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Cauchy action on filter spaces

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Abstract

A Cauchy group $(G, D, \cdot)$ has a Cauchy-action on a filter space (X, C), if it acts in a compatible manner. A new filter-based method is proposed in this paper for the notion of group-action, from which the properties of this action such as transitiveness and its compatibility with various modifications of the G-space (X, C) are determined. There is a close link between the Cauchy action and the induced continuous action on the underlying G-space, which is explored here. In addition, a possible extension of a Cauchy-action to the completion of the underlying G-space is discussed. These new results confirm and generalize some of the properties of group action in a topological context.

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1. Introduction

Group-action has applications in numerous branches of algebra, especially in finite permutation groups [6] and their primitivity. Needless to say that some of these ideas have been very cautiously applied to the topological arena where the action is named as 'continuous group action'. Motivated by the wide range of applications of continuous group-actions to various branches, a new interaction between group-action and set-theoretic topology is demonstrated in this paper. Continuous action of a convergence group on a convergence

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space already exists in the literature [16, 5]. In particular, there exists an one-to-one correspondence between the homeomorphic representations in $H(X)$ and continuous group actions of convergence groups [16], which is generally unavailable in the case of a topological space X due to the lack of an admissible topology on $H(X)$ [13]. Later, this was generalized by Boustique *et al.* [2] and applied to convergence approach spaces by Colebunders *et al.* [5].

As discussed in Section 2, there is a close link between a convergence structure (generalization of a topology) [1] and a Cauchy space (generalization of a uniform space) [8], so it is reasonable that one would anticipate the corresponding continuous action of a Cauchy group on a Cauchy space. In this context, a natural question arises–whether the theory of group action can be linked to Cauchy continuity? In this paper, this question is partially answered with an introduction of the notion of the Cauchy action in Section 3, where it is shown that an equivalence relation on the filters is a G-congruence. Since an algebraically compatible group structure on a filter space induces a Cauchy structure on the space, we need to look no further than a filter semigroup [17] having a continuous action on a filter space. In Section 5, attempts have been made to investigate such an action on a filter space and a few of its modifications, while Section 4 demonstrates the interaction between a Cauchy action and the continuous action on the corresponding G-space. The paper concludes with an extension of a Cauchy action to a larger space, namely, the completion of the original filter space in Section 6.

A word or two must be said about the underlying spaces on which this special type of action is defined. Cauchy spaces are generalizations of uniform spaces and metric spaces. If we exclude the last of the three Keller's [8] axioms for a Cauchy space, the resulting space is what we call a *filter space*. A filter semigroup is a filter space with a compatible semigroup operation defined on it. The category FIL of filter spaces with the Cauchy maps as morphisms forms a topological universe and the category CHY of Cauchy spaces is a bireflective subcategory of FIL [15]. As pointed out by Császár [4], there are three different ways of associating a convergence structure with a filter space. Out of these three ways, completions corresponding to the $\mathcal{K}$-convergence were discussed in [10]. It is well-known that when a filter space is compatible with a group operation, the three associated convergence structures γ, λ and $\mathcal{K}$ [4] coincide. So without loss of generality, Császár's $\mathcal{K}$-convergence [4] is used for defining Cauchy actions of filter semigroups in this paper.

2. Preliminaries

2.1. **Filter space.** For basic definitions and notations the reader is referred to [10], though some of the frequently used definitions and notations will be mentioned here. Let X be a non-empty set and $\mathbf{F}(X)$ denote the set of all filters on X. There is a partial order relation '$\leq$', defined on $\mathbf{F}(X)$: $\mathcal{F} \leq \mathcal{G}$ if and only if $\mathcal{F} \subseteq \mathcal{G}$. If $\mathcal{B}$ is a *base* for the filter $\mathcal{F}$, then $\mathcal{F}$ is said to be *generated* by $\mathcal{B}$, written as $\mathcal{F} = [\mathcal{B}]$. In particular, $\dot{x} = [\{x\}]$ is the fixed point filter generated by the singleton set $\{x\}$ and $\mathcal{F} \cap \mathcal{G} = [\{F \cup G \mid F \in \mathcal{F},\ G \in \mathcal{G}\}]$. If $F \cap G \neq \phi$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then $\mathcal{F} \vee \mathcal{G}$ is the filter $[\{F \cap G \mid F \in \mathcal{F},\ G \in \mathcal{G}\}]$. If there exists $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \cap G = \phi$, then $\mathcal{F} \vee \mathcal{G}$ *fails to exist*. If $A \subseteq X$ and $\mathcal{F} \in \mathbf{F}(X)$ such that $F \cap A \neq \phi$ for all $F \in \mathcal{F}$, then $\mathcal{F}_A = [\{F \cap A \mid F \in \mathcal{F}\}]$ is called the *trace* of $\mathcal{F}$ on A.

Definition 2.1. Let X be a set and $C \subseteq \mathbf{F}(X)$. The pair (X, C) is called a *filter space*, if the following conditions hold :
(I) $\dot{x} \in C,\ \forall x \in X$;
(II) $\mathcal{F} \in C$ and $\mathcal{G} \geq \mathcal{F}$ imply that $\mathcal{G} \in C$.

Any two filters $\mathcal{F}, \mathcal{G} \in C$ are said to be $C-$*linked*, written $\mathcal{F} \sim_C \mathcal{G}$, if there exists a finite number of filters $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n \in C$ such that all of the filters $\mathcal{F} \vee \mathcal{H}_1$, $\mathcal{H}_1 \vee \mathcal{H}_2, \ldots, \mathcal{H}_{n-1} \vee \mathcal{H}_n$ and $\mathcal{H}_n \vee \mathcal{G}$ exist. Note that the relation C-linked is an equivalence relation on C. The equivalence class containing $\mathcal{F} \in C$ is denoted by $[\mathcal{F}]_C$ (or $[\mathcal{F}]$, if C is clear from the context). Associated with the filter space (X, C), there is a *preconvergence* structure p_C [10] defined as

$$\mathcal{F} \xrightarrow{p_C} x \quad \text{if and only if} \quad \mathcal{F} \sim_C \dot{x}.$$

A filter space (X, C) is a *c-filter space*(respectively, *Cauchy space*), if it satisfies the following additional condition (III) (respectively, (IV)),
(III) $\mathcal{F}, x \in X$ and $\mathcal{F} \sim_C \dot{x}$ imply that $\mathcal{F} \cap \dot{x} \in C$.
(IV) $\mathcal{F}, \mathcal{G} \in C$ and $\mathcal{F} \sim_C \mathcal{G}$ imply that $\mathcal{F} \cap \mathcal{G} \in C$.
Associated with the c-filter space (X,C) (respectively, Cauchy space), there is a convergence structure q_C defined as $\mathcal{F} \xrightarrow{q_C} x$ if and only if $\mathcal{F} \cap \dot{x} \in C$. Note that even though q_C coincides with p_C when (X, C) is a c-filter space (respectively, Cauchy space), for the astute reader we will use the notation p_C and q_C for the associated pre-convergence and convergence structures, respectively.

Example 2.2. Let X be the set of real numbers, $\mathcal{F} = [\{(0, 1/n) \mid n \in N\}]$, where $(0, 1/n)$ denotes an open interval in X, and $\mathcal{L} = \{$all complements of countable sets$\}$. Define $C = \{\dot{x} \mid x \in X\} \cap \{\mathcal{H} \mid \mathcal{H} \geq \mathcal{F} \cap \dot{0}\} \cap \{\mathcal{K} \mid \mathcal{K} \geq \mathcal{L} \cap \dot{0}\}$; then (X, C) is a filter space. Note that (X, C) is not a *c*-filter space and, hence, not a Cauchy space.

For any $A \subseteq X$, we define $cl_{p_C} A = \{x \in X \mid \exists \mathcal{F} \in C$ such that $\mathcal{F} \xrightarrow{p_C} x$ and $A \in \mathcal{F}\}$ and $cl_{p_C} \mathcal{A} = [\{cl_{p_C} A \mid A \in \mathcal{A}\}]$. A filter space is said to be

(i) T_2 or *Hausdorff* if and only if $x = y$, whenever $\dot{x} \sim_C \dot{y}$,
(ii) *regular* if and only if $cl_{p_C} \mathcal{F} \in C$, whenever $\mathcal{F} \in C$,
(iii) T_3 if and only if it is T_2 and regular,
(iv) *complete* if and only if each $\mathcal{F} \in C$ p_C-converges.

If (X, C) and (X, D) are two filter spaces such that $C \subseteq D$, then C is said to be *finer* than D, written $C \geq D$.

Let $A \subseteq X$ and $C_A = \{\mathcal{G} \in \mathbf{F}(A) \mid$ there exists $\mathcal{F} \in C$ such that $\mathcal{F}_A$ exists and $\mathcal{G} \geq \mathcal{F}_A\}$. Then (A, C_A) is a filter space, called the *subspace* of (X, C). A mapping $f : (X, C) \to (Y, D)$ between the filter spaces is called a *Cauchy map* if and only if $\mathcal{F} \in C$ implies that $f(\mathcal{F}) \in D$. The map f is a *homeomorphism* if and only if it is bijective and both f and f^{-1} are continuous. Moreover, f is an *embedding* of (X, C) into (Y, D) if and only if $f : (X, C) \to (f(X), D_{f(X)})$ is a homeomorphism. A *completion* of a filter space (X, C) is a pair $((Z, K), \phi)$, where (Z, K) is a complete filter space and the map $\phi : (X, C) \to (Z, K)$ is an embedding satisfying the condition $cl_{p_K} \phi(X) = Z$. The T_2 Wyler completion of a filter space is an example of such a completion [11]; it has been applied to obtain completion and compactification of a family of extension spaces [12].

Let FIL (respectively, c-FIL, CHY) denote the category of all filter spaces (respectively, c-filter spaces, Cauchy spaces) as objects with Cauchy maps as morphisms.

2.2. **Filter semigroup.** The binary operation in a semigroup $(G, \cdot)$ can be applied to subsets and filters in an obvious way. For $A,\ B \subseteq G$, $AB = \{xy \mid x \in A,\ y \in B\}$, and for any two filters $\mathcal{F}$ and $\mathcal{G}$ in $\mathbf{F}(G)$, $\mathcal{F}\mathcal{G} = [\{FG \mid F \in \mathcal{F},\ G \in \mathcal{G}\}]$. In particular, for $x \in X$ and $\mathcal{F} \in \mathbf{F}(G)$, we define $x \cdot \mathcal{F} = \dot{x} \cdot \mathcal{F}$ is the filter generated by $\{x \cdot F \mid F \in \mathcal{F}\}$. The semigroup $(G, \cdot)$ is abelian if and only if $\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F}$, for all $\mathcal{F},\ \mathcal{G} \in \mathbf{F}(G)$. Also, for $\mathcal{F}_i, \mathcal{G}_i \in \mathbf{F}(G)$, $(i = 1, 2)$, $\mathcal{F}_1 \geq \mathcal{G}_1$ and $\mathcal{F}_2 \geq \mathcal{G}_2$ imply that $\mathcal{F}_1\mathcal{F}_2 \geq \mathcal{G}_1\mathcal{G}_2$, which shows that $\mathbf{F}(G)$ is a partially ordered semigroup. In particular, when $(G, \cdot)$ is a group, we define $\mathcal{F}^{-1} = [\{F^{-1} \mid F \in \mathcal{F}\}]$ for all $\mathcal{F} \in \mathbf{F}(G)$.

A triplet $(G, D, \cdot)$ is called a *filter semigroup*, if $(G, D) \in |FIL|$, $(G, \cdot)$ is a semigroup with identity e and $\mathcal{K},\ \chi \in D$ imply that $\mathcal{K}\chi$ and $\chi\mathcal{K} \in D$. Note that the binary operation $b : (G, D) \times (G, D) \to (G, D)$, where $b(g, h) = g \cdot h$ is a Cauchy map with respect to the Cauchy product [9] on $G \times G$. Properties of filter semigroups and their completions were investigated in [17]. A filter semigroup $(G, D, \cdot)$ is a *Cauchy group*, if $(G, \cdot)$ is a group and D is compatible with the group operations, that is, $\mathcal{K}\chi^{-1} \in D$, whenever $\mathcal{K}$ and χ are in D. Let $FILSG$ (respectively, CHG) denote the category of all objects as filter semigroups with identity element (respectively, Cauchy groups) and morphisms as Cauchy maps which are also homomorphisms.

The equivalence relation '$\sim_D$' preserves the binary operation on the filter semigroup $(G, D, \cdot)$, that is, $\mathcal{K}_i \sim_D \chi_i$ for $i = 1, 2$, implies $\mathcal{K}_1\mathcal{K}_2 \sim_D \chi_1\chi_2$ [17, Proposition 3.2]. Note that if $(G, \cdot)$ is a group and $(G, D) \in |FIL|$, then $(G, D, \cdot)$ is a Cauchy group ([7], there called pre-Cauchy group), since for any two filters $\mathcal{K}$ and $\chi \in D$, $\mathcal{K} \vee \chi$ exists implies that $\mathcal{K} \cap \chi \geq \mathcal{K}\chi^{-1}$.

Example 2.3. If (X, C) is a filter space, then the set of all Cauchy maps denoted by $C(X, X)$ is a semigroup with respect to the composition of Cauchy maps and identity map. The space $(C(X, X), D)$ is a filter semigroup, where

$$D = \{\Phi \in \mathbf{F}(C(X, X)) \mid \mathcal{F} \in C,\ p_C\text{-convergent} \Rightarrow \Phi(\mathcal{F})\, p_C\text{-convergent}\}.$$

A triplet $(G, q, \cdot)$ is called a *preconvergence group*, if $(G, \cdot)$ is a group, (G, q) is a preconvergence space [10], and $\mathcal{F} \xrightarrow{q} x$, $\mathcal{G} \xrightarrow{q} y$ imply that $\mathcal{F}\mathcal{G}^{-1} \xrightarrow{q} xy^{-1}$. A preconvergence group is a *convergence group*, if (G, q) is a convergence space.

Note that when (G, q) is a convergence group, $H(G)$ the set of all homeomorphisms on G is a complete Cauchy group.

2.3. **Continuous group action.** Let X be a set and $(G, \cdot)$ be a group with a binary operation '$\cdot$' and the identity element e. Consider a function $\mu : X \times G \longrightarrow X$, denoted by $\mu(x, g) = x^g$. For any $F \subseteq X$, $g \in G$, $F^g = \{f^g \mid f \in F\}$ and for any $A \subseteq G$, $F^A = \cup\{F^g \mid g \in A\}$. If $\mathcal{F} \in \mathbf{F}(X)$ and $g \in G$, then $\mathcal{F}^g = [\{F^g \mid F \in \mathcal{F}\}]$, and for any subset $A \subseteq G$, $\mathcal{F}^A = [\{F^A \mid F \in \mathcal{F}\}]$. If $\mathcal{K} \in \mathbf{F}(G)$, then $\mathcal{F}^{\mathcal{K}} = [\{F^A \mid F \in \mathcal{F},\ A \in \mathcal{K}\}]$.

The mapping μ is said to be a *group action* of G on X (or equivalently, X is a *G-space*), if it satisfies the following conditions:

(a1) $x^e = x, \ \ \forall x \in X$,
(a2) $(x^g)^h = x^{gh}, \forall x \in X$ and $\forall \ g, h \in G$.

This implies that for any $\mathcal{K}, \mathcal{H} \in \mathbf{F}(G)$ and $\mathcal{F} \in \mathbf{F}(X)$, $\mathcal{F}^{\dot{e}} = \mathcal{F}$ and $(\mathcal{F}^{\mathcal{K}})^{\mathcal{H}} = \mathcal{F}^{\mathcal{K}\mathcal{H}}$. In particular, for any $x \in X$ and $g \in G$, $\dot{x}^{\dot{g}} = x^{\dot{g}} = \dot{x}^g = \dot{x^g}$ is the fixed ultrafilter with base $\{x^g\}$. An action of a group G is *transitive on* X, if for any $x, y \in X$, $\exists g \in G$ such that $x^g = y$. An action is said to be *quasi-transitive* with respect to an equivalence relation $\sim$ on X, if $x, y \in X$ implies that there exists $g \in G$ such that $x^g \sim y$. Note that every transitive action is quasi-transitive with respect to '$=$'. An equivalence relation '$\sim$' on a G-space A is called a *G-congruence* [6], if for $a, b \in A$, $a \sim b \Longrightarrow a^g \sim b^g$ for all $g \in G$.

Remark 2.4. The conditions (a1) and (a2) are equivalent to $(a1')$ $\mathcal{F}^e = \mathcal{F}$ and $(a2')$ $(\mathcal{F}^g)^h = \mathcal{F}^{gh}$, respectively, for all $\mathcal{F} \in \mathbf{F}(X)$ and all $g, h \in G$. So every action μ on a set X induces a group action on $\mathbf{F}(X)$. In other words, if X is a G-space, then $\mathbf{F}(X)$ is a G-space on its own right. Hence, in this paper, the group actions are investigated more closely with respect to filters than the individual elements in a set.

For a detailed discussion on convergence structures and related notions, the reader is referred to the consolidated work of Beattie and Butzman [1].

Definition 2.5 ([16, Definition 5.1])**.** A group action of a convergence group $(G, \Lambda, \cdot)$ on a convergence space (X, q) is called a *continuous group action*, if

$$\text{(ca)} \quad \forall \mathcal{F} \xrightarrow{q} x \text{ and } \forall \mathcal{K} \xrightarrow{\Lambda} g,\ \mathcal{F}^{\mathcal{K}} \xrightarrow{q} x^g, \text{ where } \mathcal{F} \in \mathbf{F}(X) \text{ and } \mathcal{K} \in \mathbf{F}(G).$$

The name of this type of action suggests it all: the group action is continuous with respect to the product convergence [3] on $X \times G$. In this case, (X, q) is called a *G-space with a continuous action* or more precisely, *a CG-space*. Examples of CG-spaces are abundant. For instance, any convergence group is a CG-space with respect to the right-multiplication $(a, x) \to a^x = ax$ for all $a, x \in G$. The homeomorphism group $H(X)$ acts on the convergence space (X, q) with respect to the action defined as $(x, f) \to x^f = f^{-1}(x)$ for all $x \in X$ and $f \in H(X)$. A detailed discussion on continuous group action can be found in [16].

3. Cauchy action

In theory, a filter-preserving map between two filter spaces (respectively, Cauchy spaces) is named as a Cauchy map. In alliance with this terminology, a filter-preserving action is reasonable to be named as a *Cauchy action*. Throughout this section, $(G, D, \cdot)$ denotes a filter semigroup with identity e and (X, C) denotes a filter space, unless otherwise specified.

Definition 3.1. A group action $\mu : X \times G \to X$, denoted by $\mu(x, g) = x^g$ $\forall x \in X$ and $\forall g \in G$ is said to be a *Cauchy action* of G on X, if

$$\text{(Cha)} \quad \forall \mathcal{F} \in C \text{ and } \forall \mathcal{K} \in D,\ \mathcal{F}^{\mathcal{K}} \in C.$$

Note that from condition (Cha), the map $\mu : X \times G \to X$ on the product space $X \times G$, is a Cauchy map. So, in this case, it is natural to call (X, C) a *G-space with a Cauchy action*, or a *ChG-space* in short. Later in Section 4, it is shown that when (X, C) is a ChG-space, (X, p_C) is a CG-space.

Example 3.2. If a filter semigroup $(G, D, \cdot)$ has a Cauchy action on X, then it has a Cauchy action on $Y = X \times X$, where the action is defined by $(x_1, x_2)^g = (x_1^g, x_2^g)$, $\forall \ g \in G$ and $x_1, x_2 \in X$. Here it is assumed that Y has the corresponding product structure.

Example 3.3. Consider the right regular representation of the filter semigroup $(G, D, \cdot)$ on itself, defined by $x^g = xg$ for all $x,\ g \in G$. Since the group operation on G is Cauchy compatible, $\mathcal{F} \in D, \mathcal{K} \in D \Longrightarrow \mathcal{F}^{\mathcal{K}} = \mathcal{F}\mathcal{K} \in D$. Therefore, this is a Cauchy action. If $(G, D, \cdot)$ is a group, then it also follows that the left regular representation [6] of $(G, D, \cdot)$ on itself is a Cauchy action.

Theorem 3.4. *If the filter semigroup $(G, D, \cdot)$ has a Cauchy action, μ on a filter space (X, C), then there exists the finest structure C^F coarser than C such that μ is a Cauchy action on (X, C^F).*

Proof. Let $C^F = \{\mathcal{T} \in \mathbf{F}(X) \mid \exists \mathcal{F} \in C, \mathcal{K} \in D \,\text{such that}\ \mathcal{T} \geq \mathcal{F}^{\mathcal{K}}\}$. Since $\dot{x}^{\dot{e}} = \dot{x}$, it follows that (X, C^F) is a filterspace. If $\mathcal{T} \in C^F$, then $\exists \mathcal{F} \in C$ and $\mathcal{H} \in D$ such that $\mathcal{T} \geq \mathcal{F}^{\mathcal{H}}$. So for any $\mathcal{K} \in D$, $\mathcal{T}^{\mathcal{K}} \geq (\mathcal{F}^{\mathcal{H}})^{\mathcal{K}} = \mathcal{F}^{\mathcal{H}\mathcal{K}}$, which implies that $\mathcal{T}^{\mathcal{K}} \in C^F$. Since (a1) and (a2) hold, μ is, therefore, a Cauchy action on (X, C^F).

Since for any $\mathcal{F} \in C$, $\mathcal{F} = \mathcal{F}^{\dot{e}}$, it follows that C^F is coarser than C. Let (X, E) be a filter space such that E is coarser than C, and μ is a Cauchy action on (X, E). Now for any $\mathcal{T} \in C^F$, $\mathcal{T} \geq \mathcal{F}^{\mathcal{H}}$ for some $\mathcal{F} \in C$ and $\mathcal{H} \in D$. However, this implies that $\mathcal{F}^{\mathcal{H}} \in E$, since E is coarser than C and μ is a Cauchy action on (X, E). Hence C^F is finer than E, which proves the theorem. □

The next result yields that a Cauchy action of a filter semigroup $(G, D, \cdot)$ on a filter space (X, C) preserves the equivalence relation '$\sim_c$' on the filters in C.

Theorem 3.5. *Let the filter semigroup $(G, D, \cdot)$ have a Cauchy action on a filter space (X, C). If $\mathcal{F}, \mathcal{G} \in C$ with $[\mathcal{F}]_C = [\mathcal{G}]_C$, and $\mathcal{K}, \mathcal{L} \in D$ with $[\mathcal{K}]_D = [\mathcal{L}]_D$, then $[\mathcal{F}^{\mathcal{K}}]_C = [\mathcal{G}^{\mathcal{L}}]_C$.*

Proof. Let $\mathcal{F},\ \mathcal{G} \in C$ such that $\mathcal{F} \sim_C \mathcal{G}$ and $\mathcal{K}, \mathcal{L} \in D$ such that $\mathcal{K} \sim_D \mathcal{L}$. We show that $\mathcal{F}^{\mathcal{K}} \sim_C \mathcal{G}^{\mathcal{L}}$. The filter $\mathcal{F} \sim \mathcal{G}$ implies that there exists a finite number of filters $H_1,\ H_2, \ldots, H_n$ in C such that $\mathcal{F} \vee \mathcal{H}_1,\ H_1 \vee \mathcal{H}_2$, $\ldots \mathcal{H}_n \vee \mathcal{G}$ exist. Similarly, there exists a finite number of filters $\mathcal{T}_1,\ \mathcal{T}_2, \ldots, \mathcal{T}_r$ in D such that $\mathcal{K} \vee \mathcal{T}_1,\ \mathcal{T}_1 \vee \mathcal{T}_2,\ \ldots \mathcal{T}_r \vee \mathcal{L}$ exist. Assuming $r \geq n$, note that $\mathcal{F}^{\mathcal{K}}, \mathcal{G}^{\mathcal{L}},\ \mathcal{H}_i^{\mathcal{T}_i}$ for $i = 1, \ldots, n$ and $\mathcal{G}^{\mathcal{T}_j}$ for $j = n+1, \ldots r$, are in C. Also, since $(\mathcal{F} \times \mathcal{K}) \vee (\mathcal{H}_1 \times \mathcal{T}_1)$ exists, it follows that $\mathcal{F}^{\mathcal{K}} \vee\ \mathcal{H}_1^{\mathcal{T}_1}$ exists. Similarly, it can be shown that $\mathcal{H}_1^{\mathcal{T}_1} \vee \mathcal{H}_2^{\mathcal{T}_2},\ \cdots \mathcal{H}_n^{\mathcal{T}_n} \vee \mathcal{G}^{\mathcal{T}_{n+1}}$ exist. Continuing the same argument $\mathcal{G}^{\mathcal{T}_{n+1}} \vee \mathcal{G}^{\mathcal{T}_{n+2}}, \ldots, \mathcal{G}^{\mathcal{T}_{r-1}} \vee \mathcal{G}^{\mathcal{T}_r}, \mathcal{G}^{\mathcal{T}_r} \vee \mathcal{G}^{\mathcal{L}}$ exist, which implies that $\mathcal{F}^{\mathcal{K}} \sim_C \mathcal{G}^{\mathcal{L}}$. If $r < n$, then $\mathcal{F}^{\mathcal{K}} \vee\ \mathcal{H}_1^{\mathcal{T}_1}, \mathcal{H}_1^{\mathcal{T}_1} \vee \mathcal{H}_2^{\mathcal{T}_2}, \ldots \mathcal{H}_r^{\mathcal{T}_r} \vee \mathcal{H}_{r+1}^{\mathcal{L}},\ \mathcal{H}_{r+1}^{\mathcal{L}} \vee \mathcal{H}_{r+2}^{\mathcal{L}}, \ldots, \mathcal{H}_n^{\mathcal{L}} \vee \mathcal{G}^{\mathcal{L}}$ exist which also implies that $\mathcal{F}^{\mathcal{K}} \sim_C \mathcal{G}^{\mathcal{L}}$. This completes the proof. □

In the following, using Remark 2.4 and Theorem 3.5, we derive the impact of a Cauchy action on the set of filters in C.

Proposition 3.6. *Let (X, C) be a ChG-space.*

(I) *Every Cauchy action on (X,C), induces an action on the set of filters in C.*

(II) *The equivalence relation C-linked on C is a G-congruence.*

(III) *Let $\overline{C}$ be the set of all p_C-convergent filters in C. If μ is a transitive Cauchy action on (X, C), then μ is quasi-transitive with respect to the equivalence relation $\sim_C$ on $\overline{C}$.*

Proof. (I) Let $\mu : X \times G \longrightarrow X$ be a Cauchy action on (X, C) and $g \in G$. Now consider the map μ on $C \times G$. By replacing $\mathcal{K}$ with $\dot{g}$ in (Cha), $\mathcal{F}^g \in C$ for all $\mathcal{F} \in C$, since μ is a Cauchy action. Also, in view of Remark 2.4, (a1) and (a2) hold. Therefore, $\mu : C \times G \longrightarrow C$ is an action of G on C, that is, C inherits an action from the Cauchy action μ on (X, C). Proof of (II) follows directly from Theorem 3.5. To prove (III), let $\mathcal{F}, \mathcal{H} \in \overline{C}$. Then, $\exists\ x, y \in X$ such that $\mathcal{F} \sim_C \dot{x}$ and $\mathcal{H} \sim_C \dot{y}$. Since μ is transitive, $x^g = y$, for some $g \in G$. By Theorem 3.5, $\mathcal{F}^{\dot{g}} \sim_C \dot{x^g} \Longrightarrow \mathcal{F}^{\dot{g}} \sim_C \dot{y} \sim_C \mathcal{H}$, therefore, μ is quasi-transitive on $\overline{C}$ with respect to $\sim_C$. □

4. Interaction between Cauchy action and continuous action

Continuous action of a convergence group on a convergence space was discussed in [16], and later it was extended to continuous action of preconvergence semigroups on preconvergence spaces by Boustique et al. [2] and to approach spaces by Colebunders et al. [5]. In this section, the interaction between continuous group action and Cauchy action is explored with respect to some admissible convergence (Cauchy compatible convergence) structures. Note that $(G, p_D, \cdot)$ is a preconvergence semigroup (respectively, convergence group) whenever $(G, D, \cdot) \in FILSG$ [17] (respectively, Cauchy group [7]). As shown in the following theorem, it turns out that a Cauchy action always induces a continuous action on the respective preconvergence (convergence) space.

Theorem 4.1. *If the filter semigroup $(G, D, \cdot)$ has a Cauchy action on*

(i) *a filter space (X, C), then the preconvergence group $(G, p_D, \cdot)$ acts continuously on the preconvergence space (X, p_C);*
(ii) *a c-filter space (X, C), then the preconvergence group $(G, p_D, \cdot)$ acts continuously on the convergence space (X, q_C);*
(iii) *a Cauchy space (X, C), then the preconvergence group $(G, p_D, \cdot)$ acts continuously on the convergence space (X, q_C).*

Proof. (i) We only need to show that (ca) holds. Let $\mathcal{F} \xrightarrow{p_C} x$ and $\mathcal{K} \xrightarrow{p_D} g$. So, $\mathcal{F} \in C, \mathcal{F} \sim_C \dot{x}$ and $\mathcal{K} \in D$, $\mathcal{K} \sim_D \dot{g}$. Since $(G, D, \cdot)$ has a cauchy action on (X,C), $\mathcal{F}^{\mathcal{K}} \in C$ and by Theorem 3.5, $\mathcal{F}^{\mathcal{K}} \sim_C \dot{(x^g)}$, which imply that $\mathcal{F}^{\mathcal{K}} \xrightarrow{p_C} x^g$.

(ii) Assuming (X, C) is a c-filter space, let $\mathcal{F} \xrightarrow{q_C} x$ and $\mathcal{K} \xrightarrow{p_D} g$. Thus, $\mathcal{F} \cap \dot{x} \in C$, $\mathcal{K} \in D$ and $\mathcal{K} \sim_D \dot{g}$. Since $(G, D, \cdot)$ has a Cauchy action on (X, C), $\mathcal{F}^{\mathcal{K}} \in C$. Also, $\mathcal{F} \in C$ and $\mathcal{F} \sim_C \dot{x}$, which by Theorem 3.5, imply that $\mathcal{F}^{\mathcal{K}} \sim_C \dot{(x^g)}$. Since (X, C) is a c-filter space, $\mathcal{F}^{\mathcal{K}} \in C$ and $\mathcal{F}^{\mathcal{K}} \sim_C \dot{(x^g)}$ imply that $\mathcal{F}^{\mathcal{K}} \cap \dot{(x^g)} \in C$. In other words, $\mathcal{F}^{\mathcal{K}} \xrightarrow{q_C} x^g$ which proves (ii). Proof of (iii) is similar to the proof of (ii). $\square$

It is well-known that the category FIL is a topological universe. Let $CFIL$ and CHY denote the full subcategories of FIL whose objects are c-filter spaces and Cauchy spaces, respectively. Let $PCONV$ (respectively, $CONV, LIM$) denote the category of preconvergence spaces (respectively, convergence spaces, limit spaces) with continuous maps as morphisms. If $M : FIL \longrightarrow PCONV$ is defined by $M(X, C) = (X, p_C)$ for each object in FIL and $M(f) = f$ for each morphism, then M defines a functor on FIL. Note that $M\mid_{CFIL}: CFIL \longrightarrow CONV$ and $M\mid_{CHY}: CHY \longrightarrow CONV$ with $M(X, C) = (X, q_C)$ are also functors on the respective subcategories. Let $PCONV_1$ (respectively, $CONV_1, LIM_1$) denote the full subcategory of $PCONV$ (respectively, $CONV$, LIM) whose objects are in the range of the functor M. The objects of $PCONV_1$, $CONV_1$ and LIM_1) are, respectively, filter space, c-filter space and Cauchy space compatible. A characterization of objects of these subcategories can be found in [10].

A filter semigroup (respectively, Cauchy group) $(G, D, \cdot)$ is said to *act continuously* on a preconvergence (respectively, convergence) space (X, q), if $(G, q_D, \cdot)$ has a continuous group action [16] on (X, q).

Remark 4.2. Furthermore, if $(X, q) \in PCONV_1$ (respectively, $CONV_1$) and $(G, D, \cdot)$ is complete, then it has a Cauchy action on (X, C_q), with $C_q = \{\mathcal{F} \in \mathbf{F}(X) \mid \mathcal{F} \xrightarrow{q} x \text{ for some } x \in X\}$.

A preconvergence (respectively, convergence, limit) group $(G, \gamma\cdot)$ is said to *act continuously* on a filter space (respectively, *c*-filter space, Cauchy space) (X, C), if it has a continuous group action on (X, p_C) (respectively, (X, q_C)).

Remark 4.3. (a) Let $(G, \gamma, \cdot) \in PCONVG_1$ have a continuous action on a complete filter space (X, C). Then $(G, D_\gamma, \cdot)$ has a Cauchy action on (X, C). (b) Let $(G, \gamma, \cdot) \in CONVG_1$ have a continuous action on a complete c-filter space (respectively, Cauchy space) (X, C). Then $(G, D_\gamma, \cdot)$ has a Cauchy action on (X, C).

Proposition 4.4.

(a) *Let* $(G, \gamma, \cdot) \in PCONVG_1$ *act continuously on* $(X, q) \in PCONV_1$. *Then* $(G, C_\gamma, \cdot)$ *has a Cauchy action on* (X, C_q).

(b) *Let* $(G, \gamma, \cdot) \in CONVG_1$ *act continuously on* $(X, q) \in CONV_1$. *Then* $(G, \cdot, C_\gamma)$ *has a Cauchy action on* (X, C_q).

Proof. (a) Let $\mathcal{K} \in C_\gamma$ and $\mathcal{F} \in C_q$. Then $\mathcal{K} \xrightarrow{\gamma} g$ for some $g \in G$ and $\mathcal{F} \xrightarrow{q} x$ for some $x \in X$. Since $(G, \gamma, \cdot)$ acts continuously on (X, q), $\mathcal{F}^{\mathcal{K}} \xrightarrow{q} x^g$, which implies that $\mathcal{F}^{\mathcal{K}} \in C_q$. This completes the proof of (a). Proof of (b) is similar to (a). $\square$

5. Cauchy action on modifications of a ChG-space

Let C_{cp} and C_c denote, respectively, the c-filter modification and Cauchy modification [10] of C on X. In fact, C_{cp} (respectively, C_c) denotes the finest c-filter structure (respectively, Cauchy structure) on X coarser than C. The following proposition shows that these modified spaces also preserve the Cauchy action of G on the original filter space (X, C).

Theorem 5.1. *Let the filter semigroup* $(G, D, \cdot)$ *have a Cauchy action on the filter space* (X, C). *Then the following statements are true.*

(i) *If* $(G, D, \cdot)$ *is complete, then it has a Cauchy action on* (X, C_{cp}).

(ii) *Also,* $(G, D, \cdot)$ *has a Cauchy action on* (X, C_c).

Proof. Note that (a1) and (a2) hold in both cases (i) and (ii) above, so to complete the proof, we need only show that (Cha) holds. (i) Note that $C_{cp} = C \cup \{\mathcal{F} \cap \dot{x_1} \cap \dot{x_2} \cap ....\dot{x_n} \mid \mathcal{F} \in C, \mathcal{F} \sim_C \dot{x_i}, \text{ for all } i = 1, \ldots, n\}$. Let $\mathcal{K} \in D$ and $\mathcal{G} \in C_{cp}$. If $G \in C$, then $G^{\mathcal{K}} \in C \subseteq C_{cp}$. If $\mathcal{G} = \mathcal{F} \cap \dot{x_1} \cap \dot{x_2} \cap \ldots \dot{x_n}$, where $\mathcal{F} \in C$ and $\mathcal{F} \sim_C \dot{x_i}$, for all $i = 1, \ldots, n$, then $\mathcal{F}^{\mathcal{K}} \in C$. Moreover, if $(G, D, \cdot)$ is complete then there exists $t \in G$ such that $\mathcal{K} \sim_D \dot{t}$, so it follows from Theorem 3.5 that $\mathcal{F}^{\mathcal{K}} \sim_C \dot{x_i^t}$ for each i. This implies $\mathcal{G}^{\mathcal{K}} = \mathcal{F}^{\mathcal{K}} \cap \dot{x_1^t} \cap \dot{x_2^t} \cdots \dot{x_n^t} \in C_{cp}$.
(ii) Recall that $C_c = C \cup \{\mathcal{G} \mid \exists \mathcal{H}_1 \sim_C \mathcal{H}_2 \sim_C \cdots \mathcal{H}_n \in C \text{ such that } \mathcal{G} \geq \cap_{i=1}^n \mathcal{H}_i\}$. Let $\mathcal{K} \in D$ and $\mathcal{G} \in C_c$. Then there exist $\mathcal{H}_1, \ldots, \mathcal{H}_n \in C$ such that $\mathcal{H}_1 \sim_C \mathcal{H}_2 \sim_C \cdots \mathcal{H}_n$ and $\mathcal{G} \geq \cap_{i=1}^n \mathcal{H}_i$. Since $(G, D, \cdot)$ has a Cauchy action on (X, C), $\mathcal{H}_1^{\mathcal{K}}, \cdots, \mathcal{H}_1^{\mathcal{K}} \in C$ and by Theorem 3.5, $\mathcal{H}_1^{\mathcal{K}} \sim_C \mathcal{H}_2^{\mathcal{K}} \sim_C \cdots \mathcal{H}_n^{\mathcal{K}}$. Also, $\mathcal{G} \geq \cap_{i=1}^n \mathcal{H}_i$, implies $\mathcal{G} \times \mathcal{K} \geq (\cap_{i=1}^n \mathcal{H}_i) \times \mathcal{K}$. Hence,

$$\mathcal{G}^{\mathcal{K}} = \mu(\mathcal{G} \times \mathcal{K}) \geq \mu((\cap_{i=1}^n \mathcal{H}_i) \times \mathcal{K}) = \cap_{i=1}^n \mu(\mathcal{H}_i \times \mathcal{K}) = \cap_{i=1}^n \mathcal{H}_i^{\mathcal{K}}.$$

Therefore, $\mathcal{F}^{\mathcal{K}} \in C_c$, which completes the proof of the theorem. $\square$

Next, we explore whether a Cauchy action can be extended to the regular modification of the filter space (X, C). The regular modification of filter spaces was discussed in [10], and a regularity series for objects in larger categories such as FIL and c-FIL were introduced. These series are briefly described here for completeness. For $\mathcal{A} \subseteq \mathbf{F}(X)$ and $\mathcal{A}' = \{\dot{x} \mid x \in X\} \cup \mathcal{A}$, define

(I) $PC_{\mathcal{A}} = \{\mathcal{F} \in \mathbf{F}(X) \mid \exists \mathcal{G} \in \mathcal{A}' \text{ with } \mathcal{F} \geq \mathcal{G}\}$,

(II) $q\text{-}C_{\mathcal{A}} = \{\mathcal{F} \in \mathbf{F}(X) \mid \exists \mathcal{G} \in \mathcal{A}'$ and a finite number of $x_1, x_2, \cdots, x_n \in X$ with $\mathcal{G} \sim_{\mathcal{A}'} x_i$ for each $i = 1, \cdots n$, such that $\mathcal{F} \geq \mathcal{G} \cap \dot{x_1} \cap \cdots \cap \dot{x_n}\}$,

(III) $CC_{\mathcal{A}} = \{\mathcal{F} \in \mathbf{F}(X) \mid \exists$ a finite number of $\mathcal{A}'$-linked filters $\mathcal{H}_1, \cdots, \mathcal{H}_n$ such that $\mathcal{F} \geq \cap_{i=1}^{n} \mathcal{H}_i\}$.

Proposition 5.2 ([10, Proposition 1.11]). *For $(X, C) \in |FIL|$, $PC_{\mathcal{A}}$ (respectively, $q\text{-}C_{\mathcal{A}}$, $CC_{\mathcal{A}}$) is the finest filter structure (respectively, c-filter structure, Cauchy structure) on X containing $\mathcal{A}$.*

A regularity series for a filter space (X, C) leads to its regular modification. Let $r_0 C = C$, $r_1 C = PC_{\mathcal{A}_1}$, where $\mathcal{A}_1 = \{cl^n_{p_{r_0 C}} \mathcal{F} \mid \mathcal{F} \in C,\ n \in N\} \cup \{\dot{x} \mid x \in X\}$ and $r_2(C) = PC_{\mathcal{A}_2}$, where $\mathcal{A}_2 = \{cl^n_{p_{r_1 C}} \mathcal{F} \mid \mathcal{F} \in C,\ n \in N\} \cup \{\dot{x} \mid x \in X\}$. In general, $r_\beta(C) = PC_{\mathcal{A}_\beta}$, where $\mathcal{A}_\beta = \{cl^n_{p_{r_{\beta-1}(C)}} \mathcal{F} \mid \mathcal{F} \in C,\ n \in N\} \cup \{\dot{x} \mid x \in X\}$ if β is a non-limit ordinal, and $r_\beta(C) = \cup\{r_\alpha(C) \mid \alpha \leq \beta\}$ if β is a limit ordinal. From the construction, it can be shown that $C = r_0 C \geq r_1 C \geq \cdots \geq r_\beta(C) \geq r_{\beta+1}(C)$, for all ordinal numbers β. The *length* l_r of a regularity series r for a filter space (X, C) is the smallest ordinal number γ for which $r_\gamma(C) = r_{\gamma+1}(C)$. Let rC, called the *regular modification* of C, be the finest regular filter structure on X, which is coarser than C.

Lemma 5.3. *Let the filter semigroup $(G, D, \cdot)$ have a Cauchy action on the filter space $(X, r_\alpha(C))$ where α is any ordinal. Then, for any $\mathcal{F} \in \mathbf{F}(X)$, we have $cl_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}}) \leq (cl_{p_{r_{\alpha(C)}}} \mathcal{F})^{\mathcal{K}} \leq \mathcal{F}^{\mathcal{K}}$.*

Proof. The second inequality is straightforward, hence, only proof of the first inequality is given. Let $T \in cl_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}})$, then $cl_{p_{r_{\alpha(C)}}}(F^K) \subseteq T$ for some $F \in \mathcal{F}$ and $K \in \mathcal{K}$.

Claim: $(cl_{p_{r_{\alpha(C)}}} F)^K \subseteq cl_{p_{r_{\alpha(C)}}}(F^K)$. Let $y \in (cl_{p_{r_{\alpha(C)}}} F)^K \Rightarrow y = x^g$ for some $g \in K$ and $x \in cl_{p_{r_{\alpha(C)}}} F$. So, $\exists\, \mathcal{L} \overset{p_{r_{\alpha(C)}}}{\longrightarrow} x$ and $F \in \mathcal{L}$, but since $(G, D, \cdot)$ has a Cauchy action on $(X, r_{\alpha(C)})$, by Proposition 4.1, $\mathcal{L}^g \overset{p_{r_{\alpha(C)}}}{\longrightarrow} x^g = y$ and $F^g \in \mathcal{L}^g$. However, this implies $F^K \in \mathcal{L}^g$, since $F^g \subseteq F^K$. Since there exists a filter $\mathcal{L}^g \overset{p_{r_{\alpha(C)}}}{\longrightarrow} y$ and $F^K \in \mathcal{L}^g$, it follows that $y \in cl_{p_{r_{\alpha(C)}}}(F^K)$. This proves the claim. Hence for any $T \in cl_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}})$, $\exists\, F \in \mathcal{F}$ and $K \in \mathcal{K}$ such that $(cl_{p_{r_{\alpha(C)}}} F)^K \subseteq T$, which completes the proof of the lemma. □

Remark 5.4. In particular, if the filter semigroup $(G, D, \cdot)$ has a Cauchy action on the filter space (X, C'), then for any $\mathcal{L} \in \mathbf{F}(X)$, we have $cl_{p_{C'}}(\mathcal{L}^{\mathcal{K}}) \leq (cl_{p_{C'}} \mathcal{L})^{\mathcal{K}} \leq \mathcal{L}^{\mathcal{K}}$.

Lemma 5.5. *Let the filter semigroup $(G, D, \cdot)$ have a Cauchy action on the filter space $(X, r_\alpha(C))$ where α is any ordinal. Then for any $n \in N$ and $\mathcal{F} \in C$,*

$$cl^n_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}}) \leq (cl^n_{p_{r_{\alpha(C)}}} \mathcal{F})^{\mathcal{K}} \leq \mathcal{F}^{\mathcal{K}}.$$

Proof. Since $(G, D, \cdot)$ has a Cauchy action on the filter space $(X, r_\alpha(C))$, by Lemma 5.3, $cl_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}}) \leq (cl_{p_{r_{\alpha(C)}}}\mathcal{F})^{\mathcal{K}}$.

Next, let $cl^i_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}}) \leq (cl^i_{p_{r_{\alpha(C)}}}\mathcal{F})^{\mathcal{K}}$ for some $n = i$. This implies

$$cl_{p_{r_{\alpha(C)}}}\left[cl^i_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}})\right] \leq cl_{p_{r_{\alpha(C)}}}\left[(cl^i_{p_{r_{\alpha(C)}}}\mathcal{F})^{\mathcal{K}}\right].$$

Now substituting $\mathcal{L} = cl^i_{p_{r_{\alpha(C)}}}\mathcal{F}$ and $C' = r_\alpha(C)$, from Remark 5.4, it follows that $cl_{p_{C'}}(\mathcal{L}^{\mathcal{K}}) \leq (cl_{p_{C'}}\mathcal{L})^{\mathcal{K}} \leq \mathcal{L}^{\mathcal{K}}$. Therefore,

$$cl^{i+1}_{p_{r_{\alpha(C)}}}(\mathcal{F}^{\mathcal{K}}) \leq cl_{p_{r_{\alpha(C)}}}\left[(cl^i_{p_{r_{\alpha(C)}}}\mathcal{F})^{\mathcal{K}}\right] \leq \left[cl_{p_{r_{\alpha(C)}}}\left(cl^i_{p_{r_{\alpha(C)}}}\mathcal{F}\right)\right]^{\mathcal{K}} = \left(cl^{i+1}_{p_{r_{\alpha(C)}}}\mathcal{F}\right)^{\mathcal{K}}.$$

The proof is now complete by applying induction. □

Theorem 5.6. *If $(G, D, \cdot) \in |FILSG|$ has a Cauchy action on $(X, C) \in |FIL|$, then it has a Cauchy action on $(X, r_\gamma(C))$ where γ is any ordinal.*

Proof. It is clear that (a1) and (a2) hold for each $(X, r_\alpha(C))$, so we need to prove only (Cha), that is, for any ordinal α if $\mathcal{F} \in r_\alpha(C)$ and $\mathcal{K} \in D$, then $\mathcal{F}^{\mathcal{K}} \in r_\alpha(C)$. This can be proved by transfinite induction. First, let $\mathcal{H} \in r_1(C)$. Then $\exists\, \mathcal{L} \in C$ and $n \in N$ such that $\mathcal{F} \geq cl^n_{p_C}\mathcal{L}$. Using Remark 5.4, this yields

$$\mathcal{H}^{\mathcal{K}} \geq [cl^n_{p_C}\mathcal{L}]^{\mathcal{K}} \geq cl^n_{p_C}[\mathcal{L}^{\mathcal{K}}].$$

Since $(G, D, \cdot)$ has a Cauchy action on (X, C), $\mathcal{L}^{\mathcal{K}} \in C$, so that $\mathcal{H}^{\mathcal{K}} \in r_1(C)$. This implies that $(G, D, \cdot)$ has a Cauchy action on $(X, r_1(C))$.

Next assume that for a fixed ordinal α, $(G, D, \cdot)$ has a Cauchy action on $(X, r_\beta(C))$ for each ordinal $\beta < \alpha$.

Case 1 Let α be a non-limit ordinal. So, if $\mathcal{F} \in r_\alpha(C)$, then $\mathcal{F} \geq cl^n_{p_{r_{\alpha-1(C)}}}\mathcal{L}$ for some $\mathcal{L} \in C$ and $n \in N$. Therefore, for any $\mathcal{K} \in D$, $\mathcal{F}^{\mathcal{K}} \geq [cl^n_{p_{r_{\alpha-1(C)}}}\mathcal{L}]^{\mathcal{K}}$. By assumption, since $(G, D, \cdot)$ has a Cauchy action on $(X, r_{\alpha-1}(C))$, from Lemma 5.5, it follows that

$$[cl^n_{p_{r_{\alpha-1(C)}}}\mathcal{L}]^{\mathcal{K}} \geq cl^n_{p_{r_{\alpha-1(C)}}}[\mathcal{L}^{\mathcal{K}}],$$

which implies $\mathcal{F}^{\mathcal{K}} \geq cl^n_{p_{r_{\alpha-1(C)}}}[\mathcal{L}^{\mathcal{K}}]$. Since $\mathcal{L}^{\mathcal{K}} \in C$, it follows from definition that $\mathcal{F}^{\mathcal{K}} \in r_\alpha(C)$.

Case 2 Let α be a limit ordinal, then $\mathcal{F} \in r_\alpha(C)$ implies that $\mathcal{F} \in r_\beta(C)$ for some ordinal $\beta < \alpha$. By assumption since $(G, D, \cdot)$ has a Cauchy action on $(X, r_\beta(C))$, $\mathcal{F}^{\mathcal{K}} \in r_\beta(C)$, for any $\mathcal{K} \in D$. Hence, $\mathcal{F}^{\mathcal{K}} \in r_\alpha(C)$.

The proof of the proposition is now complete by transfinite induction. □

Theorem 5.6 leads to the following result.

Theorem 5.7. *$(G, D, \cdot) \in |FILSG|$ has a Cauchy action on $(X, C) \in |FIL|$, then the Cauchy action can be extended to the regular modification (X, rC).*

6. Cauchy action on completion of a ChG-space

In this section, it is assumed that (X, C) is a T_2 filter space and $(G, D, \cdot)$ is a Cauchy group, that is, as noted in Section 2.2, $(G, \cdot)$ is a group with a Cauchy-compatible group operation. If μ is a Cauchy action of $(G, D, \cdot)$ on (X, C), then in this case, from (Cha) it follows that $\forall\ \mathcal{F} \in C$ and $\forall\ \mathcal{K} \in D$, $\mathcal{F}^{\mathcal{K}^{-1}} \in C$. Note that for any $\mathcal{K} \in \mathbf{F}(G)$, $\dot{e} \geq \mathcal{K}\mathcal{K}^{-1} = \mathcal{K}^{-1}\mathcal{K}$. This yields $\mathcal{F} = \mathcal{F}^{\dot{e}} \geq \mathcal{F}^{\mathcal{K}\mathcal{K}^{-1}}$ for all $\mathcal{F} \in C$.

Let CHG denote the category of all Cauchy groups with Cauchy maps as morphisms. Since every Cauchy group is also a filter semigroup, CHG is a full subcategory of $FILSG$ [17]. Consequently, in view of Theorem 5.1, the axioms (a1), (a2) and (Cha) define a Cauchy action of the Cauchy group $(G, D, \cdot)$ on a filter space in general.

Remark 6.1. In the case when $(G, D, \cdot)$ is a Cauchy group, $\mathcal{K} \sim_D \mathcal{L}$ implies $\mathcal{K}^{-1} \sim_D \mathcal{L}^{-1}$, therefore, if it has a Cauchy action μ on a filter space (X, C), from Theorem 3.5 it follows that $[\mathcal{F}^{\mathcal{K}^{-1}}]_C = [\mathcal{G}^{\mathcal{L}^{-1}}]_C$, where $\mathcal{F}, \mathcal{G} \in C$ and $\mathcal{K}, \mathcal{L} \in D$ with $\mathcal{K} \sim_D \mathcal{L}$ and $\mathcal{F} \sim_D \mathcal{G}$. In particular, $[\mathcal{H}^{\mathcal{K}}]_C = [\mathcal{T}]_C$ implies $[\mathcal{H}]_C = [\mathcal{T}^{\mathcal{K}^{-1}}]_C$, since $\mathcal{H} = \mathcal{H}^{\dot{e}} \geq \mathcal{H}^{\mathcal{K}\mathcal{K}^{-1}}$.

Lemma 6.2. *Let the Cauchy group $(G, D, \cdot)$ have a Cauchy action on a filter space (X, C), and $\mathcal{K} \in D$ be p_D-convergent. Then $\mathcal{F}^{\mathcal{K}}$ is p_C-convergent if and only if $\mathcal{F} \in C$ is p_C-convergent.*

Proof. Let $\mathcal{K} \sim_D \dot{g}$. If $\mathcal{F} \sim_C \dot{x}$, then $\mathcal{F}^{\mathcal{K}} \sim_C \dot{x^g}$, so that $\mathcal{F}^{\mathcal{K}}$ is convergent. Next, let $\mathcal{F} \in C$ be nonconvergent. If $\mathcal{F}^{\mathcal{K}}$ is convergent, then $\mathcal{F}^{\mathcal{K}} \sim_C \dot{y}$ for some $y \in X$. However, from Remark 6.1, since $\mathcal{K}^{-1} \sim_D \dot{g^{-1}}$, we get $\mathcal{F}^{\mathcal{K}\mathcal{K}^{-1}} \sim_C \dot{y^{g^{-1}}}$. This implies that $\mathcal{F} = \mathcal{F}^{\dot{e}} \sim_C \dot{y^{g^{-1}}}$, which is a contradiction. This completes the proof of the lemma. $\square$

In particular, for any $g \in G$, $\mathcal{F}^{\dot{g}}$ is p_C-convergent if and only if $\mathcal{F} \in C$ is p_C-convergent.

The Wyler completion of a T_2 filter space (X, C) was studied in [10] which is briefly summarized here. The completion $\mathcal{W}^* = ((X^*, C^*), j)$, where $X^* = \{[\mathcal{F}] \mid \mathcal{F} \in C\}$, $j : X \longrightarrow X^*$ is defined as $j(x) = [\dot{x}]\,\forall\, x \in X$ and $C^* = \{\mathcal{A} \in \mathbf{F}(X^*) \mid \exists\, \mathcal{F} \in C$ such that $\mathcal{A} \geq j(\mathcal{F}) \cap \dot{[\mathcal{F}]}\}$.

Next, the extension problem is investigated, that is, if the Cauchy group $(G, D, \cdot)$ has a Cauchy action on (X, C), then, whether it can be extended to its completion $\mathcal{W}^* = ((X^*, C^*), j)$. Define

$$\mu^* : X^* \times G \longrightarrow X^*, \text{ by } \mu^*([\mathcal{F}], g) = [\mathcal{F}^{\dot{g}}], \text{ for all } \mathcal{F} \in C \text{ and } g \in G.$$

Note that $\mu^*([\dot{x}], g) = [\dot{x^g}]$, that is, $\mu^*(j(x), g) = j(\mu(x, g)) \forall x \in X$ and $g \in G$.

Lemma 6.3. *The map μ^* is an action of G on (X^*, C^*).*

Proof. Since μ is a Cauchy action, $\dot{x}^{\dot{g}} = \dot{x^g} \in C$ and p_C-convergent, so $[\dot{x^g}] \in X^*$. Also, if $\mathcal{F} \in C$ is nonconvergent, then by Lemma 6.2, $\mathcal{F}^{\dot{g}} \in C$ is nonconvergent which implies $[\mathcal{F}^{\dot{g}}] \in X^*$. By Theorem 3.5, μ^* is well-defined.

Next, $\mu^*([\dot{x}], e) = [\dot{x^e}] = [\dot{x}] \; \forall [\dot{x}] \in X^*$, and $\mu^*([\mathcal{F}], e) = [\mathcal{F}^{\dot{e}}] = [\mathcal{F}]$ for all nonconvergent $\mathcal{F} \in C$, therefore, (a1) holds. Similarly, it is straightforward to check that (a2) holds. $\square$

The action μ^* is said to be a *Wyler extension* of the Cauchy action μ. Note that by Lemma 6.3, every Wyler extension of a Cauchy action is an action on the Wyler completion of the filter space. Further, if $\mu^*([\mathcal{F}], \mathcal{K}) = [\mu(\mathcal{F}, \mathcal{K})]$ for all p_C-nonconvergent filters $\mathcal{F} \in C$ and $\mathcal{K} \in D$, then μ^* is called a *standard Wyler extension* of the Cauchy action μ.

Lemma 6.4. *Define a map $h : (X \times G) \longrightarrow (X^* \times G)$ by $h(x, g) = (j(x), g)$ for all $x \in X$ and $g \in G$, then the following holds:*

(i) $\mu^* \circ h = j \circ \mu$,
(ii) *for any $Y \subseteq X$ and $A \subseteq G$, $\mu^*(j(Y), A) = j(\mu(Y, A))$.*

Proof. Note that h is well-defined. Moreover, by definition, $\mu^* \circ h(x, g) = \mu^*(h(x, g)) = \mu^*(j(x), g) = \mu^*([\dot{x}], g) = [\dot{x^g}]$ by definition, and this equals $[\mu(\dot{x}, g)] = j(\mu(x, g)) = j \circ \mu(x, g)$, which proves (i). To prove (ii), recall that from (i), [16]

$$\mu^*(j(Y), A) = \bigcup_{a \in A} \mu^*(j(Y), a) = \bigcup_{a \in A} \mu^*(h(Y, a)) = \bigcup_{a \in A} j(\mu(Y, a)),$$

but this equals $j(\bigcup_{a \in A} \mu(Y, a)) = j(\mu(Y, A))$ and this completes the proof. $\square$

Theorem 6.5. *Every Cauchy action of a complete Cauchy group $(G, D, \cdot)$ on a T_2 filter space (X, C) has a standard Wyler extension which is also a Cauchy action.*

Proof. In view of Lemma 6.3, we need only to prove (Cha). Let $\mathcal{A} \in C^*$ and $\mathcal{K} \in D$. So $\mathcal{A} \geq j(\mathcal{F}) \cap [\dot{\mathcal{F}}]$ for some p_C-nonconvergent $\mathcal{F} \in C$ or $\mathcal{A} \geq j(\mathcal{H})$ for some p_c-convergent $\mathcal{H} \in C$. In the latter case, $\mu^*(\mathcal{A}, \mathcal{K}) \geq \mu^*(j(\mathcal{H}), \mathcal{K}) = j(\mu(\mathcal{H}, \mathcal{K})) = j(\mathcal{H}^{\mathcal{K}})$ by Lemma 6.4. Note that $\mu(\mathcal{H}, \mathcal{K}) = \mathcal{H}^{\mathcal{K}} \in C$, μ being a Cauchy action, and also it is p_C-convergent by Lemma 6.2. Hence, $\mu^*(\mathcal{A}, \mathcal{K}) \in C^*$.

On the other hand, if $\mathcal{A} \geq j(\mathcal{F}) \bigcap [\dot{\mathcal{F}}]$ for some p_C-nonconvergent filter $\mathcal{F} \in C$, then $\mu^*(\mathcal{A}, \mathcal{K}) \geq \mu^*(j(\mathcal{F}) \bigcap [\dot{\mathcal{F}}], \mathcal{K})$.
<u>Claim:</u> $\mu^*(j(\mathcal{F}) \bigcap [\dot{\mathcal{F}}]$ and $\mathcal{K}) \geq \mu^*(j(\mathcal{F}), \mathcal{K}) \bigcap \mu^*([\dot{\mathcal{F}}], \mathcal{K})$.
Let $T \in \mu^*(j(\mathcal{F}), \mathcal{K}) \bigcap \mu^*([\dot{\mathcal{F}}], \mathcal{K})$, then $\exists \, F \in \mathcal{F}$ and K_1, $K_2 \in \mathcal{K}$ such that $\mu^*(j(F), K_1) \bigcup \mu^*([\mathcal{F}], K_2) \subseteq T$. Let $L = K_1 \bigcap K_2$, then $L \in \mathcal{K}$ and

$$\mu^*(j(F), L) \bigcup \mu^*([\mathcal{F}], L) \subseteq \mu^*(j(F), K_1) \bigcup \mu^*([\mathcal{F}], K_2) \subseteq T.$$

However, since

$$\mu^*(j(F), L) \bigcup \mu^*([\mathcal{F}], L) = \mu^*((j(F), L) \bigcup ([\mathcal{F}], L)) = \mu^*(j(F) \bigcup [\mathcal{F}], L),$$

there exist $F \in \mathcal{F}$ and $L \in \mathcal{K}$ such that $\mu^*(j(F) \bigcup [\mathcal{F}], L) \subseteq T$. This implies that $T \in \mu^*(j(\mathcal{F}) \bigcap [\dot{\mathcal{F}}], \mathcal{K})$, which proves the claim.

From Lemma 6.4 it follows that $\mu^*(j(\mathcal{F}),\mathcal{K}) = j(\mu(\mathcal{F},\mathcal{K}))$, and since μ^* is a standard Wyler extension, $\mu^*([\mathcal{F}],\mathcal{K}) = [\mu(\mathcal{F},\mathcal{K})]$. Therefore,

$$\mu^*(j(\mathcal{F}) \bigcap [\dot{\mathcal{F}}],\mathcal{K}) \geq j(\mu(\mathcal{F},\mathcal{K})) \bigcap [\mu(\dot{\mathcal{F}},\mathcal{K})],$$

which implies that

$$\mu^*(\mathcal{A},\mathcal{K}) \geq j(\mu(\mathcal{F},\mathcal{K})) \bigcap [\mu(\dot{\mathcal{F}},\mathcal{K})].$$

Since μ is a Cauchy action on (X, C), $\mu(\mathcal{F},\mathcal{K}) \in C$, therefore, $\mu^*(\mathcal{A},\mathcal{K}) \in C^*$. This completes the proof of the theorem. □

7. Conclusion

The notion of Cauchy continuity in group-action is a natural blend of set-theoretic topology and algebra, so the scope of this work is two-fold. The additional property of a group-action, that is, Cauchy continuity leads to a vast area of research in many directions with numerous topological implications. In a nutshell, this is a minuscule attempt by the author to lay the foundation for a new area of research in general topology. The new areas may include, for instance, the extension of Cauchy action to larger spaces, their impact on normal and completely normal spaces, the quotient space, compactification of the G-space and G-equivariant maps, to name but a few. On the other hand, results related to algebraic groups acting on sets can be applied to Cauchy groups, for example, the invariant subspaces, homeomorphism groups and primitivity of a Cauchy group. Moreover, a few results, so far related to actions of finite groups, can be applied to infinite groups using filters for Cauchy actions.

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