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Infinite games and quasi-uniform box products

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Abstract

We introduce new infinite games, played in a quasi-uniform space, that generalise the proximal game to the framework of quasi-uniform spaces. We then introduce bi-proximal spaces, a concept that generalises proximal spaces to the quasi-uniform setting. We show that every bi-proximal space is a W-space and as consequence of this, the bi-proximal property is preserved under Σ-products and closed subsets. It is known that the Sorgenfrey line is almost proximal but not proximal. However, in this paper we show that the Sorgenfrey line is bi-proximal, which shows that our concept of bi-proximal spaces is more general than that of proximal spaces. We then present separation properties of certain bi-proximal spaces and apply them to quasi-uniform box products.

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1. Introduction

Jocelyn R. Bell, in [2], introduced an infinite game played in a uniform space which she called the *proximal game*. She showed that every proximal space is a W-space and the proximal property is preserved under Σ-products, countable products and closed subsets. Also, every proximal space is collectionwise normal, countably paracompact and collectionwise Hausdorff. She then used this

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game to show that the uniform box product of countably many copies of a Fort-space is collectionwise normal, countably paracompact and collectionwise Hausdorff. During the 29^{th} *Summer Conference on Topology and its Applications held in New York 2014*, Ralph Kopperman asked whether the proximal game, played in a uniform space, can be extended to generalised uniform spaces, for example, quasi-uniform spaces. In this article we answer Ralph Kopperman's question for the class of quasi-uniform spaces. In particular, we introduce infinite games played in a quasi-uniform space which generalise the proximal game.

Since for any quasi-uniform space $(X,\mathcal{U})$, $\mathcal{U}^{-1}$ is a quasi-uniformity on X and $\mathcal{U}^s$ is a uniformity on X, there are, atleast, three types of infinite games played in a quasi-uniform space that generalise the proximal game. We call the game played in $(X,\mathcal{U})$, *the left-proximal game*; the game played in $(X,\mathcal{U}^{-1})$, *the right proximal game*; and the game played in $(X,\mathcal{U}^s)$, *the* $\mathcal{U}^s$*-proximal game*. We show that a quasi-uniform space is left proximal if and only if it is right proximal. Since for any quasi-uniformity $\mathcal{U}$ on X, $\mathcal{U}^s$ is a uniformity on X, we observe that the $\mathcal{U}^s$-proximal game corresponds to the proximal game. Furthermore, we say that a space is *bi-proximal* provided it is left and right proximal, and thereafter, show that a space is bi-proximal if and only if it is $\mathcal{U}^s$-proximal. In [2, Example 5], Bell showed that the Sorgenfrey line is almost proximal but it is not proximal. However, in this paper we show that the Sorgenfrey line is bi-proximal, which proves that the bi-proximal property is more general than the proximal property. Also, we show that every bi-proximal space is a W-space, and as the consequence of this, the bi-proximal property is closed under closed subsets, Σ-products and countable products. This implies that the Sorgenfrey plane is bi-proximal. However, it is known that the Sorgenfey plane is not normal and collectionwise Hausdorff. Therefore, unlike proximal spaces, bi-proximal spaces are not, in general, collectionwise normal and collectionwise Hausdorff. Furthermore, in [9], Kunži and Watson showed that there exists a quasi-metric space which is not countably metacompact. Therefore, bi-proximal spaces are not, in general, countably metacompact.

If we restrict ourselves to the quasi-uniform spaces $(X,\mathcal{U})$ which satisfy the property that $\mathcal{U}$ and $\mathcal{U}^s$ are both compatible with the topology on X, then most topological properties satisfied by proximal spaces are also satisfied by bi-proximal spaces. We then use this fact to show that the quasi-uniform box product of countably many copies of a Fort-space is bi-proximal, and as a consequence of this, it is collectionwise normal, countably paracompact and collectionwise Hausdorff. We point out that our work is in parallel with Bell [2]. In fact, we shall adapt some ideas and techniques of [2], which will be appropriately mentioned.

2. Quasi-uniform spaces

Definition 2.1 ([8]). A quasi-uniformity on a set X is a filter $\mathcal{U}$ on $X \times X$ such that

(i) each member U of $\mathcal{U}$ contains the diagonal $\triangle = \{(x,x) : x \in X\}$ of X,
(ii) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $2V \subseteq U$ where $2V = V \circ V = \{(x,z) \in X \times X :$ there is $y \in X$ such that $(x,y) \in V$ and $(y,z) \in V\}$.

The members $U \in \mathcal{U}$ are called *entourages* of $\mathcal{U}$ and the elements of X are called *points*. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*

If $\mathcal{U}$ is a quasi-uniformity on a set X, then the filter $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ on $X \times X$ is also a quasi-uniformity on X. The quasi-uniformity $\mathcal{U}^{-1}$ is called the *conjugate* of $\mathcal{U}$. A quasi-uniformity that is equal to its conjugate is called a *uniformity*. The union of a quasi-uniformity $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ yields a subbase of the coarsest uniformity, denoted $\mathcal{U}^s$, finer than $\mathcal{U}$. If $U \in \mathcal{U}$, the elements of $\mathcal{U}^s$ are of the form $U \cap U^{-1}$ and are denoted by U^s. For $U \in \mathcal{U}$, $x \in X$ and $Z \subset X$, put $U(x) = \{y \in X : (x,y) \in U\}$ and $U(Z) = \bigcup\{U(z) : z \in Z\}$. A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets $\{U(x) : U \in \mathcal{U}\}$ is a base of neighbourhoods of any point $x \in X$.

A subset A of X belongs to $\tau(\mathcal{U})$ if and only if for each $x \in A$, there is an entourage $U \in \mathcal{U}$ such that $U(x) \subset A$. Thus for each $x \in X$ and $U \in \mathcal{U}$, $U(x)$ is a $\tau(\mathcal{U})$-neighborhood of x. Note that $U(x)$ need not be $\tau(\mathcal{U})$-open in general. However, there is always a base $\mathcal{B}$ for $\mathcal{U}$ such that for each $B \in \mathcal{B}$ and $x \in X$, $B(x) \in \tau(\mathcal{U})$.

Proposition 2.2 ([4]). *Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on X. Let $U \in \mathcal{U}$, $V \in \mathcal{V}$ and $M \subset X \times X$. Then $U \circ M \circ V$ is a neighborhood of M in the topology of $\mathcal{U}^{-1} \times \mathcal{V}$.*

Corollary 2.3 ([4]). *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then $\{U : U \in \mathcal{U}$ and U is $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$ open in $X \times X\}$ is a base for $\mathcal{U}$.*

Definition 2.4 ([12]). Let (X,τ) be a topological space. Then the family of subsets

$$\mathcal{B} = \{[A \times A] \cup [(X \setminus A) \times X] : A \in \tau\}$$

is a subbase of a quasi-uniformity on X that generates the topology on X. The quasi-uniformity generated by this subbase is called the *Pervin quasi-uniformity*.

Pervin remarked in [12] that the quasi-uniformity generating the topology is not unique.

Example 2.5. A Fort-space is the one point compactification of a discrete space. If W is a discrete space, we will denote the one point compactification X of W by $X = W \cup \{\infty\}$. If W is uncountable, we say X is an uncountable Fort-space. If we equip the Fort-space $X = W \cup \{\infty\}$ with the quasi-uniformity $\mathcal{U}$ which has subbase $\{U_F : F \subseteq W$ is finite$\}$, where $U_F = \triangle \cup [(X \setminus F) \times X]$, then $\mathcal{U}^{-1}$ has subbase $\{U_F^{-1} : F \subseteq W \text{ is finite}\}$, where $U_F = \triangle \cup [X \times X \setminus F]$. Thus $U_F \cap U_F^{-1} = \triangle \cup (X \setminus F) \times (X \setminus F)$. It follows that if $x \in F$, then $U_F^{-1}(x) = \{x\} \cup (X \setminus F)$ and $U_F(x) = \{x\}$. If $x \notin F$, then we have $U_F^{-1}(x) = X \setminus F$ and $U_F(x) = X$. Moreover, it follows that for any $x \in F$, $U_F(x) \cap U_F^{-1}(x) = [\{x\} \cup (X \setminus F)] \cap \{x\} = \{x\}$. Also, for any $x \notin F$, we have $U_F(x) \cap U_F^{-1}(x) = X \setminus F \cap X = X \setminus F$.

Definition 2.6. Let X be a set and $d : X \times X \to [0, \infty)$ be a function mapping into the set $[0, \infty)$ of nonnegative real numbers. Then d is called a *quasi-pseudometric* on X if $d(x, x) = 0$ for all $x \in X$, and $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, $z \in X$. The pair (X, d) is called a *quasi-pseudometric space*. If in addition, for any $x, y \in X$, $d(x, y) = 0 = d(y, x) \implies x = y$, then d is called a T_0-*quasi-metric* and the pair (X, d) is called a T_0-*quasi-metric space*.

Example 2.7 ([8])**.** Let d be a quasi-pseudometric on a set X. For each $\epsilon > 0$, set $U_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. Since for each $\epsilon > 0$, $2U_{\epsilon/2} \subseteq U_\epsilon$, the filter generated by the base $\{U_\epsilon : \epsilon > 0\}$ is a quasi-uniformity on X and is called the *quasi-pseudometric quasi-uniformity* $\mathcal{U}_d$ induced by d on X.

3. The proximal game played on a quasi-uniform space

This section presents infinite games, played in a quasi-uniform space, that generalise the proximal game to the quasi-uniform setting. Since for any quasi-uniform space $(X, \mathcal{U})$, $\mathcal{U}^{-1}$ is a quasi-uniformity and $\mathcal{U}^s$ is a uniformity on X, there are, atleast, three types of infinite games played in a quasi-uniform space namely; the left proximal game, the right proximal game and the $\mathcal{U}^s$-proximal game, as defined in Section 1.

Similarly to [2, Section 3], the left-proximal game is a game of perfect information that is played in a quasi-uniform space $(X, \mathcal{U})$. Following [2, Section 3], in this game, Player A, the entourage picking player, chooses elements of the quasi-uniformity $\mathcal{U}$ while Player B, the point picking Player, chooses elements of X. The first two rounds of the game are as follows:

(i) Player A chooses $U_1 \in \mathcal{U}$
Player B chooses $x_1 \in X$

(ii) Player A chooses $U_2 \in \mathcal{U}$ with $U_2 \subseteq U_1$
Player B chooses $x_2 \in U_1(x_1) \cap U_1^{-1}(x_1)$

In general, if $x_1, x_2, \cdots, x_n$ are the n choices of Player B, Player A chooses $U_{n+1} \subseteq U_n$ and then Player B must choose $x_{n+1} \in U_n(x_n) \cap U_n^{-1}(x_n)$. Then Player A wins the game if

(i) there exists $z \in X$ such that $x_1, x_2, \cdots$ $\tau(\mathcal{U}^s)$-converges to z or
(ii) $\bigcap_{i \in \mathbb{N}} U_i(x_i) \cap U_i^{-1}(x_i) = \varnothing$.

Remark 3.1. Note that Player B chooses a point x_{n+1} in $U(x_n) \cap U_n^{-1}(x_n)$ and not in $U_n(x_n)$ or $U_n^{-1}(x_n)$ only. This is because the space in which Player B chooses only elements in $U_n(x_n)$ or $U_n^{-1}(x_n)$ does not generalise the proximal game to the framework of quasi-uniform spaces.

A *left-play* of the game is a sequence $(U_1, x_1, U_2, x_2, \cdots)$, where $U_i \in \mathcal{U}$ and $x_i \in X$ are chosen according to the rules of the game. A finite sequence of points $x_1, x_2, \cdots, x_n$ is *left admissible* if for some sequences of entourages $U_1 \supseteq U_2, \cdots \supseteq U_n$ from $\mathcal{U}$, $(U_1, x_1, U_2, x_2, \cdots, U_n, x_n)$ is a left-partial play of the game. A *left strategy*, in the left-proximal game on $(X, \mathcal{U})$, is a recursively defined map w from the set of left admissible finite sequences $\mathcal{A}$ of X to $\mathcal{U}$, that is,

$$w : \mathcal{A} \to \mathcal{U}$$

such that

(i) $w(\varnothing) = X \times X$,
(ii) $x_{n+1} \in w(x_1, x_2, \cdots, x_{n-1})(x_n) \cap w(x_1, x_2, \cdots, x_{n-1})^{-1}(x_n)$, and
(iii) $w(x_1, x_2, \cdots, x_{n-1}) \supseteq w(x_1, x_2, \cdots, x_n)$.

Therefore, $w(x_1, x_2, \cdots, x_n)$ would be an element of $\mathcal{U}$ chosen by Player A if $x_1, x_2, \cdots, x_n$ are the n choices of Player B and $w(x_1, x_2, \cdots, x_n)^{-1}$ is the conjugate of $w(x_1, x_2, \cdots, x_n)$. A sequence of points $x_1, x_2, \cdots$ resulting from a left-play of a left strategy is a *left-proximal* sequence. A left-strategy is winning if

(i) every left-proximal sequence $\tau(\mathcal{U}^s)$-converges to a point $z \in X$ or
(ii) $\bigcap_{i \in \mathbb{N}} w(x_1, x_2, \cdots, x_{i-1})(x_i) \cap w(x_1, x_2, \cdots, x_{i-1})^{-1}(x_i) = \varnothing$.

Definition 3.2 (compare [2, Definition 3])**.** A quasi-uniform space $(X, \mathcal{U})$ is *left-proximal* provided Player A has a winning strategy in the left-proximal game on $(X, \mathcal{U})$. If a quasi-uniformizable space X has a compatible quasi-uniformity $\mathcal{U}$ for which X is left-proximal, we say X *is left-proximal.*

Remark 3.3. If $\mathcal{U}$ is a uniformity on X, that is, $\mathcal{U} = \mathcal{U}^{-1}$, then the left-proximal game is exactly the proximal game in the sense of [2].

In this paper, we will work with a filter base, consisting of $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$-open entourages, rather than the whole quasi-uniformity.

Lemma 3.4. *Suppose $(X, \mathcal{E})$ is a left-proximal quasi-uniform space and $\mathcal{U}$ is the filter base for $\mathcal{E}$. Then $(X, \mathcal{U})$ is left-proximal.*

Proof. Suppose $w : \mathcal{A} \to \mathcal{E}$ is a left-winning strategy in the left-proximal game on $(X, \mathcal{E})$, where $\mathcal{A}$ is the set of left-admissible finite sequences. Then the construction of the left winning strategy $v : \mathcal{A}' \to \mathcal{U}$, where $\mathcal{A}'$ is the set of left-admissible finite sequences in the left-proximal game on $(X, \mathcal{U})$ follows exactly that of [2, Lemma 3]. □

Corollary 3.5. *Suppose $(X, \mathcal{E})$ is a left-proximal quasi-uniform space and $\mathcal{U}$ is the subbase for $\mathcal{E}$. Then $(X, \mathcal{U})$ is left-proximal.*

Lemma 3.6 (compare [2, Lemma 4])**.** *Every T_0-quasi-metric space (X, d) is left-proximal with respect to the quasi-uniformity generated by the filter base $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, where $U_n = \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$.*

Proof. Suppose $x_n, x_2, \cdots x_n$ are the n choices for Player B. Then Player A chooses the entourage $w(x_1, x_2, \cdots, x_n) = U_{n+1}$, where $U_{n+1} = \{(x, y) : d(x, y) < 2^{-(n+1)}\}$. Then $x_1, x_2, \cdots$ is a d^s-Cauchy sequence. Therefore, $x_1, x_2, \cdots$ $\tau(\mathcal{U}^s)$-converges to $z \in X$ or $\bigcap_{i\in\mathbb{N}} U_i(x_i) \cap U_i^{-1}(x_i) = \varnothing$. Hence w is a Markov winning strategy (depending only on the opponent's last choice and the round number) for Player A in the left-proximal game on $(X, \mathcal{U})$. $\square$

Example 3.7. Let $\mathbb{R}$ be the set of reals equipped with the T_0-quasi-metric d, defined by $d(x, y) = x - y$ if $x \geq y$ and $d(x, y) = 1$ otherwise. Consider the quasi-uniformity generated by the base $\mathcal{U} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : d(x, y) < 2^{-n}\}$, where $n \in \mathbb{N}$. Then one can easily check that $\tau(\mathcal{U})$ is the Sorgenfrey topology on $\mathbb{R}$, which is generated by the base $\{(a, b] : a, b \in \mathbb{R}, a < b\}$ on $\mathbb{R}$. Similarly, $\tau(\mathcal{U}^{-1})$ is a topology on $\mathbb{R}$ generated by the base $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. Then $\mathbb{R}$ equipped with the Sorgenfrey T_0-quasi metric is left-proximal by Lemma 3.6.

Example 3.8. The Sorgenfrey plane has a topology generated by clopen boxes $(a, b] \times (c, d]$. If $p = (x, y) \in \mathbb{R}\times\mathbb{R}$ and $\epsilon > 0$ we write $B(p, \epsilon) = (x, x+\epsilon] \times (y, y+\epsilon]$ and call it the clopen square cornered at p with side ϵ. This topology is quasi-metrizable by the quasi-metric defined as follows: for any points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in $\mathbb{R} \times \mathbb{R}$, $\rho(p_1, p_2) = \max\{d(x_1, x_2), d(y_1, y_2)\}$, where d is the T_0-quasi-metric defined in Example 3.7. Consider the quasi-uniformity generated by the base $\mathcal{U} = \{(p_1, p_2) \in \mathbb{R} \times \mathbb{R} : \rho(p_1, p_2) < 2^{-n}\}$, where $n \in \mathbb{N}$. Then one can easily check that $\tau(\mathcal{U})$ is the basis for the topology on the Sorgenfrey plane $\mathbb{R}\times\mathbb{R}$ which is generated by the base $\{(a, b] \times (c, d] : a, b, c, d \in \mathbb{R}, a < b,\ c < d\}$. Since $(\mathbb{R} \times \mathbb{R}, \rho)$ is a quasi-metric space, it is left proximal by Lemma 3.6.

A T_0-quasi-metric space may not be left-proximal with respect to a quasi-uniformity other than the one inherited from the T_0-quasi-metric as the next example shows.

Example 3.9 (compare [2, Example 1])**.** Let W be uncountable set with the discrete topology. Then W is quasi-metrizable with the Sorgenfrey T_0-quasi-metric. Consider the quasi-uniformity generated by the subbase $\mathcal{U} = \{U_F : F \subseteq W \text{ is finite}\}$, where $U_F = [F \times F] \cup [(W \setminus F) \times W]$. Then $\mathcal{U}^{-1} = \{U_F^{-1} : F \subseteq W \text{ is finite}\}$, where $U_F^{-1} = [F \times F] \cup [W \times (W \setminus F)]$, is also a subbase generating a quasi-uniformity on W. Then W is not left proximal. To see this, suppose $w : \mathcal{A} \to \mathcal{U}$ is any strategy for Player A and $w(\varnothing) = U_{F_1}$. Then Player B chooses x_1 so that $x_1 \notin F_1$. Then $U_{F_1}(x_1) = W$ and $U_{F_1}^{-1}(x_1) = W \setminus F_1$ and so $U_{F_1}(x_1) \cap U_{F_1}^{-1}(x_1) = W \setminus F_1$. In general, if for all $k \leq n$, $w(x_1, x_2, \cdots, x_k) = U_{F_k}$, Player B chooses $x_n \notin F_1 \cup F_2, \cdots \cup F_n$, so that x_n is distinct from all previous choices of Player B. Then $\bigcap_{n\in\mathbb{N}} U_{F_n}(x_n) \cap U_{F_n}^{-1}(x_n) = W \setminus F_n \neq \varnothing$ and $x_1, x_2, \cdots$ is not $\tau(\mathcal{U}^s)$-convergent to any point in W.

We now give an example of a left-proximal space which is not quasi-metrizable.

Example 3.10 (compare [2, Example 2])**.** An uncountable Fort-space $X = W \cup \{\infty\}$ is a non quasi-metrizable space which is left proximal. To see this, consider the quasi-uniformity generated by the subbase $\mathcal{U} = \{U_F : F \subseteq W \text{ is finite}\}$, where $U_F = [F \times F] \cup [(X \setminus F) \times X]$. Then $\mathcal{U}^{-1} = \{U_F^{-1} : F \subseteq W \text{ is finite}\}$, where $U_F^{-1} = [F \times F] \cup [X \times (X \setminus F)]$, is also a subbase for a quasi-uniformity on X. Suppose Player B chooses $x_1 \in X$. If $x_1 \neq \infty$, Player A lets $w(x_1) = U_{F_1}$, where $F_1 = \{x_1\}$. Then Player B chooses x_2. If $x_2 \neq x_1$, then $U_{F_1}(x_2) = X$ and $U_{F_1}^{-1} = X \setminus F_1$ and so $U_{F_1}(x_2) \cap U_{F_1}^{-1}(x_2) = X \cap (X \setminus F_1) = X \setminus F_1$. Since $F_1 = \{x_1\}$, then $U_{F_1}(x_2) \cap U_{F_1}^{-1}(x_2) = X \setminus \{x_1\}$ and Player B cannot choose the point x_1 in the future rounds of the game. If $x_2 = x_1$, then $U_{F_1}(x_2) = \{x_1\}$ and $U_{F_1}^{-1}(x_2) = X$. Therefore, $U_{F_1}(x_2) \cap U_{F_1}^{-1}(x_2) = \{x_1\}$. This means that Player B is forced to pick x_1 at every rounds of the game. The left-winning strategy for Player A is to add Player B's last choice (as long as it is not ∞) to the finite set which determines the element of the quasi-uniformity. Precisely, if $x_1, x_2, \cdots x_n$ are the first n choices of Player B, then $w(x_1, x_2, \cdots, x_n) = U_{F_n}$, where $F_n = \{x_1, x_2, \cdots, x_n\} \setminus \{\infty\}$. Then any left proximal sequence $x_1, x_2, \cdots$ either $\tau(\mathcal{U}^s)$-converges to ∞ or is eventually constant.

The right proximal game proceeds as the left proximal game except that Player A, picks elements in the conjugate quasi-uniformity $\mathcal{U}^{-1}$, that is, elements of the form U^{-1}, where $U \in \mathcal{U}$. The winning criteria for Player A, in the right proximal game, is the same as the winning criteria for Player A in the left proximal.

Definition 3.11. A quasi-uniform space $(X,\mathcal{U})$ is *right-proximal* provided Player A has a winning strategy in the right-proximal game on $(X,\mathcal{U})$. If a quasi-uniformizable space X has a compatible quasi-uniformity $\mathcal{U}$ for which X is right-proximal, we say X *is right-proximal*.

Lemma 3.12. *A quasi-uniform space $(X,\mathcal{U})$ is left-proximal if and only if it is right proximal.*

Proof. Suppose X is left-proximal, $\mathcal{A}$ is the set of left admissible finite sequences and $\mathcal{B}$ is the set of right admissible finite sequences. Suppose $w : \mathcal{A} \to \mathcal{U}$ is a left winning strategy and $m : \mathcal{B} \to \mathcal{U}^{-1}$ is a right strategy. Let $w(\varnothing) = X \times X$ and $m(\varnothing) = X \times X$. Suppose Player B, in the left-proximal game, chooses $x_1 \in X$. Then Player A chooses $w(x_1)$. Suppose x_1 is a choice for Player B in the right proximal game. Then Player A, in the right proximal game, chooses $m(x_1)$, where $m(x_1)$ is a conjugate of $w(x_1)$. In general, if $x_1, x_2, \cdots, x_n$ are the n choices for Player B in the left proximal game and $x_1, x_2, \cdots, x_n$ is a right admissible sequence, then Player A, in the left-proximal game, chooses $w(x_1, x_2, \cdots, x_n) \subseteq w(x_1, x_2, \cdots, x_{n-1})$. Also, Player A, in the right proximal game, chooses $m(x_1, x_2, \cdots, x_n) \subseteq m(x_1, x_2, \cdots, x_{n-1})$, where $m(x_1, x_2, \cdots, x_n)$ is a conjugate of $w(x_1, x_2, \cdots, x_n)$. Now suppose Player B, in the left-proximal game, chooses

$$x_{n+1} \in w(x_1, x_2, \cdots, x_{n-1})(x_n) \cap w(x_1, x_2, \cdots, x_{n-1})^{-1}(x_n).$$

Then Player B, in the right proximal game, can choose

$$x_{n+1} \in m(x_1, x_2, \cdots, x_{n-1})(x_n) \cap m(x_1, x_2, \cdots, x_{n-1})^{-1}(x_n)$$

since $m(x_1, x_2, \cdots, x_{n-1})$ is a conjugate of $w(x_1, x_2, \cdots, x_{n-1})$. Therefore, $x_1, x_2, \cdots, x_{n+1}$ is a right admissible sequence. Since $(X, \mathcal{U})$ is left-proximal, $x_1, x_2, \cdots$ $\tau(\mathcal{U}^s)$-converges to $z \in X$. If this does not hold, then

$$\bigcap_{i \in \mathbb{N}} m(x_1, x_2, \cdots, x_{i-1})(x_i) \cap m(x_1, x_2, \cdots, x_{i-1})^{-1}(x_i)$$

$$= \bigcap_{i \in \mathbb{N}} w(x_1, x_2, \cdots, x_{i-1})(x_i) \cap w(x_1, x_2, \cdots, x_{i-1})^{-1}(x_i) = \varnothing.$$

Therefore, m is a right winning strategy. The converse follows the same argument. □

Remark 3.13. Since any left-proximal quasi-uniform space $(X, \mathcal{U})$ is right proximal by Lemma 3.12, we have that $\mathcal{A} = \mathcal{B}$, where $\mathcal{A}$ is a set of left-admissible finite sequences and $\mathcal{B}$ is a set of right-admissible finite sequences. Moreover, for any choices $x_1, x_2 \cdots, x_n$ for Player B in the left and right proximal games, Player B can choose

$$x_{n+1} \in w(x_1, x_2 \cdots, x_{n-1})(x_n) \cap m(x_1, x_2 \cdots, x_{n-1})(x_n)$$

since $w(x_1, x_2 \cdots, x_{n-1})$ is a conjugate of $m(x_1, x_2 \cdots, x_{n-1})$.

Remark 3.14. One can easily show that the Sorgenfrey line and the Fort-space are right proximal. However, the uncountable space W is not right proximal with respect to the Pervin quasi-uniformity.

The $\mathcal{U}^s$-proximal game proceeds as the left proximal game except that Player A, picks elements in the symmetrised uniformity $\mathcal{U}^s$, that is, elements of the for $U \cap U^{-1}$, where $U \in \mathcal{U}$. The winning criteria for Player A, in the $\mathcal{U}^s$-proximal game, is the same as the winning criteria for Player A in the left proximal.

Definition 3.15. A quasi-uniform space $(X, \mathcal{U})$ is $\mathcal{U}^s$*-proximal* provided Player A has a winning strategy in the $\mathcal{U}^s$-proximal game on $(X, \mathcal{U})$. If a quasi-unifomizable space X has a compatible quasi-uniformity $\mathcal{U}$ for which X is $\mathcal{U}^s$-proximal, we will say X *is* $\mathcal{U}^s$*-proximal*.

Remark 3.16. Since for any quasi-uniform space $(X, \mathcal{U})$, $\mathcal{U}^s$ is a uniformity on X, then the $\mathcal{U}^s$-proximal game corresponds to the proximal game.

We now present bi-proximal spaces, a concept that generalises proximal spaces to the framework of quasi-uniform spaces. These are spaces that posses a winning strategy for Player A in the left and right proximal games.

Definition 3.17. Let $(X, \mathcal{U})$ be a quasi-uniform space. We say $(X, \mathcal{U})$ is *bi-proximal* provided it is left and right-proximal. If a quasi-uniformizable topological space X has a compatible quasi-uniformity $\mathcal{U}$ for which X is bi-proximal, we will say X *is bi-proximal*.

Remark 3.18. One can easily show that the Sorgenfrey line and the Fort-space are bi-proximal. However, the uncountable space W is not bi-proximal with respect to the Pervin quasi-uniformity. Moreover, the Sorgenfrey line is an example of a space which is bi-proximal but not proximal in the sense of Bell [2]. Therefore, bi-proximal spaces are more general than proximal spaces. Also, the Sorgenfrey line is first countable, hence, it is a W-space in the sense of [5]. As we will show later, all bi-proximal spaces are, in fact, W-spaces.

Theorem 3.19. *A quasi-uniform space* $(X,\mathcal{U})$ *is* $\mathcal{U}^s$*-proximal if and only if it is bi-proximal.*

Proof. Suppose $(X,\mathcal{U})$ is $\mathcal{U}^s$-proximal, $\mathcal{M}$ is a set of $\mathcal{U}^s$-admissible finite sequences and $\beta : \mathcal{M} \to \mathcal{U}^s$ is the $\mathcal{U}^s$-winning strategy. Then we define the winning strategy $w : \mathcal{A} \to \mathcal{U}$ for Player A in the left-proximal game, where $\mathcal{A}$ be a set of left admissible finite sequences.

Suppose Player B, in the $\mathcal{U}^s$-proximal game, chooses x_1 and Player A chooses $\beta(x_1)$. Suppose x_1 is a choice for Player A in the left proximal game. Then Player A in the left proximal game chooses $w(x_1)$, where $\beta(x_1) = w(x_1) \cap w(x_1)^{-1}$. Suppose Player B, in the $\mathcal{U}^s$-proximal game, has chosen $x_1, x_2, \cdots x_n$ according to the rules of the $\mathcal{U}^s$-proximal game and Player A chooses $\beta(x_1, x_2, \cdots x_n) \subseteq \beta(x_1, x_2, \cdots x_{n-1})$. Suppose $x_1, x_2, \cdots x_n$ is a left admissible finite sequence. Then Player A, in the left-proximal game, chooses $w(x_1, x_2, \cdots x_n) \subseteq w(x_1, x_2, \cdots x_{n-1})$, where

$$\beta(x_1, x_2, \cdots x_n) = w(x_1, x_2, \cdots x_n) \cap w(x_1, x_2, \cdots x_n)^{-1}.$$

Suppose Player B, in the $\mathcal{U}^s$-proximal game, chooses

$$x_{n+1} \in \beta(x_1, x_2, \cdots, x_{n-1})(x_n).$$

Then

$$x_{n+1} \in w(x_1, x_2, \cdots, x_{n-1})(x_n) \cap w(x_1, x_2, \cdots, x_{n-1})^{-1}(x_n)$$

and Player B, in the left-proximal game can choose x_{n+1}. Thus $x_1, x_2, \cdots, x_n, x_{n+1}$ is a left admissible finite sequence. Since β is a $\mathcal{U}^s$ winning strategy, then $x_1, x_2, \cdots$ $\tau(\mathcal{U}^s)$-converges to $z \in X$. If this does not hold, then

$$\bigcap_{i \in \mathbb{N}} \beta(x_1, x_2, \cdots, x_{i-1})(x_i) =$$

$$\bigcap_{i \in \mathbb{N}} (w(x_1, x_2, \cdots, x_{i-1})(x_i) \cap w(x_1, x_2, \cdots, x_{i-1})^{-1}(x_i)) = \varnothing.$$

Therefore, $(X,\mathcal{U})$ is left-proximal and by Lemma 3.12, $(X,\mathcal{U})$ is right proximal.

Conversely, suppose $(X,\mathcal{U})$ is a bi-proximal quasi-uniform space. Suppose $w : \mathcal{A} \to \mathcal{U}$ and $m : \mathcal{A} \to \mathcal{U}^{-1}$ are the left and right winning strategies respectively, where $\mathcal{A}$ is a set of left and right admissible finite sequences. Then we need to define the $\mathcal{U}^s$-winning strategy $\beta : \mathcal{M} \to \mathcal{U}^s$ for Player A in the $\mathcal{U}^s$-proximal game, where $\mathcal{M}$ is a set of $\mathcal{U}^s$-admissible finite sequences.

Suppose Player B, in the left and right-proximal games, chooses $x_1 \in X$. Then Player A, in the left proximal game chooses $w(x_1)$, whereas Player A, in the right proximal game, chooses $m(x_1)$, where $m(x_1)$ is a conjugate of $w(x_1)$. Then Player A, in the $\mathcal{U}^s$-proximal game chooses $\beta(x_1)$, where $\beta(x_1) = w(x_1) \cap m(x_1)$. In general, if Player B, at stage n, in the left and right proximal games, has chosen $x_1, x_2, \cdots, x_n$ and that $x_1, x_2, \cdots, x_n$ is a $\mathcal{U}^s$-admissible finite sequence. Then Player A, in the left-proximal game, chooses $w(x_1, x_2, \cdots, x_n) \subseteq w(x_1, x_2, \cdots, x_{n-1})$, whereas Player A, in the right-proximal game, chooses $m(x_1, x_2, \cdots, x_n) \subseteq m(x_1, x_2, \cdots, x_{n-1})$, where $m(x_1, x_2, \cdots, x_n)$ is a conjugate of $w(x_1, x_2, \cdots, x_n)$. Then Player A in the $\mathcal{U}^s$-proximal game chooses $\beta(x_1, x_2, \cdots, x_n) = w(x_1, x_2, \cdots, x_n) \cap m(x_1, x_2, \cdots, x_n)$. Now, Player B, in the left and right proximal games chooses $x_{n+1} \in w(x_1, x_2, \cdots, x_{n-1})(x_n) \cap m(x_1, x_2, \cdots, x_{n-1})(x_n)$. Then Player B, in the $\mathcal{U}^s$-proximal game, can choose $x_{n+1} \in \beta(x_1, x_2, \cdots, x_n)(x_n)$ and $x_1, x_2, \cdots, x_n, x_{n+1}$ is a $\mathcal{U}^s$-admissible finite sequence. Since $(X,\mathcal{U})$ is bi-proximal, we have

$$\bigcap_{i\in\mathbb{N}} \beta(x_1, x_2, \cdots, x_{i-1})(x_i) =$$

$$\bigcap_{i\in\mathbb{N}} w(x_1, x_2, \cdots, x_{i-1})(x_i) \cap m(x_1, x_2, \cdots, x_{i-1})(x_i) = \varnothing.$$

If not, then $x_1, x_2, \cdots$ $\tau(\mathcal{U}^s)$-converges to $z \in X$. Therefore, $(X,\mathcal{U})$ is $\mathcal{U}^s$-proximal. □

Remark 3.20. In a bi-proximal space, the set of left admissible finite sequences, right-admissible finite sequences and $\mathcal{U}^s$-admissible finite is the same, that is $\mathcal{M} = \mathcal{A}$. Therefore, we simply call the set $\mathcal{A}$, the set of admissible finite sequences. The same is true for proximal sequences.

If we restrict the winning strategy for Player A in the left-proximal game to only require convergence, then we say Player A *absolutely wins* the left-proximal game. Also, If we restrict the winning strategy for Player A in the right-proximal game to only require convergence, then we say Player A *absolutely wins* the right-proximal game. Thus a space is *absolutely bi-proximal* provided Player A has absolutely left and right winning strategies in the left and right proximal games. For T_0-quasi-metric spaces, it turns out that an absolutely bi-proximal space is bicomplete. A T_0-quasi-metric space (X, d) is said to be bicomplete if the metric space (X, d^s) is complete.

Lemma 3.21. *A T_0-quasi-metric space (X, d) is bi-proximal if and only if is bicomplete.*

Proof. Suppose (X, d) is bicomplete. Then (X, d^s) is complete and by [2, Lemma 5] (X, d) is $\mathcal{U}^s$-proximal. Conversely, suppose (X, d) is bi-proximal with respect to the quasi-uniformity generated by the filter base having elements of the form $U_n = \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$. Then (X, d) is $\mathcal{U}^s$-proximal by Theorem 3.19 and (X, d^s) is complete by [2, Lemma 5]. Therefore, (X, d) is bicomplete. □

We now present the weakening of the left proximal game, the right proximal game and the $\mathcal{U}^s$-proximal game. In these games, Player A wins the game if he forces Player $B's$ choices to cluster.

Definition 3.22. Let $(X,\mathcal{U})$ be a quasi-uniform space and $w : \mathcal{A} \to \mathcal{U}$ be a left-strategy for Player A in the left proximal game. Player A *almost wins* the left-proximal game if for every left-proximal sequence $x_1, x_2, \cdots$, either

(i) $x_1, x_2, \cdots$ has a $\tau(\mathcal{U}^s)$-accumulation point or
(ii) $\bigcap_{i\in\mathbb{N}} w(x_1, x_2, \cdots, x_{i-1})(x_i) \cap w(x_1, x_2, \cdots, x_{i-1})^{-1}(x_i) = \varnothing$.

If Player A has an almost left winning strategy, we say the space is *almost-left proximal*. If X is a quasi-uniformizable space and there exists a quasi-uniformity $\mathcal{U}$ for which X is almost left proximal, we say X *almost left-proximal*.

Definition 3.23. Let $(X,\mathcal{U})$ be a quasi-uniform space and $m : \mathcal{A} \to \mathcal{U}^{-1}$ be a right-strategy for Player A in the right proximal game. Player A *almost wins* the right-proximal game if for every right-proximal sequence $x_1, x_2, \cdots$, either

(i) $x_1, x_2, \cdots$ has a $\tau(\mathcal{U}^s)$-accumulation point or
(ii) $\bigcap_{i\in\mathbb{N}} m(x_1, x_2, \cdots, x_{i-1})(x_i) \cap m(x_1, x_2, \cdots, x_{i-1})^{-1}(x_i) = \varnothing$.

If Player A has an almost right winning strategy, we say the space is *almost-right proximal*. If X is a quasi-uniformizable space and there exists a quasi-uniformity $\mathcal{U}$ for which X is almost right proximal, we say X *almost right-proximal*.

Definition 3.24. Let $(X,\mathcal{U})$ be a quasi-uniform space and $\beta : \mathcal{A} \to \mathcal{U}$ be a $\mathcal{U}^s$-strategy for Player A in the $\mathcal{U}^s$-proximal game. Player A *almost wins* the $\mathcal{U}^s$-proximal game if for every $\mathcal{U}^s$-proximal sequence $x_1, x_2, \cdots$, either

(i) $x_1, x_2, \cdots$ has a $\tau(\mathcal{U}^s)$-accumulation point or
(ii) $\bigcap_{i\in\mathbb{N}} \beta(x_1, x_2, \cdots, x_{i-1})(x_i) = \varnothing$.

If Player A has an almost $\mathcal{U}^s$-winning strategy, we say the space is *almost $\mathcal{U}^s$-proximal*. If X is a quasi-uniformizable space and there exists a quasi-uniformity $\mathcal{U}$ for which X is almost $\mathcal{U}^s$-proximal, we say X *almost $\mathcal{U}^s$-proximal*.

Definition 3.25. A quasi-uniform space is *almost bi-proximal* provided it is almost left and almost right proximal. A quasi-uniformizable space X is almost bi-proximal provided there is a quasi-uniformity for which X is almost bi-proximal.

Remark 3.26. It is clear that every bi-proximal space is almost bi-proximal. However, the converse is not necessarily true. Also, one can use the arguments from Lemma 3.12 to show that a space is almost left-proximal if and only if it is almost right proximal. In addition, one can use arguments from Theorem 3.19 to show that a space is almost bi-proximal if and only if it is almost $\mathcal{U}^s$-proximal.

4. Basic properties of bi-proximal spaces

We begin this section by showing that every bi-proximal space is a W-space. W-spaces, introduced by Gruenhage in [5], are generalized first countable spaces. They have many nice properties, for example, every W-space is a Frechet-Urysohn space. Also, subspaces and Σ-products of W-spaces are again W-spaces.

Definition 4.1 ([5])**.** A W-space is a topological space (X, τ) in which Player 1 has a winning strategy in the following infinite game:

(i) Player 1 chooses $V_1 \in \tau$ with $x \in V_1$
Player 2 chooses $x_1 \in V_1$

(ii) Player 1 chooses $V_2 \in \tau$ with $x \in V_2$
Player 2 chooses $x_2 \in V_2$

A strategy for Player A is winning if the sequence $x_1, x_2, \cdots$ converges to x. This game is called the *Gruenhage game at x*.

Lemma 4.2 (compare [2, Lemma 6])**.** *Every bi-proximal space is a W-space.*

Proof. Suppose $\mathcal{U}$ is filter base generating the quasi-uniformity on X, consisting of $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$ open entourages in $X \times X$, witnessing that X is bi-proximal. Suppose Player A, in the left-proximal game, picks elements of $\mathcal{U}$ and Player B, the point picking Player, picks points in X. Similarly, suppose Player A, in the right proximal game, picks elements of $\mathcal{U}^{-1}$ while Player B picks elements of X. Let Player 1, in the Gruenhage game, pick $\tau(\mathcal{U})$-open sets while Player 2 pick points in X. Suppose $w : \mathcal{A} \to \mathcal{U}$ and $m : \mathcal{A} \to \mathcal{U}^{-1}$ are left and right winning strategies for Player A in the left and right proximal games on $(X,\mathcal{U})$, where $\mathcal{A}$ is the set of admissible finite sequences. Fix a point $x \in X$ and let N_x denote the $\tau(\mathcal{U})$-open neighbourhoods of x. Then following the proof of [2, Lemma 6] with modifications, one can use w and m to inductively construct a winning strategy $\delta : \mathcal{A}' \to N_x$ in the Gruenhage game at x, where $\mathcal{A}'$ is a set of admissible finite sequences in the Gruenhage game at x. □

Lemma 4.3 (compare [2, Lemma 7])**.** *Suppose $(X,\mathcal{U})$ is a bi-proximal space. Then a $\tau(\mathcal{U}^s)$-closed subspace of X is bi-proximal.*

Proof. Suppose $(X,\mathcal{U})$ is a bi-proximal quasi-uniform space, $\beta : \mathcal{A} \to \mathcal{U}$ is a $\mathcal{U}^s$-winning strategy and $K \subseteq X$, where K is $\tau(\mathcal{U}^s)$-closed. Define a filter base generating a quasi-uniformity on K by $\mathcal{U}_K = \{U \cap (K \times K) : U \in \mathcal{U}\}$. Then $\mathcal{U}^s_K = \{V \cap V^{-1} : V \in \mathcal{U}_K\}$ is a filter base generating a uniformity on K. Define $\beta_K : F(K) \to \mathcal{U}^s_K$ by $\beta_K(x_1, x_2, \cdots, x_n) = \beta(x_1, x_2, \cdots, x_n) \cap (K \times K)$. Then β_K is a $\mathcal{U}^s_K$-winning strategy in $(K,\mathcal{U}_K)$ since if $\bigcap_{i\in\mathbb{N}} \beta(x_1, x_2, \cdots x_i)(x_{i+1}) \neq \varnothing$, then there is $z \in X$ such that $x_1, x_2, \cdots$ $\tau(\mathcal{U}^s)$-converges to z and $z \in K$ since K is $\tau(\mathcal{U}^s)$-closed. By Theorem 3.19, $(K,\mathcal{U}_K)$ is bi-proximal. □

Remark 4.4. A $\tau(\mathcal{U}^s)$-open subspace of a quasi-uniform space $(X,\mathcal{U})$ need not have a winning strategy in the bi-proximal game played with the subspace of quasi-uniformity generated by $\mathcal{U}$ and a subspace of a bi-proximal space need not be bi-proximal.

Definition 4.5. Suppose $(X_n,\mathcal{U}_n)$ is a quasi-uniform space for each $n \in \mathbb{N}$. Then the product quasi-uniformity on $\prod_{n\in\mathbb{N}} X_n$ is generated by the subbase $\check{\mathcal{U}} = \{\check{U}_n : U_n \in \mathcal{U}_n\}$, where for each $n \in \mathbb{N}$ and each $U_n \in \mathcal{U}_n$,

$$\check{U}_n = \left\{(x,y) \in \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n : (x(n),y(n)) \in U_n\right\}$$

Remark 4.6. It was observed by Stoltenberg in [14] that $\check{\mathcal{U}}$ induces the Tychonov product topology on $\prod_{n\in\mathbb{N}} X_n$.

For a fixed $z \in \prod_{n\in\mathbb{N}} X_n$, a Σ-product with base point z is the set

$$Z = \{x \in \prod_{n\in\mathbb{N}} X_n : |\{\beta : x(\beta) \neq z(\beta)\}| \leq \omega\}.$$

Theorem 4.7. *A Σ-product of bi-proximal spaces is bi-proximal.*

Proof. Suppose $(X_n,\mathcal{U}_n)$ is a quasi-uniform space for each $n \in \mathbb{N}$, Z is the Σ-product with base point z and $\beta_n : \mathcal{A}_n \to \mathcal{U}_n$ is a $\mathcal{U}_n^s$-winning on $(X_n,\mathcal{U}_n)$, where $\mathcal{A}_n$ is a set of finite admissible sequences in X_n. Then $(Z,\check{\mathcal{U}})$ is $\check{\mathcal{U}}^s$-proximal by [2, Theorem 8]. Therefore, $(Z,\check{\mathcal{U}})$ is bi-proximal by Theorem 3.19. □

Corollary 4.8. *Let $(X_n,\mathcal{U}_n)$ be a bi-proximal quasi-uniform space for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ and each $U_n \in \mathcal{U}_n$, let*

$$\hat{U}_n = \left\{(x,y) \in \prod_{n\in\mathbb{N}} X_n \times \prod_{n\in\mathbb{N}} X_n : (x(n),y(n)) \in U_n\right\}$$

and define $\hat{\mathcal{U}} = \{\hat{U}_n : U_n \in \mathcal{U}_n\}$. Then $\left(\prod_{n\in\mathbb{N}} X_n,\hat{\mathcal{U}}\right)$ is bi-proximal.

Proof. Since $(X,\mathcal{U}_n)$ is bi-proximal for each $n \in \mathbb{N}$, then $(X,\mathcal{U}_n)$ is $\mathcal{U}_n^s$-proximal by Theorem 3.19 for each $n \in \mathbb{N}$. Therefore, $\left(\prod_{n\in\mathbb{N}} X_n,\hat{\mathcal{U}}\right)$ is $\hat{\mathcal{U}}^s$-proximal by [2, Corollary 1]. Hence $\left(\prod_{n\in\mathbb{N}} X_n,\hat{\mathcal{U}}\right)$ is bi-proximal by Theorem 3.19. □

5. Separation and covering properties

A topological space (X,τ) is said to be collectionwise normal (respectively, collectionwise Hausdorff) provided that each discrete collection of closed sets (respectively, each closed discrete point set) can be simultaneously separated by a pairwise disjoint collection of open sets [15].

Bell [2] showed that proximal spaces are collectionwise normal and collectionwise Hausdorff. However, this does not hold, in general, for bi-proximal spaces. In Example 3.8, we showed that the Sorgenfrey plane is bi-proximal. However, the Sorgenfrey plane is not collectionwise normal and collectionwise Hausdorff. For this reason, we restrict our discussion to only those bi-proximal spaces X for which $\mathcal{U}$ is a compatible quasi-uniformity for which X is bi-proximal and $\mathcal{U}^s$ is a compatible uniformity for which X is proximal.

Theorem 5.1. *A bi-proximal space X for which for which $\mathcal{U}$ is a compatible quasi-uniformity for which X is bi-proximal and $\mathcal{U}^s$ is a compatible uniformity for which X is $\mathcal{U}^s$-proximal is collectionwise normal.*

Proof. Since X is bi-proximal, then it is $\mathcal{U}^s$-proximal by Theorem 3.19. This implies that X is collectionwise normal by [2, Theorem 10]. □

Theorem 5.2. *An almost bi-proximal space X for which $\mathcal{U}$ is a compatible quasi-uniformity for which X is almost bi-proximal and $\mathcal{U}^s$ is a compatible uniformity for which X is almost $\mathcal{U}^s$-proximal is collectionwise Hausdorff.*

Proof. Since X is almost bi-proximal, then it is almost $\mathcal{U}^s$-proximal by Remark 3.26. This implies that X is collectionwise Hausdorff by [2, Theorem 12]. □

A topological space (X, τ) is *countably paracompact* if every countable open cover has a locally finite open refinement and countably *metacompact* if every countable open cover has a point finite open refinement [15].

We use the following definition for a countably metacompact space.

Definition 5.3 ([6])**.** A topological space (X, τ) is countably metacompact if and only if for any descending sequence $\{F_i\}_{i\in\mathbb{N}}$ of nonempty closed sets such that $\bigcap_{i\in\mathbb{N}} F_i = \varnothing$, there exists a descending sequence $\{G_i\}_{i\in\mathbb{N}}$ of nonempty open sets such that $F_i \subset G_i$ for each $i \in \mathbb{N}$ and $\bigcap_{i\in\mathbb{N}} cl(G_i) = \varnothing$.

In Theorem 11, Bell [2] showed that all proximal spaces are countably metacompact. This implies that all metric spaces are countably metacompact since they are proximal. However, in [9], Künzi, and Watson showed that there exists a quasi-uniformity which is not countably metacompact. This shows that bi-proximal spaces are not, in general, countably metacompact. For this reason, in the next result, we restrict our discussion to only those bi-proximal spaces X for which $\mathcal{U}$ is a compatible quasi-uniformity for which X is bi-proximal and $\mathcal{U}^s$ is a compatible uniformity for which X is proximal.

Theorem 5.4. *An almost bi-proximal space X for which $\mathcal{U}$ is a compatible quasi-uniformity for which X is almost bi-proximal and $\mathcal{U}^s$ is a compatible uniformity for which X is almost $\mathcal{U}^s$-proximal is countably metacompact.*

Proof. Since X is almost bi-proximal, then it is almost $\mathcal{U}^s$-proximal by Theorem 3.26. This implies that X is countably metcompact by [2, Theorem 11]. □

6. Application to quasi-uniform box products

The quasi-uniform box product, introduced in [11], is a topology, on the product of countably many copies of a quasi-uniform space, which is finer than the Tychonov product topology but coarser than the uniform box product. It is generated by a quasi-uniformity, called the constant quasi-uniformity, whose symmetrised uniformity coincides with the constant uniformity in the sense of Bell [1]. In this section, we discuss some separation and covering properties of the quasi-uniform box product of countably many copies of the uncountable Fort-space.

Definition 6.1 ([11])**.** Let $(X,\mathcal{U})$ be a quasi-uniform space and $\prod_{n\in\mathbb{N}} X$ be the product of countably many copies of X. For each $U\in\mathcal{U}$, let

$$\overline{U}=\left\{(x,y)\in\prod_{n\in\mathbb{N}}X\times\prod_{n\in\mathbb{N}}X:\ \text{for all } n\in\mathbb{N},\ \ (x(n),y(n))\in U\right\}.$$

Then $\overline{\mathcal{U}}=\{\overline{U}:U\in\mathcal{U}\}$ is a filter base generating a quasi-uniformity on $\prod_{n\in\mathbb{N}} X$. The quasi-uniformity $\overline{\mathcal{U}}$ is called the *constant quasi-uniformity* on the product $\prod_{n\in\mathbb{N}} X$, the topology $\tau(\overline{\mathcal{U}})$ is called the *constant quasi-uniform topology* on $\prod_{n\in\mathbb{N}} X$ and the pair $\left(\prod_{n\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$ is called the *quasi-uniform box product.*

Definition 6.2 (compare [2, Definition 7])**.** Let $X=W\cup\{\infty\}$ be uncountable Fort-space and X^ω denote the Tychonov product of countably many copies of X. For each finite set $F\subseteq W$, let $U_F=\triangle\cup[(X\setminus F)\times X]$ and for each $k\in\mathbb{N}$, define

$$U_{F,k}=\{(x,y)\in X^\omega\times X^\omega:\forall n<k,\,(x(n),y(n))\in U_F\}.$$

Then $\mathcal{U}=\{U_{F,k}:F\subseteq W \text{ is finite and } k\in\mathbb{N}\}$ is a subbase for a quasi uniformity on X^ω compatible with the topology.

Remark 6.3. Note that $\mathcal{U}^{-1}=\{U^{-1}_{F,k}:F\subseteq W \text{ is finite and } k\in\mathbb{N}\}$, where

$$U^{-1}_{F,k}=\{(x,y)\in X^\omega\times X^\omega:\forall n<k,\,(x(n),y(n))\in U^{-1}_F\}$$

and $U^{-1}_F=\triangle\cup[X\times(X\setminus F)]$, is a subbase for a quasi-uniformity on X^ω. Furthermore, $\mathcal{U}^s=\{U_{F,k}\cap U^{-1}_{F,k}:F\subseteq W \text{ is finite and } k\in\mathbb{N}\}$, where

$$U^s_{F,k}=\{(x,y)\in X^\omega\times X^\omega:\forall n<k,\,(x(n),y(n))\in U^s_F\}$$

and $U^s_F=\triangle\cup[(X\setminus F)\times(X\setminus F)]$, is a uniformity base compatible with the topology on X^ω [2].

For any $x\in X^\omega$, we have

$$U_{F,k}(x)=\prod_{x(n)\in F,n<k}\{x(n)\}\times\prod_{x(n)\notin F,n<k}X\times\prod_{n\geq k}X.$$

Similarly, for any $x\in X^\omega$, we have

$$U^{-1}_{F,k}(x)=\prod_{x(n)\in F,n<k}\Big(\{x(n)\}\cup X\setminus F\Big)\times\prod_{x(n)\notin F,n<k}(X\setminus F)\times\prod_{n\geq k}X.$$

Furthermore,

$$U^s_{F,k}(x)=U_{F,k}(x)\cap U^{-1}_{F,k}(x)=\prod_{x(n)\in F,n<k}\{x(n)\}\times\prod_{x(n)\notin F,n<k}(X\setminus F)\times\prod_{n\geq k}X.$$

Notice that we may assume that some finite set occurs in the first $k-1$ coordinates. Let $\left(\prod_{n\in\mathbb{N}}X^\omega,\overline{\mathcal{U}}\right)$ be the quasi-uniform box product and fix $x\in\prod_{n\in\mathbb{N}}X^\omega$. We use round brackets $x(n)$ to refer to the nth coordinate in

$\prod_{n\in\mathbb{N}} X^\omega$, that is, $x = (x(1), x(2), \cdots)$, where $x(n) \in X^\omega$. We use square brackets to refer to the ith coordinate of $x(n)$, that is, $x(n) = (x(n)[1], x(n)[2], \cdots)$, where $x(n)[i] \in X$. For each finite subset $F \subseteq W$ and each $k \in \mathbb{N}$, we have

$$\overline{U}_{F,k}(x) = \prod_{n\in\mathbb{N}} U_{F,k}(x(n)),$$

where,

$$U_{F,k}(x) = \prod_{x(n)[i]\in F, n<k} \{x(n)[i]\} \times \prod_{x(n)[i]\notin F, i<k} X \times \prod_{i\geq k} X.$$

Similarly,

$$\overline{U}^{-1}_{F,k}(x) = \prod_{n\in\mathbb{N}} U^{-1}_{F,k}(x(n)),$$

where,

$$U^{-1}_{F,k}(x(n)) = \prod_{x(n)[i]\in F, n<k} \Big(\{x(n)[i]\} \cup X \setminus F\Big) \times \prod_{x(n)[i]\notin F, i<k} (X \setminus F) \times \prod_{i\geq k} X.$$

Furthermore,

$$\overline{U}^{s}_{F,k}(x) = \prod_{n\in\mathbb{N}} U^{s}_{F,k}(x(n)).$$

where,

$$U^{s}_{F,k}(x) = \prod_{x(n)[i]\in F, n<k} \{x(n)[i]\} \times \prod_{x(n)[i]\notin F, i<k} (X \setminus F) \times \prod_{i\geq k} X,$$

which corresponds to the basic open and closed set in the sense of [2].

Theorem 6.4. *The quasi-uniform box product* $\left(\prod_{n\in\mathbb{N}} X^\omega, \overline{\mathcal{U}}\right)$ *is bi-proximal, where* $X = W \cup \{\infty\}$, $\mathcal{U} = \{U_F : F \subseteq W \text{ is finite}\}$ *with* $U_F = \triangle \cup [X \setminus F \times X]$ *and* X^ω *is the Tychonov product of countably many copies of a Fort-space* X*.*

Proof. Since $\left(\prod_{n\in\mathbb{N}} X^\omega, \overline{\mathcal{U}}\right)$ is $\overline{\mathcal{U}}^s$-proximal by [2, Theorem 13], then by Theorem 3.19, $\left(\prod_{n\in\mathbb{N}} X^\omega, \overline{\mathcal{U}}\right)$ is bi-proximal. □

Corollary 6.5. *(Compare* [2, Corollary 5]*) The quasi-uniform box product* $\left(\prod_{n\in\mathbb{N}} T, \overline{\mathcal{U}}\right)$ *is collectionwise normal, countably paracompact and collectionwise Hausdorff, where* T *is the Tychonov product of countably many Fort-spaces and* $\mathcal{U} = \{U_F : F \subseteq W \text{ is finite}\}$ *with* $U_F = \triangle \cup [X \setminus F \times X]$*.*

Proof. Since T is the Tychonov product $T = \prod_{n\in\mathbb{N}} X_n$, where each X_n is a Fort-space, T is a $\tau(\overline{\mathcal{U}}^s)$-closed subspace of the Tychonov product $\prod_{n\in\mathbb{N}} Y$, where Y is a Fort-space such that $|Y| > \sup\{|X_n| : n \in \mathbb{N}\}$. □

Since $\left(\prod_{n\in\mathbb{N}} X^\omega, \overline{\mathcal{U}}\right)$, where X is a Fort-space and $\mathcal{U} = \{U_F : F \subset W \text{is finite}\}$, where $U_F = \triangle \cup [(X \setminus F) \times X]$, contains the quasi-uniform box product $\left(\prod_{n\in\mathbb{N}} X, \mathcal{U}\right)$ as a $\tau(\overline{\mathcal{U}}^s)$-closed subspace [2], we have the following:

Theorem 6.6. *The quasi-uniform box product* $\left(\prod_{n\in\mathbb{N}} X, \mathcal{U}\right)$, *where* $X = W \cup \{\infty\}$ *and* $\mathcal{U} = \{U_F : F \subset W \textit{is finite}\}$, *where* $U_F = \triangle \cup [(X \setminus F) \times X]$, *is collectionwise normal, countably paracompact and collectionwise Hausdorff.*

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