there exists n_ω such that

$$|f(\omega) - \psi_n(\omega)| < \varepsilon \quad \text{for all } n \geq n_\omega. \tag{3.1.2}$$

In other hand, since $\omega \in Supp\ f = \cup_k Supp\ \psi_k$, there exists k_ω such that $\omega \in Supp\ \psi_{k_\omega} = \big(\cup_j A_j^{k_\omega}\big) \cup Z^{k_\omega}$. As $\omega \notin Z^{k_\omega}$, there exists j_{k_ω} such that $\omega \in A_{j_{k_\omega}}^{k_\omega} \subset \cup_{i=1}^n \cup_{j=1}^n A_j^i$ for all $n \geq k_\omega, j_{k_\omega}$. From this and (3.1.2), for all $n \geq \tilde{n}_\omega = \max\{n_\omega, k_\omega, j_{k_\omega}\}$, it follows that

$$|f(\omega) - \varphi_n(\omega)| = |f(\omega) - \psi_n(\omega)| < \varepsilon.$$

Then, $f \geq f - \varphi_n \downarrow 0$ ν–a.e. This implies that $\|f - \varphi_n\|_\nu \to 0$ as $f \in \big(L_w^1(\nu)\big)_a$. Since $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$ is dense in $L^1(\nu)$, we have that $f \in L^1(\nu)$. The same holds for a general function f by taking positive and negative parts.

Suppose now that $\big(L_w^1(\nu)\big)_a = L^1(\nu)$. Let $B \in \mathcal{R}^{loc}$ with $\|\nu\|(B) < \infty$ and satisfying that $\|\nu\|(B_n) \to 0$ whenever $(B_n) \subset B$ and $B_n \downarrow$ with $\cap_n B_n$ being ν-null. Let us see that $\chi_B \in \big(L_w^1(\nu)\big)_a$. Note that $\chi_B \in L_w^1(\nu)$ as $\|\nu\|(B) < \infty$. Let $f_n \in L_w^1(\nu)$ be such that $\chi_B \geq f_n \downarrow 0$ ν–a.e. Then,

$$\bigcup_{N\geq 1} \bigcap_{k\geq 1} \bigcup_{n\geq k} \Big\{\omega \in \Omega :\ |f_n(\omega)| > \frac{1}{N}\Big\} = \bigcup_{N\geq 1} \bigcap_{k\geq 1} \Big\{\omega \in \Omega :\ |f_k(\omega)| > \frac{1}{N}\Big\}$$

is a ν–null set. In particular, for each fixed $N \geq 1$, the set $\cap_{k\geq 1}\big\{\omega \in \Omega :\ |f_k(\omega)| > \frac{1}{N}\big\}$ is ν–null. Since, $\big\{\omega \in \Omega :\ |f_k(\omega)| > \frac{1}{N}\big\} \subset Supp\ f_n \subset B$ and $\big\{\omega \in \Omega :\ |f_k(\omega)| > \frac{1}{N}\big\} \downarrow$ with $\cap_{k\geq 1}\big\{\omega \in \Omega :\ |f_k(\omega)| > \frac{1}{N}\big\}$ being ν-null, then $\|\nu\|(\big\{\omega \in \Omega :\ |f_k(\omega)| > \frac{1}{N}\big\}) \to 0$.

Given $\varepsilon > 0$, take N_ε such that $N_\varepsilon \geq \frac{2\|\nu\|(B)}{\varepsilon}$. Noting that $f_n = f_n\chi_B \leq \chi_B \leq 1$, we have that

$$\begin{aligned}
\|f_n\|_\nu &\leq \left\|f_n\chi_{\{\omega\in\Omega:\,|f_n(\omega)|\leq\frac{1}{N_\varepsilon}\}}\right\|_\nu + \left\|f_n\chi_{\{\omega\in\Omega:\,|f_n(\omega)|>\frac{1}{N_\varepsilon}\}}\right\|_\nu \\
&\leq \frac{1}{N_\varepsilon}\|\nu\|(B) + \|\nu\|\Big(\Big\{\omega\in\Omega:\ |f_n(\omega)| > \frac{1}{N_\varepsilon}\Big\}\Big) \\
&\leq \frac{\varepsilon}{2} + \|\nu\|\Big(\Big\{\omega\in\Omega:\ |f_n(\omega)| > \frac{1}{N_\varepsilon}\Big\}\Big) \leq \varepsilon
\end{aligned}$$

for all large enough n.

So, $\chi_B \in \big(L^1_w(\nu)\big)_a = L^1(\nu)$. Then, there exists $(\varphi_n) \subset \mathcal{S}(\mathcal{R})$ such that $\varphi_n \to \chi_B$ in norm and ν–a.e., that is, there exists a ν–null set Z such that $\varphi_n(\omega) \to \chi_B(\omega)$ for each $\omega \in \Omega\backslash Z$. Hence, $B = Supp\ \chi_B \subset \big(\cup_n Supp\ \varphi_n\big) \cup Z$, and so $B = \big(\cup_n B \cap Supp\,\varphi_n\big) \cup (B\cap Z)$, where $B\cap Supp\,\varphi_n \in \mathcal{R}$ (as $Supp\,\varphi_n \in \mathcal{R}$) and $B \cap Z$ is ν–null. Therefore ν is weakly locally σ–finite. □

The following example shows that locally σ–finiteness and weakly locally σ–finiteness are not equivalent conditions on ν.

Example 3.8 *Let* $\Gamma := (0,+\infty)$*, the* δ*–ring* $\mathcal{R} := \{A \subset \Gamma : A \text{ is finite}\}$ *and* $\nu : \mathcal{R} \to c_0(\Gamma)$ *defined by* $\nu(A) := \sum_{\gamma\in A}\frac{1}{\gamma}\chi_{\{\gamma\}} = \sum_{i=1}^n \frac{1}{\gamma_i}\chi_{\gamma_i}$ *when* $A \in \mathcal{R}$ *with* $A = \{\gamma_1,\gamma_2,\ldots,\gamma_n\} \subseteq (0,+\infty)$*. Clearly,* ν *is a vector measure, but it is not locally* σ*–finite since we can find* $B \in \mathcal{R}^{loc} = 2^\Gamma$ *with* $\|\nu\|(B) < +\infty$ *such that* $B \neq \big(\cup_{n\geq1} A_n\big) \cup N$*, for every* $A_n \in \mathcal{R}, n \in \mathbb{N}$ *and every* ν*–null set* $N \in \mathcal{R}^{loc}$*. Indeed, let* $B = [2,+\infty) \in \mathcal{R}^{loc}$*. Since* $\sup_{\gamma\in\Gamma}|\nu(A)(\gamma)| = \sup_{1\leq i\leq m}\{\frac{1}{\gamma_i}, 0\} \leq \frac{1}{2}$*, when* $A = \{\gamma_1,\gamma_2,\ldots,\gamma_m\} \in \mathcal{R}\cap 2^{[2,+\infty)}$*, then*

$$\frac{1}{2}\|\nu\|(B) \leq \sup\{\|\nu(A)\|_{c_0(\Gamma)} : A \in \mathcal{R}\cap 2^B\} \leq \frac{1}{2},$$

therefore we get that $\|\nu\|(B) < +\infty$*. On the other hand, as in Example 3.2 a set* $N \in \mathcal{R}^{loc}$ *is* ν*–null if and only* $N = \emptyset$*. Since* B *cannot be expressed as a countable union of finite sets,* ν *is not locally* σ*–finite.*

In order to see that $(L^1_w(\nu))_a = L^1(\nu)$ *let us first show that* $L^1_w(\nu) = \gamma l^\infty(\Gamma)$ *and* $L^1(\nu) = \gamma c_0(\Gamma)$*:*

To see that $L^1_w(\nu) = \gamma l^\infty(\Gamma)$*, fix* $x^* = (\beta_\gamma)_{\gamma\in\Gamma} \in (c_0(\Gamma))^* = l^1(\Gamma)$*. Clearly for all* $A \in \mathcal{R}$ *the scalar measure* $(x^*\nu)(A) = x^*(\nu(A)) = \sum_{\gamma\in A}\beta_\gamma\frac{1}{\gamma}$ *and for all*

$B \in \mathcal{R}^{loc}$ its variation $|x^*\nu|(B) = \sum_{\gamma\in B} |\beta_\gamma|\frac{1}{\gamma}$. Hence

$$\int_B |f|\, d|x^*\nu| = \sum_{\gamma\in B} |f(\gamma)||\beta_\gamma|\frac{1}{\gamma},$$

for all $B \in \mathcal{R}^{loc}$, in particular for $B = \Gamma$, and then $f \in L^1(|x^*\nu|)$ if and only if $\sum_{\gamma\in\Gamma} |f(\gamma)||\beta_\gamma|\frac{1}{\gamma} < \infty$, and so $f \in L^1_w(\nu)$ if and only if $f\frac{1}{\gamma} \in l^\infty(\Gamma)$.

Let us show now that $L^1(\nu) = \gamma c_0(\Gamma)$. Indeed, given $\varphi \in \mathcal{S}(\mathcal{R})$ and $B \in \mathcal{R}^{loc}$ it is easy to see that $\int_B \varphi\, d\nu = (\varphi\frac{1}{\gamma}\chi_B)_{\gamma\in\Gamma}$. Then, by [8, Proposition 2.3], $f \in L^1(\nu)$ if it is a limit ν–a.e. of a sequence $(\varphi_n)_{n\geq 1}$ in $\mathcal{S}(\mathcal{R})$ such that $\int_B \varphi_n\, d\nu$ converges in $c_0(\Gamma)$, with $B \in \mathcal{R}^{loc}$. Moreover, in this case

$$\int_B f\, d\nu = \lim_{n\to\infty} \int_B \varphi_n\, d\nu = \lim_{n\to\infty} (\varphi_n\frac{1}{\gamma}\chi_B) = (\lim_{n\to\infty} \varphi_n)\frac{1}{\gamma}\chi_B = f\frac{1}{\gamma}\chi_B.$$

Taking in particular $B = \Gamma$ we obtain that $f \in L^1(\nu)$ if and only if $f\frac{1}{\gamma} \in c_0(\Gamma)$. Moreover, if $f \in L^1(\nu)$, since ν is a positive measure, [18, Lemma 3.13] gives

$$\|f\|_\nu = \|\int_\Gamma |f|\, d\nu\|_{c_0(\Gamma)} = \big\||f|\frac{1}{\gamma}\big\|_{c_0(\Gamma)} = \|f\frac{1}{\gamma}\|_{c_0(\Gamma)}.$$

Finally, let us show that $(L^1_w(\nu))_a = L^1(\nu)$. To see this, let $f \in \gamma l^\infty(\Gamma) \setminus \gamma c_0(\Gamma)$, then there is an $\varepsilon > 0$ such that $H = \{\gamma \in \Gamma : |\gamma^{-1}f(\gamma)| \geq \varepsilon\}$ is not finite. Let $\{\gamma_1, \gamma_2, \ldots\} \subset H$; we have that $|\gamma_i^{-1}f(\gamma_i)| \geq \varepsilon$ for all $i \in \mathbb{N}$. If we define the sequence $(f_n)_{n\geq 1}$ in $\gamma l^\infty(\Gamma)$ given by

$$f_n(\gamma) = \begin{cases} \varepsilon\gamma, \text{ if } \gamma = \gamma_i,\ i \geq n \\ 0, \text{ in other case} \end{cases}$$

it is clear that $f_n \downarrow 0$ and $|\gamma_i^{-1}f(\gamma_i)| \geq \varepsilon = (\gamma_i^{-1})\gamma_i\varepsilon \geq \gamma_i^{-1}f_n(\gamma_i)$ for all $i \geq n$, from which $|f(\gamma)| \geq f_n(\gamma)$ for all $n \in \mathbb{N}$ and $\gamma \in \Gamma$. But $\|f_n\|_{\gamma l^\infty(\Gamma)} = \|\gamma^{-1}f_n\|_{l^\infty(\Gamma)} = \sup_{\gamma\in\Gamma} |\gamma^{-1}f_n(\gamma)| = \varepsilon$ for all $n \in \mathbb{N}$. Therefore $\|f_n\|_{\gamma l^\infty(\Gamma)} \nrightarrow 0$ and $f \notin (\gamma l^\infty(\Gamma))_a$. Consequently, $(\gamma l^\infty(\Gamma))_a \subseteq \gamma c_0(\Gamma)$ and the equality is obtained, since the opposite inclusion always holds.

3.2 Fatou type properties of $L^1_w(\nu)$.

In the case of σ–algebras it is known that the "σ–Fatou completion" of $L^1(\nu)$ is $L^1_w(\nu)$ (see the comments just before Proposition 2.4 in [7, p.191]).

This can be easily extended to the case of σ–finite vector measures on δ–rings. However, this property does not hold any more under just weak σ–finiteness assumptions on ν. In the following example we show a vector measure that is locally σ–finite and there is a σ–Fatou ideal that strictly contains $L^1(\nu)$ and is strictly contained in $L^1_w(\nu)$.

Example 3.9 *Let Γ be an uncountable index set, and consider as in Example 3.2 the δ–ring $\mathcal{R}$ of the finite subsets of Γ. Let $\nu : \mathcal{R} \to l^\infty(\Gamma)$ be the vector measure given by $\nu(A) := \sum_{\gamma \in A_0} \chi_{\{\gamma\}}$. The corresponding spaces of integrable functions can be easily calculated and are $L^1(\nu) = c_0(\Gamma)$ and $L^1_w(\nu) = l^\infty(\Gamma)$ (see [8, Example 2.2]). Let us write $l^\infty_0(\Gamma)$ for the Banach lattice of the bounded functions $h : \Gamma \to \mathbb{R}$ of countable support. It has the σ–Fatou property. However, $L^1(\nu) \subsetneqq l^\infty_0(\Gamma) \subsetneqq L^1_w(\nu)$.*

As we have said in Preliminaries, it is also known that the space $L^1_w(\nu)$ of weakly integrable functions with respect to a vector measure on a δ–ring has the σ–Fatou property. When the δ–ring is actually a σ–algebra, the space $L^1_w(\nu)$ has also the Fatou property. Indeed, the space $L^1_w(\nu)$ is a Banach function space on a finite positive measure space (see Section 2) and since it has the σ–Fatou property, by [19, Theorem 112.3], $L^1_w(\nu)$ has the weak Fatou property. Now, [7, Proposition 2.1] yields that $L^1_w(\nu)$ has the Fatou property. In the general case, this is an open question. In this section we provide a partial answer.

Recently, it has been proved that every Banach lattice with the Fatou property such that its σ–order continuous part is order dense in it, is order isometric to a $L^1_w(\nu)$ space with respect to a vector measure on a δ–ring ([9, Theorem 8]). A further reading allows us to notice that the measure that is used in the constructive proof of this result satisfy a good decomposition property that is isolated and treated here. As we show it is in a sense the key for the Fatou property to be satisfied for the space $L^1_w(\nu)$.

Let us define now a decomposition property for the vector measure that will be shown to be in a sense equivalent to the fact that $L^1_w(\nu)$ has the Fatou property. We will show first in Remark 3.10 that it is always possible to obtain a disjoint decomposition of Ω using elements of $\mathcal{R}$ union a (non necessarily measurable) ν–null set (notice that this notion can be defined for any subset of Ω in the following way: $B \subset \Omega$ is ν–null if and only if $\sup_{A\in\mathcal{R}} \|\nu\|(A) = 0$).

Remark 3.10 *Consider a δ–ring $\mathcal{R}$ of subsets of Ω and a vector measure ν on it. Then there is a class of pairwise disjoint sets $\{A_i : i \in I\} \subseteq \mathcal{R}$ and a disjoint ν–null subset $N \subseteq \Omega$ such that $A = \cup_{i\in I} A_i \cup N$. To show this we use Zorn's Lemma. Consider the family $\mathcal{Z}$ of all disjoint classes of subsets of $\mathcal{R}$. Take the order given by the inclusion of classes. If we consider a chain $\{F_j \in \mathcal{Z} : j \in J\}$, $J \subset I$, then clearly $F_0 = \cup_{j\in J} F_j$ belongs to $\mathcal{Z}$. Then there is a maximal element F of $\mathcal{Z}$. Let us show that $\Omega \setminus \cup_{A\in F} A$ is a ν–null set. If this is not the case, there is an element $A \in \mathcal{R}$ such that $\|\nu\|(A) > 0$. But this implies in particular that there is a set $A \in \mathcal{R}$ that is disjoint to the elements of F. This contradicts the maximality of F.*

Such kind of decomposition properties for δ–rings are already known. The one that is normally used in the setting of vector measure integration is due to Brooks and Dinculeanu [4, Theorem 3.1], see also for instance [8, Theorem 3.6]. However for the following definition we need a maximal decomposition as the one above.

Definition 3.11 *A vector measure ν over a δ–ring $\mathcal{R}$ of subsets of an abstract set Ω is said to be $\mathcal{R}$–decomposable if there exists a maximal decomposition of Ω as in Remark 3.10 given by $(\Omega_\alpha)_{\alpha\in\Delta}$ in $\mathcal{R}$ and N ν–null such that*

(1) for every arbitrary family $(A_\alpha)_{\alpha\in\Delta}$ of elements or $\mathcal{R}$ such that $A_\alpha \subset \Omega_\alpha$ for all $\alpha \in \Delta$, the union $\cup_{\alpha\in\Delta} A_\alpha$ is in $\mathcal{R}^{loc}$, and

(2) for every arbitrary family of ν–null sets $(Z_\alpha)_{\alpha\in\Delta}$ in $\mathcal{R}$ such that $Z_\alpha \subset \Omega_\alpha$ for all $\alpha \in \Delta$, the union $\cup_{\alpha\in\Delta} Z_\alpha$ is ν–null.

Example 3.12 *It can be easily seen that the property given above depends on the particular maximal decomposition that we consider, even for the case of finite measures on σ–algebras. Take the Lebesgue measure space $([0,1],\mathcal{L},\mu)$. Obviously, it is $\mathcal{L}$–decomposable; just take the maximal decomposition given by the single set $\{[0,1]\}$. Notice also that $\mathcal{L}$ is a σ–algebra and so $\mathcal{L}^{loc}=\mathcal{L}$. Consider now the maximal decomposition given by the single points, i.e. $\{\{\gamma\}:\gamma\in[0,1]\}$, and the μ–null set $N=\emptyset$. Take a non Lebesgue measurable set $A\subset[0,1]$. Then $A=\cup_{\gamma\in A}\{\gamma\}$, but it is not in $\mathcal{L}^{loc}$. So (1) in Definition 3.11 is not satisfied. Moreover, $\{\{\gamma\}:\gamma\in[0,1]\}$ is a family of μ–null sets, but $\mu\big(\{\{\gamma\}:\gamma\in[0,1]\}\big)=\mu([0,1])=1$, that also contradicts (2).*

Remark 3.13 *All the classical examples of vector measures are $\mathcal{R}$-decomposable; consequently the corresponding space $L^1_w(\nu)$ has the Fatou property. This is the case when the vector measure is actually defined on a σ–algebra; clearly, σ–finiteness of ν implies that ν is $\mathcal{R}$–decomposable. Moreover, discrete vector measures are $\mathcal{R}$–decomposable as well, i.e. if Γ is an abstract uncountable set, $\mathcal{R}=\{A\subset\Gamma\,:\,A \text{ is finite }\}$, $\mathcal{R}^{loc}=2^\Gamma$ and $\nu:\mathcal{R}\to X$ for a Banach space X.*

Theorem 3.14 *Let $\mathcal{R}$ be a δ–ring of subsets of Ω, X a Banach space and $\nu:\mathcal{R}\to X$ an $\mathcal{R}$–decomposable vector measure. Then $L^1_w(\nu)$ has the Fatou property and $L^1(\nu)$ is an order dense ideal in it.*

Proof. For every $I\subset\Delta$ finite, consider $\Omega_I=\cup_{\alpha\in I}\Omega_\alpha$ and the σ–algebra $\Sigma_I=\big\{\cup_{\alpha\in I}A_\alpha:\,A_\alpha\in\Sigma_\alpha \text{ for all } \alpha\in I\big\}$ of parts of Ω_I and $\Sigma_\alpha=\mathcal{R}\cap\Omega_\alpha$. Note that $\Omega_I\subset\Omega$ and $\Sigma_I\subset\mathcal{R}$. Denote by $\nu_I\colon\Sigma_I\to X$ the restriction of ν to Σ_I. Since ν_I is a vector measure defined on a σ–algebra, $L^1_w(\nu_I)$ has the Fatou property.

For each $f\in\mathcal{M}(\mathcal{R}^{loc})$, denote by f^I the function resulting from the restriction of f to Ω_I. Of course, $f^I\in\mathcal{M}(\Sigma_I)$. For every $x^*\in X^*$, it follows

$$\int_{\Omega_I}|f^I|\,d|x^*\nu_I|=\int_\Omega|f|\chi_{\Omega_I}\,d|x^*\nu|. \tag{3.2.1}$$

Indeed, for every $A \in \Sigma_I$ we have that $|x^*\nu_I|(A) = |x^*\nu|(A)$ and so it is routine to check that (3.2.1) holds for $f \in \mathcal{S}(\mathcal{R}^{loc})$. For a general f the result follows by applying the monotone convergence theorem. Then, for every $f \in L^1_w(\nu)$ we have that $f\chi_{\Omega_I} \in L^1_w(\nu)$ and so $f^I \in L^1_w(\nu_I)$ with $\|f^I\|_{\nu_I} = \|f\chi_{\Omega_I}\|_\nu$. Note that if Z is a ν–null set then $Z \cap \Omega_I$ is ν_I–null.

Consider $0 \leq f \in L^1_w(\nu)$ and choose $(\varphi_n)_{n\geq 1} \subset \mathcal{S}(\mathcal{R}^{loc})$ such that $0 \leq \varphi_n \uparrow f$. For each $n \geq 1$ and $I \subset \Delta$ finite, we define $\xi_{(n,I)} = \varphi_n \chi_{\Omega_I} \in \mathcal{S}(\mathcal{R})$. Then $(\xi_{(n,I)})_{(n,I)} \subset L^1(\nu)$ is an upwards directed system $0 \leq \xi_{(n,I)} \uparrow f$. Moreover $\sup_{(n,I)} \xi_{(n,I)} = f$ and consequently $L^1(\nu)$ is order dense in $L^1_w(\nu)$. Indeed, $\xi_{(n,I)}(\omega) \leq \varphi_n(\omega) \leq f(\omega)$, $\omega \in \Omega$, then $\xi_{(n,I)} \leq f$ and so $\xi_{(n,I)} \leq f$ ν–a.e. Furthermore, let $h \in L^1_w(\nu)$ with $\xi_{(n,I)} \leq h$ ν–a.e, then $h \geq 0$ except on a ν–null set $M \in \mathcal{R}^{loc}$ and for every $n \geq 1$ and $I \subset \Delta$ finite $\xi_{(n,I)} \leq \tilde{h} := h\chi_{\Omega \setminus M}$ except on a ν–null set $Z_{(n,I)}$ in Ω_I. In particular, for every $n \geq 1$ and $\alpha \in \Delta$, $\xi_{(n,\{\alpha\})} \leq \tilde{h}$ except on a ν–null set $Z_{(n,\{\alpha\})}$ in Ω_α. Note that $\cup_{\alpha\in\Delta} Z_{(n,\{\alpha\})} \in \mathcal{R}^{loc}$ is ν–null, so $Z := \cup_{n\geq 1} \cup_{\alpha \in \Delta} Z_{(n,\{\alpha\})} \in \mathcal{R}^{loc}$ is ν–null. For every $\omega \in \Omega \setminus Z \cup N$, there is just one $\alpha \in \Delta : \omega \in \Omega_\alpha$ and then $\xi_{(n,\{\alpha\})}(\omega) = \varphi_n(\omega) \leq \tilde{h}(\omega)$ for all $n \geq 1$, so $f(\omega) \leq \tilde{h}(\omega)$. Hence, $f \leq \tilde{h}$ ν–a.e. and also $f \leq h$ ν–a.e.

Let $(f_\tau)_\tau \subset L^1_w(\nu)$ be an upwards directed system $0 \leq f_\tau \uparrow$ ν–a.e. such that $\sup_\tau \|f_\tau\|_\nu < \infty$. Then, $(f^I_\tau)_\tau \subset L^1_w(\nu_I)$ is an upwards directed system $0 \leq f^I_\tau \uparrow \nu_I$–a.e. and $\sup_\tau \|f^I_\tau\|_{\nu_I} = \sup_\tau \|f_\tau \chi_{\Omega_I}\|_\nu \leq \sup_\tau \|f_\tau\|_\nu < \infty$. Since $L^1_w(\nu_I)$ has the Fatou property, there exists $f_I = \sup_\tau f^I_\tau$ in $L^1_w(\nu_I)$ and $\|f_I\|_{\nu_I} = \sup_\tau \|f^I_\tau\|_{\nu_I}$.

Now from each $I = \{\alpha\}$ with $\alpha \in \Delta$ we construct the function $f\colon \Omega \to \mathbb{R}$ given by $f(\omega) = f_{\{\alpha\}}(\omega)$, $\omega \in \Omega_\alpha$. Remark that it is well defined as the family $(\Omega_\alpha)_{\alpha\in\Delta}$ is a disjoint family. Since ν is $\mathcal{R}$–decomposable, $f^{-1}(B) = \cup_{\alpha\in\Delta}(f_{\{\alpha\}})^{-1}(B)$ is in $\mathcal{R}^{loc}$ for all Borel set B on $\mathbb{R}$ and then we have that $f \in \mathcal{M}(\mathcal{R}^{loc})$. Noting also that in this case $\cup_{\alpha\in\Delta} Z_\alpha$ is ν–null whenever $Z_\alpha \subset \Omega_\alpha$ is ν_α–null for all $\alpha \in \Delta$ and so by a similar argument to the one used above, we have that $f = \sup_\tau f_\tau$. Furthermore, for every $I \subset \Delta$ finite, $f_I = f^I$ ν–a.e. Let us see now that $f \in L^1_w(\nu)$. Fix $x^* \in X^*$. For every $I \subset \Delta$ finite,

$$\sum_{\alpha\in I} \int_{\Omega_\alpha} f \, d|x^*\nu| = \int_\Omega f\chi_{\Omega_I} \, d|x^*\nu| = \int_{\Omega_I} f_I \, d|x^*\nu_I| \leq \|f_I\|_{\nu_I} \leq \sup_\tau \|f_\tau\|_\nu < \infty.$$

Then $\sum_{\alpha\in\Delta}\int_{\Omega_\alpha} f\,d|x^*\nu_\alpha|$ converges. Hence, there exists a countable set $J\subset\Delta$ such that $\int_{\Omega_\alpha} f\,d|x^*\nu| = 0$ for all $\alpha\in\Delta\backslash J$ and so $f=\sum_{\alpha\in J} f\chi_{\Omega_\alpha}$ $|x^*\nu|$–a.e. By the monotone convergence theorem $f\in L^1(|x^*\nu|)$ and

$$\int_\Omega f\,d|x^*\nu| = \sum_{\alpha\in\Delta}\int_{\Omega_\alpha} f\,d|x^*\nu| \text{ for all } x^*\in E^*.$$

Therefore, $f\in L^1_w(\nu)$ and $\|f\|_\nu\le\sup_\tau\|f_\tau\|_\nu$. The equality follows, as $\|f_\tau\|_\nu\le\|f\|_\nu$ for all τ. Consequently, $L^1_w(\nu)$ has the Fatou property. □

In the converse direction, it is also possible to give a partial answer. We need first the following result with no further conditions on the measure.

Proposition 3.15 *Let $\mathcal{R}$ be a δ–ring of subsets of Ω, X a Banach space and ν a vector measure. If the space $L^1_w(\nu)$ has the Fatou property and $L^1(\nu)$ is an order dense ideal in it, then $(L^1_w(\nu))_a = L^1(\nu)$.*

Proof. Clearly $L^1(\nu)\subset(L^1_w(\nu))_a = (L^1_w(\nu))_{an} = \{f\in L^1_w(\nu) : |f| > f_\tau\downarrow 0 \Rightarrow \|f_\tau\|_\nu\downarrow 0\}$ as $L^1_w(\nu)$ has the Fatou property. Let $0\le f\in(L^1_w(\nu))_{an}$. By the order density of $L^1(\nu)$ there exists an upwards directed system $0\le f_\tau\uparrow f$ ν-a.e. so such that $f\ge f-f_\tau\downarrow 0$ ν-a.e. Then $\|f-f_\tau\|_\nu\downarrow 0$. Hence, for every $n\ge 1$, there exists an index τ_n such that $\|f-f_{\tau_n}\|_\nu<\frac{1}{2^n}$ and completeness of $L^1(\nu)$ yields that f_{τ_n} converges to f in norm in $L^1(\nu)$. □

Under the requirements of the preceding proposition, there exists a δ–ring $\tilde{\mathcal{R}}$ of parts of an abstract set $\tilde{\Omega}$ and a $\tilde{\mathcal{R}}$–decomposable vector measure $\tilde{\nu}:\tilde{\mathcal{R}}\to L^1(\nu)$ such that the operator integration $I_{\tilde{\nu}}$ is an order isometry between $L^1(\tilde{\nu})$ and $L^1(\nu)$. Moreover, $I_{\tilde{\nu}}$ can be extended to $L^1_w(\tilde{\nu})$ and this extension is an order isometry from $L^1_w(\tilde{\nu})$ to $L^1_w(\nu)$ (see [9, Theorem 4 and Theorem 8]). Remark also that if ν is $\mathcal{R}$–decomposable, the conditions in proposition above hold, and so $(L^1_w(\nu))_a = L^1(\nu)$ and ν is weakly locally σ–finite.

Theorem 3.14 and the remarks above give in such a way a characterization of the Fatou property of $L^1_w(\nu)$.

Corollary 3.16 *Let $\mathcal{R}$ be a δ–ring of subsets of Ω, X a Banach space and $\nu:\mathcal{R}\to X$ a vector measure. If ν is $\mathcal{R}$–decomposable, then $L^1_w(\nu)$ has the Fatou* *property and $L^1(\nu)$ is an order dense ideal in it. In the converse direction, if the space $L^1_w(\nu)$ has the Fatou property and $L^1(\nu)$ is an order dense ideal in it, then there exists a δ–ring $\tilde{\mathcal{R}}$ and a $\tilde{\mathcal{R}}$–decomposable vector measure $\tilde{\nu} : \tilde{\mathcal{R}} \to L^1(\nu)$ such that $L^1_w(\nu)$ and $L^1_w(\tilde{\nu})$ are order isometric as well as $L^1(\nu)$ and $L^1(\tilde{\nu})$.*

Remark 3.17 *Even in the case of L^1 spaces on a scalar measure space $(\Omega, \Sigma, \lambda)$, the corollary above find applications. It allows to prove that the space $L^1(\lambda)$ can always be represented as a $L^1(\nu)$ of an $\mathcal{R}$–decomposable measure. Actually, consider $L^1(\lambda)$ with $\lambda : \Sigma \to \mathbb{R}$ a scalar measure. It is σ–order continuous and has the σ–Fatou property. By [19, Theorem 113.4] it is then order continuous and has the weak Fatou property. Consequently, it has the Fatou property. Corollary 3.16 yields that there exists a δ–ring $\mathcal{R}$ and an $\mathcal{R}$–decomposable vector measure $\nu : \mathcal{R} \to L^1(\lambda)$ such that $L^1(\lambda)$ and $L^1(\nu)$ are order isometric, since $L^1(\lambda) = L^1_w(\lambda)$ is trivially true for a scalar measure λ.*

We finish this section introducing the last new requirement on ν that we will need to establish a new representation theorem in the next chapter.

The construction of Brooks and Dinculeanu used in the proof of Theorem 3.5 motivates the following definition.

Definition 3.18 *We say that a family $\{A_\alpha \in \mathcal{R} : \alpha \in \Delta\}$ of non ν–null sets is a σ–local decomposition of $\mathcal{R}$ with respect to ν if it satisfies that*

1) $A_\alpha \cap A_\beta$ is ν–null for each $\alpha \neq \beta$, and

2) for each $B \in \mathcal{R}$, $B = \cup_{\alpha \in I}(B \cap A_\alpha) \cup N$ where I is countable and $N \in \mathcal{R}$ is ν–null.

The existence of a maximal σ–local decomposition of $\mathcal{R}$ with respect to ν for any vector measure is proved in [4, Theorem 3.1]. Let us show two more examples of such structures without the maximality requirement.

Example 3.19 *(1) In the example $E = \sum_{l^\infty} L^1(m_n)$ that we have explained in Chapter 2, the class of sets $\{[n, n+1] : n \in \mathbb{N} \cup \{0\}\}$ is obviously a σ–local decomposition of $\mathcal{R}$ with respect to the measure ν defined in (2.2.1).*

(2) Consider a non countable set of indexes I, a disjoint family of order continuous Banach function spaces $X_i(\mu_i)$ over probability measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ and the space $E := \sum_{l^\infty(I)} X_i(\mu_i)$. Take the δ–ring $\mathcal{R}_0 := \{\cup_{i=1}^n A_i : A_i \in \Sigma_i, 1 \leq i \leq n \in \mathbb{N}\}$ and the vector measure $\nu_0 : \mathcal{R}_0 \to E$ given by $\nu_0(B) := \sum_{i=1}^n \chi_{A_i}$, $B = \cup_{i=1}^n A_i$. Then the class $\{\Omega_i : i \in I\}$ is a σ–local decomposition of $\mathcal{R}_0$ with respect to ν_0.

Definition 3.20 *We shall say that the measure ν is locally σ–integrable if there exists a σ–local decomposition $\{A_\alpha\}_{\alpha \in \Delta}$ of $\mathcal{R}$ with respect to ν such that for every $f \in L^1_w(\nu)$, $\mathrm{supp}(f) = \cup_{n \geq 1}(\mathrm{supp}(f) \cap A_{\alpha_n})$ ν–a.e., $\alpha_n \in \Delta$ for all $n \in \mathbb{N}$, and $f\chi_{A_{\alpha_n}} \in L^1(\nu)$.*

Clearly, each measure ν that is locally σ–integrable is locally σ–finite with respect to $\mathcal{R}$, which implies that in this case $(L^1_w(\nu))_a = L^1(\nu)$. However, these properties are not equivalent. The vector measure given in [8, Example 2.2] provides an example of a locally σ–finite measure that is not locally σ–integrable. Let us explain this. Consider the set $[0, \infty)$ and the vector measure $\nu_0 : \mathcal{P}_f \to l_\infty([0, \infty))$ given by $\nu_0(A) := \chi_A$, where $\mathcal{P}_f$ is the δ–ring of finite parts of $[0, \infty)$ and $A \in \mathcal{P}_f$. It is shown in [8] that in this case $L^1_w(\nu_0) = l_\infty([0, \infty))$ and $L^1(\nu_0) = c_0([0, \infty))$. To see that the measure is locally σ–finite, it is enough to notice that for every $B \subset [0, \infty)$, $\|\nu_0\|(B) = 0$ if and only if B is countable. However, ν_0 cannot be locally σ–integrable, since $\chi_{[0,\infty)} \in l_\infty([0, \infty))$ and its support is not countable.

Regarding the structure of the space of weakly integrable functions with respect to a locally σ–integrable vector measure ν, notice that in this case $L^1_w(\nu)$ can be written as a (pointwise) direct sum of a family of disjoint ideals, each of them being order isometric to $L^1(\nu_A)$, where each ν_A is the vector measure ν restricted to a σ–algebra as $A \cap \mathcal{R}$ for some $A \in \mathcal{R}$. Namely, ν is locally σ–integrable if and only if each $0 \leq f \in L^1_w(\nu)$, $f = \sup_{n\geq 1} \sum_{k=1}^{n} f\chi_{A_{\alpha_k}}$ with $f\chi_{A_{\alpha_k}} \in L^1(\nu)$ for a fixed σ–local decomposition of $\mathcal{R}$ with respect to ν.

Chapter 4

Representation theorems for Banach lattices with the σ–Fatou property

Let E be a Dedekind σ–complete Banach lattice. Then for each $0 \leq x \in E$, the projection $P_x : E \to P_x(E)$ given by $P_x(e) := \sup_{n\geq 1}(nx \wedge e)$ if $e \geq 0$ and $P_x(e) = P_x(e^+) - P_x(e^-)$ for the general case can be defined. On the other hand, as it was already mentioned in the Introduction, since the σ–order continuous part E_a of E is an ideal of a Dedekind σ–complete Banach lattice, E_a is also Dedekind σ–complete and E_a is then order continuous (see for instance [13, Proposition 1.a.8]) and can be then written as an unconditional sum (non necessarily countable) of a family of pairwise disjoint ideals $\{E_\alpha\}_{\alpha\in\Delta}$, each of them with a weak unit $e_\alpha > 0$. Concretely, each $x \in E_a$ can be written as $x = \sum_{\alpha\in\Delta} x_\alpha$, $x_\alpha \in E_\alpha$ being this decomposition unique with a countably family of $x_\alpha \neq 0$, and this series converges unconditionally ([13, Proposition 1.a.9]).

If we read the proof of this result carefully, it can be seen that for the order continuous Banach lattice E_a there is a family of pairwise disjoint positive elements $\{e_\alpha\}_{\alpha\in\Delta}$ such that $E_\alpha = P_{e_\alpha}(E_a)$ is an ideal (a band in fact) for which each e_α is a weak unit, and for $0 \leq x \in E_a$, there is a countably set of subindexes α, $\{\alpha_1, \alpha_2, \ldots\}$ such that $\sum_{n=1}^{\infty} P_{e_{\alpha_n}}(x)$ converges unconditionally to x. In fact, the family $\{e_\alpha\}_{\alpha\in\Delta}$ can always be found and it can be assumed to be maximal if necessary. This construction motivates the following

Definition 4.1 *Let F be a Dedekind σ–complete Banach lattice. We say that a family $\{e_\alpha\}_{\alpha\in\Delta}$ of disjoint positive elements of F is a σ–local decomposition of F if it satisfies that for every $0 \leq x \in F$ there is a countable set of indexes $(\alpha_k)_{k\geq 1} \subset \Delta$ such that $\sup_{n\geq 1}\sum_{k=1}^{n} P_{e_{\alpha_k}}(x) = x$.*

Using the representation explained above, each order continuous Banach lattice E is order isomorphic and isometric to a space of integrable functions with respect to a vector measure on a δ–ring (see the Introduction). Let us recall again the details that are necessary to establish the notation that will be used in the rest of the section. Since E is order continuous, it can be written as a direct unconditional sum of ideals $\{E_\alpha : \alpha \in \Delta\}$ of E that are order continuous and have weak unit e_α. For each E_α there is a set Ω_α, a σ–algebra Σ_α and a vector measure $\nu_\alpha : \Sigma_\alpha \to E_\alpha$ such that $L^1(\nu_\alpha)$ is order isomorphic and isometric to E_α. Concretely, each E_α is order isomorphic and isometric to a Banach function space $X(\lambda_\alpha)$ with respect to a probability space $(\Omega_\alpha, \Sigma_\alpha, \lambda_\alpha)$ that associates the weak unit e_α of the first space with the weak unit χ_{Ω_α} of the second one. If we consider the vector measure $A \in \Sigma_\alpha \mapsto \nu_\alpha(A) = \chi_A \in E_\alpha$ —identifying E_α with $X(\lambda_\alpha)$—, then the integration map from $L^1(\nu_\alpha)$ to E_α is an order isometry —that is, in fact, the identity map—.

If we take now $\Omega = \cup_{\alpha\in\Delta}\big(\{\alpha\} \times \Omega_\alpha\big)$ and the δ–ring $\mathcal{R}$ of parts of Ω consisting of sets $\cup_{\alpha\in\Delta}(\{\alpha\} \times A_\alpha)$ satisfying that $A_\alpha \in \Sigma_\alpha$ for all $\alpha \in \Delta$ and there exists a finite set $I \subset \Delta$ such that A_α is ν_α-null for all $\alpha \in \Delta\backslash I$, the set function $\nu\colon \mathcal{R} \to E$ defined by

$$\nu\big(\cup_{\alpha\in\Delta}\left(\{\alpha\} \times A_\alpha\right)\big) = \sum_{\alpha\in\Delta} \nu_\alpha(A_\alpha).$$

$$A = \cup_{i\in I} A_i \in \mathcal{R} \mapsto \nu(A) = \sum_{i\in I} \nu_i(A_i) \in E$$

is a vector measure and the operator integral is a bijection that preserves the order and the norm between the spaces $L^1(\nu)$ and E. Note that for every $\alpha \in \Delta$, I_ν carries $\chi_{\{\alpha\}\times\Omega_\alpha}$ in e_α.

Theorem 4.2 *Let E be a σ–Fatou Banach lattice with a σ–local decomposition of E, $\{e_\alpha\}_{\alpha\in\Delta}$, such that $\{P_{e_\alpha}(E) : \alpha \in \Delta\} \subseteq E_a$. Then there is a vector measure ν defined on a δ–ring with values on E_a^+ such that there exists and order isometry T from E into a sublattice, $T(E)$, of $L_w^1(\nu)$. Moreover $T(P_{e_\alpha}(e)) = P_{\chi_{\{\alpha\}\times\Omega_\alpha}}(T(e))$, for all $\alpha \in \Delta$ and $e \in E^+$.*

Proof. Since a σ–Fatou Banach lattice is in particular Dedekind σ–complete then the existence of a σ–local decomposition for E makes sense. Since E_a is an order continuous Banach lattice there is a vector measure defined on a δ–ring, $\mathcal{R}$, and with values in E_a^+ such that E_a and $L^1(\nu)$ are order isometric. Let us call $T : E_a \to L^1(\nu)$ to this order isometry.

Let us extend T to E^+. In order to do that let $e \in E^+$ and define $e_n = \sum_{k=1}^n P_{e_{\alpha_k}}(e)$. Recall that, by the comments just above, this decomposition is unique. By hypothesis

$$e = \sup_{n\geq 1} e_n = \sup_{n\geq 1} \sum_{k=1}^{n} P_{e_{\alpha_k}}(e)$$

with $(e_n)_n \subset E_a$. Let us define

$$T(e) := \sup_{n\geq 1} T(e_n).$$

T is well defined. Indeed, since $(e_n)_n \subset E_a$, $e_n \uparrow \sup_n e_n = e$ and T is an order isomorphism in E_a then $T(e_n) \uparrow$. On the other hand T is also an isometry on E_a so

$$\sup_{n\geq 1} \|T(e_n)\|_\nu = \sup_{n\geq 1} \|e_n\| \leq \|e\|.$$

Hence, by the σ–Fatou property of $L_w^1(\nu)$, there exists $\sup_{n\geq 1} T(e_n) \in L_w^1(\nu)$ and $\|\sup_{n\geq 1} T(e_n)\|_\nu = \sup_{n\geq 1} \|T(e_n)\|_\nu$.

Clearly, T is positive in E^+ since it is so in E_a.

T is an isometry in E^+ since E is σ–Fatou and hence

$$\|T(e)\|_\nu = \|\sup_{n\geq 1} T(e_n)\|_\nu = \sup_{n\geq 1} \|T(e_n)\|_\nu = \sup_{n\geq 1} \|e_n\| = \|e\|.$$

A direct computation shows that T is linear in E^+.

Moreover, since $T: E^+ \to L^1_w(\nu)^+$ is additive, there is a unique positive and linear map $T: E \to L^1_w(\nu)$ that extends this operator. Note that in fact for $e \in E$ then $e = e^+ - e^-$ and $T(e) := T(e^+) - T(e^-)$.

Let us show that T is a lattice homomorphism. By using [17, Lemma 1.3.11], to see this, it is enough to see that $T(e^+) \wedge T(e^-) = 0$ for all $e \in E$. Let $e, \widetilde{e} \in E^+$ such that $e \wedge \widetilde{e} = 0$ and $(e_n)_{n\geq 1} \subset E_a$ y $(\widetilde{e}_m)_{m\geq 1} \subset E_a$ sequences such that $e = \sup_{n\geq 1} e_n$ and $\widetilde{e} = \sup_{m\geq 1} \widetilde{e}_m$. Since $e \wedge \widetilde{e} = 0$, we have that $e_n \wedge \widetilde{e}_m = 0$, for all $n, m \geq 1$. Hence, since T is a lattice homomorphism, we obtain that,

$$0 = T(0) = T(e_n \wedge \widetilde{e}_m) = T(e_n) \wedge T(\widetilde{e}_m),$$

in E_a, and therefore

$$T(e) \wedge T(\widetilde{e}) = \big(\sup_{n\geq 1} T(e_n)\big) \wedge \big(\sup_{m} T(\widetilde{e}_m)\big) = \sup_{n\geq 1}\sup_{m\geq 1} \big(T(e_n) \wedge T(\widetilde{e}_m)\big) = 0.$$

Since $e^+ \wedge e^- = 0$ for all $\forall e \in E$ the result holds for the general case.

Finally $T: E \to T(E)$ is an isometric lattice isomorphism. Indeed, in this setting, by [13, Theorem 1.17], we have that $|T(e)| = T(|e|)$ for all $e \in E$ and then T is an isometry in E, since for all $e \in E$,

$$\|T(e)\|_\nu = \|\,|T(e)|\,\|_\nu = \|T(|e|)\|_\nu = \|\,|e|\,\| = \|e\|,$$

and so we obtain that T is injective and bicontinuous, so $T: E \to T(E)$ is an isometric lattice isomorphism since the inverse map is also positive (see for instance [13, page 2]).

It remains to prove that $T(P_{e_\alpha}(e)) = P_{\chi_{\Omega_\alpha}}(T(e))$, for all $\alpha \in \Delta$ and $e \in E^+$. Let $e \in E^+$, $(\alpha_k)_{k\geq 1}$ the corresponding countable family of indexes given by the σ–local decomposition. First notice that if α is not in $(\alpha_k)_{k\geq 1}$ the result holds trivially by the construction of T. So fix $k \in \mathbb{N}$. By definition $P_{e_{\alpha_k}}(e) = \sup_{m\geq 1}(me_{\alpha_k} \wedge e) \in E_a$. Since $me_{\alpha_k} \wedge e \uparrow \sup_{m\geq 1}(me_{\alpha_k} \wedge e) = P_{e_{\alpha_k}}(e)$ in E_a and this space is order continuous, $(me_{\alpha_k} \wedge e)_{m\geq 1}$ converges in norm in E_a and the limit is $\sup_{m\geq 1}(me_{\alpha_k} \wedge e)$. Then, by the continuity of T in E_a,

$$T(P_{e_{\alpha_k}}(e)) = T\big(\sup_{m\geq 1}(me_{\alpha_k} \wedge e)\big) = T\big(\lim_{m\to\infty}(me_{\alpha_k} \wedge e)\big) = \lim_{m} T(me_{\alpha_k} \wedge e).$$

On the other hand, each increasing sequence in a Banach lattice that converges in the norm converges also in the order (see [19, Theorem 100.4]) therefore

$$\lim_{m\to\infty} T(me_{\alpha_k} \wedge e) = \sup_{m\geq 1} \big(T(me_{\alpha_k}) \wedge T(e)\big) =$$
$$\sup_{m\geq 1} \big(m\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}} \wedge T(e)\big) = P_{\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}}(T(e)),$$

as $T(e_{\alpha_k}) = \chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}$. □

We shall call a measure ν as the one that is given in Theorem 4.2 a *representing measure* for E. As it has been shown, each representing measure ν has an associated operator T. In the rest of the section we use this notation without further explanations.

The following theorem is a representation theorem for Banach lattices with a σ–local decomposition by means of a locally σ–integrable representing measure.

Theorem 4.3 *Let E be a σ–Fatou Banach lattice with a σ–local decomposition of E, $\{e_\alpha\}_{\alpha\in\Delta}$, such that $\{P_{e_\alpha}(E) : \alpha \in \Delta\} \subseteq E_a$. Then the following assertions are equivalent:*

1) There is a measure ν representing for E such that E is order isometrically isomorphic to $L^1_w(\nu)$.

2) There is a measure ν representing for E that is locally σ–integrable.

Proof. Let us start by showing that 1) implies 2). Suppose that there is a measure ν representing for E such that E is order isometrically isomorphic to $L^1_w(\nu)$ and let $f \in L^1_w(\nu)^+$. Then there exists $e \in E^+$ such that $T(e) = f$. By the hypothesis, e can be written as $e = \sup_{n\geq 1} \sum_{k=1}^n P_{e_{\alpha_k}}(e)$ so by Theorem

$$\begin{aligned} f &= T(e) = T\big(\sup_{n\geq 1}\sum_{k=1}^n P_{e_{\alpha_k}}(e)\big) = \sup_{n\geq 1} T\big(\sum_{k=1}^n P_{e_{\alpha_k}}(e)\big) = \sup_{n\geq 1}\sum_{k=1}^n T\big(P_{e_{\alpha_k}}(e)\big) \\ &= \sup_{n\geq 1}\sum_{k=1}^n P_{\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}}(T(e)) = \sup_{n\geq 1}\sum_{k=1}^n P_{\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}}(f) = \sup_{n\geq 1}\sum_{k=1}^n f\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}, \end{aligned}$$

where $f\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}$ is an element of $L^1(\nu)$ since it is the image of the element $P_{e_{\alpha_k}}(e)$ belonging to E_a by T. Therefore $\mathrm{supp}(f) \subseteq \cup_{k=1}^\infty \{\alpha_k\} \times \Omega_{\alpha_k}$. The extension to all functions is straightforward.

Let us prove now that 2) implies 1). Assume that the representing measure ν for E is locally σ–integrable and take $f \in L^1_w(\nu)^+$. Then it can be written as $f = \sum_{k\geq 1} f\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}} = \sup_{n\geq 1}\sum_{k=1}^n f\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}$ with $f\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}} \in L^1(\nu)$. Thus $f = \sup_{n\geq 1} f_n$ where $f_n = \sum_{k=1}^n f\chi_{\{\alpha_k\}\times\Omega_{\alpha_k}}$ and so $f_n \uparrow f$ with $(f_n)_{n\geq 1} \subset L^1(\nu)$. For each $n \geq 1$, consider $\widetilde{e}_n = T^{-1}(f_n) \in E_a$, and define $e := \sup_{n\geq 1}\widetilde{e}_n$. Note that $T^{-1}(f_n) \uparrow$ in $E_a \subset E$ with

$$\|T^{-1}(f_n)\| = \|f_n\|_\nu \leq \|f\|_\nu < \infty,$$

therefore the σ–Fatou property of E gives the existence of $\sup_{n\geq 1}\widetilde{e}_n \in E$ with

$$\|\sup_{n\geq 1}\widetilde{e}_n\| = \sup_{n\geq 1}\|\widetilde{e}_n\|,$$

and the definition above makes sense. We have to see now that $T(e) = f$. First of all note that, for each $k \geq 1$, since $\widetilde{e}_k \leq \sup_{k\geq 1}\widetilde{e}_k = e$ then $f_k = T(\widetilde{e}_k) \leq T(e)$ and hence

$$T(e) \wedge f_k = f_k, \qquad k \geq 1. \tag{4.0.1}$$

In order to see that $T(e) = f$ let us show now that $\|T(e) - f\|_\nu = 0$. Since $e_n \leq e$, we obtain that

$$e_n = e_n \wedge e = e_n \wedge (\sup_{k\geq 1}\widetilde{e}_k) = \sup_{k\geq 1}(e_n \wedge \widetilde{e}_k)$$

and therefore

$$T(e_n) = T\big(\sup_{k\geq 1}(e_n\wedge\widetilde{e}_k)\big) = \sup_{k\geq 1} T\big(e_n\wedge\widetilde{e}_k\big) = \sup_{k\geq 1}\big(T(e_n)\wedge T(\widetilde{e}_k)\big) = \sup_{k\geq 1}\big(T(e_n)\wedge f_k\big).$$

On the other hand $T(e)$, $T(e_n)$, f and f_n belong to $L^1_w(\nu)$, so $T(e)$, $T(e_n)$, f and f_n belong to $L^1(|x^*\nu|)$ for all $x^* \in X^*$. Therefore, applying the Monotone Convergence Theorem for the scalar measure $|x^*\nu|$, the σ–Fatou property of $L^1_w(\nu)$ and (4.0.1) we obtain

$$\begin{aligned}
\int_\Omega T(e)\,d|x^*\nu| &= \int_\Omega \big(\sup_{n\geq 1} T(e_n)\big)\,d|x^*\nu| = \sup_{n\geq 1}\int_\Omega T(e_n)\,d|x^*\nu| \\
&= \sup_{n\geq 1}\int_\Omega \sup_{k\geq 1}(T(e_n)\wedge f_k)\,d|x^*\nu| = \sup_{n\geq 1}\sup_{k\geq 1}\int_\Omega \big(T(e_n)\wedge f_k\big)\,d|x^*\nu| \\
&= \sup_{k\geq 1}\int_\Omega \sup_{n\geq 1}(T(e_n)\wedge f_k)\,d|x^*\nu| = \sup_{k\geq 1}\int_\Omega (\sup_{n\geq 1} T(e_n))\wedge f_k\,d|x^*\nu| \\
&= \sup_{k\geq 1}\int_\Omega \big(T(e)\wedge f_k\big)\,d|x^*\nu| = \sup_{k\geq 1}\int_\Omega f_k\,d|x^*\nu| \\
&= \int_\Omega f\,d|x^*\nu|, \quad x^*\in X^*.
\end{aligned}$$

Therefore, $\int (T(e)-f)\,d|x^*\nu| = 0$. Taking the supremum on $x^*\in B_{X^*}$, we obtain that $\|T(e)-f\|_\nu = 0$, and so $f = T(e)$. The result for arbitrary functions is then direct. □

Remark 4.4 *Note that the σ–local decomposition for E is necessary in the theorem, since $L^1_w(\nu)$ always has a σ–local decomposition whenever ν is locally σ–integrable.*

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