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MULTI-CRITERIA DECISION METHODS TO SUPPORT THE MAINTENANCE MANAGEMENT OF COMPLEX SYSTEMS

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To my father, my mother and Nino
The rock upon which I stand

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Three years ago I undertook a long and difficult, but beautiful journey, called a double PhD. It has been so challenging and undoubtedly the most important personal achievement of my whole life so far. I compared myself with something I thought was much bigger than me. Actually, in order to accomplish it successfully, I just needed to fully involve my heart. I did it without saving my strengths. What I have learnt during my doctorate will remain with me forever. Not only did I further develop my professional skills and my attitude to scientific research, but above all I made real my passion for discovery. Discovering the world, meeting new people, appreciating diversity, considering various manners of working and living, learning new languages, facing my weaknesses and getting over them. All of the things listed above unquestionably influenced my personal development. And personal growth is an endless process. It goes without saying that this doctoral thesis is the result of all my efforts and love. I dedicate it…

To my father and my mother, the most important people in my life. I wish I could be even half as good person and engineer as they are. All that I am I owe to my parents. To my brother Nino, who is for me the best example of courage and who has had right answers to my doubts since we were just children. To my grandma Wilma, a great woman to whom I owe most of my personality. To all my family, especially to my grandparents Maria and Mario, and to my cousins/sisters Paola, Luisa and Carla.

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Silvia

ABSTRACT

This doctoral thesis proposes using multi-criteria decision making (MCDM) methods as a strategic tool to support maintenance management of complex systems.

The development of this doctoral thesis is framed within a cotutelle (co-tutoring) agreement between the *Università degli Studi di Palermo* (UNIPA) and the *Universitat Politècnica de València* (UPV), within their respective programmes of doctorates in 'Technological Innovation Engineering' and 'Mathematics'. Regarding this thesis, these programmes are closely linked through the topic of MCDM, providing crucial tools to manage maintenance of real complex systems by applying in-depth mathematical analyses.

The purpose of this connection is to robustly take into account uncertainty in attributing subjective evaluations, collecting and synthetizing judgments attributed by various decision makers, and dealing with large sets of elements characterising the faced issue. The main topic of the present doctoral work is the management of maintenance activities to increase the levels of technological innovation and performance of the analysed complex systems. All kinds of systems can be considered as objects of study, including production systems and service delivery systems, among others, by evaluating their real contexts.

Thus, this doctoral thesis proposes facing maintenance management through the development of three tightly linked main research lines.

- The first is the core and illustrates most of the methodological aspects of the thesis. It refers to the use of MCDM methods for supporting strategic maintenance decisions, and dealing with uncertainty affecting data/evaluations even when several decision makers are involved (experts in maintenance).
- The second line develops reliability analyses for real complex systems (also in terms of human reliability analysis) on the basis of which any maintenance activity must be implemented. These analyses are approached by considering the reliability configuration of both the components belonging to the system under study and the specific features of the operational environment.
- The third research line focuses on important methodological aspects to support maintenance management, and emphasises the need to monitor the performance of maintenance activities and evaluate their effectiveness using suitable indicators.

A wide range of real real-world case studies has been faced to evaluate the effectiveness of MCDM methods in maintenance and then prove the usefulness of the proposed approach.

SOMMARIO

La presente tesi di dottorato propone l'utilizzo dei metodi decisionali multi-criterio (MCDM) quale strumento strategico per supportare la gestione della manutenzione di sistemi complessi.

Lo sviluppo di questa tesi di dottorato è regolato da un accordo di cotutela stipulato tra l'*Università degli Studi di Palermo* (UNIPA) e l'*Universitat Politècnica de València* (UPV), nell'ambito dei rispettivi programmi di dottorato in "Ingegneria dell'Innovazione Tecnologica" e "Matematica". In relazione alla presente tesi, tali programmi sono strettamente correlati attraverso il topic MCDM, il quale fornisce strumenti cruciali per gestire la manutenzione di sistemi complessi reali applicando approfondite analisi matematiche.

Lo scopo di tale collaborazione consiste nel trattare in maniera robusta l'incertezza caratterizzante l'espressione di valutazioni soggettive, nel raccogliere e sintetizzare i giudizi attribuiti dai diversi decisori, nonché nel trattare ampi insiemi degli elementi che caratterizzano la tematica affrontata. Il tema principale del presente lavoro di dottorato è la gestione delle attività di manutenzione con lo scopo di migliorare i livelli di innovazione tecnologica e di performance dei sistemi complessi analizzati. Tutte le tipologie di sistema possono essere considerate quale oggetto di studio, inclusi i sistemi produttivi e di servizi, tra gli altri, valutando i rispettivi contesti reali.

La presente tesi di dottorato propone dunque di affrontare la gestione della manutenzione attraverso lo sviluppo di tre linee di ricerca, tra esse strettamente correlate.

- La prima linea costituisce il corpo principale e illustra la maggior parte degli aspetti metodologici della tesi. Si riferisce all'utilizzo dei metodi MCDM per supportare decisioni manutentive strategiche e per trattare l'incertezza che affetta dati/valutazioni anche quando più decisori sono coinvolti (esperti in manutenzione).
- La seconda linea sviluppa analisi affidabilistiche per sistemi complessi reali (anche in termini di analisi dell'affidabilità umana) sulla base delle quali deve essere implementata una generica attività manutentiva. Tali analisi sono approcciate considerando sia la configurazione affidabilistica dei componenti appartenenti al sistema oggetto di studio sia le specifiche caratteristiche dell'ambiente operativo.
- La terza linea di ricerca si focalizza su importanti aspetti metodologici a supporto della gestione della manutenzione, ed enfatizza il bisogno di monitorare la performance delle attività manutentive e di valutare la loro efficacia utilizzando appropriati indicatori.

Un'ampia gamma di casi studio reali è stata affrontata al fine di valutare l'efficacia dei metodi MCDM in tema di manutenzione e quindi validare l'utilità dell'approccio proposto.

RESUMEN

Esta tesis doctoral propone el uso de métodos de toma de decisiones multi-criterio (MCDM, por sus iniciales en inglés) como herramienta estratégica para apoyar la gestión del mantenimiento de sistemas complejos.

El desarrollo de esta tesis doctoral se enmarca dentro de un acuerdo de cotutela entre la Università degli Studi di Palermo (UNIPA) y la Universitat Politècnica de València (UPV), dentro de sus respectivos programas de doctorado en 'Ingeniería de Innovación Tecnológica' y 'Matemáticas'. Estos programas están estrechamente vinculados a través del tópico MCDM, ya que proporciona herramientas cruciales para gestionar el mantenimiento de sistemas complejos reales utilizando análisis matemáticos serios.

El propósito de esta sinergia es tener en cuenta de forma sólida la incertidumbre al atribuir evaluaciones subjetivas, recopilar y sintetizar juicios atribuidos por varios responsables de la toma de decisiones, y tratar con conjuntos grandes de esos elementos. El tema principal del presente trabajo de doctorado es el gestionamiento de las actividades de mantenimiento para aumentar los niveles de innovación tecnológica y el rendimiento de los sistemas complejos. Cualquier sistema puede ser considerado objeto de estudio, incluidos los sistemas de producción y los de prestación de servicios, entre otros, mediante la evaluación de sus contextos reales.

Esta tesis doctoral propone afrontar la gestión del mantenimiento a través del desarrollo de tres líneas principales de investigación estrechamente vinculadas.

- La primera es el núcleo, e ilustra la mayoría de los aspectos metodológicos de la tesis. Se refiere al uso de métodos MCDM para apoyar decisiones estratégicas de mantenimiento, y para hacer frente a la incertidumbre que afecta a los datos/evaluaciones, incluso cuando están involucrados varios responsables (expertos en mantenimiento) en la toma de decisiones.
- La segunda línea desarrolla análisis de fiabilidad para sistemas complejos reales (también en términos de fiabilidad humana) sobre cuya base se debe implementar cualquier actividad de mantenimiento. Estos análisis consideran la configuración de fiabilidad de los componentes del sistema en estudio y las características específicas del entorno operativo.
- La tercera línea de investigación aborda aspectos metodológicos importantes de la gestión de mantenimiento y enfatiza la necesidad de monitorizar el funcionamiento de las actividades de mantenimiento y de evaluar su efectividad utilizando indicadores adecuados.

Se ha elaborado una amplia gama de casos de estudio del mundo real para evaluar la eficacia de los métodos MCDM en el mantenimiento y así probar la utilidad del enfoque propuesto.

RESUM

Aquesta tesi doctoral proposa l'ús de mètodes de presa de decisions multi-criteri (MCDM, per les seves inicials en anglès) com a eina estratègica per donar suport a la gestió del manteniment de sistemes complexos.

El desenvolupament d'aquesta tesi doctoral s'emmarca dins d'un acord de cotutela entre la Università degli Studi di Palermo (UNIPA) i la Universitat Politècnica de València (UPV), dins dels seus respectius programes de doctorat en 'Enginyeria d'Innovació Tecnològica' i ' Matemàtiques '. Aquests programes estan estretament vinculats a través del tòpic MCDM, ja que proporciona eines crucials per gestionar el manteniment de sistemes complexos reals utilitzant anàlisis matemàtics profunds.

El propòsit d'aquesta sinergia és tenir en compte de forma sòlida la incertesa en atribuir avaluacions subjectius, recopilar i sintetitzar judicis atribuïts per diversos responsables de la presa de decisions, i tractar amb conjunts grans d'aquests elements en els problemes plantejats. El tema principal del present treball de doctorat es la gestió de les activitats de manteniment per augmentar els nivells d'innovació tecnològica i el rendiment dels sistemes complexos. Qualsevol sistema pot ser considerat objecte d'estudi, inclosos els sistemes de producció i els de prestació de serveis, entre d'altres, mitjançant l'avaluació dels seus contextos reals.

Aquesta tesi doctoral proposa afrontar la gestió del manteniment mitjançant el desenvolupament de tres línies principals d'investigació estretament vinculades.

- La primera és el nucli, i il·lustra la majoria dels aspectes metodològics de la tesi. Es refereix a l'ús de mètodes MCDM per donar suport a decisions estratègiques de manteniment, i per fer front a la incertesa que afecta les dades/avaluacions, fins i tot quan estan involucrats diversos responsables (experts en manteniment) en la presa de decisions.
- La segona línia desenvolupa anàlisis de fiabilitat per a sistemes complexos reals (també en termes de fiabilitat humana) sobre la qual base s'ha d'implementar qualsevol activitat de manteniment. Aquestes anàlisis consideren la configuració de fiabilitat dels components del sistema en estudi i les característiques específiques de l'entorn operatiu.
- La tercera línia d'investigació aborda aspectes metodològics importants de la gestió de manteniment i emfatitza la necessitat de monitoritzar el funcionament de les activitats de manteniment i d'avaluar la seva efectivitat utilitzant indicadors adequats.

S'ha elaborat una àmplia gamma de casos d'estudi del món real per avaluar l'eficàcia dels mètodes MCDM en el manteniment i així provar la utilitat de l'enfocament proposat.

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ACRONYM LIST

AHP	Analytic Hierarchy Process
AIJ	Aggregation of Individual Judgments
AIP	Aggregation of Individual Priorities
AMC	Annual Maintenance Cost vs annual maintenance budget
ANP	Analytic Network Process
APJ	Absolute Probability Judgement
ASEP	Accident Sequence Evaluation Program
ATHEANA	A Technique for Human Error Analysis
BRC	British Retail Consortium
CAHR	Connectionism Assessment of Human Reliability
CESA	Commission Errors Search and Assessment
CEF	Component efficiency
CHEP	Conditioned Human Error Probability
CI	Consistency Index
CODA	Conclusions from occurrences by descriptions of actions
CR	Consistency Ratio
CREAM	Cognitive reliability and error analysis method
DEMATEL	Decision-Making Trial and Evaluation Laboratory
DM	Decision-Maker
DS	Dempster-Shafer
DSS	Decision Support System
EJBM	Expert Judgement Based Method
ELECTRE	ELimination Et Choix Traduisant la REalité
EOC	Errors Of Commission
EOO	Errors Of Omission
ER	Equipment reliability
FAHP	Fuzzy Analytic Hierarchy Process
FCPM	Fuzzy Pairwise Comparison Matrix
FGM	First Generation Methods
FM	Facility Management
FMEA	Failure Mode and Effects Analysis
FMECA	Failure Mode, Effects and Criticality Analysis
FTOPSIS	Fuzzy Technique for Order of Preference by Similarity to Ideal Solution

GMM	Geometric Mean Method
GPR	**G**round Penetrating Radar
HEART	Human Error Assessment & Reduction Technique
HEP	Human Error Probability
HF/E	Human Factors and Ergonomics
HFACS	Human Factors Analysis and Classification System
HFACS-MA	Human Factors Analysis and Classification System for Maritime Accidents
HMS	Human Management System
HPC	High-Performance Computer
HRA	Human Reliability Analysis
HRM	Human Resource Management
HRMS	Human reliability management system
ICT	Information and Communication Technology
IFS	International Food Standard
IIoT	Industrial Internet of Things
IoT	Internet of Things
IWDS	Industrial Water Distribution System
IWSS	Intermittent Water Supply System
JHEDI	Justified Human Error Data Information
KPI	Key Performance Indicator
MCDM	Multi-Criteria Decision Making
MERMOS	Assessment method for performance of safety operation
MOA	Multi-Objective Algorithm
MOA	Maintenance and Repair Organization
MTTF	Mean Time To Failure
NA	Number of alarms
NARA	Nuclear Action Reliability Assessment
NDT	Non-Destructive Techniques
NI	Number of interventions
NPA	Not Publicly Available
NSGA-II	Non-Dominated Sorting Genetic Algorithm II
P2P	Peer-to-Peer
PA	Publicly Available
PC	Paired comparisons

PCM	Pairwise Comparison Matrix
PM-FD	Process Monitoring and Fault Detection
PSF	Performance Shaping Factor
PU	Pressure Uniformity
QFD	Quality Function Development
RCM	Reliability-Centred Maintenance
RI	Random Index
RPN	Risk Priority Number
SA	Simulated Annealing
SC	Schedule compliance
SGM	Second Generation Methods
SLIM-MAUD	Success likelihood index methodology, Multi-Attribute Utility Decomposition
SPAR-H	Standardized Plant Analysis Risk-Human
SW	System wear
TD	Total downtime
TFN	Triangular Fuzzy Numbers
TGM	Third Generation Methods
THERP	Technique for Human Error Rate Prediction
TOPSIS	Technique for Order of Preference by Similarity to Ideal Solution
TrFN	Trapezoidal Fuzzy Numbers
WAMM	Weighted Arithmetic Mean Method
WGMM	Weighted Geometric Mean Method
WSS	Water Supply System

SUMMARY OF FIGURES

SUMMARY OF TABLES

INTRODUCTION

A general overview of this doctoral dissertation is briefly presented. The academic conditions under which the dissertation was developed are then described. Lastly, the objectives pursued by the thesis are stated, together with the methodologies used to achieve the objectives.

General overview and contribution

Industries used to consider maintenance as a simple set of technical-economic activities, with the main objective of reducing the costs of operations as much as possible. There was no real perception of the important relationship between system maintenance, safety, security, and availability.

In contrast, system availability is now earnestly pursued, because of the associated demanding investments related to system utilisation. Thus, the role of maintenance is continuously growing in importance in order to enhance the competitive capabilities of industries and businesses (ISO 55000:2014).

The progressive evolution of maintenance is explained by the passage from maintenance being considered as a simple repair process, to assuming the role of a complex management procedure dedicated to continuous improvement. Effective maintenance management enables the achievement of important goals (Certa *et al.*, 2013a) related to the reduction of direct and indirect maintenance costs, enhanced reputation, improved safety and security levels, and the reduction of environmental impacts.

Management of maintenance activities increases the levels of technological innovation and performance of complex systems. All kinds of systems can be considered as objects of study, including production systems (Liu *et al*., 2015; Bertolini *et al.*, 2006) and service delivery systems (Antonovsky *et al*., 2016; Jun and Huibin, 2012; Bosse *et al*., 2016), among others, by evaluating their real contexts (Sidibé *et al*., 2016; Ee *et al*., 2015).

This doctoral thesis proposes facing maintenance management through the development of three tightly linked main research lines.

- The first is the core and provides most of the methodological aspects of the thesis. It refers to the use of multi-criteria decision-making (MCDM) methods for supporting strategic maintenance decisions, and dealing with uncertainty affecting data/evaluations even when several decision makers are involved (experts in maintenance).
- The second line develops reliability analyses for real complex systems (also in terms of human reliability analysis) on the basis of which any maintenance activity must be implemented (Koning *et al*., 2009; Aven, 2016a). These processes are approached by considering the reliability configuration of both the components belonging to the system under study and the specific features of the operational environment.
- The third research line focuses on important methodological aspects of maintenance management, and emphasises the need to monitor the performance of maintenance activities and evaluate their effectiveness using suitable indicators.

Development framework

The development of this doctoral thesis is framed within a *cotutelle* (co-tutoring) agreement between the *Università degli Studi di Palermo* (UNIPA) and the *Universitat Politècnica de València* (UPV), within their respective programmes of doctorates in 'Technological Innovation Engineering' and 'Mathematics'. Regarding this thesis, these programmes are closely linked through the topic of MCDM (providing crucial tools to optimise real complex systems by applying in-depth mathematical analyses). The purpose is to robustly take into account human uncertainty in attributing evaluations, collecting and synthetizing judgments attributed by various decision makers, and dealing with large sets of those subjective elements.

The cotutelle of the doctoral thesis, which leads to the achievement of a double PhD degree was conducted during a period of traineeship at the UPV during the first year of the doctorate, within the UNIPA Erasmus+ programme for PhD students.

The research activity has been developed by spending around the same periods of time in the two universities, specifically, at the *Dipartimento dell'Innovazione Industriale e Digitale* (DIID) of the UNIPA, and at the *Instituto Universitario de Matemática Multidisciplinar* (IMM) of the UPV.

Moreover, part of this doctoral thesis was developed within a second Erasmus+ traineeship, this time promoted by the UPV, during a three-month period spent at the *Energy and Design of Environments Department* (EDEn) of the University of Bath (UK). This traineeship was useful to learn elements of mathematical programming and parallel computing to manage high-memory-demanding complex problems.

These issues were further expanded and practically applied during a final visiting period at *IngeniousWare GmbH* in Karlsruhe, Germany, whose core business consists in creating innovative software solutions for companies and professionals. During this period, several aspects of a multi-criteria decision-making method were programmed, and a website was developed to provide worldwide companies with a friendly support framework for their decision-making processes by taking into account numerous factors.

Table 1 presents the phases of research which were formally planned and performed during a specific academic year or throughout the duration of the doctorate.

Table 1. Development of the research activity

Phase of research		Academic Year		
		2015 2016	**2016 2017**	**2017 2018**
1	Literature review and definition of objectives	✓	✓	✓
2	Course attendance in the UNIPA doctoral programme	✓		✓
3	Course attendance in the UPV doctoral programme		✓	
4	Reliability analysis of complex systems	✓		
5	Mathematical analysis of MCDM methods to support maintenance	✓	✓	✓
6	Maintenance monitoring through performance indicators			✓
7	Development of real case studies	✓	✓	✓
8	Result formalisation by building scientific products	✓	✓	✓

Objectives

The objectives of this doctoral thesis, collected in the Table below, are structured as:

- general objective (or main goal of the research),
- intermediate objectives (related to the various phases of the research), and
- specific objectives (directly linked with the intermediate objectives).

Table 2. Definition of objectives

General objective	
Proposing the use of MCDM methods as a strategic tool to support maintenance management of complex systems	
Intermediate objectives	**Specific objectives**
• Carrying out a detailed study of literature contributions broadly focused on the themes of maintenance management and MCDM methods.	• Collecting a wide number of opinions and procedures related to the application of MCDM methods in the field of interest.
	• Studying and undertaking the specific steps to apply MCDM methods by comparing approaches proposed by various authors.
	• Evaluating the state of the art to propose possible answers to cutting-edge issues and innovative approaches to various real problems.
	• Detecting the possible presence of research gaps in the existing literature to define new directions of study and integrate the use of MCDM methods within the context of maintenance management.
• Carrying out mathematical analyses on the framework of the AHP technique from different perspectives to support decision making processes.	• Exploiting expert single or team-based judgments about the mutual importance of maintenance-based aspects.
	• Improving judgment consistency by mathematically manipulating matrices of pairwise comparison judgments.
	• Taking into account the vagueness characterising human judgment, often expressed by means of linguistic variables, through the support of fuzzy concepts.
	• Managing missing and incomplete information due to uncertainty by decision makers in formulating their opinions using graph theory.
	• Estimating uncertain expert judgments through probability theory.

	• Examining clustering techniques to deal with large set of elements related to decision making problems that could be grouped into clusters.
	• Developing a new website that proposes an AHP-based tool for professionals and/or firms to help make the management of their generic decision-making processes easier.
• Focusing on other MCDM methods considered as helpful to support maintenance decisions, and prepare hands-on case studies.	• Selecting the best option(s) among various possibilities, representing the best trade-off among the various considered criteria.
	• Ranking alternatives to solve maintenance decision making problems.
	• Integrating multi-criteria and multi-objective perspectives to rank solutions belonging to a Pareto front.
• Analysing a wide range of real complex production and/or service delivery systems.	• Elaborating real-world case studies to evaluate the effectiveness of MCDM methods in maintenance and then prove the usefulness of the proposed approach.
• Leading reliability analyses of complex systems by means of advanced qualitative/quantitative techniques.	• Selecting the main parameters and functions involved in such kinds of analyses.
	• Analysing relations among components of complex systems in terms of reliability configurations.
	• Estimating reliability and availability of complex systems, drivers for implementing suitable maintenance activities aimed at increasing system functionality.
	• Taking into account the importance of human factors in maintenance.
	• Applying techniques of human reliability analysis aimed at quantitatively evaluating the risk of human error.
	• Evaluating the degree of interdependency existing among the considered elements to identify those most influencing the others.

• Implementing maintenance interventions and monitoring the level of quality of the choices undertaken through the support of MCDM methods.	• Deciding about the scheduling of maintenance interventions and the implementation of suitable maintenance policies by seeking to optimise costs and production.
	• Integrating maintenance management with the innovative blockchain technology to optimise the process of control of system states.
	• Analysing useful key performance indicators in the maintenance field.
	• Selecting a set of suitable indicators among the plethora existing in the literature.

Methodologies

The main hypothesis of this research consists in providing analysts or maintenance experts with effective tools to improve the organisation of various maintenance activities. In this way, it is possible to offer innovative perspectives through the dissemination of results and propose solutions to companies operating in various sectors. The proposed research offers a scientific contribution to an issue – maintenance management – considered of great importance in the literature since industries now compete in a global market by optimising the organisation of their processes. The main role is taken on by complex systems, and managing their maintenance means globally improving operational conditions and production. The possibility of pursuing this kind of optimisation can be real through the use of MCDM methods (a wide-open field of research currently discussed in the developing literature). MCDM methods are thus the main methodological elements of this thesis.

MCDM methods are particularly useful in supporting various kinds of decision problems (Nikas *et al.*, 2018; Certa *et al*., 2013b; 2015; Carpitella *et al*., 2018c; 2018d) and, as expressed by Kumar *et al*. (2017), their crucial role is widely recognised. Mulliner *et al*. (2016) recommend these methods for successful outcomes. Various evaluation criteria, sometimes conflicting with each other, must be considered for making sound decisions. These authors consider the support given by MCDM methods as valuable and capable of managing both qualitative and quantitative aspects when an evaluation concerning a set of alternatives is required. Moreover, a strategic integration among various MCDM methods aims to exploit their strengths and make the results of analyses more reliable. This kind of integration is supported in the literature (Zanakis *et al*., 1998), and applied in several operational contexts (Mousavi-Nasab and Sotoudeh-Anvai, 2017; Løken, 2007; Wang *et al*., 2016; Certa *et al*., 2013a).

However, the process of maintenance management does not involve merely the selection of the most suitable maintenance policy or the implementation of intervention scheduling. Additionally, the phase of monitoring must be an essential part of the process and carefully conducted to confirm the quality of the choices made. Effective control enables modifying or adjusting the implemented solutions if they do not guarantee good performance. The monitoring process for maintenance management has been approached based on suitable maintenance key performance indicators (KPIs), especially referring to the following three clusters of aspects: economic; technical; and organisational. However, since the related literature presents a plethora of indicators, it is necessary to select the most representative. This aspect has been tackled again with the help of MCDM methods.

Finally, novel developments explicitly designed to ease application in complex problems (including such features as uncertain judgment and large size) have been produced within this research. In the case of AHP, the linearisation technique developed by Benítez *et al.* (2011a) is used to elaborate on: i) estimation of missing judgments making use of graph theory (Benítez *et al.*, 2018a); ii) treatment of uncertain judgments using probability theory (Benítez *et al.*, 2017); and iii) clustering techniques to reduce the size of problems with too many options for reasonable human judgment ability (Benítez *et al.*, 2018b). These aspects ae treated within specific sections of this thesis.

Thesis organization

The present doctoral thesis is organised as follows.

After this introduction, part I explores the application of MCDM methods to manage the various aspects considered within the present doctoral thesis. The methods analysed in part I are AHP, two variants of ELECTRE, and TOPSIS. Regarding AHP, also applied in its fuzzy version, new results are given that address uncertainty-based and large size-based features. Moreover, practical case studies have been developed by underlining both the effectiveness of these methods in supporting maintenance strategies and advancements made in the existing literature.

Reliability analysis and maintenance monitoring are developed in Part II. After selecting the most significant parameters and defining some of the most relevant reliability configurations for complex systems, advanced techniques of reliability analysis such as

FMEA and FMECA are practically applied to a real-world case study, according to the related standard. Moreover, a proposal to overcome some drawbacks of the traditional risk priority number (RPN) calculation is implemented using an MCDM-based approach. Part II also underlines the importance of the human factor, an aspect that is common to all the themes and methods that have been the object of study so far and play a key role in maintenance. To deal with this issue, the topic of human reliability analysis and some of the relative techniques are considered in a real case study.

Part II also gives special attention to predictive maintenance policies, implemented by means of surveillance systems (typically composed of sensors) to monitor wear on critical components. Regarding this kind of maintenance policy, it is proposed to link its implementation with the prompt action of maintenance crews using blockchain technology, which is helpful in recording the related interactions, managing data, and information flow. Moreover, the use of appropriate KPIs is discussed for leading the monitoring process and continuously increasing the level of technological innovation.

Lastly, conclusions and various proposals for possible future developments of this doctoral work are proposed.

Closing the document there are two lists of references: namely, the list of general references used within the thesis, and the list of the scientific production developed during the elaboration of this doctoral dissertation, integrated by published papers in well reputed scientific journals and contributions to highly ranked international congresses.

PART I

DECISION-SUPPORT MODELS FOR COMPLEX SYSTEM MAINTENANCE

Maintenance management of complex systems is a function of utmost importance in industry (Lopes *et al.*, 2016). The literature supports the evidence that attention has to be paid to all the phases of the process of maintenance management. Furthermore, having the implementation of maintenance activities a direct impact on complex system performance, it has to perfectly respond to system features, after having conducted an in-depth reliability analysis.

The strong relationship existing among maintenance, security and availability of systems is unarguable, and a structured decision-making approach is very useful when working in this field. The field of maintenance management of complex systems may be solidly supported by MCDM methods because of the ability of these methods to consider a wide variety of qualitative factors that play an important role in this special operational field.

Specifically, the present Part I of the thesis will focus on such techniques as the Analytic Hierarchy Process (AHP) (Saaty, 1977; 1980; 1994), the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) (Hwang and Yoon, 1981), and the ELimination Et Choix Traduisant la REalité (ELECTRE) (Figueira *et al.*, 2005). Additionally, applications to real-world case studies are conducted, including feedback from experts, whose judgments have been collected and interpreted. Multi-criteria decision methods have also been combined with techniques of multi-objective mathematical programming, aimed at modelling operational constraints characterizing the problem under analysis.

This part is organised as follows. Chapter 1 presents the AHP methodology, the linearization technique to improve consistency, and a fuzzy extension of AHP; emphasis is always placed on practical applications within the research field of interest. Afterwards, various kinds of mathematical analyses are applied for managing uncertainty affecting evaluations, and related results are also presented in terms of real-world case studies. In particular, chapter 2, which present new aspects developed within this thesis, considers graph and probability theories within the framework of the AHP to deal with uncertainty enabling to estimate missing and unclear judgments; besides, clustering techniques are applied to group large sets of elements, thus helping simplify some complex problems. Chapter 3 describes the work carried out during two international traineeships – the first one in UK and the second one in Germany – related to the implementation of a website proposing an AHP tool to individuals and companies as support for their decision-making processes. Chapter 4 deals with other MCDM method applications, namely ELECTRE I, ELECTRE III, and TOPSIS, aimed at managing maintenance of complex systems or problems. This section also shows the possibility of integrating multi-objective and multi-criteria approaches to select, with relation to a set of evaluation criteria, the alternative representing the best trade-off among the optimal solutions belonging to the Pareto front resulting from a multi-objective optimisation problem.

Chapter 1

The AHP for maintenance management

In the literature (Homenda *et al.*, 2016; Liu *et al.*, 2016) a decision-maker (DM) is defined as an actor or stakeholder that takes and influences decisions with his/her own evaluation of arguments and his/her own personal and professional background. Among the wide number of MCDM methods existing in the literature (Sipahi and Timor, 2010), the most popular (Petruni *et al.*, 2017; Kolahi *et al.*, 2018; Szulecka and Monges Zalazar, 2017; Aşchilean *et al.*, 2017) is the AHP technique, developed by Saaty (1977; 1980; 2000, 2008c) on the basis of the concept of pairwise comparisons between pair of elements (Saaty, 2008a), namely criteria or alternatives. The AHP easily carries out a ranking of decision alternatives (Chen *et al.*, 2014) and enables to calculate the vector of weights of the involved elements on the basis of those pairwise comparisons.

The AHP has also been deeply investigated with relation to consensus aspects in decision groups (Blagojevic *et al.*, 2016; Certa *et al.*, 2015; Delgado-Galván *et al.*, 2014). As asserted by Vargas *et al.* (2017), the AHP is particularly suitable for group decision-making scenarios. Certa *et al.* (2013b) develop a case study in which a team of experts is involved to select the best maintenance plan focused on a multi-component system. Cheng *et al.* (2016) analyse the issue of group decision making by highlighting the lack of exhaustiveness in traditional models to characterise dynamics in forming judgments. With this perspective, the authors consider the possibility of modelling the process of dynamic spreading of opinions on the basis of the "opinion acceptability" factor. Chen and Tsai (2016) develop a new multi-attribute decision-making method by proposing the combination of operators based on the geometric mean and eventually demonstrating the robustness of this method. Also, according to Zhang (2016b), preference relations could not respect the properties of reciprocity, especially if expressed by a decision-making group.

Several aspects have been deeply investigated in the AHP context. In this chapter we focus on techniques aimed at improving consistency (Benítez *et al.*, 2011a; 2012a; 2014a) of stakeholders' judgement and considering feedbacks from the experts (Benítez *et al.*, 2011b). Many authors (Massanet *et al.*, 2016; Xu *et al.*, 2016; Zhang, 2016a) believe the lack of consistency as generally due to the fact that decision makers, when expressing their judgments through preference relations, often make errors in the very formulation of their opinions. The literature presents several works of research aimed at increasing consistency of judgments (Pandeya and Kumar, 2016; Wang and Tong, 2016), what represents a relevant common factor of the AHP-based applications. As underlined by Karanik *et al.* (2016), this aspect is fundamental to apply the AHP method in a reliable way. The authors deal with the difficulty in making consistent an inconsistent matrix. Certa *et al.* (2015) apply the AHP method by involving a team of experts expressing judgments about the efficacy of an academy training course for graduate people. The authors underline the primary role of consistency both for individual and group decisions. Berrittella *et al.* (2008) measure consistency of judgments within a decision-making group through the measure proposed by Saaty (1980; 2000).

The AHP has been successfully applied in many fields and problems (Saaty, 1994; Partovi, 2006; Melon *et al.*, 2008; Huang *et al.*, 2008), especially to support industrial processes as, for instance, shown by Lolli *et al.* (2017) in the manufacturing field, and by Seiti *et al.* (2017) in the production field. Given the possibility of integrating the AHP with other techniques (Ho, 2008; Ortiz-Barrios *et al.*, 2017), a plethora of applications is discussed in the literature. Just to get a glimpse, Vaidya and Kumar (2006) present a wide literature review related to the AHP technique. They collect a sample of 150 papers on AHP and classify the applications into the following contexts: selection, evaluation, benefit-cost analysis, allocations, planning and development, priority and ranking, decision making, forecasting, and quality function development (QFD). There's a plethora of materials in the literature about the AHP and its applications. Next, the specific aspects needed for this thesis are addressed.

Starting from the study of the existing literature, the application of the AHP for solving complex real problems must be supported by sound mathematical foundations aimed at increasing consistency of human judgments given by experts and synthetized in pairwise comparisons matrices (Saaty, 2003; Benítez *et al.*, 2012a; Stewart, 2001). This is indeed a key point of the AHP, since the quality of decision directly depends on the consistency of the judgments (Bulut, 2012; Hillerman *et al.*, 2017).

With the objective of having a good understanding of how AHP can practically support maintenance management of complex systems, the linearisation process (Benítez *et al.*, 2011a) related to mathematical manipulation of pairwise comparison matrices (Meyer, 2001; Benítez *et al.*, 2013; 2011b; 2012b) is presented as the mathematical base to treat the AHP issues addressed in the thesis.

To note, it is impossible to achieve a complete degree of consistency when expressing judgments, due to the lack of human thinking. For this reason, tools aimed at increasing consistency (Finan and Hurley, 1997; Franek and Kresta, 2014; Wang and Chen, 2008; Aznar and Guijarro, 2008) are necessary. Moreover, the Fuzzy Analytic Hierarchy Process (FAHP), that is the fuzzy evolution of the AHP, has been proposed as a way to manage situations affected by uncertainty using linguistic variables.

1.1. Making decisions by collecting opinions from maintenance experts

As already underlined, the AHP represents a suitable tool for making decisions through the concept of pairwise comparison judgments. Its application enables convergence to a shared choice among various decision makers who have to express their preference judgments on the elements (criteria, sub-criteria and alternatives) under comparison.

The AHP decomposes the decision problem into sets of elements, according to several common characteristics and levels that correspond to the common characteristics of the elements. The first step to apply the AHP technique consists thus in breaking down the problem and representing it by means of a hierarchical structure (Saaty and Vargas, 1994).

The topmost level of this structure is the "focus" of the problem or main goal; the intermediate levels correspond to criteria (C_1, C_2, …, C_n) and sub-criteria that the upper level criteria may have, while the lowest level contains the decision alternatives (A_1, A_2, …, A_m). If each element of each level depends on all the elements of the upper level, then the hierarchy is complete; otherwise, it is considered incomplete.

The following figure shows a typical graphical example of a complete hierarchical structure representing the decomposition of a generic complex decision-making problem considering four criteria and five alternatives.

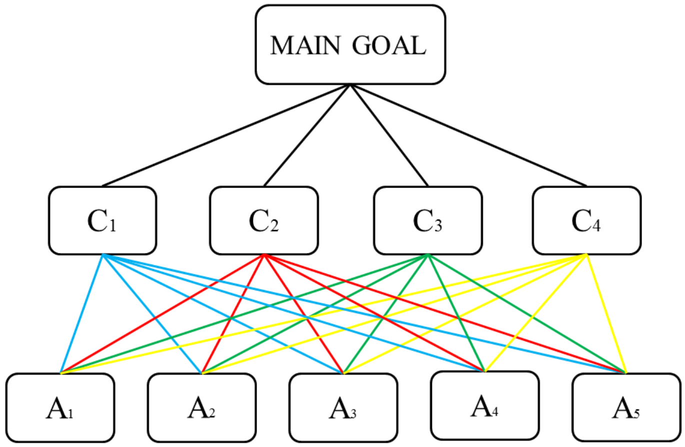


Figure 1.1. Example of an AHP hierarchical structure

The elements of each level are pairwise compared with respect to a specific element in the immediate upper level by means of grades and numerical values from one of the various scales available in the literature, among which the most used is the nine-point Saaty scale (Saaty, 1977).

Table 1.1. Saaty scale

Numerical values	**Pairwise comparisons**
1	Equal importance of two elements
3	Moderate importance of one element over another
5	Strong importance of one element over another
7	Very strong importance of one element over another
9	Extreme importance of one element over another
2, 4, 6, 8	Intermediate values
Reciprocals	Used for inverse comparisons
Decimal values	Used to express intermediate importance

Performing such a comparison for a set of n elements yields an $n \times n$ matrix $A = (a_{ij})$, known as pairwise comparison matrix (PCM), whose (positive) entries must adhere to two important properties, namely, $a_{ii} = 1$ (homogeneity) and $a_{ji} = 1/a_{ij}$ (reciprocity), $i, j = 1, \dots, n$. Such a (positive) matrix is said reciprocal. In fact, homogeneity derives from reciprocity, since for $i = j$, $a_{ij}a_{ji} = 1$, using $a_{ij} > 0$, gives $a_{ii} = 1$. However, it is customary, for the sake of clarity, to present these properties separately.

The problem for reciprocal matrix A becomes one of producing for the n elements (criteria or alternatives) under comparison, a set of numerical values $w_1, \dots, w_n$ reflecting the priorities of the compared elements according to the elicited judgments. If all judgments are completely consistent, the relations between weights w_i and judgments a_{ij} are simply given by $w_i/w_j = a_{ij}$ $(i, j = 1, \dots, n)$, and matrix A is then said to be consistent.

Theorem 1 (Benítez *et al.*, 2012a) provides equivalent conditions for a reciprocal matrix A to be consistent. Firstly, some notations are provided. It will be assumed that n-dimensional real vectors are column vectors. The superscript $^{\mathrm{T}}$ denotes the matrix transposition. For a given $n \times m$ matrix A, let us write $[A]_{ij}$ its (i, j) entry. The mapping between $n \times m$ positive matrices defined by $[J(A)]_{ij} = 1/[A]_{ij}$ will play an important role in the sequel.

Theorem 1. Let $A = (a_{ij})$ be an $n \times n$ positive matrix. The following statements are equivalent.

(i) There exists a positive n-vector $\mathbf{x}$ such that $A = J(\mathbf{x})\mathbf{x}^{\mathrm{T}}$.

(ii) There exists a positive vector, $\mathbf{w} = [w_1, \dots, w_n]^{\mathrm{T}}$, such that $a_{ij} = w_i/w_j$, for $i, j = 1, \dots, n$.

(iii) $a_{ij}a_{jk} = a_{ik}$ holds for all $i, j, k = 1, \dots, n$.

Note that (ii) implies reciprocity since $a_{ij}a_{ji} = (w_i/w_j)\,(w_i/w_j) = 1$. As a result, consistency implies reciprocity, while the reciprocal statement is, in general, not true. It is easy to find reciprocal matrices, of order $n > 2$, which are not consistent.

For a consistent PCM, the leading eigenvalue (which is easily proven to be equal to n) and its corresponding (Perron or principal) eigenvector provide information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector (Saaty 2008). Vector $\mathbf{w}$ in (ii) is not unique, however, it is an eigenvector corresponding to the eigenvalue n, whose associated eigenspace has dimension one. Thus, $\mathbf{w}$ may be taken as any of the normalized columns of A. From condition (i), A has rank one. As any consistent matrix has rank one (Benítez *et al.*, 2012a), any of its normalized rows and, in particular, the normalized vector of the geometric means of the rows, also provides the priority vector (also note that eigenvalues different than n vanish).

However, some degree of inconsistency is always expected, because of the natural lack of consistency of human judgment, and, as a result, in general, the reciprocal PCM A is not consistent. As shown in (Saaty, 2003) the eigenvector is necessary for obtaining priorities. The hypothesis that the estimates of these values are small perturbations of the ‘‘right’’ values guarantees a small perturbation of the eigenvalues (see, e.g. Stewart, 2001). For non-consistent matrices, one has to solve the problem known as eigenvalue problem, that is $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the priority vector.

The AHP theory developed by Saaty provides a measure of the inconsistency in each set of judgments. The consistency of the judgmental matrix can be determined by means of the so-called consistency ratio (CR), defined as:

$$CR = CI/RI; \tag{1.1}$$

where CI is called the consistency index, and RI is the random index.

For matrices of order n, CI is defined as:

$$CI = \frac{\lambda_{\max} - n}{n-1}; \tag{1.2}$$

interpreted as the average of the other (all except $\lambda_{\max}$) eigenvalues.

Furthermore, Saaty (2000) provided average consistencies (RI values) of randomly generated matrices (see Table 1.2).

Table 1.2. Random index values

n	2	3	4	5	6	7	8	9	10
RI	0	0.52	0.89	1.11	1.25	1.35	1.4	1.45	1.49

In general, a CR value of 0.1 or less implies acceptable consistency (observe that $CR = 0$ is equivalent to matrix consistency). Such a threshold is usually taken as 0.08 for matrices of size four and 0.05 for matrices of size three (also observe that reciprocal matrices of order 2 are consistent). If the CR value is greater than these thresholds, the judgments may not be reliable and should be reconsidered. Judgment modifications can be performed either using an improving consistency tool or asking for a new elicitation.

Regarding the number of elements that can be simultaneously compared, Saaty (1980) argues that to maintain a reasonable consistency of pairwise comparisons, the number of considered factors should be less than or equal to nine, however it will be demonstrated in the following (see Chapter 2, section 2.3) that this number can be higher.

Moreover, when a group of differently weighted decision makers is involved, it will be necessary to aggregate experts’ opinions to eventually produce a final consensus priority vector. There are various aggregation procedures for obtaining a group priority vector supporting a decision-making process, as stressed, for example, by Blagojevic *et al.* (2016). In general, two different methods can be applied to obtain such an aggregated result, namely the aggregation of individual judgments (AIJ) and the aggregation of individual priorities (AIP). In the case of AIJ, the individual comparison matrices are merged into one, so that the group normally becomes a 'new individual'; in contrast, in the AIP technique, individuals act with different value systems, producing alternative individual priorities (Forman and Peniwati, 1998) that are eventually merged into one priority vector.

Additionally, it is suggested to aggregate either judgments or priorities of different experts by means of the (weighted) geometrical mean (Delgado-Galván *et al.*, 2014), since it assures, in the case of AIJ, the reciprocity of the aggregated pairwise comparison judgments.

A numerical example of an AHP application (Carpitella *et al.*, 2017a) involving a group of three decision makers is presented next. Let's consider the same problem represented by means of the hierarchical structure of Figure 1.1. The main goal consists in obtaining the ranking of five decision alternatives under the evaluation of four criteria.

A team of three decision makers, D_1, D_2 and D_3, is involved in calculating the vector of criteria weights. In particular, the experts are assumed to have different weights in the decision-making process, respectively 40%, 35% and 25%.

The numerical evaluations translating comparisons between pairs of criteria are given in the following table, with the related consistency values.

Table 1.3. Criteria evaluations issued by the decision makers

D_1	**C_1**	**C_2**	**C_3**	**C_4**	***CR***
C_1	1	5	4	1	0.0724
C_2	1/5	1	3	1/5	
C_3	¼	1/3	1	1/5	
C_4	1	5	5	1	

D_2	**C_1**	**C_2**	**C_3**	**C_4**	***CR***
C_1	1	3	3	1	0.0394
C_2	1/3	1	2	1/5	
C_3	1/3	1/2	1	1/4	
C_4	1	5	4	1	

D_3	C_1	C_2	C_3	C_4	CR
C_1	1	1/3	1/6	1/4	0.0495
C_2	3	1	1/3	2	
C_3	6	3	1	3	
C_4	4	1/2	1/3	1	

The pairwise comparison judgements are aggregated into a single matrix (AIJ) by means of the weighted geometrical mean; the criteria weights (Perron vector) are also given in % in Table 1.4.

Table 1.4. Aggregated matrix and criteria weights

	C_1	C_2	C_3	C_4	Weights
C_1	1	2.125	1.634	0.707	**28.70%**
C_2	0.471	1	1.503	0.356	**16.43%**
C_3	0.612	0.665	1	0.426	**14.92%**
C_4	1.414	2.812	2.350	1	**39.95%**

Table 1.5 presents the (consensus) alternatives' evaluations related to the considered criteria. The last two columns, respectively, give the local priorities and the values of consistency ratios CR. In particular, the judgments' consistency is verified, because all the CR values do not surpass the threshold of 0.1.

Table 1.5. Evaluation of alternatives with respect to criteria, local priorities and CR

C_1	A_1	A_2	A_3	A_4	A_5	Local priorities	CR
A_1	1	5	4	2	1/3	0.2383	0.0748
A_2	1/5	1	1	1/3	1/6	0.0579	
A_3	1/4	1	1	1/3	1/3	0.0755	
A_4	1/2	3	3	1	1/6	0.1387	
A_5	3	6	3	6	1	0.4896	

C_2	A_1	A_2	A_3	A_4	A_5	Local priorities	CR
A_1	1	1/3	½	1/4	7	0.1162	
A_2	3	1	2	1	9	0.3231	
A_3	2	1/2	1	2	7	0.2620	0.0708
A_4	4	1	½	1	9	0.2710	
A_5	1/7	1/9	1/7	1/9	1	0.0278	

C_3	A_1	A_2	A_3	A_4	A_5	Local priorities	CR
A_1	1	6	5	4	1/4	0.2672	
A_2	1/6	1	½	1/2	1/7	0.0461	
A_3	1/5	2	1	3	1/5	0.1011	0.0838
A_4	1/4	2	1/3	1	1/6	0.0640	
A_5	4	7	5	6	1	0.5217	

C_4	A_1	A_2	A_3	A_4	A_5	Local priorities	CR
A_1	1	7	3	7	1/5	0.2449	
A_2	1/7	1	¼	1	1/7	0.0430	
A_3	1/3	4	1	3	1/5	0.1143	0.0809
A_4	1/7	1	1/3	1	1/7	0.0448	
A_5	5	7	5	7	1	0.5530	

On the basis of the criteria weights, the global score for each alternative has been obtained by applying the weighted sum of their local priorities. The vector **s** of scores is obtained as multiplication of matrix LP whose columns are the vectors of local priorities and vector **w** of criteria weights:

$$\mathbf{s} = LP \cdot \mathbf{w}. \tag{1.3}$$

In the analysed case, the scores of the five alternatives are then calculated as:

$$\mathbf{s} = \begin{bmatrix} 0.2383 & 0.1162 & 0.2672 & 0.2449 \\ 0.0579 & 0.3231 & 0.0461 & 0.0430 \\ 0.0755 & 0.2620 & 0.1011 & 0.1143 \\ 0.1387 & 0.2710 & 0.0640 & 0.0448 \\ 0.4896 & 0.0278 & 0.5217 & 0.5530 \end{bmatrix} \begin{bmatrix} 0.2870 \\ 0.1643 \\ 0.1492 \\ 0.3995 \end{bmatrix}. \tag{1.4}$$

The final ranking is shown in Table 1.6.

Table 1.6. Ranking of alternatives

Position	**Alternative**	**Score**
1st	A_5	0.4438
2nd	A_1	0.2252
3rd	A_3	0.1255
4th	A_4	0.1118
5th	A_2	0.0938

The AHP methodology may be complemented with techniques for consistency improvement (Benítez *et al.*, 2011a; 2012a; 2013), including the necessary feedback with the expert(s) (Benítez *et al.*, 2011b). The next paragraph succinctly presents some basic elements.

1.2. Linearisation: a technique to improve consistency of judgments

When consistency for a matrix is not satisfactory, it is necessary to improve it. Finan and Hurley (1997) stated that additional artificial manipulation to increase consistency will improve, on average, the reliability of the analysis. So, if consistency is unacceptable, it should be improved. The literature proposes several ways to improve consistency, mostly based on optimization. After discussing the nonlinear nature of some of those methods, the linearisation technique implemented by Benítez *et al.* (2011a) is herein described as an orthogonal projection mechanism over a certain vector space, and a simple formula implementing this technique for reciprocal matrices is also presented.

Broadly speaking, optimization methods to improve consistency are based in Saaty's proposal (2003) based on perturbation theory to modify the most inconsistent judgments in the matrix while adhering to some constraints. Thus, in general, slight modifications of the comparison matrix entries are sought, while trying to maintain the main properties of the comparison matrix, namely homogeneity, reciprocity and consistency. Aznar and Guijarro (2008) propose a goal programming method using relative deviations to force changes in the comparisons' values so that the target values differ as little as possible from the original values, while approximately taking homogeneity into account and preserving reciprocity and consistency. A slight modification of this method that reduces the number of decision variables and constraints is used by Delgado-Galván *et al.* (2010). However, Benítez *et al.* (2012a) provide an optimization process that has the important advantage of depending only on n decisional variables – the number of compared elements. The solution makes use of the previously presented Theorem 1 to solve the problem. Find

$$\min\{\|A - J(\mathbf{y})\mathbf{y}^T\|_F : \mathbf{y} \text{ a real n} - \text{vector}\}; \tag{1.5}$$

where $\mathbf{y}$ is a positive n-vector, $\|\cdot\|_F$ is the matrix Frobenius norm. Note that $\|A\|_F = [\text{tr}(A^{\text{T}}A)]^{1/2}$, where $\text{tr}(\cdot)$ stands for the trace of a matrix, and the 1-norm for n-vectors is $\|\mathbf{y}\|_1 = |y_1| + \cdots + |y_n|$.

To solve this optimization problem one may use, for instance, Lagrangian multipliers. However, this is still a non-linear optimization problem. The linearisation technique transforms the consistency improvement problem into a linear one.

The linearisation technique provides the closest consistent matrix to a given non-consistent matrix by using an orthogonal projection on a certain linear space. This method provides a direct way of achieving consistency, in contrast with methods relying on non-linear optimization, which are iterative by nature.

The inspiration for the linearisation methods comes from the following example. Let us consider the PC matrices:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 2 \\ 1/2 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 8 \\ 1/8 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 1 & 9 \\ 1/9 & 1 \end{bmatrix}. \tag{1.6}$$

These four matrices are reciprocal (consistent, since they are 2×2) and correspond to four situations in which one must choose the best choice between two elements.

Using the Frobenius norm we have:

$$\|A_1 - B_1\|_F = 1.118, \|A_2 - B_2\|_F = 1.001. \tag{1.7}$$

This, somehow, shows that A_1 and B_1 resemble in a similar way as A_2 and B_2 do. This is not intuitive, since A_1 reflects the fact that both criteria are equally important, while B_1 gives double importance to the first over the second. In contrast, A_2 and B_2 show similar importance for both criteria.

From an intuitive viewpoint the distance between A_1 and B_1 should be much higher than the distance between A_2 and B_2. Taking the example further, to allocate 100 euro between two competing options, the allocations obtained from these four matrices would be the ones given in the following table.

Table 1.7. Allocation for various PC matrices

Amount allocated to the…	A_1	A_2	B_1	B_2
… first option	50	66.3	88.9	90
… second option	50	33.3	11.1	10

It is possible to observe that the change from $\boldsymbol{A_1}$ to $\boldsymbol{B_1}$ allocations is much higher than from $\boldsymbol{A_2}$ to $\boldsymbol{B_2}$, as intuitively expected. So, just the Frobenius norm is not a good way to measure distances between matrices for this problem.

However, by taking logarithms one can observe a more reasonable jump between 1 and 2 than between 8 and 9, since $\mathbf{log(2) - log(1) \approx 0.693}$ and $\mathbf{log(8) - log(9) \approx 0.118}$. To conclude the example: we can conjecture that a new way to measure distances $\mathbf{d}(\boldsymbol{A_1}, \boldsymbol{B_1})$ between the pairwise comparison matrices $\boldsymbol{A_1}$ and $\boldsymbol{B_1}$ could be computed as:

$$\mathrm{d}(A_1, B_1) = \|\mathrm{LOG}(A_1) - \mathrm{LOG}(B_1)\|_F; \quad (1.8)$$

where $\mathrm{LOG}(\cdot)$ is the matrix operator that associates the entries of a positive matrix with their logarithms, $[\mathrm{LOG(X)}]_{i,j} = \log([\mathrm{X}_{ij}])$. With this definition $\mathrm{d}(A_1, B_1) \approx 0.98$, while $\mathrm{d}(A_2, B_2) \approx 0.17$, which confirms the intuition that the distance between A_1 and B_1 should be much higher than the one between A_2 and B_2.

Another advantage of using the map LOG is that methods of linear algebra can be used to improve consistency by solving an approximation problem in terms of the orthogonal projection, p_n, of $\mathrm{LOG}(A)$ onto a linear subspace of the space of $n \times n$ matrices.

To complete the details, let us define this subspace as $\{\mathrm{LOG}(A): A$ is a positive $n \times n$ consistent matrix$\}$, which can be proved to be an $(n - 1)$-dimensional linear subspace of the space of $n \times n$ matrices. The complete process of getting consistency through linearisation can be described by the following scheme:

$$A \xrightarrow{\mathrm{LOG}} \mathrm{LOG}(A) \overset{p_n}{\rightarrow} p_n\big(\mathrm{LOG}(A)\big) \overset{\mathrm{E}}{\rightarrow} A^C; \quad (1.9)$$

producing A^C, the closest consistent matrix to A; the operator E is defined for any matrix, X, by $[\mathrm{E}(X)]_{i,j} = \exp([X]_{i,j})$.

The first and third steps are trivial. So, only calculating $p_n\big(\mathrm{LOG}(A)\big)$, the orthogonal projection of $\mathrm{LOG}(A)$ onto the mentioned linear space is needed. The solution is guaranteed through standard linear algebra (Meyer, 2001), this projection being given by the formula in the following result (Benítez *et al.*, 2011a).

Theorem 2. Let A be a positive $n \times n$ matrix. Then

$$p_n\big(\mathrm{LOG}(A)\big) = \frac{1}{2n} \sum_{i=1}^{n-1} \frac{\mathrm{tr}(\mathrm{LOG}(A)^{\mathrm{T}} \phi_n(\mathbf{y}_i))}{\|\mathbf{y}_i\|^2} \phi_n(\mathbf{y}_i),$$

where $\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of the orthogonal complement to span $\{\mathbf{1}_n\}$, where $\mathbf{1}_n$ is the n-vector $[1, \dots, 1]^{\mathrm{T}}$, and $\phi_n(\mathbf{x})$ is the map that associates to a vector $\mathbf{x} = (x_1, \cdots, x_n)^T$ the matrix whose (i, j) entry is $x_i - x_j$.

The following result (Benítez *et al.*, 2011a) shows that the calculations involved in the Fourier expansion given by the previous expression of the closest matrix are straightforward.

Theorem 3. Let $(Y_n)_{n=2}^{\infty}$ be the sequence of matrices defined as follows:

$$Y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, Y_{n+1} = \begin{bmatrix} Y_n & 1_n \\ 0 & -n \end{bmatrix}, n \geq 2.$$

Then for every $n \geq 2$, the columns of Y_n are orthogonal and belong to the orthogonal complement of span $\{\mathbf{1}_n\}$.

The formulas given in theorems 2 and 3 are extremely simple and require few operations. The implementation of these formulas in conventional spreadsheets is really simple. However, matrix environments such as Matlab or Octave are deemed more appropriate. The related Matlab or Octave codes implementing these formulas are given in the next box. Note that $\phi_n(\mathbf{x})$ is easily calculated as $\phi_n(\mathbf{x}) = \mathbf{x}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{x}^T$.

```
function y = y(n)
% This function calculates matrices Y in Theorem 3
y = zeros(n,n-1);
for k=1:n-1
      y(1:k,k)=ones(k,1);
      y(k+1,k)=-k;
end

function matrix = matrix(A)
% Calculates sought consistent matrix in Theorem 2
B = log(A);
[n m] = size(A);
Y = y(n);
X = zeros(size(A));
for i = 1:n-1
      phiy = Y(:,i)*ones(1,n)-ones(n,1)*Y(:,i)';
      factor = trace(B'*phiy)/(i+i^2);
      X = X + factor*phiy;
end
X = X/(2*n);
matrix = exp(X);
```

In (Benítez *et al.*, 2013) the authors show that, for reciprocal matrices, the projection can be obtained with great simplicity by using the formula:

$$p_n\big(\mathrm{LOG}(A)\big) = \frac{1}{n}[(\mathrm{LOG}(A)U_n) - (\mathrm{LOG}(A)U_n)^{\mathrm{T}}]; \tag{1.10}$$

where $U_n = \mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$.

Since this formula involves just sums, computational efficiency is guaranteed and integration in any AHP-based decision support system, including conventional spreadsheets, is straightforward. Let us consider the following reciprocal matrix as an example:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1/3 & 1 & 2 \\ 1 & 1/2 & 1 \end{bmatrix}. \tag{1.11}$$

By using the Saaty's criterion of consistency we get that $CI/RI \simeq 0.35 \gg 0.1$. According to this criterion, the consistency of matrix A is not acceptable. We then modify matrix A to improve its consistency. We can apply this last formula to get:

$$p_3\big(\mathrm{L}(A)\big) = \begin{bmatrix} 0 & 0.501 & 0.597 \\ -0.501 & 0 & 0.096 \\ -0.597 & -0.096 & 0 \end{bmatrix}. \tag{1.12}$$

Now the consistent matrix closest to A is:

$$\mathrm{E}\left(p_3\big(\mathrm{LOG}(A)\big)\right) = \begin{bmatrix} 1 & 1.65 & 1.82 \\ 0.61 & 1 & 1.10 \\ 0.55 & 0.91 & 1 \end{bmatrix}. \tag{1.13}$$

However, maybe the experts can consider that this new matrix does not represent their opinions. For example, $[A]_{1,2} = 3 > 1 = [A]_{1,3}$, while in the new matrix, the entry $(1,2)$ is lower than the entry $(1,3)$.

It is important to note that matrix $\mathrm{E}(p_n(\mathrm{LOG}(A)))$ should *a priori* be never the last matrix, unless it definitely reflects the thoughts of the expert. As this may be not the case, some balance between expert judgments and synthetic consistency obtained by the strict application of the linearisation method given by Theorems 2 and 3 must be achieved.

Therefore, after computing the closest consistent matrix given by the linearisation method, it is necessary for the expert to be able to modify the new matrix.

However, as explained next, the process of getting the new priority vector is simple and there is no need to start calculations from scratch.

Let us suppose that a reciprocal matrix A is obtained from a stakeholder judgment and the consistent matrix $A^C = \mathrm{E}(p_n(\mathrm{LOG}(A)))$ closest to A is calculated. Perhaps this actor does not completely agree that the entries in A^C fully represent his or her judgment. If the stakeholder decides to change, let us say, the entry a_{rs} in A^C, which compares criteria r and s (where $r \neq s$ and $1 \leq r, s \leq n$), another reciprocal, probably non-consistent, matrix B is obtained. The entries of B compared with the entries of A^C verify: $b_{rs} = \alpha\, a_{rs}$ and $b_{sr} = \alpha^{-1} a_{sr}$ for some $\alpha > 0$, and $b_{ij} = a_{ij}$ in the remaining entries.

Let us denote by $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ the standard basis of $\mathbf{R}^n$. The relationship between matrices $\mathrm{LOG}(A)$ and $\mathrm{LOG}(B)$ is:

$$\mathrm{LOG}(B) = \mathrm{LOG}(A) + \log\alpha(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}). \tag{1.14}$$

Using now the linearity of the projection one can state the following result (Benítez *et al.*, 2011b).

Theorem 4. Let A be a positive $n \times n$ matrix and let A^C be the consistent matrix closest to A. If B is defined by the previous formula (1.14), and B^C is the consistent matrix closest to B, then:

$$B^c = A^c \otimes \mathrm{E}\left(\frac{\log\alpha}{n}(\mathbf{e}_r - \mathbf{e}_s)\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n(\mathbf{e}_r - \mathbf{e}_s)^{\mathrm{T}}\right).$$

$\otimes$ is the Hadamard (component-wise) matrix product.

Following a feedback procedure, by repeating both steps, a matrix representing a reasonable trade-off between consistency and expert opinion will be eventually obtained.

1.3. Weighting elements in a fuzzy environment

As previously underlined, the AHP easily carries out ranking of decision alternatives. The method is able to calculate the vector of weights of involved criteria on the basis of the opinions formulated by a single expert or a group of decision makers. Regarding opinion formulation, Cid-López *et al.* (2016) emphasize that linguistic terms provide experts with an element of support in expressing judgments. The authors develop a linguistic multi-criteria decision-making model in the Information and Communication Technology (ICT) field. Gupta and Mohanty (2016) express decision makers' judgments through linguistic terms with the aim to better represent real situations. They implement a new methodology to collect and aggregate various points of view within a given time horizon. Jin *et al.* (2016) also consider that experts prefer to give their opinions by means of linguistic variables. Ekel *et al.* (2016) propose aggregating information coming from different sources by referring to practical decision-making problems developed in the field of power engineering.

However, Büyüközkan *et al.* (2011) observe the inability of the AHP in correctly reflecting the vagueness of the decision makers' perception and thus, in many real cases, linguistic assessment is necessary, instead of just crisp numbers, to represent the real situation.

The fuzzy set theory represents a valid support to manage uncertainty affecting human judgments. Indeed, linguistic variables can be expressed through fuzzy numbers rather than crisp values, and have associated a degree of membership, $\mu(x)$, varying between 0 and 1.

There are various types of fuzzy numbers. The most common ones are triangular fuzzy numbers (TFN) and trapezoidal fuzzy numbers (TrFN) (Zimmermann, 1985; Kubler *et al.*, 2016).

A generic TFN $\tilde{n}$ is defined by three numerical values, a, b and c, respectively called the lower, the medium and the upper value of the fuzzy number, where $a \leq b \leq c$. A generic TrFN $\tilde{m}$ is defined by four numerical values, d, e, f and g, respectively called the lower, the two medium and the upper values of $\tilde{m}$; here $d \leq e \leq f \leq g$:

$$\tilde{n} = (a,\ b,\ c); \tag{1.15}$$

$$\tilde{m} = (d,\ e,\ f,\ g). \tag{1.16}$$

Their membership functions, $\mu_{\tilde{n}}(x)$ and $\mu_{\tilde{m}}(x)$, are expressed as follows and represented in Figure 1.2.

$$\mu_{\tilde{n}}(x) = \begin{cases} \frac{x-a}{b-a} & \text{for } a \leq x \leq b \\ \frac{x-c}{c-b} & \text{for } b \leq x \leq c \\ 0 & \text{otherwise} \end{cases}; \tag{1.17}$$

$$\mu_{\tilde{m}}(x) = \begin{cases} \frac{x-d}{e-d} & \text{for } d \leq x \leq e \\ 1 & \text{for } e \leq x \leq f \\ \frac{x-g}{f-g} & \text{for } f < x < g \\ 0 & \text{otherwise} \end{cases}. \tag{1.18}$$

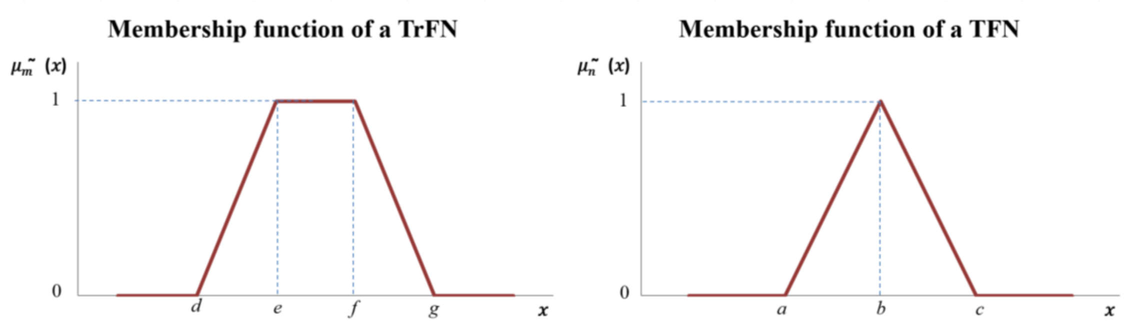


Figure 1.2. Membership functions for TrFNs and TFNs

Algebraic operations can be accomplished among fuzzy numbers. For instance, considering two TFNs $\tilde{n}_1$ and $\tilde{n}_2$:

$$\tilde{n}_1 \oplus \tilde{n}_2 = (a_1 + a_2,\ b_1 + b_2,\ c_1 + c_2); \tag{1.19}$$

$$\tilde{n}_1 \odot \tilde{n}_2 = (a_1 \times a_2,\ b_1 \times b_2,\ c_1 \times c_2); \tag{1.20}$$

$$\tilde{n}_1{}^{-1} = (\tfrac{1}{c_1},\ \tfrac{1}{b_1},\ \tfrac{1}{a_1}). \tag{1.21}$$

A further development of the AHP method consists in a fuzzy extension. The FAHP, firstly proposed by Van Laarhoven and Pedrycz (1983), takes advantage of the fuzzy set theory (Zadeh, 1965; Klir and Yuan, 1995) for adequately managing uncertainty often characterizing judgments expressed by experts.

Kubler *et al.* (2016) present a wide review of many applications of FAHP. The authors analyse 190 papers published between the years 2004 and 2016 and classify them on the basis of their main features and application fields. According to the survey carried out by the authors, the FAHP is commonly used in the literature for calculating criteria weights and then it is combined with other MCDM methods (Kutlu and Ekmekçioğlu, 2012; Büyüközkan and Çifçi, 2012; Kaya and Kahraman, 2011; Ka, 2011), to rank the alternatives under evaluation.

The FAHP method is considered helpful in risk evaluations, as shown by Hsu *et al.* (2016). They deal with the risk assessment related to operational safety for dangerous goods in airfreights, presenting a case study taking place in Taiwan. However, as assumed by Wang and Chen (2008), this method presents some weaknesses with relation to the number of pairwise judgments to express respect to a particular criterion, that is the difficulty to have consistent pairwise comparisons matrices.

The application of the FAHP method can be summarized through this three following steps (Durán and Aguiló, 2006):

- building the hierarchy structure that represents the problem under analysis;
- collecting fuzzy pairwise comparisons with relation to decision alternatives with respect to each evaluation criterion;
- ranking alternatives to prioritize them or to select the best one.

Concerning the collection of fuzzy pairwise comparisons, the purpose is to build a fuzzy pairwise comparison matrix (FPCM), $\tilde{X}$. In this matrix, the linguistic judgments attributed by the expert(s) correspond to fuzzy numbers. For example, given a number n of criteria (or alternatives) to be pairwise compared, one can build the square, reciprocal matrix:

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \cdots & \tilde{x}_{nn} \end{bmatrix}; \tag{1.22}$$

in which a generic element $\tilde{x}_{ij}$ expresses the degree of preference of criterion (or alternative) i with respect to criterion (or alternative) j with a certain level of uncertainty. Moreover, reciprocity implies that for each pairwise comparison judgment $\tilde{x}_{ij} = (x_1,\ x_2,\ x_3)$ one has that $\tilde{x}_{ji} = (\frac{1}{x_3},\ \frac{1}{x_2},\ \frac{1}{x_1})$. Once made up the FPCM $\tilde{X}$, several approaches are tackled in the literature to obtain the relative weights. In particular, Chang (1996) proposes to derive crisp weights from the matrix, by exploiting the extent analysis method.

As said before, linguistic variables are used by an analyst (or decision maker) to express pairwise comparisons about importance between two elements. In particular, these variables refer to the fuzzy version of the Saaty scale (1977), shown in Figure 1.3, and can be stated as: equal (EQ), moderate (M), strong (S), very strong (VS) and extreme (EX) importance. The associated TFNs are respectively: (1,1,2), (2,3,4), (4,5,6), (6,7,8) and (8,9,9). The TFNs (1,2,3), (3,4,5), (5,6,7) and (7,8,9) correspond to the intermediate values.

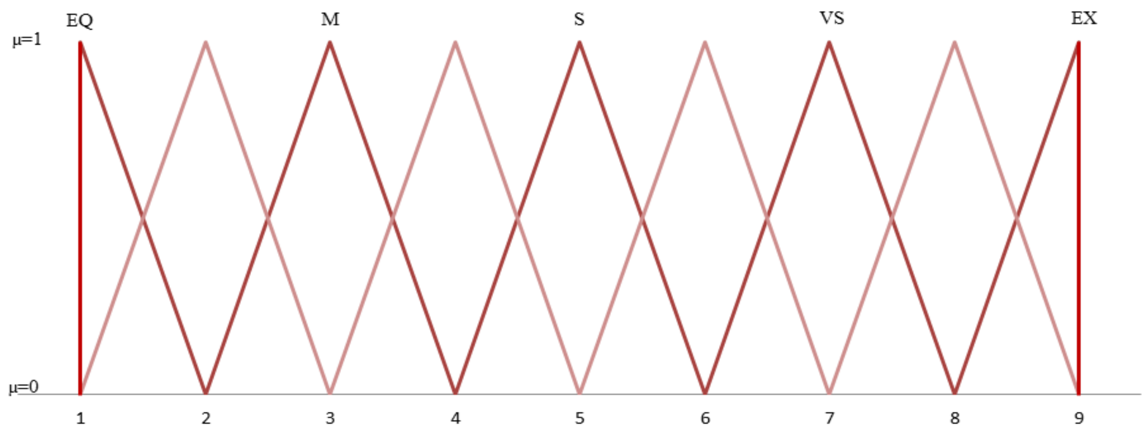


Figure 1.3. Fuzzy version of the Saaty scale

The value of fuzzy synthetic extent with relation to the i^{th} element of matrix $\tilde{X}$ can be calculated as follows:

$$S_i = \sum_{j=1}^{m} \tilde{x}_{ij} \odot \left[\sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{x}_{ij}\right]^{-1}; \quad (1.23)$$

being, in our case, $n = m$ because the FPCM $\tilde{X}$ is a square matrix.

Let us consider two fuzzy pairwise comparisons, e.g. two TFNs noted as $\tilde{n}_1 = (a_1, b_1, c_1)$ and $\tilde{n}_2 = (a_2, b_2, c_2)$. We are interested in establishing the degree of possibility that $\tilde{n}_1 \geq \tilde{n}_2$, defined as (Kutlu and Ekmekçioğlu, 2012):

$$V(\tilde{n}_1 \geq \tilde{n}_2) = \mu(x^*) = \begin{cases} 1 & \text{if } b_1 \geq b_2 \\ 0 & \text{if } a_2 \geq c_1 \\ \frac{a_2 - c_1}{(b_1 - c_1) - (b_2 - a_2)} & \text{otherwise} \end{cases}; \quad (1.24)$$

where x^* is the ordinate of the highest intersection point P between $\mu_{\tilde{n}_1}$ and $\mu_{\tilde{n}_2}$, as we can observe in Figure 1.4. In order to compare the two TFNs $\tilde{n}_1$ and $\tilde{n}_2$, it is necessary to calculate both values $V(\tilde{n}_1 \geq \tilde{n}_2)$ and $V(\tilde{n}_2 \geq \tilde{n}_1)$.

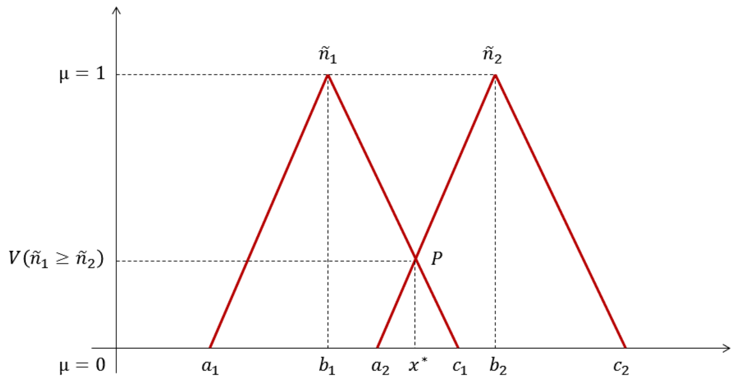


Figure 1.4. Representation of the degree of possibility that $\tilde{n}_1 \geq \tilde{n}_2$

Furthermore, the possibility degree that a fuzzy number $\tilde{n}$ is greater than k fuzzy numbers $\tilde{n}_i (i = 1 \dots k)$ corresponds to:

$$V(\tilde{n} \geq \tilde{n}_1, \tilde{n}_2, \dots, \tilde{n}_k) = V[(\tilde{n} \geq \tilde{n}_1) \text{ and } (\tilde{n} \geq \tilde{n}_2) \text{ and} \dots \text{and} (\tilde{n} \geq \tilde{n}_k)] =$$
$$= \min V(\tilde{n} \geq \tilde{n}_i), \quad i = 1 \dots k. \tag{1.25}$$

Then, it is possible to link each criterion (or alternative) X_i considered in the FPCM $\tilde{X}$ to the relative value of fuzzy synthetic extent and to define:

$$x^{*\prime}(X_i) = \min V(S_i \geq S_k); \tag{1.26}$$

for $k = 1 \dots n, k \neq i$. The vector of crisp and not normalised weights is lastly given by:

$$W' = \left(x^{*\prime}(X_1), x^{*\prime}(X_2), \dots, x^{*\prime}(X_n)\right)^T. \tag{1.27}$$

Let us observe that these obtained weights have to be normalized with respect to their total so that their sum equals one; the vector of normalized crisp weights will be:

$$W = \left(x^*(X_1), x^*(X_2), \dots, x^*(X_n)\right)^T. \tag{1.28}$$

The last operation consists in checking the CR of the collected comparisons. To such an aim, each fuzzy value $\tilde{x}_{ij}$ of the matrix is defuzzified and transformed into a crisp value x_{ij} by means of the graded mean integration approach:

$$G(\tilde{x}_{ij}) = x_{ij} = \frac{x_1 + 4x_2 + x_3}{6}. \tag{1.29}$$

After having defuzzified each value of the matrix, consistency can be easily verified with the proper threshold (Saaty, 1977).

The following case study presents an application of the FAHP to support a decision-making problem of image mining processing analyses aimed at improving maintenance of water networks (Carpitella *et al.*, 2018a). The hierarchical structure representing the whole problem is given in the figure below, even if, at this stage, we are just interested in calculating the vector of criteria weights. The case study will be completed in Chapter 4, Section 4.1.

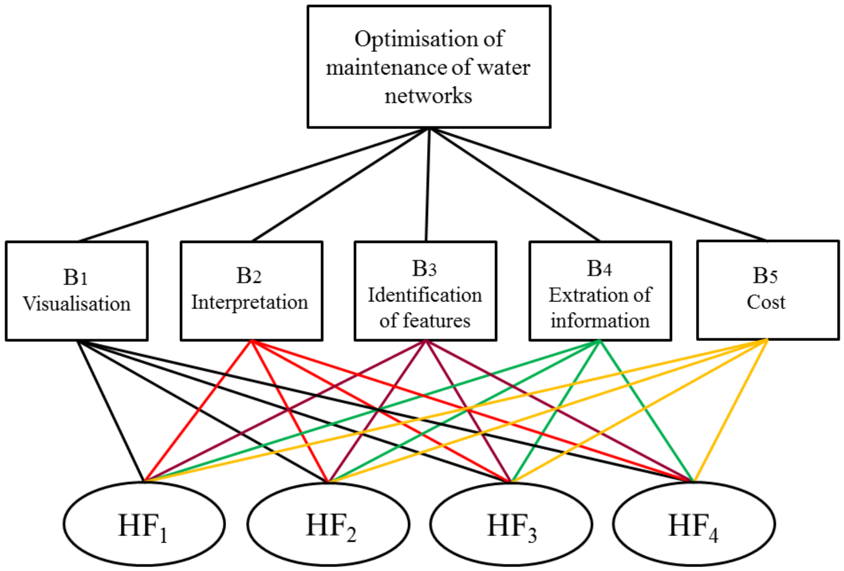


Figure 1.5. Hierarchical structure representing the problem

The five criteria (B_1, B_2, B_3, B_4, B_5) on the basis of which the four alternatives (HF_1, HF_2, HF_3, HF_4,) - that are GPR (ground penetrating radar) images resulting from four different techniques of data processing analyses - will be eventually evaluated are: visualization, interpretation, identification of features, extraction of information and affordability. The FAHP technique effectively enables to manage uncertainty of evaluations. In particular, an expert in the field of GPR analysis was asked to draw up the FPCM matrix (five first columns in the following table) to pairwise compare criteria and attribute judgments through the linguistic variables previously defined.

Table 1.8. Fuzzy Pairwise Comparison Matrix

$\tilde{X}$	**B1**	**B2**	**B3**	**B4**	**B5**	**weights**
B1	(1, 1, 2)	(1, 2, 3)	(3, 4, 5)	(1, 2, 3)	(1, 2, 3)	0.2934
B2	$(\frac{1}{3}, \frac{1}{2}, 1)$	(1, 1, 2)	(2, 3, 4)	(1, 2, 3)	$(\frac{1}{3}, \frac{1}{2}, 1)$	0.2226
B3	$(\frac{1}{5}, \frac{1}{4}, \frac{1}{3})$	$(\frac{1}{4}, \frac{1}{3}, \frac{1}{2})$	(1, 1, 2)	(1, 2, 3)	$(\frac{1}{3}, \frac{1}{2}, 1)$	0.1380
B4	$(\frac{1}{3}, \frac{1}{2}, 1)$	$(\frac{1}{3}, \frac{1}{2}, 1)$	$(\frac{1}{3}, \frac{1}{2}, 1)$	(1, 1, 2)	$(\frac{1}{3}, \frac{1}{2}, 1)$	0.1109
B5	$(\frac{1}{3}, \frac{1}{2}, 1)$	(1, 2, 3)	(1, 2, 3)	(1, 2, 3)	(1, 1, 2)	0.2351

The values of fuzzy synthetic extent for each criterion can be calculated by using the corresponding formula:

$$S_1 = (7.00, 11.00, 16.00) \odot \left(\frac{1}{51.82}, \frac{1}{32.58}, \frac{1}{21.12}\right) = (0.14, 0.34, 0.76); \quad (1.30)$$

$$S_2 = (4.67, 7.00, 11.00) \odot \left(\frac{1}{51.82}, \frac{1}{32.58}, \frac{1}{21.12}\right) = (0.09, 0.21, 0.52); \quad (1.31)$$

$$S_3 = (2.78, 4.08, 6.83) \odot \left(\frac{1}{51.82}, \frac{1}{32.58}, \frac{1}{21.12}\right) = (0.05, 0.13, 0.32); \quad (1.32)$$

$$S_4 = (2.33, 3.00, 6.00) \odot \left(\frac{1}{51.82}, \frac{1}{32.58}, \frac{1}{21.12}\right) = (0.05, 0.09, 0.28); \quad (1.33)$$

$$S_5 = (4.33, 7.50, 12.00) \odot \left(\frac{1}{51.82}, \frac{1}{32.58}, \frac{1}{21.12}\right) = (0.08, 0.23, 0.57). \quad (1.34)$$

These values have to be compared and the relative degrees of possibility, summarized in the following table, are calculated by means of the proper formula.

Table 1.9. Degrees of possibility to compare values of fuzzy synthetic extent

$V(S_1 \geq S_2)$	1	$V(S_2 \geq S_1)$	0.7586	$V(S_3 \geq S_1)$	0.4704	$V(S_4 \geq S_1)$	0.3778	$V(S_5 \geq S_1)$	0.8013
$V(S_1 \geq S_3)$	1	$V(S_2 \geq S_3)$	1	$V(S_3 \geq S_2)$	0.7229	$V(S_4 \geq S_2)$	0.6126	$V(S_5 \geq S_2)$	1
$V(S_1 \geq S_4)$	1	$V(S_2 \geq S_4)$	1	$V(S_3 \geq S_4)$	1	$V(S_4 \geq S_3)$	0.8739	$V(S_5 \geq S_3)$	1
$V(S_1 \geq S_5)$	1	$V(S_2 \geq S_5)$	0.9661	$V(S_3 \geq S_5)$	0.6959	$V(S_4 \geq S_5)$	0.5922	$V(S_5 \geq S_4)$	1

The components $W' = \left(x^{*\prime}(B_1), x^{*\prime}(B_2), x^{*\prime}(B_3), x^{*\prime}(B_4), x^{*\prime}(B_5)\right)^T$ of the non-normalized vector of weights are calculated as follows:

$$x^{*\prime}(\mathrm{B}_1) = V(S_1 \geq S_2, S_3, S_4, S_5) = \min(1; 1; 1; 1) = 1; \quad (1.35)$$

$$x^{*\prime}(\mathrm{B}_2) = V(S_2 \geq S_1, S_3, S_4, S_5) = \min(0.7586; 1; 1; 0.9661) = 0.7586; \quad (1.36)$$

$$x^{*\prime}(\mathrm{B}_3) = V(S_3 \geq S_1, S_2, S_4, S_5) = \min(0.4704; 0.7229; 1; 0.6959) = 0.4704; \quad (1.37)$$

$$x^{*\prime}(\mathrm{B}_4) = V(S_4 \geq S_1, S_2, S_3, S_5) = \min(0.3778; 0.6126; 0.8739; 0.5922) = 0.3778; \quad (1.38)$$

$$x^{*\prime}(\mathrm{B}_5) = V(S_5 \geq S_1, S_2, S_3, S_4) = \min(0.8013; 1; 1; 1) = 0.8013. \quad (1.39)$$

The obtained normalized vector of weights (given in the last column of table 2.8) is: $\mathbf{w} = (0.2934, 0.2226, 0.1380, 0.1109, 0.2351)^{\mathrm{T}}$. The last step consists in verifying consistency upon having defuzzified the FPCM by means of the graded mean integration approach. In our case, consistency is perfectly acceptable, the CR index being equal to 0.0639.

Chapter 2

New developments

This chapter presents new developments associated to the development of this thesis related to the integration of mathematical analyses within the framework of the AHP.

In general, in scientific problem-solving practice, especially in the case of complex problems (Hennissen *et al.*, 2017, La Rocca *et al.*, 2017), the classical separation between objective and subjective, quantifiable and qualitative, tangible and intangible, rational and emotional, etc., more than frequently does not occur. On the contrary, the neutrality of values demanded by theory-driven science is an unrealistic hypothesis (Söderbaum, 1999; Kaufmann, 1999). This is especially clear in decision-making where subjective factors, not easily quantifiable, intangible, etc., such as aspects associated with human behaviour, which are the key players in decision-making processes, are especially present. Actually, it is absolutely essential to incorporate the human factor into theoretical models (Dittrich, 2016), especially when facing high complexity problems (De Tombe, 2001). It is imperative that the chosen problem-solving methodology combines the quantifiable, objective, tangible and rational of classical science with the qualitative, subjective, intangible and emotional of human behaviour (Kunz, 2015). Only in this way will it be possible to achieve an objective treatment of the subjectivity (Keeney, 1992), so that an adequate rational treatment of the emotionality can be achieved.

To contribute to narrow that abovementioned gap between theory and application, several mathematical tools within the AHP context are herein developed, since sound mathematical foundation of multi-criteria decision methods is fundamental because enables to effectively apply methods tailored to specific operational contexts.

A specific focus is dedicated to explore the treatment of incomplete or uncertain judgements that could characterize pairwise comparisons matrices (Benítez *et al.*, 2014b; 2015), because not always the involved experts can express their judgements in evaluating specific aspects or criteria. Indeed, some judgments could miss, that is, incomplete judgments may characterize comparison matrices. It represents an issue currently discussed in the literature, occurring when decision makers prefer not to express their opinion concerning the importance of an element with respect to another (Benítez *et al.*, 2014b; 2015). To contemplate this possibility, missing information is treated within the framework of consistency by using ideas and results from the graph theory (Bapat, 2011).

Moreover, elements of the probability theory (Klir and Yuan, 1995) are integrated into the study with the purpose of managing uncertainty of data and expert evaluations. Indeed, the probability theory appears to be a useful support to take into consideration cases in which decision makers have doubts when expressing their judgments. These judgments are modelled as random variables, for which the expectancy represents the most credible value, the variance the degree of uncertainty and the covariance the degree of interdependency.

Lastly, a clustering technique is studied that helps reduce the size of some complex problems making them more manageable and sometimes revealing hidden relationships between various considered elements. These elements could be then grouped into clusters on the basis of their degree of similarity to make it easier to handle the problem.

The following subsections describe in detail the perspectives from which the AHP technique has been approached within the present doctoral thesis. In particular, each analysed decision problem is presented through a real-world case study, focused on maintenance optimisation of production or service delivery systems.

2.1. Estimating missing judgments through graph theory

Decision-making driven by a very well-defined decision structure and integrated by objective elements may be relatively easy. However, when subjectivity permeates the decision-making environment things become harder. If, in addition, the decision-making context is plagued with uncertainty and/or incomplete information, decision-making may turn into a task of great complexity. As underlined by Floricel *et al.* (2016), complexity is an intrinsic factor of any field and environment. The authors approach this factor both in its structural and dynamical shape and stress the need to model complexity with the aim to better manage project planning and strategies. In fact, complexity is usually determined and impacted by the presence of uncertain or incomplete information regarding the process under analysis. Significant losses, especially in terms of costs and time (Qazi *et al.*, 2016), may derive when the main complex aspects are not correctly faced or taken into account. However, frequently, it is natural that some of the decision makers involved are not familiar enough with all the issues to make appropriate judgment elicitation, or simply some comparisons cannot be performed. There are several reasons for an actor to provide incomplete information. In particular, in (Harker, 1987) three reasons are provided, namely, reducing time to perform judgment, unwillingness to issue a certain opinion, and lack of sureness about a given opinion. A forth reason can be added, namely there is no available information to build a given comparison. In any case, it is necessary to pay special attention in managing uncertainty or absence characterising data and/or judgments of experts, because this aspect influences evaluations of decisions expressed under different criteria (Carpitella *et al.*, 2016). Specifically, it is necessary to formulate decisional models with a solid scientific basis, capable of managing the intrinsic subjective and not well-informed nature of decisions. This formulation should aim to make decisions as objective as possible, even if the decision-making process cannot be totally objective or there is a lack of information.

With this recognition, flexible decision-making methods are useful to consider a wide variety of aspects, *i.e.* various criteria and alternatives, since decision on one alternative with the best objective value is affected by various multiple, frequently conflicting criteria. The final selection of the alternative is usually made with the help of inter and intra-attribute comparisons, which may involve explicit or implicit trade-off (Huang and Yung, 1981).

In this context, Wang and Xu (2016) underline that it is rarely possible to avoid incomplete preference relations in decision making groups. For this reason, they seek the support of experts in the phase of expression of their preference through an interactive algorithm based on consistency, for evaluating the missing entries of matrices.

About the issue of incomplete information characterizing matrices in AHP applications, many authors express their opinions. Srdjevic *et al.* (2014) propose a method to fill in gaps in matrices. Starting by the knowledge of two consolidated methodologies (Harker, 1987; Van Uden, 2002) used to generate missing data in comparisons matrices, the authors propose the first-level transitive rule method. It consists in, firstly, screening matrix entries in the neighbourhood of a missing one, and, secondly, scaling and geometric averaging of screened entries to fill in the gap. By presenting numerical examples, it is shown the coherence of results. Bozóki *et al.* (2016) deal with the theme of incomplete PCMs by applying the eigenvector method (Saaty, 1977) and the logarithmic least squares method (Crawford and Williams, 1985) to obtain the relative weights. The authors address a ranking of professional tennis players over the last 40 years using an obviously incomplete history of results of matches between top tennis players. Ergu *et al.* (2016) stress the need to improve the consistency ratio of matrices related to emergency management. To such an aim, they propose a model to quickly estimate missing comparisons in an incomplete matrix by extending the geometric mean induced bias matrix method (Ergu *et al.*, 2012). The literature also proposes to estimate incomplete judgments by specially focalizing on uncertainty management. With this perspective, as emphasized in (Certa *et al.*, 2013c), Hua *et al.* (2008)

propose an innovative approach to solve multi-attribute decision making problems with incomplete information. They integrate the AHP method with the Dempster-Shafer (DS) theory of evidence (1976), using a mixed DS-AHP approach (Beynon *et al.*, 2000). This method permits to deal with uncertainty of experts and to determine preference relations among all decision alternatives by comparing their belief intervals. Dong *et al.* (2015) estimate missing preference information by carrying out a consistent recovery method. They focus on multi-criteria group decision-making problems in which preference alternatives are expressed by fuzzy triangular numbers.

Given the importance in the literature of the issue of incomplete judgments, i.e. missing entries, that could characterise AHP pairwise comparison matrices and following the line initiated by Benítez *et al.* (2015; 2014b), this section is aimed at building consistent information from an incomplete body of pairwise comparisons.

The purpose consists in studying the system obtained in Theorem 1 of Benítez *et al.* (2015) in terms of a graph related to an incomplete pairwise comparison matrix. The degree of freedom of the set of solutions is computed in terms of the number of connected components of this graph (see Theorem 3 of Appendix A). As a trivial corollary of this latter result, it is obtained that the solution to the problem is unique if and only if this graph is connected (see Corollary 2 of Appendix A); this result of uniqueness was first obtained by Bozóki *et al.* (2010). It is noteworthy that the proof of Theorem 3 of Appendix A follows a completely different approach than the given by this last paper. Furthermore, when the solution is not unique, non-singular linear systems are always obtained, in contrast with the linear systems obtained by Benítez *et al.*, (2015). More importantly, in addition to get the priority vector and level of consistency based on the known entries, the interest consists also in building the complete PCM, since optimal values of the unknown entries may be informative (Bozóki *et al.*, 2010). This step is crucial in the necessary trade-off process (between synthetic consistency and personal judgment) with the experts. The number of missing entries in a comparison matrix with elicited entries is, generally, small in practical problems (frequently reduced to one or two above the main diagonal). However, in applications like ranking tennis or chess players using incomplete tournament results, may obviously produce higher numbers of missing entries. As a result, addressing the general situation provides the completion methodology with wide generality. To show the performance of the results given in Appendix A, it is proposed the usage of a theoretical matrix with a large number of missing entries and an associated graph with two non-connected components that exhibits the claimed generality, and various other matrices corresponding to a real case of decision-making with one or two missing entries. Let us finally note that in the case of just one missing entry (and its symmetrical), the results provided in Appendix A reduce to the Van Uden's rule, which gives the solution explicitly with no need to solve any linear system of equations.

The necessary prerequisites and the development of the main technical results, including proofs of various theorems, a synthetic example and two illustrative comparisons with other methods, have been collected in Appendix A with the purpose of not distracting the expositive flow line of the chapter. Specifically, Appendix A includes:

1. problem setting;
2. some review of graph theory;
3. main results;
4. synthetic example;
5. comparison with other methods.

Based on Appendix A's contents, a case study and the solution obtained are herein presented (Benítez *et al.*, 2018a). The case study refers to an industrial layout reorganisation problem involving materials handling – specifically the reorganisation of storage space in a factory. This reorganisation seeks the best arrangement (using various criteria) for shelving to store pallets of finished products and cardboards. Moreover, a path for the transit of people and forklifts (*i.e.*, lines to transport the goods) must be defined by considering the available space inside the storage facility. The AHP technique is applied to select the best option from a set of three layout proposals (LP_1, LP_2, LP_3), evaluated on the basis of five criteria (C_1, C_2, C_3, C_4, C_5).

The considered and mutually independent criteria are: safety & security; cost; innovation; transport; and placement. The first criterion considers the aspect of safety and security at the workplace for the stakeholders of the storage facility. The second criterion refers to the cost of implementing a specific layout. The third criterion regards the innovative character of each alternative in terms of broad flexibility for enhancing the storage conditions (for example, by creating spaces for the employees to communicate and so better integrate operations). The fourth criterion is related to the movement of goods in the storage area on forklifts and managing the pedestrian areas crossed by employees and visitors inside the facility. The fifth criterion considers how a specific layout alternative may facilitate the placement of materials on shelves with the aim of distributing pallets of finished products and cardboard in various sectors of the shelves on the basis of their uses (and thus avoiding mixing materials).

The hierarchical structure of the problem is shown in Figure 2.1.

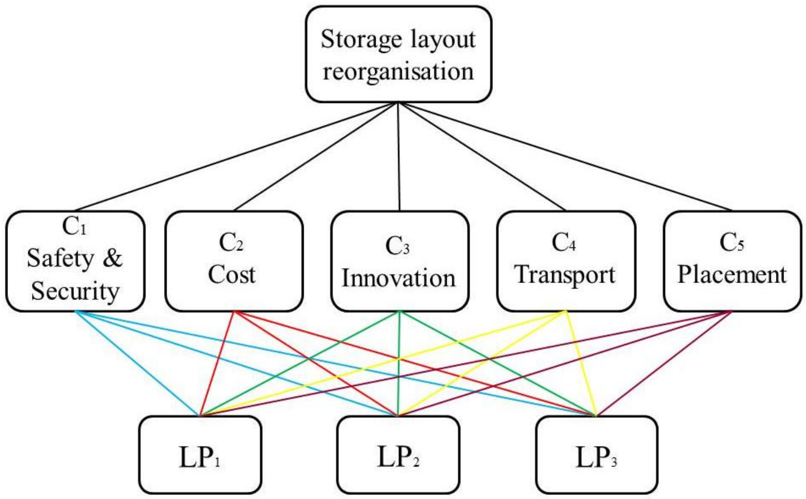


Figure 2.1. Hierarchy structure of the storage layout reorganisation problem

Figure 2.2 shows the (feasible) schemes of the three layout proposals. The shelves to be arranged are highlighted as grey blocks numbered from one to five. Others blocks represent fixed elements in the facility. The topmost parts of the plants are the production areas of the firm that communicate with the storage and so more than two shelves cannot be allocated in this area (e.g. shelves 1 and 5 in LP_2 in Figure 2.2). Observe that shelf 2 may be divided into two halves.

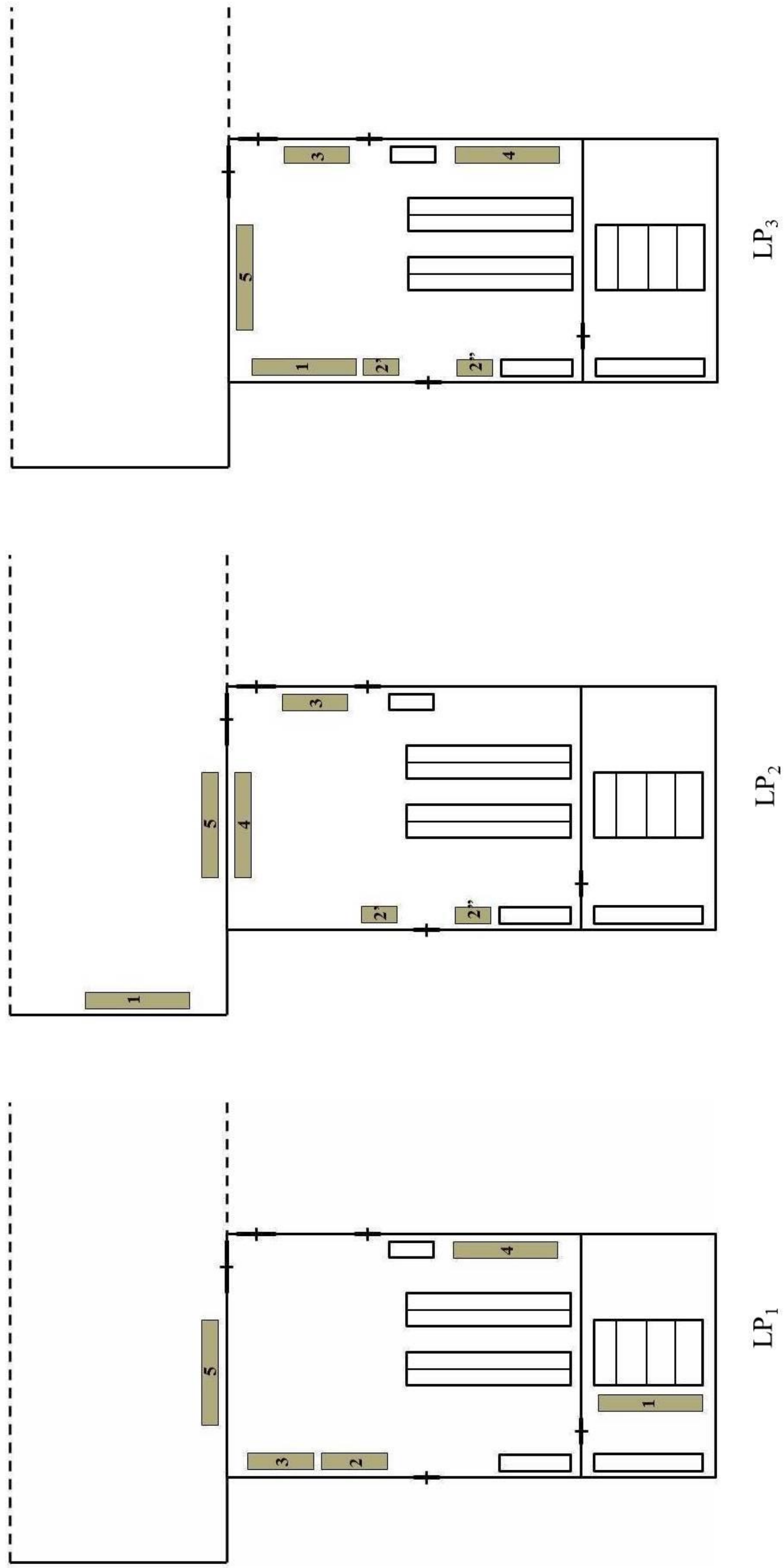


Figure 2.2. Layout proposals LP$_1$, LP$_2$ and LP$_3$

The following tables show the relative evaluations of the alternatives with respect to the criteria.

In each table, the last two columns give, respectively, the normalised local priorities (the Perron vectors of the matrices), and the consistency indices (CR). Note that all the relative judgments are consistent because none of the CR indices exceed the value of 0.05, that is the threshold for matrices of size 3×3 (Saaty, 1977).

Table 2.1. Evaluation of alternatives with respect to criteria, local priorities and CR value

C_1	**LP_1**	**LP_2**	**LP_3**	**Local Priorities**	$\boldsymbol{CR}$
LP_1	1	4	4	0.667	
LP_2	1/4	1	1	0.167	0
LP_3	1/4	1	1	0.167	

C_2	**LP_1**	**LP_2**	**LP_3**	**Local Priorities**	$\boldsymbol{CR}$
LP_1	1	1/2	1/5	0.122	
LP_2	2	1	1/3	0.23	0.00352
LP_3	5	3	1	0.648	

C_3	**LP_1**	**LP_2**	**LP_3**	**Local Priorities**	$\boldsymbol{CR}$
LP_1	1	6	6	0.75	
LP_2	1/6	1	3	0.125	0
LP_3	1/6	1/3	1	0.125	

C_4	**LP_1**	**LP_2**	**LP_3**	**Local Priorities**	$\boldsymbol{CR}$
LP_1	1	1/2	1/4	0.136	
LP_2	2	1	1/3	0.238	0.0176
LP_3	4	3	1	0.625	

C_5	LP_1	LP_2	LP_3	Local Priorities	CR
LP_1	1	2	5	0.582	0.0036
LP_2	1/2	1	3	0.309	
LP_3	1/5	1/3	1	0.109	

In addition to the calculation of the local priorities of alternatives, it is necessary to evaluate the vector of criteria weights. A decision group composed of three experts (D_1, D_2, D_3) was involved to this purpose. We will assume that the experts have the same weight in the decision process. Their roles are the following: consultant; chief of health and safety, and an employee representative. These decision-makers are involved in the management of the storage area from different – but complementary – perspectives. However, in formulating the judgements, the experts prefer not to express some evaluations. The following matrices (table 2.2) show the incomplete pairwise comparisons judgments.

Table 2.2. Evaluation of criteria with respect to experts, local priorities and CR value

D_1	C_1	C_2	C_3	C_4	C_5
C_1	1	7	1	4	5
C_2	1/7	1	1/3	1/3	*
C_3	1	3	1	4	3
C_4	1/4	3	1/4	1	2
C_5	1/5	*	1/3	1/2	1

D_2	C_1	C_2	C_3	C_4	C_5
C_1	1	5	3	3	2
C_2	1/5	1	*	2	*
C_3	1/3	*	1	3	1/2
C_4	1/3	1/2	1/3	1	1
C_5	1/2	*	2	1	1

D3	C1	C2	C3	C4	C5
C1	1	5	1	2	1
C2	1/5	1	*	1/3	*
C3	1	*	1	1/2	1/3
C4	1/2	3	2	1	1
C5	1	*	3	1	1

Since the presence of missing information often affects these kind of practical problems, the main difficulty of consistent completion regards the achievement of reliable values reflecting experts' opinions and preferences. Specifically, the experts were unwilling to give their judgements about the following pairwise comparison: C_2/C_5. In other terms, they preferred not to express any opinion comparing cost and placement. Moreover, experts D_2 and D_3 did not give their judgements about another pairwise comparison, C_2/C_3. In fact, they did not wish to express a judgement comparing cost and the pursuance of innovation. With relation to this last missing comparison, although the decision maker D_1 expressed his opinion by assigning a numerical value, he could not be totally exhaustive for evaluating the mentioned comparison. Indeed, opinions of each single decision maker need to be balanced with the others and, to such an aim, the relative missing judgments must be calculated.

It is simple to check that the graphs corresponding to these matrices have only one connected component. According to Corollary 2 (Appendix A), the completions of these matrices are unique in the sense of Theorem 1 (Appendix A). Van Uden's rule (2002) can be used for the first matrix, since only one upper-diagonal entry is unknown. The completion obtained is:

$$a_{25} = \sqrt[3]{\frac{a_{21}a_{23}a_{24}}{a_{51}a_{53}a_{54}}}. \tag{2.1}$$

The value of $\boldsymbol{\theta}$ for the second matrix is $\boldsymbol{\theta} = [0.900, -0.297, -0.099, -0.578, 0.074]^{\mathrm{T}}$. This vector gives the best completion of the second matrix: $a_{23} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_3) =$ 0.82019 and $a_{25} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_5) = 0.68980$.

For the third matrix we get $\boldsymbol{\theta} = [0.461, -1.014, -0.194, 0.220, 0.528]^{\mathrm{T}}$, $a_{23} =$ $\exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_3) = 0.44068$ and $a_{25} = \exp(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_5) = 0.21394$.

By using these values, it is possible to build the respective completions with the calculated entries in bold (shown in Table 2.3). The completed matrices were then shared with the team of decision makers, who did not show reasons to disagree with the assigned values, confirming the coherence of the found results.

Table 2.3. Completed matrices

D_1	C_1	C_2	C_3	C_4	C_5
C_1	1	7	1	4	5
C_2	1/7	1	1/3	1/3	**0.78090**
C_3	1	3	1	4	3
C_4	1/4	3	1/4	1	2
C_5	1/5	**1.28058**	1/3	1/2	1

D_2	C_1	C_2	C_3	C_4	C_5
C_1	1	5	3	3	2
C_2	1/5	1	**0.82019**	2	**0.68980**
C_3	1/3	**1.21922**	1	3	1/2
C_4	1/3	1/2	1/3	1	1
C_5	1/2	**1.44991**	2	1	1

D_3	C_1	C_2	C_3	C_4	C_5
C_1	1	5	1	2	1
C_2	1/5	1	**0.44068**	1/3	**0.21394**
C_3	1	**2.26923**	1	1/2	1/3
C_4	1/2	3	2	1	1
C_5	1	**4.67609**	3	1	1

To build a blend of these matrices the AIJ technique is used. This approach agrees with the one proposed by Guitouni and Martel (1998), since the experts in our case study act together in a complementary manner and so combining individual judgments into a group judgment is recommended. To aggregate the individual priorities into group priorities, the geometric mean method (GMM) is used. Following these observations, the blended comparison matrix of criteria is shown in the table below, in which the last column shows the priority vector.

Table 2.4. Aggregated matrix and criteria weights

	C_1	C_2	C_3	C_4	C_5	Weights
C_1	1	5.593	1.442	2.884	2.154	**38.4 %**
C_2	0.179	1	0.494	0.606	0.487	**8.43 %**
C_3	0.693	2.025	1	1.817	0.794	**20.61 %**
C_4	0.347	1.651	0.550	1	1.260	**14.82 %**
C_5	0.464	2.055	1.260	0.794	1	**17.73 %**

Once the priority vectors for criteria and alternatives have been built, we aggregate the results through the distributive method and the final ranking of layout proposals is obtained.

Table 2.5. Ranking of layout alternative

Position	Alternative	Score
1^{st}	LP_1	0.5442
2^{nd}	LP_3	0.2564
3^{rd}	LP_2	0.1993

The layout proposal LP_1 was recognised to provide the best trade-off among all considered criteria, and the involved decision group, having previously agreed concerning completed matrices, eventually backed the selection as well. In particular, the application of the graph theory supports the goodness of the solution, this method being particularly advantageous in the manufacturing field (Rao Venkata, 2013). By adopting this solution, four of the five shelves (1 to 4) are arranged into the storage area, and the fifth shelf is placed in the production area. This solution permits safe management of the available spaces and is well-balanced between the two departments. In fact, this arrangement enables an optimisation of the placement of pallets of finished products and cardboards according to the logistic strategies adopted by the organisation. At the same time, transport can be improved by establishing dedicated paths for people (employees and visitors) and forklifts (materials transport) inside the storage department. Lastly, the selected layout proposal creates a special area (box) between the two doors in the upper right side of the storage area. This box can be used for employee meetings aimed at integrating the workforce and enhancing the level of communication inside the organisation.

2.2. Probability theory applied to deal with uncertain judgment

With relation to the objective of strengthening the level of competitiveness and innovation of decision-making processes (Jabbarova, 2016), various degrees of difficulty may characterize the achievement of an effective solution. It is the case of the maintenance problem object dealt with in this section, that is based on a current research that treats the present topic (Benítez *et al.*, 2018, under review). In general, the most important problems to be resolved are often the most complex as well; and, when facing a highly complex problem, making a decision that represents the best trade-off among all the involved factors is not straightforward.

Substantial cognitive and technical skills are needed to carry out optimal evaluations (Karanik *et al.*, 2016). According to Matos (2007), one of the most common causes of such complexity derives from uncertainty and vagueness in making forecasts, or in attributing judgments concerning certain aspects of the decision to be made. The author underlines that contradictory conclusions may appear after changing methods and paradigms. As asserted by Yager and Kreinovich (1999), benefits related to a certain decision frequently depend on situations beyond our control, even when rigorous and reliable decision-making procedures are followed. Johnson *et al.* (2016) accept that decisions are not often derived from a condition when the evidence is available. In fact, decision-makers may infer the most likely solution while being ignorant about relevant features concerning the problem under analysis. Regarding this aspect, Shah *et al.* (2016) observe that the literature mainly stresses how human judgment usually tends to underestimate the probability of negative consequences, being sometimes unrealistically overoptimistic. However, the authors apply five tests to observe this phenomenon without identifying traces of bias due to a general human tendency to optimism, thus confirming the vast complexity of human cognition. Proper methodologies should support this cognition, especially in the presence of missing information. For example, Soroudi *et al.* (2017) face a problem of renewable electricity supply and highlight uncertainties due to the extremely volatile nature of wind power. In particular, they develop the Information Gap Decision Theory to properly handle unknown events. Regarding problems of multi-criteria nature, Pereira *et al.* (2015) state the absolute need to formally model uncertainty with the support of a mathematical perspective, in contrast to the traditional and deterministic approach of many multi-criteria methods. In this context, Liu *et al.* (2011) suggest undertaking decision-making problems by representing the relative attributes by means of uncertain linguistic variables in terms of fuzzy numbers. They develop a decision support method to solve practical problems with interval probabilities. Yan *et al.* (2017) undertake a probabilistic interpretation of weights by implementing a linguistic decision rule through the concepts of random preference and stochastic dominance.

More generally, the literature shows plenty of efforts towards the optimization and modelling of uncertain contexts using various probabilistic approaches (Magyar *et al.*, 2016; Giang, 2015; Narens, 2016; Yu and Mao, 2017; Zhang *et al.*, 2017 a). In a vast number of real situations and practical problems, it would be more appropriate to speak of "a probably good solution" rather than "the best solution". As stressed by Biedermann *et al.* (2017), a probabilistic approach helps unveil decision-maker uncertainty about an unknown quantity or event, even if the personal interpretation of probability cannot be avoided. In this regard, Costello and Watts (2016) develop a model to represent how people estimate conditional probabilities. Moreover, Izhakian (2017) underlines the factor of ambiguity, whose degree may be interpreted as the volatility of probability. The author proposes a model to deal with uncertain event probabilities.

The probability theory can be therefore a useful tool for estimating judgments of uncertain experts within the framework of the AHP method (Benítez *et al.*, 2018, under review). Indeed, as shown by Hughes (2009), the probability theory fundamentals perfectly fit the properties of the AHP. Some probabilistic concepts of interest in AHP are described in Appendix B, which presents the following issues:

1. the definition of a random reciprocal matrix;
2. the geometric expectation and AHP;
3. the geometric variance, the geometric covariance and AHP;
4. Chebyshev's inequalities and their applications in AHP;
5. the log-normal distribution and AHP;

On the basis of what is presented in Appendix B, a case study focused on maintenance management of an industrial water distribution system is herein presented.

This case study refers to a manufacturing firm that must decide about implementing one or more of five maintenance actions (MA_1, MA_2, MA_3, MA_4, MA_5) aimed at keeping the industrial water distribution system (IWDS) that feeds the company factories, under suitable operational conditions. Consequently, the aim is to minimize the plant shutdown risk. These actions must be prioritized for the purpose of finding a suitable trade-off between improving the plant condition, while not shouldering the simultaneous implementation of numerous interventions. The AHP technique is applied to obtain the final ranking of actions. These maintenance actions belong to the following categories of maintenance policies: preventive, corrective, and predictive. The description of the actions focused on the IWDS in relation to their policy categories is provided in Table 2.6 below.

Table 2.6. Description of the maintenance actions to be ranked

Policy	ID Alternative	Maintenance action description
Preventive	MA_1	Electric pump redundancy
Corrective	MA_2	Preliminary supply of "special parts" (such as valves, fittings, and pipes), to make eventual substitution interventions faster
	MA_3	Intensifying plant flexibility by increasing the number of disconnection points in the water network for closing those parts to be maintained, and avoiding plant shutdown
	MA_4	Creation of water storage, in case of sudden interruption of the water service
Predictive	MA_5	Implementation of a tele-surveillance system for the water feeding, to monitor parameters such as temperature, flowrate, and pressure

Those maintenance actions are evaluated by means of four criteria (C_1, C_2, C_3, C_4). The evaluation criteria considered are, respectively: security; cost; productivity; and hygiene. The first criterion refers to the plant's compliance with the regulations in force. The second criterion regards the cost for implementing an action and facing a possible plant shutdown. The third criterion is related to the fulfilment of production standards and then to the need to keep the system available. Lastly, the fourth criterion evaluates the hygienic conditions for drinking water supply to the personnel and plant sanitation.

The hierarchical structure of the problem is represented in the figure below.

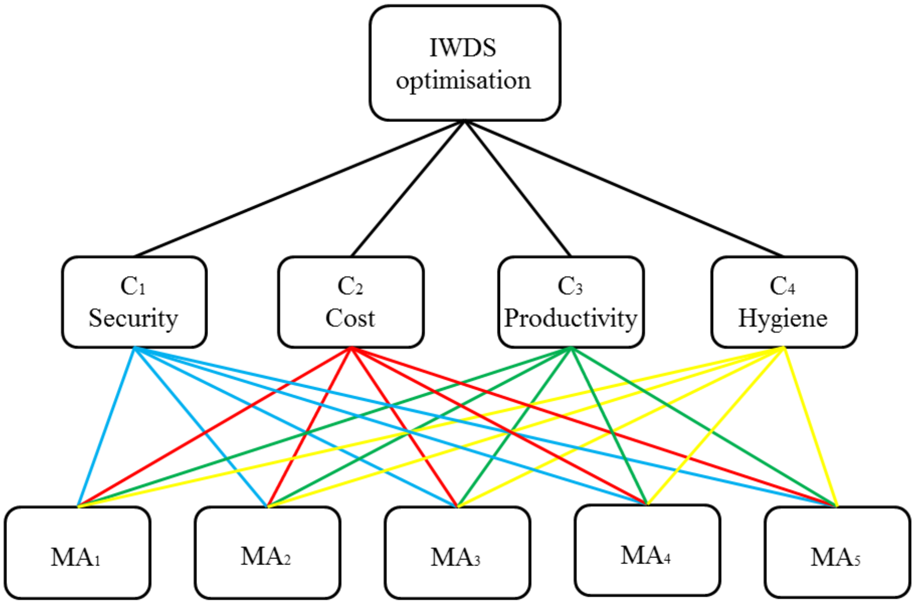


Figure 2.3. Hierarchy structure of the IWDS optimisation problem

The vector of criteria weights is obtained by involving a decision group, whose components (D_1, D_2, D_3) are assumed to have different weights in the decision process.

Table 2.7 shows the roles of each decision maker and their weights, whereas Table 2.8 reports the pairwise comparison judgments of the criteria, collected in three random reciprocal matrices. In formulating their judgements, the experts had doubts in assigning some evaluations. Particularly, experts D_1 and D_3 doubted in expressing a clear opinion about the pairwise the comparison C_1/C_4, that is to say, between security and hygiene.

Table 2.7. Roles and weights of the decision makers

Decision maker	**Role**	**Weight**
D_1	Technician	40%
D_2	Quality Manager	35%
D_3	Productivity Manager	25%

Table 2.8. Decision-makers' random reciprocal matrices of criteria evaluations

D_1	C_1	C_2	C_3	C_4
C_1	1	5	4	X_1
C_2	1/5	1	3	1/5
C_3	1/4	1/3	1	1/5
C_4	X_1^{-1}	5	5	1

D_2	C_1	C_2	C_3	C_4
C_1	1	3	3	1
C_2	1/3	1	$\boldsymbol{X_2}$	1/5
C_3	1/3	$\boldsymbol{X_2^{-1}}$	1	1/4
C_4	1	5	4	1

D_3	C_1	C_2	C_3	C_4
C_1	1	1/3	1/6	$\boldsymbol{X_3}$
C_2	3	1	1/3	2
C_3	6	3	1	3
C_4	$\boldsymbol{X_3^{-1}}$	1/2	1/3	1

Specifically, D_1 doubted between the values of 1 and 2, whereas D_3 doubted between 0.20 and 0.25. Moreover, expert D_2 doubted between the values 2 and 3 to be assigned to the pairwise comparison C_2/C_3, related to the aspects of cost and productivity. For these reasons, we consider three random reciprocal matrices, two of them with random entry a_{14} and the other with random entry a_{23}, in addition to their relative reciprocal entries a_{14}^{-1} and a_{23}^{-1}. These entries are positive random variables. Let A_i be the reciprocal matrix provided by the i^{th} expert and let X_1, X_2, and X_3 be the random variables a_{14}, a_{23}, and a_{14} for the experts D_1, D_2, and D_3, respectively. We assume that these random variables are continuous and uniformly distributed on the aforementioned intervals, specifically, $X_1 \sim U(1,2)$, $X_2 \sim U(2,3)$, and $X_3 \sim U(0.2,0.25)$. To deal with random reciprocal matrices, as explained in Appendix B, it is more appropriate to use geometric mean and variance, instead of their arithmetic counterparts. It is simple to check what is given in the following table.

Table 2.9. Geometric expectation and variance of random variables

Random variable	Geometric expectation	Geometric variance
X_1	$\mathrm{G}(X_1) \sim 1.472$	$\mathrm{Var}_g(X_1) = 0.0391$
X_2	$\mathrm{G}(X_2) \sim 2.483$	$\mathrm{Var}_g(X_2) = 0.0136$
X_3	$\mathrm{G}(X_3) \sim 0.225$	$\mathrm{Var}_g(X_3) = 0.00414$

Theorem 2 (Appendix B) is applied to calculate the geometric expectations and Theorems 3 and 4 (Appendix B) to obtain the geometric variances and covariances. Let B_i be

the closest consistent matrix to A_i and let $\mathbf{x}_i$ be a priority vector of B_i. We have that there exists $C_1 > 0$ such that:

$$\mathrm{G}(\mathbf{x}_1) = C_1 \begin{bmatrix} \sqrt[4]{1 \cdot 5 \cdot 4 \cdot G(X_1)} \\ \sqrt[4]{{}^{1}/_{5} \cdot 1 \cdot 3 \cdot {}^{1}/_{5}} \\ \sqrt[4]{{}^{1}/_{4} \cdot {}^{1}/_{3} \cdot 1 \cdot {}^{1}/_{5}} \\ \sqrt[4]{G(X_1^{-1}) \cdot 5 \cdot 5 \cdot 1} \end{bmatrix} \simeq C_1 \begin{bmatrix} 2.329 \\ 0.5886 \\ 0.3593 \\ 2.030 \end{bmatrix}; \tag{2.2}$$

and analogously,

$$\mathrm{G}(\mathbf{x}_2) \simeq C_2 [1.732\ 0.6379\ 0.4280\ 2.115]^{\mathrm{T}}; \tag{2.3}$$

and

$$\mathrm{G}(\mathbf{x}_3) \simeq C_3 [0.3342\ 1.189\ 2.711\ 0.9282]^{\mathrm{T}}; \tag{2.4}$$

for some $C_2, C_3 > 0$.

Furthermore, for each decision maker, it is possible to obtain the following matrices G_i representing, respectively, the geometric means for the entries of the consistent matrices that are closer to the given reciprocal random matrices A_i. In other words, the entry (r, s) of G_i is the geometric expectation of the entry (r, s) of B_i.

$$D_1 \to \mathrm{G}_1 = \mathrm{G}(\mathbf{x}_1) J(\mathrm{G}(\mathbf{x}_1))^{\mathrm{T}} \begin{bmatrix} 1 & 3.9573 & 6.4824 & 1.1472 \\ 0.2527 & 1 & 1.6381 & 0.2899 \\ 0.1543 & 0.6105 & 1 & 0.1770 \\ 0.8717 & 3.4494 & 5.6504 & 1 \end{bmatrix}; \tag{2.5}$$

$$D_2 \to \mathrm{G}_2 = \mathrm{G}(\mathbf{x}_2) J(\mathrm{G}(\mathbf{x}_2))^{\mathrm{T}} \begin{bmatrix} 1 & 2.7154 & 4.0468 & 0.8190 \\ 0.3683 & 1 & 1.4903 & 0.3016 \\ 0.2471 & 0.6710 & 1 & 0.2024 \\ 1.2209 & 3.3153 & 4.9509 & 1 \end{bmatrix}; \tag{2.6}$$

$$D_3 \to \mathrm{G}_3 = \mathrm{G}(\mathbf{x}_3) J(\mathrm{G}(\mathbf{x}_3))^{\mathrm{T}} \begin{bmatrix} 1 & 0.2810 & 0.1233 & 0.3601 \\ 3.5584 & 1 & 0.4387 & 1.2812 \\ 8.1114 & 2.2796 & 1 & 2.9205 \\ 2.7774 & 0.7805 & 0.3424 & 1 \end{bmatrix}; \tag{2.7}$$

The resemblance of the figures in these matrices with the respective original judgments is very noticeable. The matrices of variances, one for each decision maker, are computed by denoting $\omega_i = \mathrm{Var}_g(X_i)$. For the decision maker D_1:

$$\mathrm{Var}_g(\mathbf{x}_1) = \begin{bmatrix} \mathrm{Var}_g\left(\sqrt[4]{20 \cdot X_1}\right) \\ \mathrm{Var}_g\left(\sqrt[4]{3/_{25}}\right) \\ \mathrm{Var}_g\left(\sqrt[4]{1/_{60}}\right) \\ \mathrm{Var}_g\left(\sqrt[4]{25/_{X_1}}\right) \end{bmatrix} = \begin{bmatrix} \omega_1/_{16} \\ 0 \\ 0 \\ \omega_1/_{16} \end{bmatrix}; \qquad (2.8)$$

and analogously for D_2 and D_3:

$$\mathrm{Var}_g(\mathbf{x}_2) = [0 \;\; \omega_2/_{16} \;\; \omega_2/_{16} \;\; 0]^{\mathrm{T}}, \; \mathrm{Var}_g(\mathbf{x}_3) = [\omega_3/_{16} \;\; 0 \;\; 0 \;\; \omega_3/_{16}]^{\mathrm{T}}. \qquad (2.9)$$

Let $\sum_g(\mathbf{x}_1)$ be the geometric variance-covariance matrix of the random vector $\mathbf{x}_1$. By doing similar computations as in the example given in section 4 of Appendix B, one has:

$$\sum\nolimits_g(\mathbf{x}_1) = \frac{\omega_1}{16} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}; \qquad (2.10)$$

$$\sum\nolimits_g(\mathbf{x}_2) = \frac{\omega_2}{16} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \qquad (2.11)$$

$$\sum\nolimits_g(\mathbf{x}_3) = \frac{\omega_3}{16} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.12)$$

Finally, by denoting with V_i the matrix whose (r,s) entry is the geometric variance of the (r,s) entry of B_i, then again by performing similar computations as in the previous example (section 4 of Appendix B) yields:

$$V_1 = \frac{\omega_1}{16} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix}; \qquad (2.13)$$

$$V_2 = \frac{\omega_2}{16} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 1 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}; \qquad (2.14)$$

$$V_3 = \frac{\omega_3}{16} \begin{bmatrix} 0 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 4 & 1 & 1 & 0 \end{bmatrix}. \qquad (2.15)$$

By considering $\omega_1 = 0.0391$, $\omega_2 = 0.0136$, and $\omega_3 = 0.00414$, and using some specific values of u (as in the numerical example of section 4, Appendix B), Theorem 5 of Appendix B is used to calculate the lower bounds of the probability for each considered variable. This is shown in Table 2.10.

Table 2.10. Lower bounds of the probability

Reference random matrix	Random variable of the closest consistent matrix	Value of u	Interval of variable	Lower bound of the probability
A_1	b_{14}	0.7	$[0.5697, 2.3103]$	0.980
		0.3	$[0.8499, 1.5449]$	0.891
		0.15	$[0.9874, 1.3329]$	0.565
A_2	b_{23}	0.7	$[0.7400, 3.0011]$	0.993
		0.3	$[1.1041, 2.0117]$	0.962
		0.15	$[1.2827, 1.7315]$	0.849
A_3	b_{14}	0.7	$[0.1789, 0.7251]$	0.998
		0.3	$[0.2667, 0.4860]$	0.988
		0.15	$[0.3099, 0.4183]$	0.954

The probabilities that the considered variables do not belong to the indicated intervals are almost negligible. For example, the probability that X_2 (corresponding to b_{23} for expert D_2) does not belong to the interval $[1.1041, 2.0117]$ is lower than $1 - 0.962 = 0.0377$. This confirms the goodness of the evaluations.

Note that, although the study has been performed only for those variables originally introducing randomness in the original matrices A_i, similar calculations should be performed for all the random entries of matrices B_i that can be identified by the non-vanishing positions of the corresponding matrices V_i.

After having shared results with the decision-makers, who agreed with the final composition of the three matrices, their entries are aggregated in a single matrix using the geometric mean. The corresponding priority vector is given in the last column of Table 2.11.

Table 2.11. Aggregated matrix and criteria weights

	C_1	C_2	C_3	C_4	Weights
C_1	1	1.791	2.041	0.763	**29.77%**
C_2	0.558	1	1.140	0.426	**16.63%**
C_3	0.490	0.877	1	0.374	**14.50%**
C_4	1.310	2.346	2.675	1	**39.01%**

The following tables give the evaluations of the problem alternatives related to the considered criteria. The last two columns, respectively, give the local priorities, given by their corresponding Perron vectors, and the values of the consistency ratios CR. In particular, the consistency of the judgment is verified, because the CR values do not surpass the threshold of 0.1.

Table 2.12. Evaluation of alternatives respect to the criteria, local priorities and $\boldsymbol{CR}$ value

C_1	MA_1	MA_2	MA_3	MA_4	MA_5	Local priorities	CR
MA_1	1	5	4	2	1/3	0.2383	
MA_2	1/5	1	1	1/3	1/6	0.0579	
MA_3	1/4	1	1	1/3	1/3	0.0755	0.0748
MA_4	1/2	3	3	1	1/6	0.1387	
MA_5	3	6	3	6	1	0.4896	

C_2	MA_1	MA_2	MA_3	MA_4	MA_5	Local priorities	CR
MA_1	1	1/3	1/2	1/4	7	0.2283	
MA_2	2	1	2	1	9	0.2897	
MA_3	1/5	1/2	1	2	7	0.1747	0.0708
MA_4	4	1	1/2	1	9	0.2843	
MA_5	1/7	1/9	1/7	1/9	1	0.0230	

C_3	MA_1	MA_2	MA_3	MA_4	MA_5	Local priorities	CR
MA_1	1	6	5	4	1/4	0.2672	
MA_2	1/6	1	1/2	1/2	1/7	0.0461	
MA_3	1/5	2	1	3	1/5	0.1011	0.0838
MA_4	1/4	2	1/3	1	1/6	0.0640	
MA_5	4	7	5	6	1	0.5217	

C_4	MA_1	MA_2	MA_3	MA_4	MA_5	Local priorities	CR
MA_1	1	7	3	7	1/5	0.2449	
MA_2	1/7	1	1/4	1	1/7	0.0430	
MA_3	1/3	4	1	3	1/5	0.1143	0.0809
MA_4	1/7	1	1/3	1	1/7	0.0448	
MA_5	5	7	5	7	1	0.5530	

On the basis of criteria priorities, the global score for each alternative has been obtained by applying the weighted sum of the respective local priorities, and the final ranking is shown in the Table 2.13.

Table 2.13. Ranking of maintenance actions

Position	Alternative	Score
1st	MA_5	0.4424
2nd	MA_1	0.2248
3rd	MA_3	0.1254
4th	MA_4	0.1130
5th	MA_2	0.0944

The ranking gives the prioritization values for the five maintenance actions starting from the MA_5 alternative, which corresponds to the predictive maintenance policy.

Moreover, it is interesting to note that the corrective policies (MA_3, MA_4 and MA_2) have no relevant priorities in minimizing the plant shutdown risk for the industrial water distribution system feeding the industrial plants of the firm, and the relative interventions may be postponed.

It is clear the role of the obtained ranking in pursuing technological innovation and structuring a long-term strategy of maintenance for the organization.

2.3. A clustering technique for problem size reduction

In highly complex problems, the number of elements to be compared may be very large. One of the issues limiting pairwise comparisons' applicability to large-scale decision problems is the so-called curse of dimensionality, that is a large number of pairwise comparisons needs to be elicited from a decision maker. As an example, the number of comparison elements (criteria or alternatives) should be, according to (Saaty, 1977) at most seven to obtain a reasonable and consistent pairwise comparison matrix. Unfortunately, many decision problems far exceed this maximum threshold.

There are various ways to reduce the size of numerous comparisons. One approach is based on decomposition methodologies (Shen *et al.*, 1992; Islam *et al.*, 1997; 2006; Triantaphyllou, 1995; Ishizaka, 2008; 2012). These methodologies overcome this limit by decomposing a complex decision-making problem into smaller parts to make easier its understanding (Wright, 1985). Unfortunately, these methods also have disadvantages. For example, when a set of elements is decomposed into subsets, the obtained relative weights of the elements are valid just within those subsets. The validity of such weights is not guaranteed at the moment of aggregation. To overcome this problem, a pivot element is arbitrarily selected and assigned to all subsets and used as a basis for comparing the criteria across all disjoint subsets. The global weights can then be estimated as in (Shen *et al.*, 1992; Triantaphyllou, 2000; Ishizaka, 2012). However, pivot element selection is a challenging issue as the decisions should reduce the number and inconsistency of the pairwise comparisons. These methodologies also lack guidelines for assigning the decision elements for respective subsets since they are made arbitrarily. This does not guarantee that the elements within subsets are independent. This may prevent users from addressing such types of decompositions. Finally, the number of subsets must be known a priori and are subject to decision makers' biases and judgement errors. To overcome these problems, Jalao (2014) proposes a method to reduce the number and inconsistency of the pairwise comparisons in a large-scale decision set by using a binary integer programming model that segments pairwise comparison elements into smaller mutually exclusive subsets. However, it may happen that a large comparison matrix has already been produced and needs to be further treated.

By taking into account the decision maker's cognitive ability, Saaty and Ozdemir (2003; 2005) consider that the maximum number of elements simultaneously handled should be seven plus two because of the general limitations on human thinking. This limit is known in the literature (Miller, 1956) as "channel capacity", a measure of our ability to process information, with relation to the number of items that can be held in short term memory at any time. Miller states that his magic number is referred to items characterised by a single aspect or attribute, and this can be true for a series of tasks. However, when more attributes are included, then we can remember more, depending on our familiarity and the complexity of the subject.

Marnell (2014) affirms that Miller's paper discusses what he defines as span of immediate memory (also known as the capacity of our short-term memory), and clarifies that "the capacity of our short-term memory might well be relevant to our ability to comprehend material at the atomic level ... but at the molecular level ... its relevance is doubtful", and that "short-term memory is the very stuff of Miller's paper, especially its role in judgment, attention and recall". Marnell claims that this theory "needs to be radically updated to bring it into line with current knowledge in cognitive psychology", and quotes Baddeley (2007) concluding that "a limit of 7±2 is yesterday's guesstimate. Today it is 4±1 for unrelated items and 15 for... [related concepts]". However, in special cases comparison matrices with more than the traditional 7±2 elements may be valid, for example when an expert with a recognised experience in his field compares an elevated number of items and there is no clear possibility of clustering the items following some homogeneity criteria as suggested by Saaty and Odzemir (2003).

To take into account these aspects, it can be useful to develop such a compressing or merging technique so that certain elements may be synthesized to produce a new comparison matrix that: (i) gathers some elements into clusters, while maintaining the experience and the perception of the experts and also the consistency, and (ii) reduces the size of the problem, thus making it more manageable.

Appendix C provides the body of theory upon which the present section is based: including suitable definitions, detailed proofs of lemmas and theorems, as well as synthetic illustrative examples. In particular, Appendix C gives:

1. clustering of entries in reciprocal matrices;
2. clustering of entries in consistent matrices.

On the basis of what is illustrated in Appendix C, a case study focused on the transition from an intermittent to a continuous water supply system is addressed (Benítez *et al.*, 2018b; Ilaya-Ayza, 2016). A real intermittent water supply system (IWSS), one of the subsystems of the water supply system of the city of Oruro (Bolivia), is analysed.

Intermittent water supply operation and maintenance management in developing countries are mainly based on the experience of water company personnel, mainly derived from manual (in contrast to automatic) operation, and using simple offer-demand analyses (Ilaya-Ayza *et al.*, 2016). In addition, the collection of quantitative information is deficient. So, by using a sector-operation-difficulty-related qualitative criterion (Ilaya-Ayza *et al.*, 2017) this limitation can be overcome. In addition, this criterion adds the experience of water company technicians into the decision-making process to improve the water system.

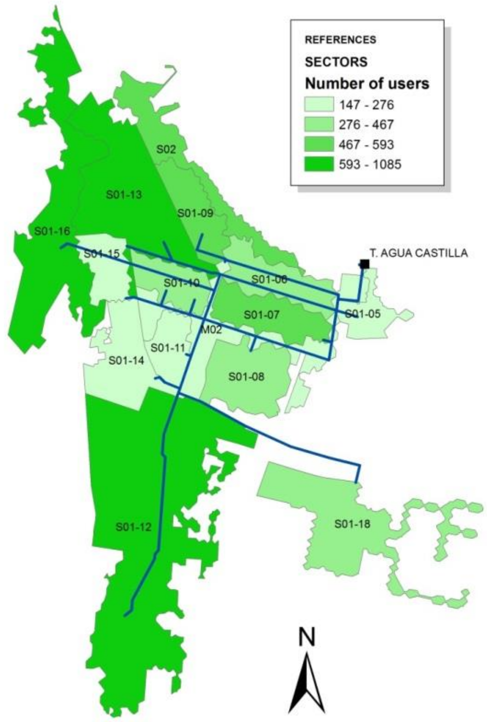


Figure 2.4. Studied IWSS, south area of Oruro, Bolivia

One way to minimize the difficulties resulting from IWSSs is to enhance an adequate system of technical management. In this case study, the use of qualitative data, regarding the network sectors in which the system is divided, is analysed as an interesting alternative for improving planning, operation, and maintenance in IWSSs. The studied IWSS (see Figure 2.4 below) is one of the subsystems of the water distribution system of the city of Oruro (Bolivia).

This subsystem is located in the southern part of the city, consists of 15 sectors fed by a single tank, and supplies water to 37,700 inhabitants. Each sector has a specific supply schedule and specific operation tasks, such as valve manoeuvring, which are manually performed.

As said, one of the qualitative criteria of interest in an IWSS, and the subject of this study, is the difficulty of operation of each of the sectors. This variable depends on several factors such as the availability of sectorization valves, the certainty of their existence, whether existing valves are operating, whether they are visible or buried, the difficulty in working for operators, consumer complaints, and others. Being a complex qualitative criterion, technicians and water company experts who understand the operation of the network and its sectors were consulted. This also ensures active management in the process of improvement.

The AHP is used to rank the alternatives (obviously, the alternatives are the 15 sectors in which the network is currently sectorized), with respect to the single criterion 'difficulty of operation'. The pairwise comparison process has been led by asking the experts to analyse the ease of operation of a sector with respect to another. Despite the large number of elements for comparison, the panel was able to develop a coherent and reliable comparison matrix (see the table below). The consistency ratio for this matrix is 5.8%.

Table 2.14. Comparison matrix for the qualitative criterion: ease of operation for sectors

	S01 -05	S01 -06	S01 -07	S01 -08	S01 -10	S01 -11	S01 -09	S01 -13	S01 -14	S01 -15	S01 -16	S02	M02	S01 -12	S01 -18	Eigen-vector
S01-05	1	5	3	1	3	1	1	1	1	1	1	3	7	1	1	0.0915
S01-06	1/5	1	1/3	1/3	1/3	1/3	1/5	1/3	1/3	1/3	1/3	5	5	1/3	1/3	0.0350
S01-07	1/3	3	1	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	1/3	7	1/3	1/3	0.0317
S01-08	1	3	3	1	1	1	1	1	1	1	1	3	7	1	1	0.0786
S01-10	1/3	3	3	1	1	1	1	1	1	1	1	1	7	1	1	0.0698
S01-11	1	3	3	1	1	1	1	1	1	1	1	1	7	1	1	0.0736
S01-09	1	5	3	1	1	1	1	1	1	1	1	3	7	3	1	0.0895
S01-13	1	3	3	1	1	1	1	1	1	1	3	3	7	3	1	0.0922
S01-14	1	3	3	1	1	1	1	1	1	1	3	3	7	3	1	0.0922
S01-15	1	3	3	1	1	1	1	1	1	1	3	3	7	3	1	0.0922
S01-16	1	3	3	1	1	1	1	1/3	1/3	1/3	1	1	5	1	1/3	0.0572
S02	1/3	1/5	3	1/3	1	1	1/3	1/3	1/3	1/3	1	1	5	1	1/3	0.0405
M02	1/7	1/5	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/7	1/5	1/5	1	1/5	1/7	0.0100
S01-12	1	3	3	1	1	1	1/3	1/3	1/3	1/3	1	1	5	1	1/3	0.0535
S01-18	1	3	3	1	1	1	1	1	1	1	3	3	7	3	1	0.0922

As explained before, the possibility of reducing the volume of this information may be deemed interesting, for example if posterior comparisons regarding alternatives were necessary. In addition, some patterns presented by the matrix clearly suggest the possibility of reducing the matrix size by grouping sectors, a situation that was not initially obvious.

The identification of these groups will enable to develop strategies for improvement in the technical management based on differentiated areas. Various proposals were presented to the technicians who manage the system.

The most successful proposal used ideas provided by the technicians to guide the observation of patterns in the matrix of comparisons. Thus, sectors S01-09, S01-13, S01-14 and S01-15 were initially selected to be grouped. Theorem 4 (Appendix C), by using the permutation given by (S01-05, S01-06, S01-07, S01-08, S01-10, S01-11, S01-16, S02, M02, S01-12, S01-18, S0109, S01-13, S01-14, S01-15), gives a new comparison matrix where the last four sectors are grouped (under the identification 09-15).

The matrix, which corresponds to the sector order (S01-05, S01-06, S01-07, S01-08, S01-10, S01-11, S01-16, S02, M02, S01-12, S01-18, 09-15), is given by:

1	5	3	1	3	1	1	3	7	1	1	1
0.2	1	0.333	0.333	0.333	0.333	0.333	5	5	0.333	0.333	0.383
0.333	3	1	0.333	0.333	0.333	0.333	0.333	7	0.333	0.333	0.347
1	3	3	1	1	1	1	3	7	1	1	0.858
0.333	3	3	1	1	1	1	1	7	1	1	0.763
1	3	3	1	1	1	1	1	7	1	1	0.804
1	3	3	1	1	1	1	1	5	1	0.333	0.625
0.333	0.2	3	0.333	1	1	1	1	5	1	0.333	0.442
0.143	0.2	0.143	0.143	0.143	0.143	0.2	0.2	1	0.2	0.143	0.109
1	3	3	1	1	1	1	1	5	1	0.333	0.585
1	3	3	1	1	1	3	3	7	3	1	1.008
1	2.614	2.885	1.165	1.311	1.244	1.601	2.262	9.161	1.711		1

The priority vector given by Theorem 4 (Appendix C) is: $[0.126, 0.048, 0.044, 0.108, 0.096, 0.101, 0.079, 0.056, 0.014, 0.074, 0.127, 0.126]^{\mathrm{T}}$.

For this matrix, the Perron vector, corresponding to the Perron eigenvalue $\lambda = 13.2$, is: $[0.127, 0.053, 0.045, 0.104, 0.088, 0.095, 0.085, 0.060, 0.013, 0.084, 0.131, 0.116]^{\mathrm{T}}$, giving the values $CI = 0.1069$ and $CR = 7.03\%$, which is satisfactory from the consistency point of view. Continuing in the same line, a new clustering was performed, taking the latter as a starting point. In this case, sectors S01-08, S01-10 and S01-11 were the candidates for a new grouping. Again, using Theorem 4 (Appendix C) on the permutation of the previous matrix given by (S01-05, S01-06, S01-07, S01-16, S02, M02, S01-12, S01-18, 09-15, S01-08, S01-10, S01-11), a new comparison matrix was obtained with the last three sectors grouped (under the name 08-11), given by:

1	5	3	1	3	7	1	1	1	1.238
0.2	1	0.333	0.333	5	5	0.333	0.333	0.383	0.474
0.333	3	1	0.333	0.333	7	0.333	0.333	0.347	0.429
1	3	3	1	1	5	1	0.333	0.625	0.774
0.333	0.2	3	1	1	5	1	0.333	0.442	0.548
0.143	0.2	0.143	0.2	0.2	1	0.2	0.143	0.109	0.135
1	3	3	1	1	5	1	0.333	0.585	0.724
1	3	3	3	3	7	3	1	1.008	1.284
1	2.614	2.885	1.601	2.262	9.161	1.711	0.993	1	1.239
0.808	2.11	2.33	1.292	1.826	7.397	1.381	0.801	0.807	1

In this matrix the corresponding order is (S01-05, S01-06, S01-07, S01-16, S02, M02, S01-12, S01-18, 09-15, 8-11). The priority vector given by Theorem 4 (Appendix C) is: $[0.159, 0.061, 0.055, 0.099, 0.070, 0.017, 0.093, 0.160, 0.159, 0.128]^{\mathrm{T}}$.

For this matrix, the Perron vector, corresponding to the Perron eigenvalue $\lambda = 11.1$, is: $[0.149, 0.070, 0.061, 0.101, 0.070, 0.015, 0.100, 0.173, 0.144, 0.116]^{\mathrm{T}}$, giving the values $CI = 0.125$ and $CR = 8.40\%$, which is still satisfactory from the consistency point of view.

This proposal was positively considered by the technicians. In addition to the acceptable values of CR, the technicians appreciated that both clusters, the initial 09-15 and the subsequent 08-11, have an interesting technical interpretation, which is based on the proximity to the source of clusters 08-11 and 09-15 (which are at successive rings, consecutively further away from the source).

As a result, areas or groups of sectors with similar operating characteristics are defined based on the opinion of water company technicians. This poses a new scenario (see the Figure 2.5) that will enable a better planning in the operation and maintenance tasks of the system, such as reorganizing the manual work of the operators, who hourly open and close the valves to supply water, and the prioritization of maintenance tasks.

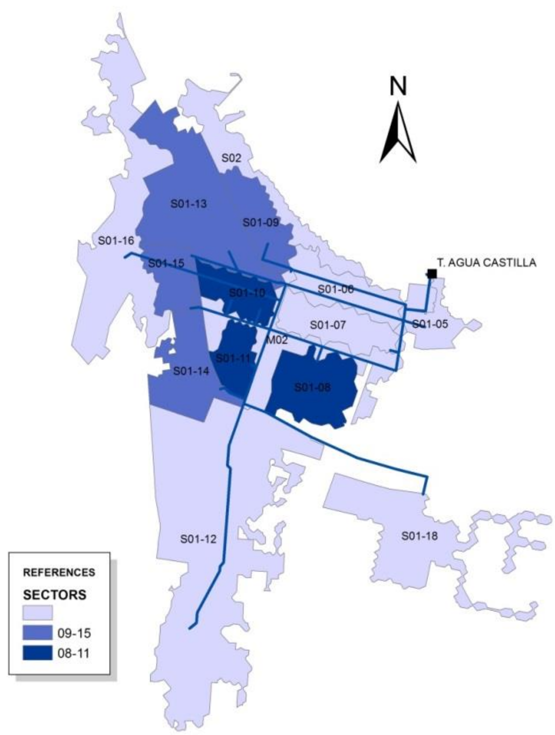


Figure 2.5. Studied IWSS, after cluster identification

Therefore, it becomes a very useful tool for the technical management of intermittent water supply systems, and is a fundamental criterion for a future transition to a continuous water supply (Ilaya-Ayza, 2016; Ilaya-Ayza *et al.*, 2016; 2017).

Several attempts were made, since there was no clear idea about which sectors to group and which not to group (except for some visual patterns identified in the original matrix). It was the final decision of the experts, after checking the various results produced, that was consolidated.

As another example, if the technique had to be used to merge companies in shared markets to avoid the whole process of starting pairwise comparisons from scratch once an alliance has been produced, the decision would consider companies inside and outside of the alliance. Conditions would probably be clear from the beginning, provided that the alliance process among companies is clear-cut. In general, an a priori cluster structure with the best number of clusters may not exist and, consequently, it would be a very interesting line of research for future study.

Regarding the procedure for obtaining the clusters of the considered elements, details depend on the problem in hand and there are no general guidelines. In the two provided examples, completely different procedures are followed. In the case of the water supply system, it has been a trial and error procedure in close contact with the system experts.

Chapter 3

Implementation of a website for worldwide companies

The content of this section has been developed during two international traineeships focused on different aspects of programming.

The first traineeship was carried out during a three-month stay in the United Kingdom, at the *Energy and Design of Environments* (EDEn) Department of the University of Bath. This period was particularly helpful to be introduced into advanced programming, learning how to use various advanced software packages for data analysis, and elements of parallel computing. This last aspect dealt with the management of complex and high-memory-demanding decision-making problems by accessing and accomplishing several simulations through the High-Performance Computer (HPC) of the University of Bath (Balena system).

The theme of programming was further expanded and practically applied to the field of MCDM methods during a two-week visiting period to the company *IngeniousWare GmbH*, located in Karlsruhe, Germany, whose core business consists in creating innovative software solutions for companies and professionals. This second traineeship aimed at undertaking an international project, initially focused on the AHP but potentially extendable to other MCDM methods, as a joint research venture in collaboration with the mentioned German firm. Besides UNIPA and UPV, the *Centre de Recherche en Automatique de Nancy* (CRAN) of the University of Lorraine (France) is also involved.

This project's main goal is to enhance the performance of real complex systems in the fields of production and services. The main effort aims to exploit the strengths of multi-criteria decision-making methods for discovering optimisation possibilities and make them tangible.

During the traineeship in Germany, a website was developed with the goal of providing worldwide companies with a friendly support framework for their decision-making processes by taking into account numerous factors. In particular, several aspects of the AHP method were programmed using the C# language, within the Visual Studio environment. The URL address of this website is:

https://ahp.imaxweb.de/

The work was basically organised through the following phases, related to the front-end and to the back-end management systems.

- Phase 1 – Designing an attractive welcome page about the AHP as a product to be proposed to companies, by highlighting the strengths and the flexibility of the method in terms of group decision-making method. The related scientific production is shown to demonstrate the experience of the research team in the field.
- Phase 2 – Defining a common procedure to acquire pairwise comparison judgments to weight elements on the basis of a scale of linguistic evaluations. This phase also refers to the process of checking consistency and, when necessary, producing the closest consistent matrix by means of the linearisation method. In this case, the system is able to propose a consistent solution to decision-makers, and interact with them to achieve an agreement about all the evaluations. This step is crucial to get a final adjusted matrix that effectively reflects experts' opinions and simultaneously fulfils the objective of keeping consistency within the allowed limit.
- Phase 3 – Formalising the hierarchical structure and the input data of a complex decision-making problem expressed by the user, who is in charge of formulating the main goal, the evaluation criteria and the set of possible alternatives to be eventually ranked. Furthermore, the same user has to formally indicate the group of experts (or a single decision maker) to weight evaluation criteria, and the mutual importance of each expert. Lastly, it is again up to the user to express comparison judgments between pair of alternatives, aiming at achieving their local priorities.
- Phase 4 – Establishing a feedback-based relationship with the experts by acquiring their pairwise comparison judgments to obtain the final vector of criteria weights. This phase terminates by sharing the final ranking of alternatives as the output of the offered service. In this way, companies can rely on a structured and reliable methodology to make many business decisions. Moreover, the ranking allows to be aware of which position is occupied by a given alternative, and may thus represent a strategic tool for planning medium and long term implementation of solutions.

3.1. Designing the welcome page

The website has been divided into the following sections: WELCOME, AHP, SERVICES, RESEARCH, ABOUT, CONTACT, REGISTER/LOGIN.

The first section, "WELCOME", presents the title and the main topic of the website.

Figure 3.1. “Welcome” section

The “AHP” section immediately states the objective of the website and continues by providing a brief introduction about this MCDM method in terms of its strengths and application usage.

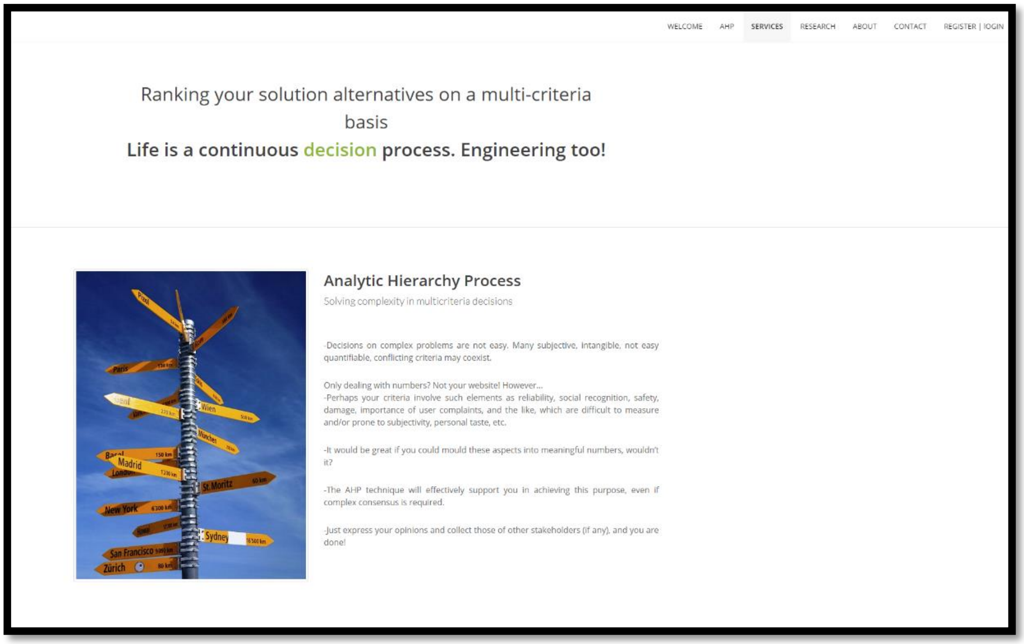


Figure 3.2. “AHP” section

The “SERVICES” section is aimed at describing the method, clearly communicating its objectives and explaining why and how it could be supportive. This section has been divided into three main parts to illustrate how the method works:

1. HIERARCHICAL STRUCTURE CREATION,
2. INTRODUCTION OF EXPERT JUDGMENT,
3. RANKING ALTERNATIVES.

Figure 3.3. “SERVICES” and “RESEARCH” sections

The “RESEARCH” section shows the team’s recently published and ongoing papers on AHP for demonstrating the experience of the research team in the field. The green button “More…” links to a page reporting the most relevant and recent contributions to international conferences and other team’s previously published papers on AHP.

Most relevant/recent contributions to congresses on AHP

Benítez, J., Carpitella, S., Certa A., Izquierdo J., La Fata, C.M. Some consistency issues in multi-criteria decision making. Proceedings of the 2017 Summer School “Francesco Turco”, Palermo, Italy, September 13-15, 411-418.

Benítez, J., Carpitella, S., Certa, A., Ilaya-Ayza, A.E., Izquierdo, J. Some awkward issues on pairwise comparison matrices. Proceedings of the 2017 International Conference on Mathematical Modeling in Engineering & Human Behaviour, Valencia, Spain, July 17-19.

Carpitella, S., Carpitella, F., Benítez, J., Certa, A., Izquierdo, J. Prioritization of maintenance actions in water distribution systems. Proceedings of the 2017 Congress of Numerical Methods in Engineering CMN, Valencia, Spain, July 3-5.

Certa, A., Enea, M., Galante, G., Lupo, T., A multi-decision makers approach to select the maintenance plan for a multi-component system Proceeding of the 2013 ISSAT International Conference on Reliability and Quality in Design, Honolulu, Hawai, USA, August 5-7.

Previously published papers on AHP

Izquierdo, J., Benítez, J., Berenguer, A., Lago-Alonso, C., I decide, therefore I am (relevant!). A project-based learning experience in linear algebra. Computer Applications in Engineering Education, 24(3), 481-492, 2016.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R., Consistent completion of incomplete judgments in decision making using AHP. Journal of Computational and Applied Mathematics, 290, 412-422, 2015.

Certa, A., Enea, M., Hopps, F., A multi- criteria approach for the group assessment of an academic course: a case study. Studies in Educational Evaluation, 44, 16-22, 2015.

Benítez, J., Izquierdo, J., Pérez-García, R., Ramos-Martínez, E., A simple formula to find the closest consistent matrix to a reciprocal matrix. Applied Mathematical Modelling, 38(15-16), 3968–3974, 2014.

Delgado-Galván, X., Izquierdo, J., Benítez, J., Pérez-García, R., Joint Stakeholder Decision-Making on the Management of the Silao-Romita Aquifer Using AHP. Environmental Modelling & Software, 51, 310-322, 2014.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R., Improving consistency in AHP decision-making processes. Applied Mathematics and Computation, 219, 2432-2441, 2012.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R., An approach to AHP decision in a dynamic context. Decision Support Systems, 53, 499-506, 2012.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R., Achieving Matrix Consistency in AHP through Linearization. Applied Mathematical Modelling, 35, 4449-4457, 2011.

Benítez, J., Delgado-Galván, X., Gutiérrez-Pérez, J.A., Izquierdo, J., Balancing Consistency and Expert Judgment in AHP. Mathematical and Computer Modelling, 54, 1785-1790, 2011.

Certa, A., Enea, M., Galante, G., La Fata, C.M., Multi-objective Human Resources Allocation in R&D Projects Planning. International Journal of Production Research, 47(13), 3503-3523, 2009.

Berrittella, M., Certa, A., Enea, M., Zito, P., Transport policy and climate change: how to decide when experts disagree. Environmental Science & Policy, 11(4), 307-314, 2008.

Figure 3.4. Link to more research items

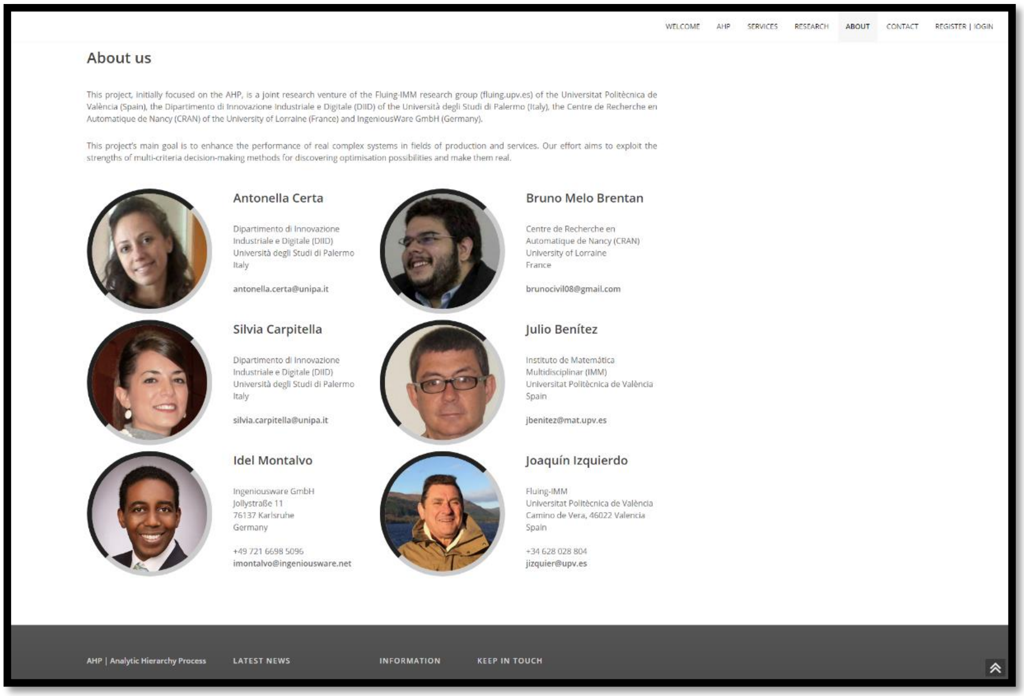


Figure 3.5. "About" section

The "ABOUT" section is dedicated to the research team, and shows photos, roles and contacts.

Some of the used images have been downloaded as free images for website or other use, and are opportunely referenced.

Lastly, the website allows users to register for accessing the service by means of the section "REGISTER/LOGIN". Once the user (also called chief of the project from now on) is registered, an automatic email is sent to his/her address to confirm the registration and activate the account. It is then possible to access through the LOGIN page.

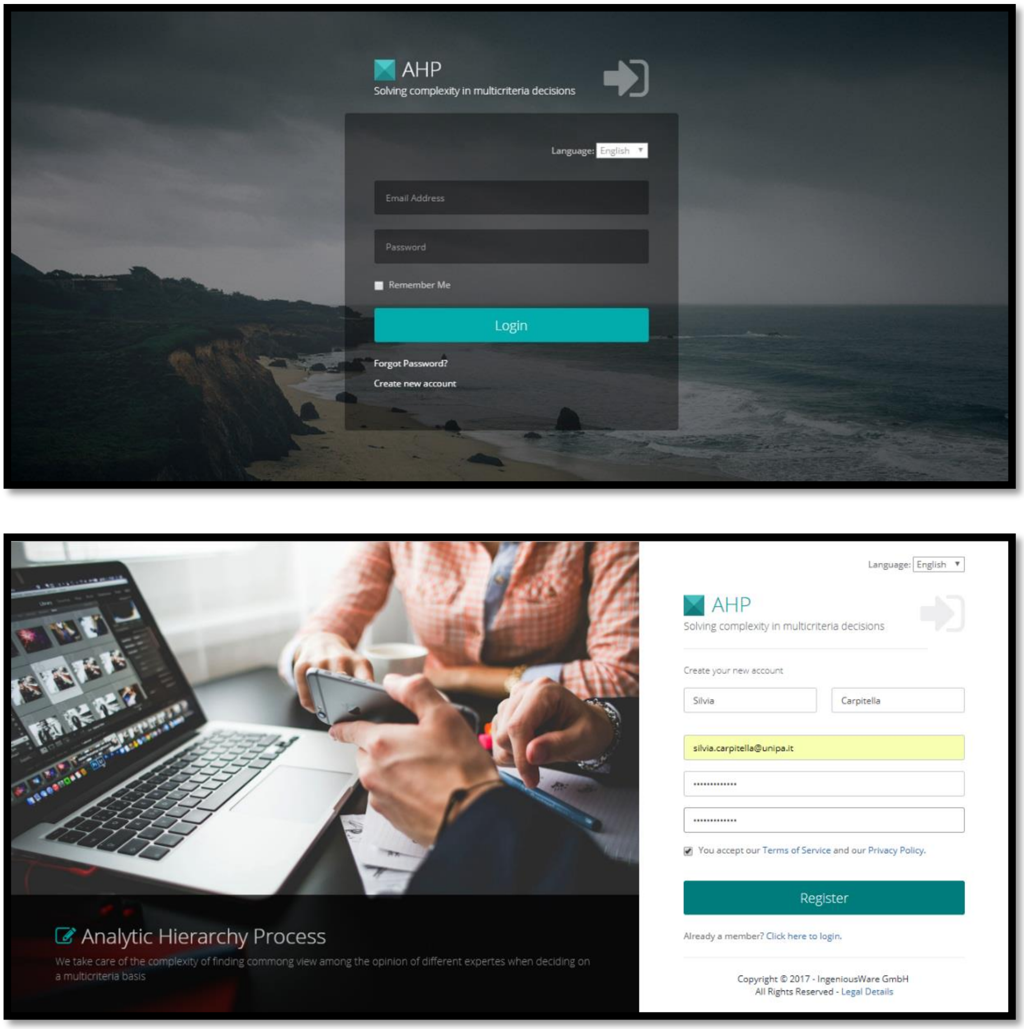


Figure 3.6. REGISTER/LOGIN pages

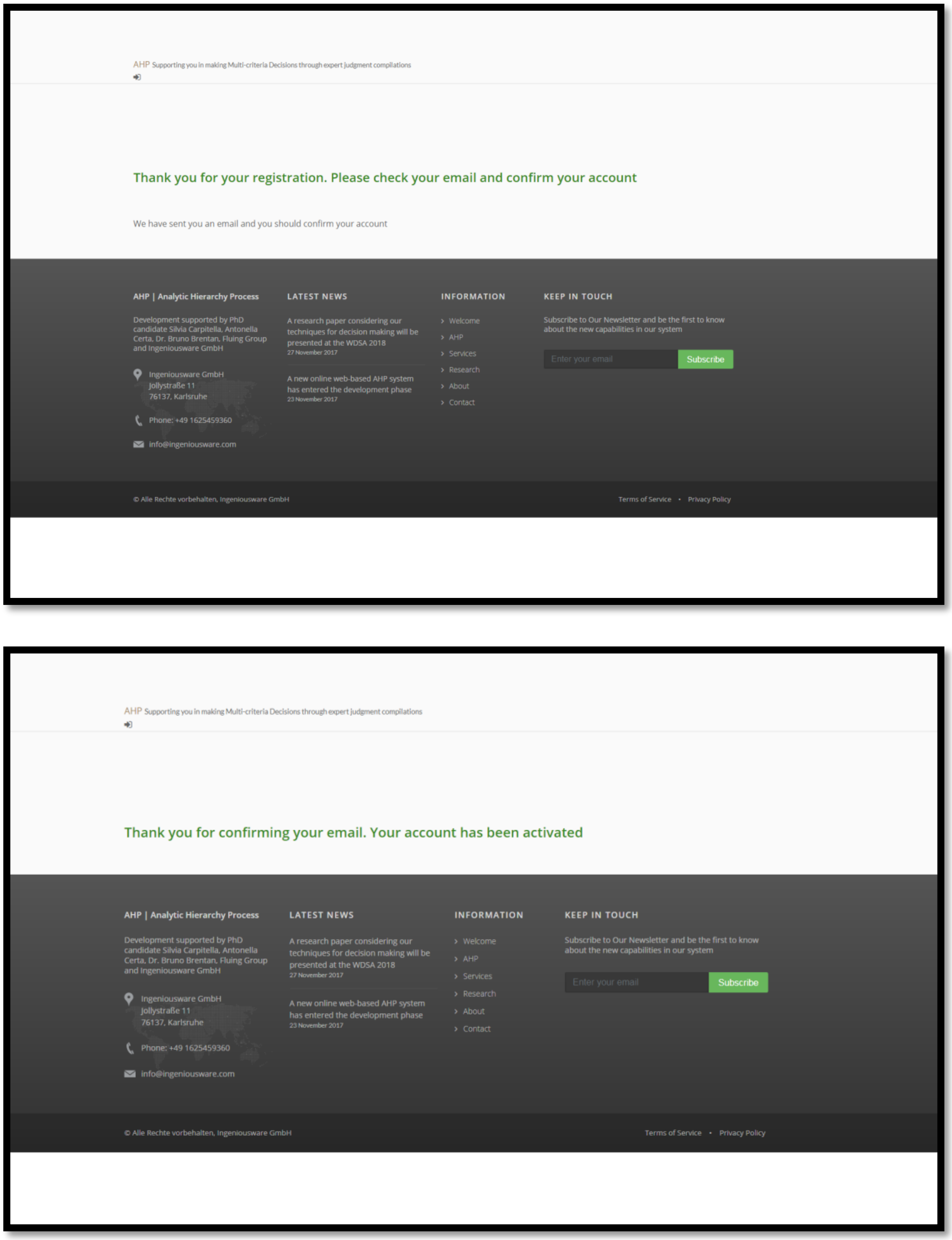


Figure 3.7. Confirm email/Confirm registration pages

Before describing in detail the development of the processes led by chief and experts, the next subsection presents general tools, which are applied in various of those processes.

3.2. Procedures for collecting pairwise comparisons and checking consistency

Once the registration has been completed, the AHP procedure is initiated. The general process, graphically synthetized in the boxes marked in bold in the chart of Figure 3.8, will be described in detail through the next subsections of the present paragraph.

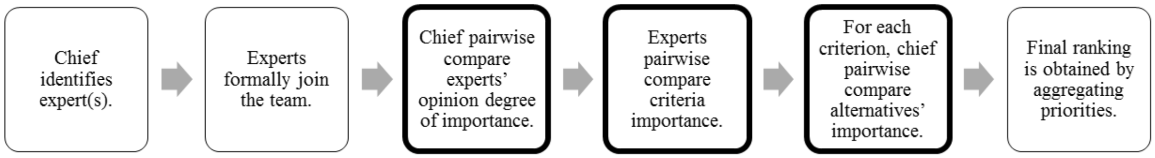


Figure 3.8. Scheme of general process

The flowchart in Figure 3.9 presents a scheme of the process of weighting elements (experts, criteria, alternatives). Observe that trapezoids represent inputs, rectangles correspond to procedures, rhombuses stand for if-else decisions and rounded figures for outputs.

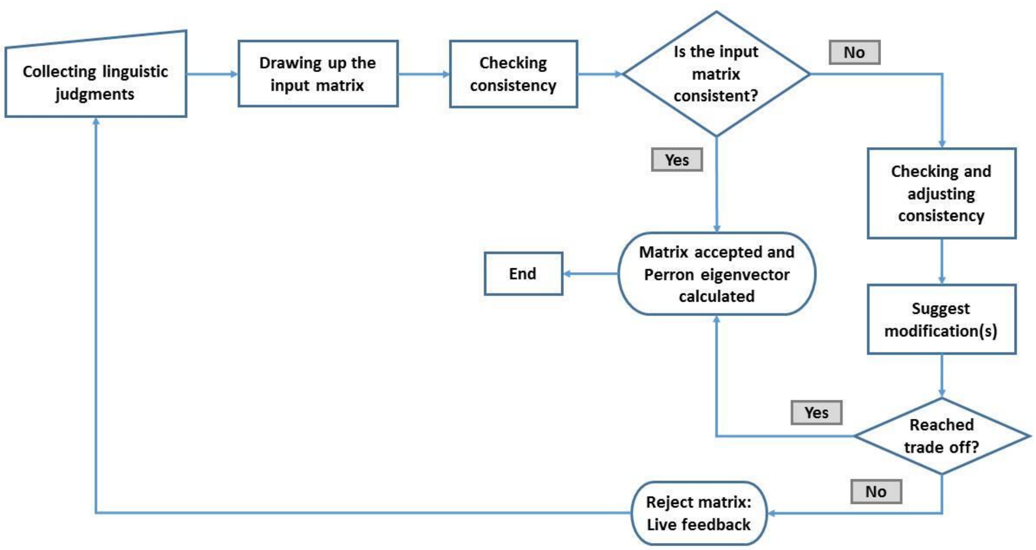


Figure 3.9. Process of weighting elements

The collection of pairwise comparison judgments (upper-left trapezoid), and the procedures (top rectangles downstream the trapezium) for calculating the corresponding PCM, and checking judgment consistency are aimed at weighting the main elements of the problem, that is to say, at establishing their relative importance.

The elements to be weighted are: experts (by the chief), criteria (by the experts) and alternatives (again by the chief). To such an aim, comparison judgments between pair of elements have to be expressed in each case by the relative decision maker (chief or expert/s), on the basis of the scale of linguistic evaluations given in Table 3.1.

Table 3.1. Linguistic scale

Lower importance
Equal importance
From equal to moderate importance
Moderate importance
From moderate to strong importance
Strong importance
From strong to very strong importance
Very strong importance
From very strong to extreme importance
Extreme importance

Collecting linguistic judgments: Each pair of elements is shown, page by page, in the dashboard, together with this evaluation linguistic scale. Regarding each pair, the decision maker has simply to click on the corresponding linguistic value he/she wants to assign. When clicking the box corresponding to "Lower importance" (whose corresponding numerical value is understood by the system as comprised between 0 and 1), the pair will be inverted. For instance, with relation to the pair C_1 – C_2, if the decision maker considers C_1 less important than C_2, the pair will be inverted to C_2 – C_1, and the decision maker will have to evaluate the importance of C_2 over C_1, by clicking one box from the one called "From equal to moderate importance" to the one called "Extreme importance".

Drawing up the input matrix: Once a decision maker has finished with the evaluations, the related matrix is filled in by using the 9-point Saaty scale (first rectangle to the left in Figure 3.9). If a pair of elements had to be inverted (see previous paragraph) the corresponding numerical value from 2 to 9 given to the inverted pair is correspondingly calculated as its reciprocal (between $1/9$ and $1/2$).

Checking consistency: The next step (second rectangle) regards the process of checking consistency, which will be followed by a possible negotiation with the decision maker about those values that could be adjusted to get a consistency ratio CR within the allowed threshold proposed by Saaty. This process is represented in the diagram in Figure 3.10.

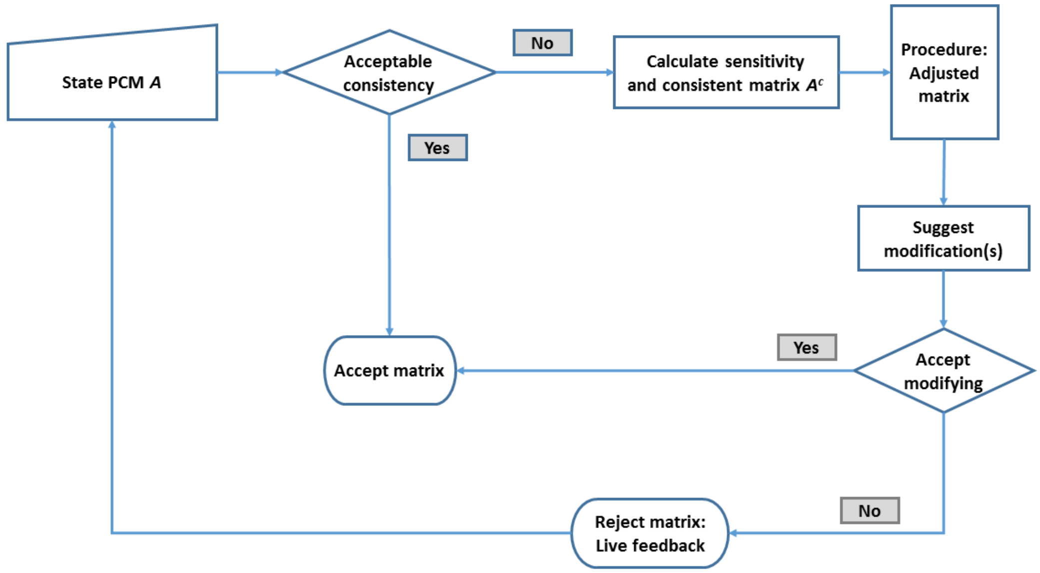


Figure 3.10. Process of checking and adjusting consistency

For an input PCM acceptably consistent there is no point to contact the decision maker again (the 'Yes' leg of the first rhombus in Figure 3.10). In this case the matrix is assumed as definitive and the process goes back to the central rounded box of Figure 3.10, and the process stops after calculating the weights. However, for a non-acceptably consistent input matrix ("No" leg of same rhombus), the process can be summarised by the following steps:

- Calculate sensitivity and consistent matrix A^c: Includes two calculations:
  - obtaining the closest consistent matrix for each decision maker by applying the linearisation process (this process has been widely described in Chapter 1, Section 1.2);
  - ranking judgments according to the higher impact on consistency, by leading a sensitivity analysis (described at the end of the present bullet list). Naturally, just the values over the main diagonal, whose number is equal to $M = \frac{n\times(n-1)}{2}$, will be considered in the ranking (called sensitivity ranking from now on);
- Procedure: Adjusted matrix: This iterative procedure is described in the diagram of Figure 3.11 and is shortly described here:
  - The process starts by calculating the "adjusted matrix" B, all whose elements but one correspond to the input matrix: the element expressing the judgment occupying the first position in the sensitivity ranking is assigned the corresponding value in the closest consistent matrix;

- calculating consistency again (central rhombus in Figure 3.11); if matrix B is consistent the process stops ('Yes' leg of that rhombus). If matrix B continues to be inconsistent, the iteration consists in changing the element corresponding to the judgment occupying the second position in the sensitivity ranking with the corresponding value in the closest consistent matrix;
- these previous steps, together with the updating of a control parameter are repeated until consistency is assured.

The process to achieve the adjusted matrix has been designed to make the adjusted matrix as adherent as possible to the input matrix, by changing as fewer as possible evaluations previously given by the decision makers.

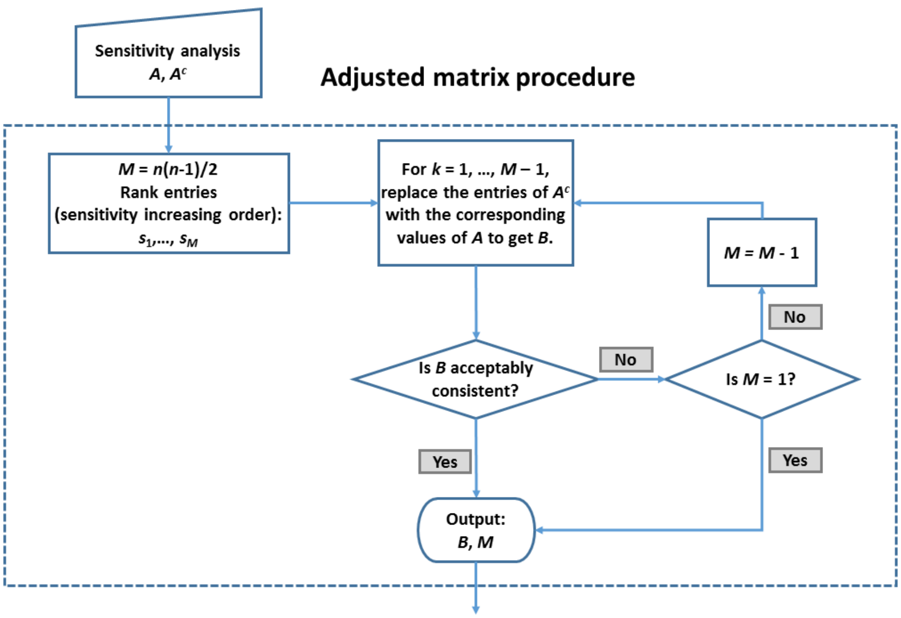


Figure 3.11. Adjusted matrix procedure

- Suggest modification(s): The modified judgments will be proposed to the corresponding decision maker, who will be invited to agree with the final evaluations. In case of disagreement ('No' leg of lower rhombus in Figure 3.10), he/she will be asked to elicit new evaluations, so that new matrices may be drawn up.

At the point in which a decision maker has associated a final adjusted consistent matrix, the process goes back to the central rounded box of Figure 3.9, and the process stops after calculating the weights the corresponding vector of priorities can be calculated.

Sensitivity analysis: The sensitivity method applied to rank the most "influencing" judgments is presented next.

Starting from an $n \times n$ PCM A, the method consists in calculating a second matrix D giving the partial derivatives of λ_{max} with respect to the entries of A, thus identifying which entries are more sensible to increase consistency. These partial derivatives are given by the following formula (Section 1.1, Stewart, 2011):

$$D = w\mathrm{v}^{\mathrm{T}} - A^2 * \mathrm{v}w^{\mathrm{T}}, \qquad (3.1)$$

where: w represents the Perron eigenvector, associated with the value of λ_{max}; v represents the left Perron eigenvector of A, that is the (right) Perron eigenvector of the transpose of A, also associated with λ_{max}, and normalized such that $\mathrm{v}\, w^{\mathrm{T}} = 1$; $*$ is the Hadamard (entry-wise) product. The Hadamard product operates on identically-shaped matrices and produces a third matrix of the same dimensions, whose elements (i, j) correspond to the product of elements (i, j) of the original two matrices.

The values corresponding to the partial derivatives allow to rank the corresponding entries of matrix A and then know which one has bigger influence in consistency. For illustration, an example is given now.

Let's consider the following PCM, A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1/2 & 1 & 4 \\ 1/3 & 1/4 & 1 \end{bmatrix};$$

The corresponding matrix of partial derivatives of λ_{max}, calculated through (3.1), is

$$D = \begin{bmatrix} 0 & -0{,}4437318 & 0{,}66559788 \\ 0{,}11093295 & 0 & -0{,}8874636 \\ -0{,}0739553 & 0{,}05546647 & 0 \end{bmatrix}.$$

This means that the pairwise comparison corresponding to entry (2,3) influences consistency most, respectively followed by comparisons (1,3) and (1,2). In particular, the consistency can be improved by decreasing the comparison value (2,3), by increasing (1,3) and by decreasing (1,2).

After presenting those general tools, the next sections describe more in detail the development of the processes led by chief and experts.

3.3. Formalising the hierarchy structure and the input data

Upon logging in, the chief can enter his/her own dashboard, by means of which various tasks may be accomplished. First of all, the decision-making problem has to be set in terms of a hierarchical structure, that is by defining the goal, the criteria and the alternatives. To such an aim, the system requires to formally fill in the following fields:

- main field of the decision-making problem;
- goal of the decision-making problem;
- number of evaluation criteria (limited to nine);
- name of evaluation criteria;
- number of alternatives (limited to nine);
- name of alternatives.

The chief has the responsibility to choose the decision-making team (maybe reduced to a single expert) whose opinions are necessary to calculate the vector of criteria weights. In particular, the user is prompted to indicate the name and email of at least one expert (maximum nine) who will provide opinions about the relative importance between pairs of evaluation criteria. The user can choose if joining the team evaluating criteria or not. Alternatively, he/she can choose to proceed to the evaluation by his/her own (in this case just his/her name and contact have to be given in the corresponding field).

Moreover, a degree of importance in percentage has to be assigned to each expert, with the aim to highlight their weight in the decision-making problem. To such an aim, the chief is asked to express judgments of pairwise comparisons between pairs of experts by using the linguistic scale in Table 3.1. The system calculates the related vector of criteria weights (normalized to one). If the chief indicates a single expert, the system assigns him/her a weight of 100% by default. For two experts, consistency is obvious. The system checks consistency of judgments (for a number of experts higher or equal to three), and they have to be elicited again in the case of non-consistency by means of the procedure described in section 1.2.

In terms of programming, the AHP library was created and classes, corresponding to the levels of the hierarchical structure, namely "`ChiefExperts`" "`Experts`", "`DecisionCriteria`" and "`SolutionAlternative`", have been added:

```
namespace AHPLibrary

{

    public class ChiefExperts
    {
        public Guid KeyId { get; set; }

        public int IdOrder { get; set; }

        public string Name { get; set; }
    }

    public class Experts
    {
        public Guid KeyId { get; set; }

        public int IdOrder { get; set; }

        public string Name { get; set; }

        public string Email { get; set; }
    }

    public class DecisionCriteria
    {
        public Guid KeyId { get; set; }

        public int IdOrder { get; set; }

        public string Name { get; set; }
    }

    public class SolutionAlternative
    {
        public Guid KeyId { get; set; }

        public int IdOrder { get; set; }

        public string Name { get; set; }
    }
}
```

Moreover, all the methods implemented to iterate the procedure are developed within the AHP library and two other classes, "`ahpDecision`" and "`ahpOpinion`", are defined to such an aim. Specifically, "`ahpDecision`" considers the whole structure of the problem and all the elements involved, whereas the class "`ahpOpinion`" defines all the necessary methods to accomplish the process of giving opinions through pairwise comparison judgments.

The following figures present some parts of these classes as example.

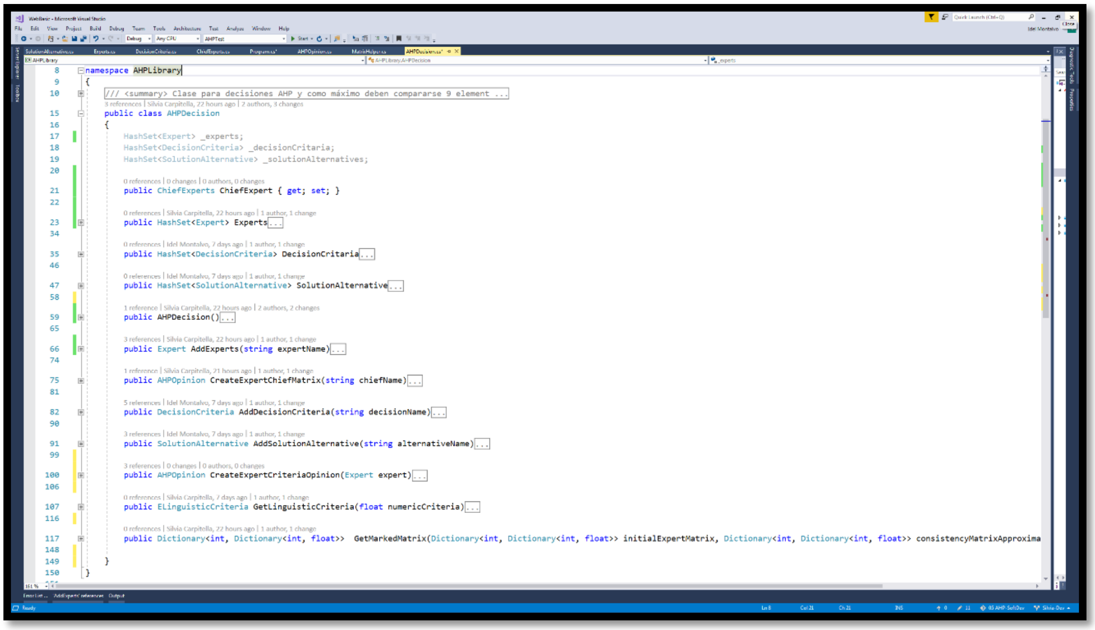


Figure 3.12. "ahpDecision" class

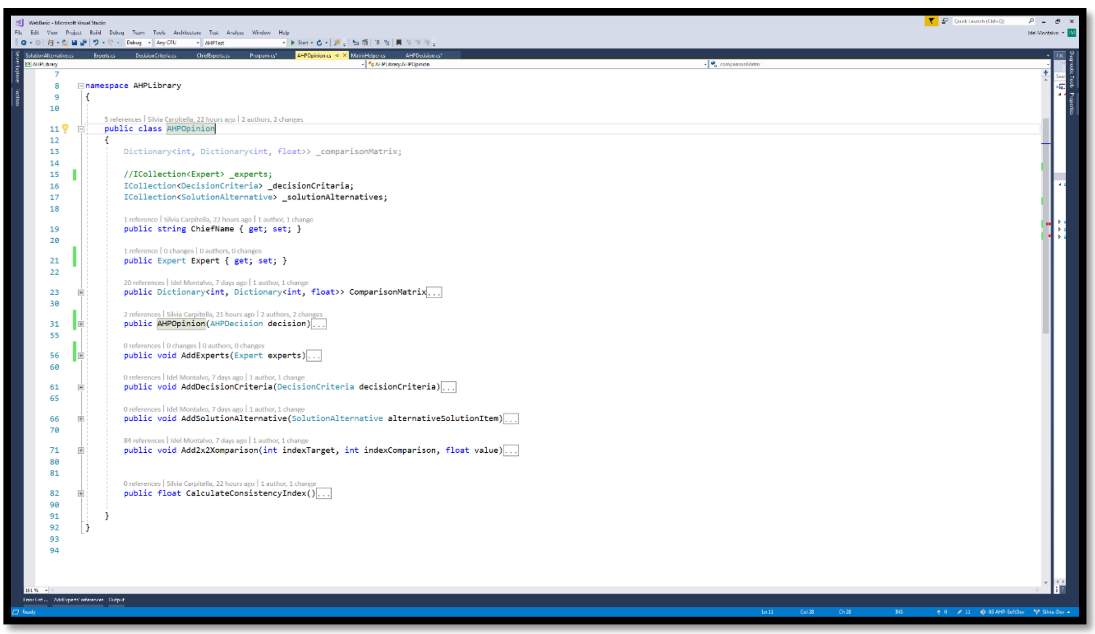


Figure 3.13. "ahpOpinion" class

For instance, the method "`Add2x2Xomparison`" has been defined within the class "`ahpOpinion`" to identify the contents of the cells of the various matrices in the main program.

An AHP test has been developed within the application to prove the main calculations by means of a synthetic example. The related main program has diverse regions, dedicated to the main actors of the process, specifically chief and experts.

The content of the first part of the region dedicated to the chief for evaluating the vector of weights is provided next with relation to a set of three experts.

```
List<Dictionary<int,    Dictionary<int,    float>>>    ExpertChiefMatrix    =    new
List<Dictionary<int, Dictionary<int, float>>>();


#region chief
var ahpOpinion = ahpDecision.CreateExpertChiefMatrix("Chief");

#region completing opinion of chief about importance of experts

ahpOpinion.Add2x2Xomparison(0, 0, 1);
ahpOpinion.Add2x2Xomparison(0, 1, 2);
ahpOpinion.Add2x2Xomparison(0, 2, 4);

ahpOpinion.Add2x2Xomparison(1, 0, 0.5f);
ahpOpinion.Add2x2Xomparison(1, 1, 1);
ahpOpinion.Add2x2Xomparison(1, 2, 3);

ahpOpinion.Add2x2Xomparison(2, 0, 0.25f);
ahpOpinion.Add2x2Xomparison(2, 1, 1/3f);
ahpOpinion.Add2x2Xomparison(2, 2, 1);

#endregion completing opinion of expert 1

Console.WriteLine("Input matrix for chief weighting experts: \n");

for (int I = 0; I < ahpOpinion.ComparisonMatrix.Count; I++)
{
   int countX = ahpOpinion.ComparisonMatrix[I].Values.Count;

   for (int J = 0; J < countX; J++)
   {
       Console.Write(string.Concat(" ",
       ahpOpinion.ComparisonMatrix[I][J].ToString("#.##").PadRight(4, ' ')));
    }

       Console.WriteLine();
}
```

The string “`ahpOpinion.Add2x2Xomparison(0, 0, 1);`” states that the cells corresponding to the first row (marked with 0) and the first column (marked with 0 as well) have associated a numerical evaluation equal to 1, and so on. The letter “f”, standing for `float`, appears to indicate decimal numbers. Moreover, the method “`ComparisonMatrix`” has been defined to return the experts’ PCM once it is filled in by the chief.

The region also contemplates how to calculate consistency and the vector of weights (expressing the importance of the experts). This is done for a number of experts higher than or equal to three. In this case, if consistency is not achieved, the system calculates the closest consistent matrix by using the linearisation method. Note that “`MatrixHelper`” reports how to accomplish the various iterations of the main program in terms of matrix calculations (for example by establishing such methods as "`GetApproximatedConsistencyMatrix`", "`GetMatrixConsistencyRate`", "`GetEigenVector`", "`NormalizeVector`", and so on).

The remaining part of the region for the chief is given next, until the calculation of the closest consistent matrix, if necessary.

```
Console.WriteLine("\n");

var matrixHelper = new MatrixHelper();


Dictionary<int, float> normalizedWeights = new Dictionary<int, float>();
bool isValid = false;
var consistencyRate =
matrixHelper.GetMatrixConsistencyRate(ahpOpinion.ComparisonMatrix, out
normalizedWeights, out isValid);

Console.WriteLine("Consistency Information about chief weighting experts: \n");
Console.WriteLine("Consistency Rate = " + consistencyRate.ToString("#.####"));
Console.WriteLine("Criteria Weights:");
for (int I = 0; I < normalizedWeights.Count; I++)
{
       Console.WriteLine(normalizedWeights[I].ToString("#.##"));
}
Console.WriteLine("\n");
if (isValid0) Console.WriteLine("The matrix for chief is consistent!!");
else Console.WriteLine("The matrix for chief is NOT consistent!!");
Console.WriteLine("\n");

var result =
matrixHelper.GetApproximatedConsistencyMatrix(ahpOpinion.ComparisonMatrix);

Console.WriteLine("Approximated consistency matrix for chief weighting experts:
\n");

for (int I = 0; I < result.Count; I++)
{
       int countX = result[I].Values.Count;

       for (int J = 0; J < countX; J++)
       {
          Console.Write(string.Concat(" ",result[I][J].ToString("#.##").
          PadRight(4,' ')));
       }

       Console.WriteLine();
}

Console.WriteLine("\n");

#endregion chief
```

Note that the previous region does not consider the process of feedback exchange to adjust consistency so far.

After having set up the AHP test as a start-up project and clicked the command "start", the following result will be produced by prompting the necessary commands.

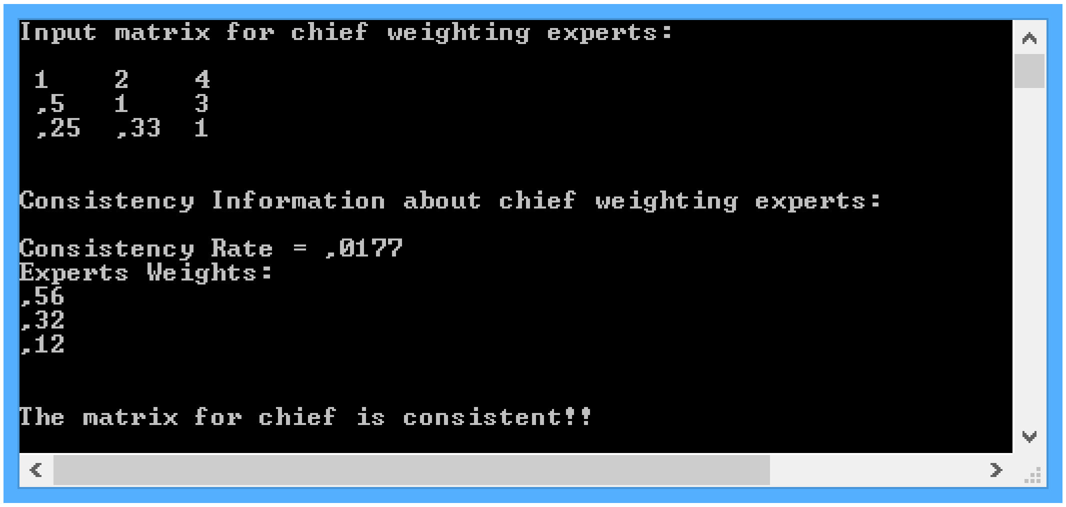


Figure 3.14. Example of consistent input matrix for chief

As it is possible to observe by the reported test, the weights are equal to 56%, 32% and 12% respectively for the first, the second and the third experts.

If the chief attributes inconsistent judgments, the closest consistent matrix is calculated. Let us consider as an example the following (non-consistent) matrix:

```
ahpOpinion.Add2x2Xomparison(0, 0, 1);
ahpOpinion.Add2x2Xomparison(0, 1, 2);
ahpOpinion.Add2x2Xomparison(0, 2, 0.2f);

ahpOpinion.Add2x2Xomparison(1, 0, 0.5f);
ahpOpinion.Add2x2Xomparison(1, 1, 1);
ahpOpinion.Add2x2Xomparison(1, 2, 4);

ahpOpinion.Add2x2Xomparison(2, 0, 5);
ahpOpinion.Add2x2Xomparison(2, 1, 0.25f);
ahpOpinion.Add2x2Xomparison(2, 2, 1);
```

It would lead to the following result:

Figure 3.15. Example of non-consistent input matrix for chief

It is necessary to share the new linguistic judgments, namely those considerably different from the input ones, with the chief. However, the new values could not reflect his/her real opinion and it is fundamental to achieve a trade-off between the practical experience and the synthetic consistent matrix. The system manages the feedback exchange with the chief (evaluating experts and alternatives), exactly as it does with the experts (evaluating criteria). In the next section, this process is shortly described for the latter case, being identical for the other two cases, with the logical changes.

Once all steps of this phase are accomplished, interactions with experts are launched to obtain the vector of criteria weights and the aggregation process, called in Section 3.5, is undertaken. However, if a trade-off is not reached, according to Figure 3.9, a live feedback may be proposed and the entire process should be started from scratch.

3.4. Establishing a feedback-based relationship with the experts and final ranking

As underlined before, the vector of criteria weights is calculated by collecting pairwise comparison judgments from experts in the field. The advantage consists in acquiring different but complementary perspectives about a given decision-making problem, trying to make the evaluation as fairer as possible. Once the decision-making team has been formalised by the user, the following email will be sent to the provided addresses.

*Dear [**name of the expert**],*

*We are contacting you because of your well-established experience in the field of [**main field of the decision-making problem**]. We need your contribution for solving a decision-making problem related to [**definition of the decision-making problem**], according to [**number of evaluation criteria**] evaluation criteria. The considered criteria are: [**name of evaluation criteria**].*

We would like to know your opinion about the importance of these criteria to find a solution representing the best trade-off for the mentioned problem. In particular, we ask you to express simple judgments of pairwise comparisons between each pair of criteria by using the following linguistic scale:

Lower importance
Equal importance
From equal to moderate importance
Moderate importance
From moderate to strong importance
Strong importance
From strong to very strong importance
Very strong importance
From very strong to extreme importance
Extreme importance

We would be grateful if you were so kind to collaborate with us by clicking the corresponding box for each pairwise comparison, and it will take a really little bit of your time.

*Please click the following link to start: [**link**]*

Looking forward to receiving your opinions, we thank you in advance for your precious collaboration.

Best regards

It is important to note that experts deal with just linguistic evaluations and not with numbers. This is done to make more understandable how to assign linguistic evaluations, and then to make easier the process of collecting judgments. The system will automatically translate these variables to numbers (those of the Saaty scale).

Once the expert has accepted to take part in the survey by registering and logging in to the system, each pair of criteria appears page by page in the dashboard, close to the related evaluation linguistic scale. To evaluate each pair of elements, the expert has simply to click on the corresponding linguistic value he wants to assign.

In case of uncertainty about one or more pairwise comparisons, the experts have also the possibility of not to give an answer. The entry of the related incomplete matrix will be consistently completed by using the technique based on the graph theory presented in Chapter 2, Section 2.1.

The system simultaneously translates all the expressed linguistic assessments to numbers and fills in one matrix for each expert (input matrix). After the chief, each expert has his or her own region assigned in the main program. The procedure for obtaining the matrix from the experts weighting criteria is the same of what was expressed for the chief weighting experts and as can be told for the chief evaluating the alternatives with respect each criterion.

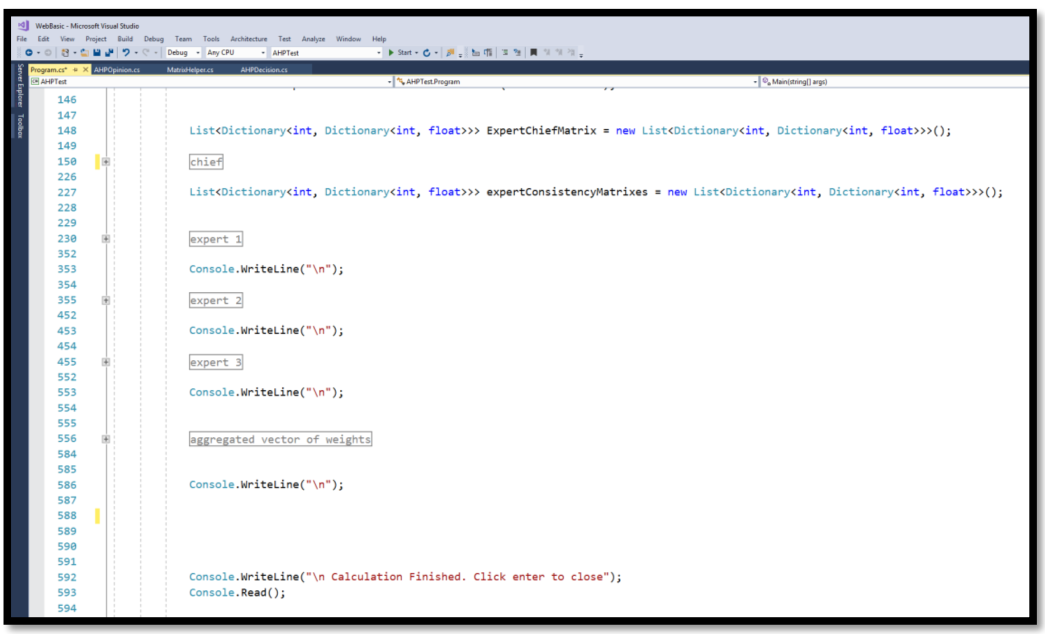


Figure 3.16. Window of the main program of the AHP test

The following screen shows an example in which three experts give non consistent judgments when pairwise evaluating a set of five alternatives. The related closest consistent matrices are shown as well.

```
Input matrix for expert 1:

 1    2    3    4    5
 ,5   1    ,2   4    3
 ,33  5    1    ,33  3
 ,25  ,25  3    1    4
 ,2   ,33  ,33  ,25  1


Consistency Information about expert 1:

Consistency Rate = ,3624
Criteria Weights:
,36
,19
,23
,18
,05


The matrix for expert 1 is NOT consistent!!


Closest consistent matrix for expert 1:

 1    2,51 2,35 2,76 7,36
 ,4   1    ,94  1,1  2,93
 ,43  1,07 1    1,17 3,13
 ,36  ,91  ,85  1    2,67
 ,14  ,34  ,32  ,37  1


Input matrix for expert 2:

 1    4    7    2    1
 ,25  1    3    1    6
 ,14  ,33  1    2    ,25
 ,5   1    ,5   1    ,25
 1    ,17  4    4    1


Consistency Information about expert 2:

Consistency Rate = ,3517
Criteria Weights:
,36
,28
,07
,1
,19


The matrix for expert 2 is NOT consistent!!


Closest consistent matrix for expert 2:

 1    1,66 4,72 3,89 1,84
 ,6   1    2,85 2,35 1,11
 ,21  ,35  1    ,82  ,39
 ,26  ,43  1,21 1    ,47
 ,54  ,9   2,57 2,12 1


Input matrix for expert 3:

 1    ,5   3    2    1
 2    1    ,25  4    3
 ,33  4    1    5    2
 ,5   ,25  ,2   1    ,33
 1    ,33  ,5   3    1


Consistency Information about expert 3:

Consistency Rate = ,2449
Criteria Weights:
,26
,24
,31
,06
,13


The matrix for expert 3 is NOT consistent!!


Closest consistent matrix for expert 3:

 1    ,87  ,74  3,25 1,43
 1,15 1    ,85  3,73 1,64
 1,35 1,17 1    4,37 1,93
 ,31  ,27  ,23  1    ,44
 ,7   ,61  ,52  2,27 1



 Calculation Finished. Click enter to close
```

Figure 3.17. Example of non-consistent input matrices for three experts

In such cases, the procedures described in the section 3.2 of this chapter are implemented to achieve the adjusted matrix. Once consistency is achieved, the system sends a final mail to the experts by asking for their approval on the modified judgments (translated from numbers to linguistic variables to be presented to the expert). If the experts accept the final evaluation, the process of getting criteria weights can be initiated. However, in the case in which the expert is not inclined to change his/her opinions, the matrix is kept inconsistent. If the expert denies to modify all the proposed judgments, the system will send an email to the chief of the project, by inviting him/her to find an agreement with the expert and eventually repeat the whole procedure of evaluation. This takes back to the lower block of Figure 3.9.

At the point in which each expert has associated a final adjusted consistent matrix, a vector of priorities can be calculated. To implement this method, the following property has been included in the class "`MatrixHelper`":

```
private float errorAcceptableEigenValue = (float)(3E-5);
```

It means that the process to achieve the vector of criteria weights iterates until the tolerance of `3E-5` between each value of the eigenvector and the former one is overcome.

A vector of weights is finally calculated for each expert. These vectors are eventually aggregated into a single priority vector by taking into account the importance of each expert previously expressed by the chief. It is herein preferred to aggregate the priorities and not the judgments of matrices to take into account situations in which evaluations could diverge too much. In such cases, aggregation of judgments would not be representative enough.

As already said, the aggregated vector of weights can be obtained either through AIJ or AIP. The aggregation can be led in both cases by means of the weighted arithmetic mean method (WAMM) or the weighted geometric mean method (WGMM), whose advantages and disadvantages are discussed by Ossadnik *et al.* (2016).

These authors assert that the AIJ (WAMM) has to be excluded from any application since it would violate the indispensable condition of reciprocity and generate inconsistent group matrices (even for perfect consistent individual judgments). The geometric variant of the AIJ (WGMM) is suitable for certain decision problems even if using just individual judgments. This method improves the collective consistency level, then the quality of decision, but a strong condition has to be respected: decision makers should agree to be considered as a synergistic unit and give up their individual evaluations in favour of the collective group preferences. The authors also demonstrate that the AIP procedure, and in particular the WGMM, is more suitable to be used as rational group decision support, given its great potential in supporting decisions with diverging or conflicting goals and simultaneously guaranteeing the fulfilment of the power conditions. For these reasons, the AIP-WGMM is used in the present context to achieve the aggregated vector of weights.

The last step of the process consists in evaluating alternatives and sharing the final ranking with the chief of the project. Once evaluation criteria have been weighted, the same process is repeated to obtain local priorities of alternatives. In this case, the chief has to make the evaluation by pairwise comparing them under each criterion. The process of checking consistency and negotiating about possible judgments to be adjusted is thus undertaken by directly communicating with the chief of the project.

The local priorities of alternatives are again calculated and, for each alternative, and have to be aggregated using equation (1.3) using the criteria weights already determined.

The presented website implementation can be useful to propose the usage of the AHP methodology to support worldwide companies and professional in managing complex decision-making problems. However, depending on the problem to be faced, other MCDM methods can be approached, as shown in the next chapter. Also for these techniques, a further development of website application may be considered in the future.

Chapter 4

Practical maintenance application of other MCDM methods

Beyond the AHP methodology and the diverse perspectives from which it has been approached, this doctoral thesis also contemplates the application of other MCDM techniques, which can greatly help in the field of maintenance management and risk assessment. Specifically, the decision-making methods belonging to the ELECTRE family and the TOPSIS are proposed and applied in the present chapter to make decisions in various real complex systems. The following two subsections deal with the ELECTRE family, whereas the last uses TOPSIS.

4.1. Outranking decision-making methods

Originally born in France at the end of the 1960s (Roy, 1968), the ELECTRE methods are fundamentally based on the so called outranking approach (Roy, 1991), seeking to establish outranking relations by pairwise comparing alternatives. These relations need to be examined and confirmed by means of two tests, namely the concordance and the discordance tests, aimed at calculating the concordance and discordance indices.

On one hand, the concordance index C_{ij} quantitatively expresses, referring to a specific criterion, the agreement degree about the fact that alternative A_i outranks or has been evaluated equal to alternative A_j .

On the other hand, the discordance index D_{ij} quantitatively expresses, referring to a specific criterion, the agreement degree about the fact that alternative A_i has a worst score compared to alternative A_j .

The ELECTRE methods generally require the preliminary collection of the following input data: 1) set of alternatives, $A_i, i = 1, \cdots, n$, to be evaluated; 2) evaluation criteria, $B_k, k = 1, \cdots, m$; 3) vector of criteria weights, $w_k, k = 1, \cdots, m$; 4) numerical evaluation of alternatives with respect to the considered criteria, $u_k(A_i)$. After having collected and possibly normalised input data in a dedicated matrix, the development of procedure is organised in two phases:

1st PHASE: developing of an outranking relation characterising the pairwise comparisons between alternatives and accomplishment of the concordance and discordance tests;

2nd PHASE: exploitation of the outranking relation in order to obtain the final result (selection of a single option, ranking of a set of alternatives or classification of alternatives into clusters) by means of a specific rule.

ELECTRE methods support analysts in a wide range of decision making problems (Abedi *et al.*, 2012; Hatami-Marbini and Tavana, 2011; Hokkanen *et al.*, 1995; Jun *et al.*, 2014; Proulx *et al.*, 2007). Various versions have been proposed (Rogers *et al.*, 2013); the main ones (Figueira *et al.*, 2013) are mentioned in the Figure 4.1, with their relative objectives.

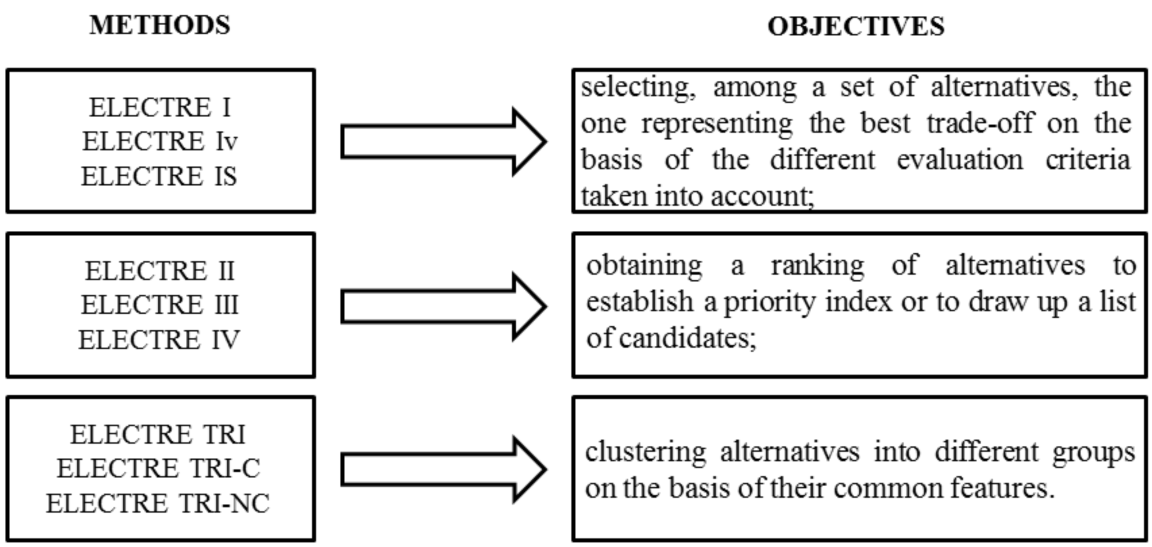


Figure 4.1. ELECTRE methods and objectives

Among the different versions proposed, the present doctoral thesis presents the practical comparison, focused on maintenance, between two methodologies. This comparison refers to a current research led to treat the present topic (Carpitella *et al.*, 2018a). First, the ELECTRE I is herein applied to lead towards the selection of the best option belonging to a set of alternatives, that is, the option representing the best trade-off on the basis of the evaluation of the involved criteria. Second, the ELECTRE III (Certa *et al.*, 2013a; Certa *et al.*, 2009; La Scalia *et al.*, 2015) is proposed to get the ranking (Vincke, 1992) of various considered alternatives, with the aim to provide useful information to optimise maintenance management.

As shown by Govindan and Jepsen (2016), ELECTRE III is the most used method of the ELECTRE family and the main fields of application are natural resources and environmental management, energy management, and water management.

The following maintenance-based case refers to a real water distribution system and is implemented to compare results, in terms of first alternative obtained by respectively applying ELECTRE I and III.

4.2. A maintenance-based comparison

A comparison between the results of the two methodologies ELECTRE I (Carpitella *et al.*, 2017b) and III (Carpitella *et al.*, 2018a) is presented with relation to a real case study focused on maintenance management of water supply systems (WSSs). Such kind of complex systems support many daily human activities and, for this reason, their full availability has to be assured. Indeed, a sudden disruption of hydraulic supply may cause enormous inconveniences. For this reason, the activities of maintenance for WSSs have critical importance, and interventions have to be adequately planned and implemented. Firstly, various parts of the networks need to be monitored and kept under control. Since most of the infrastructure is buried, non-destructive techniques (NDTs) are essential tools of inspection to explore and obtain information about the underground without damaging the infrastructure. NDTs make it easy the inspection of possible damages and the overall evaluation of WSSs, with the aim of optimizing maintenance and costs.

A wide number of NDTs are reported in the literature (Liu and Kleiner, 2013; Hao *et al*., 2012) to locate damages in WSSs, the most popular being acoustic methods, thermography and GPR (Demirci *et al*., 2012; Dong *et al*., 2011). In particular, the GPR technique is more effective than the acoustic methods in locating water leaks occurring in plastic pipes (Bimpas *et al*., 2010). It is also more flexible than thermography approaches because it can be used in all the seasons of the year without being affected by temperature variations (Ayala-Cabrera *et al*., 2013a). Moreover, the GPR technique reveals to be a useful tool in easily exploring hidden elements (Gurbuz *et al*., 2012; Hoarau *et al*., 2017; Forte and Pipan, 2017) by means of radargrams.

Radargrams provide graphical representation of contrasts existing between specific elements and the surrounding medium, due to their different dielectric characteristics (Crocco *et al*., 2010). The main difficulty in using radargrams derives from the big volume of information and the complexity of data interpretation, being necessary a high level of ability and experience by the involved personnel (Ayala-Cabrera *et al*., 2011; Thomson *et al*., 2009). For this reason, a plethora of processes and analysis methods have been developed. These methods filter and mine GPR images to improve data visualization, with the aim of effectively identifying abnormal situations occurring underground.

In this context, the ELECRE I and III methodologies are both applied to a set of four GPR images resulting from the application of four different data processing techniques. In the first case, the purpose consists in selecting the best option whereas, in the second case, the ranking of alternatives is built with the aim to prioritize techniques of data processing to prevent and discover eventual damages or water losses occurring in buried pipes. Figure 4.2 presents the set of alternatives to be evaluated.

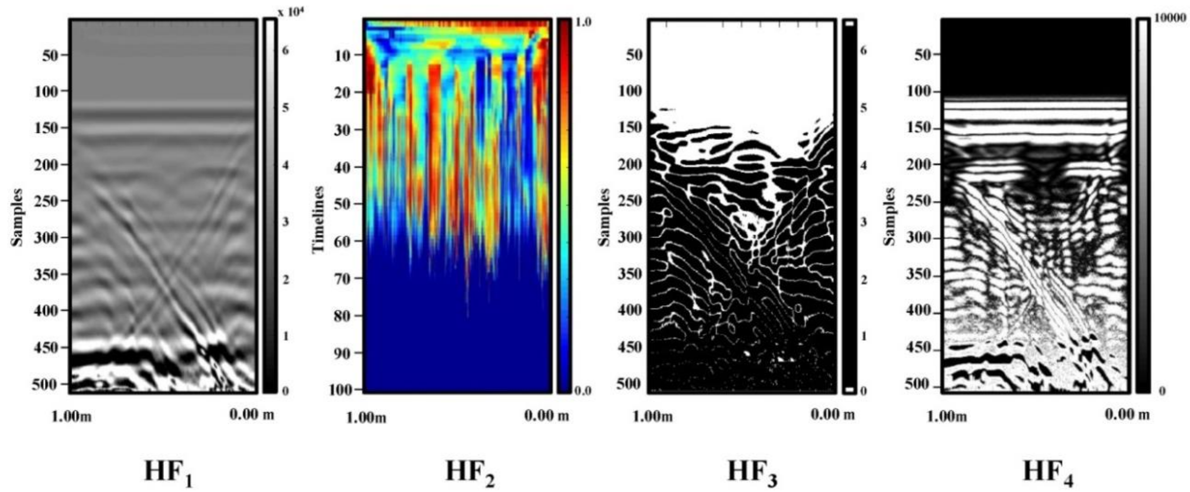


Figure 4.2. GPR images: raw image (HF_1), and images resulting from multi-agent system (HF_2), subtraction method (HF_3), and variance filter (HF_4) techniques

The set of four GPR images (HF_1, HF_2, HF_3, HF_4) represents the set of four outputs of the data processing techniques briefly described, in sequence, in the next paragraph.

The analysis of raw images (Hunaidi and Giamou, 1998) obtained from a preliminary inspection, despite not being a proper method, is widely used to identify various features in the networks (Ocaña-Levario, 2014). The multi-agent system (Ayala-Cabrera *et al*., 2013b; Shoham and Leyton-Brown, 2009) uses a multi-agent-based system to identify elliptical shapes related to abnormal conditions in the system. The subtraction method (Ayala-Cabrera *et al*., 2014) proposes a subtraction between two GPR images in order to discover hidden features in the explored area. Lastly, the variance filter (Ocaña-Levario *et al.*, 2018) applies the so-called variance filter to raw GPR images to analyse data variability.

First scenario: ELECTRE I

The considered alternatives are evaluated on the basis of four criteria (B_1, B_2, B_3, B_4), all to be maximized, that respectively are: visualisation, interpretation, identification of features, and extraction of information. The criteria weights were determined with the support of an expert. Table 4.1 gives the normalised input data.

Table 4.1. Normalised input data

	B_1	**B_2**	**B_3**	**B_4**
weights	**40%**	**30%**	**20%**	**10%**
HF_1	0,3	0,2	0,6	0,5
HF_2	0,8	0,5	0,7	0,8
HF_3	0,7	0,7	0,9	0,5
HF_4	0,8	0,7	0,8	0,5

For the sake of conciseness, the steps of the 1st PHASE are not completely reported. Tables 4.2 and 4.3 respectively show results of correlation and non-discordance tests, by assuming the following thresholds $C^* = 0.6$ and $D^* = 0.45$.

Table 4.2 Correlation test matrix

$\boldsymbol{T_C(HF_i, HF_j)}$	**HF_1**	**HF_2**	**HF_3**	**HF_4**
HF_1	-	0	0	0
HF_2	1	-	0	0
HF_3	1	0	-	1
HF_4	1	1	1	-

Table 4.3 Non-discordance test matrix

$\boldsymbol{T_D(HF_i, HF_j)}$	**HF_1**	**HF_2**	**HF_3**	**HF_4**
HF_1	-	0	0	0
HF_2	1	-	1	1
HF_3	1	1	-	1
HF_4	1	1	1	-

Table 4.4 lastly gives the outranking matrix, which is the output of the 2nd PHASE, closing the application of the whole ELECTRE I method. The best GPR image result corresponds to alternative HF_4, coming from the method based on the variance filter. This alternative outranks all the others, representing the best compromise among all considered

criteria. Moreover, we can observe that HF_4 outranks HF_3 and vice versa, whereas HF_1 does not outrank any alternative but is outranked by all the others.

Table 4.4. Outranking matrix

$S(HF_i, HF_j)$	**HF₁**	**HF₂**	**HF₃**	**HF₄**
HF_1	-	0	0	0
HF_2	1	-	0	0
HF_3	1	0	-	1
HF_4	1	1	1	-

Second scenario: ELECTRE III

We are now interested in carrying out a further in-depth analysis of the four data processing methods and in drawing up a ranking of resulting GPR images by considering a fifth evaluation criterion, namely the affordability of the analyses. ELECTRE III provides decision makers with a ranking of alternatives and, consequently, with a proper support to optimize maintenance of WSSs, taking also into account data uncertainty by means of the use of appropriate thresholds.

The vector of criteria weights, calculated by means of the FAHP technique, is the same reported in the example considered in Chapter 1, Section 1.3. The input data required to apply the ELECTRE III methodology were collected with the support of the expert mentioned in the former section and are given in Table 4.5. The scale of evaluations of alternatives under the various criteria is one-to-ten.

Table 4.5. Input data of the ELECTRE III

	B_1	**B_2**	**B_3**	**B_4**	**B_5**
weights	0.2934	0.2226	0.1380	0.1109	0.2351
I_k - S_k - V_k	2-4-6	1-3-5	1-2-3	1-2-3	1-3-5
HF_1	3	2	6	5	8
HF_2	8	5	7	8	4
HF_3	7	7	9	5	6
HF_4	8	7	8	5	6

The output of the 1st PHASE of ELECTRE III is the outranking credibility matrix $\delta(\mathrm{HF}_i, \mathrm{HF}_j)$, which enables to calculate the minimal value of outranking credibility, that is, $\delta_0 = 0.85$, with the purpose of building the Boolean matrix $T(\mathrm{HF}_i, \mathrm{HF}_j)$.

Table 4.6. Outranking credibility matrix

$\delta(HF_i, HF_j)$	HF_1	HF_2	HF_3	HF_4
HF_1	-	0	0	0
HF_2	0,7649	-	0,63315	0,77155
HF_3	0,88245	0	-	1
HF_4	0,88245	0	1	-

The last step of the 2nd PHASE consists in determining the qualification of alternatives $q(HF_i)$ for the final ranking to be built. These results are given in Table 4.7. Since the two distillation procedures do not give the same ranking, sub-distillation between HF_1 and HF_2 is necessary.

Table 4.7. Qualification of alternatives

Alternatives	$q(HF_i)$
HF_1	-2
HF_2	0
HF_3	1
HF_4	1

Table 4.8. Ascending distillation results

Alternatives	Position
HF_2, HF_3, HF_4	**1°**
HF_1	**2°**

Table 4.9. Descending distillation results

Alternatives	Position
HF_3, HF_4	**1°**
HF_2	**2°**
HF_1	**3°**

The final ranking is given in Table 4.10. Both alternatives HF_3 and HF_4 occupy the first position of the ranking. It means that there is not a significant difference among them. Then, under the perspective of the considered criteria, the application of the subtraction method or of the variance filter is indifferent for supporting and optimizing maintenance activities of WSSs.

Table 4.10. Final ranking

Alternatives	Position
HF_3, HF_4	**1^{st}**
HF_2	**2^{nd}**
HF_1	**3^{rd}**

4.3. A combined multi-objective and multi-criteria approach

This section proposes the application of another MCDM method, the TOPSIS, to support maintenance management of complex systems. In this regard, water distribution systems are considered and the problem analysed regards the issue of optimal pump scheduling, fundamental in optimising operation of such networks.

Specifically, a combined approach of a multi-objective optimization technique, namely a genetic algorithm, and a MCDM method, namely TOPSIS, is proposed.

Considering the complexity of water networks and the highly non-linear nature of the hydraulic equations describing them, hydraulic models coupled with optimization algorithms have been widely applied to design optimal operation strategies. Several works in the literature propose solutions for the optimal pump scheduling problem. Among the used techniques, Linear Programming (Jowitt and Xu, 1990), Dynamic Programming (Jowitt and Germanopoulos, 1992) and evolutionary algorithms, such as Genetic Algorithms (Farmani *et al.*, 2007) and Particle Swarm Optimization (Brentan and Luvizotto, 2014) can be highlighted.

Despite single-objective optimization is able to find interesting solutions, problem-solving using it may be complex, mainly for the application of bio-inspired algorithms, and requires special attention to the constraints. Constraints can be managed through penalty functions, which artificially penalize the objective function when constraints are violated. However, penalty functions are hard to be selected and can affect directly the performance of the optimization process. In contrast, multi-objective optimization introduces a new perspective and the constraints of the problem may be treated as objectives to meet.

Multi-objective algorithms (MOAs) have been widely applied in urban hydraulics. Instead of a single solution, the final response of an MOA is a set of solutions, the so-called Pareto front, which water utility staff can use as an aid in decision-making. Considering the problem of optimal scheduling of pumps and valves in water distribution networks, leak reduction by pressure management and minimal pressure of the system are conflicting objectives. This is also the case for energy saving, which requires lower pump operation.

Finally, tank level oscillations are also considered. Despite an MOA can be useful to propose various optimal operation solutions, the final decision, which have to be picked out from the Pareto front, may be problematic for the operators. With this perspective, this section suggests a combined approach to first find the Pareto front of non-dominated solutions and, then, rank them on the basis of a set of weighted criteria to aid decision-making. In particular, the Non-Dominated Sorting Genetic Algorithm (NSGA-II) is herein applied to solve the multi-objective problem. This problem may be stated in terms of several functions. First, the energy cost, F_1, for the pump system given by:

$$F_1 = \sum_{i=1}^{N_p} \sum_{t=1}^{P_e} \frac{Q(\alpha_{i,t})H(\alpha_{i,t})\gamma}{\eta_{i,t}} \Delta t c_t; \tag{4.1}$$

where:

- N_p is the number of pumps in the system working during time horizon P_e;
- $Q(\alpha_{i,t})$ is the pumped flow;
- $H(\alpha_{i,t})$ is the hydraulic head for pump i operated under status α at time step t, with efficiency $\eta_{i,t}$;
- γ is the specific weight of water;
- Δt is the time step, one hour in this work;
- c_t is the energy cost at time step t.

As an operational problem, the solution of the pump scheduling is constrained by the minimum pressure P_{min} in the system; the oscillation of tank levels between their maximum, $T_{k,max}$, and minimum, $T_{k,min}$; and the number of status switches during the operation horizon.

To avoid penalty functions for treating these constraints, new objectives, F_2, F_3, F_4, respectively, can be written to complete the multi-objective optimization process. For the three mentioned constraints the objective functions are:

$$F_2 = \sum_{j=1}^{N_n} \sum_{t=1}^{P_e} |P_{j,t} - P_{min}|; \tag{4.2}$$

$$F_3 = \sum_{k=1}^{N_t} \sum_{t=1}^{P_e} |T_{k,t} - T_{k,min}| + \sum_{i=1}^{N_t} \sum_{t=1}^{P_e} |T_{k,t} - T_{k,max}|; \tag{4.3}$$

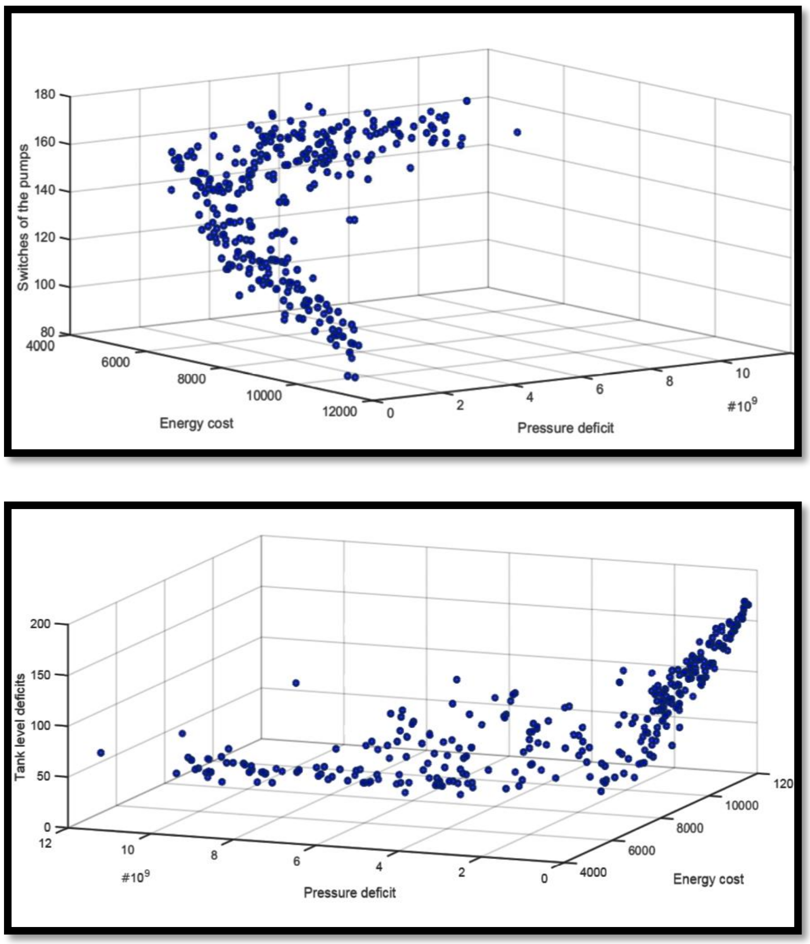
$$F_4 = \sum_{i=1}^{N_p} \sum_{t=1}^{P_e} s_{i,t}; \tag{4.4}$$

where:

- $P_{j,t}$ is the pressure at demand node j, for a water network having N_n demand nodes and N_t tanks;
- $T_{k,t}$ is the water level in tank k;

- $s_{i,t}$ is the number of status switches for pump i during the time horizon.

The NSGA-II is a genetic algorithm development for multi-objective problems, proposed in (Deb *et al.*, 2002). In each iteration, NSGA-II improves the fitness of a population of candidate solutions to a Pareto front according to various objective functions. Through evolutionary strategies (e.g. crossover, mutation and elitism), the population is organized by Pareto dominance. Similarly, sub-groups on the Pareto front are appropriately evaluated, what eventually promotes a diverse front of non-dominated solutions. Figure 4.3 presents a 3D representation of the Pareto front considering the schedule of the pumps.



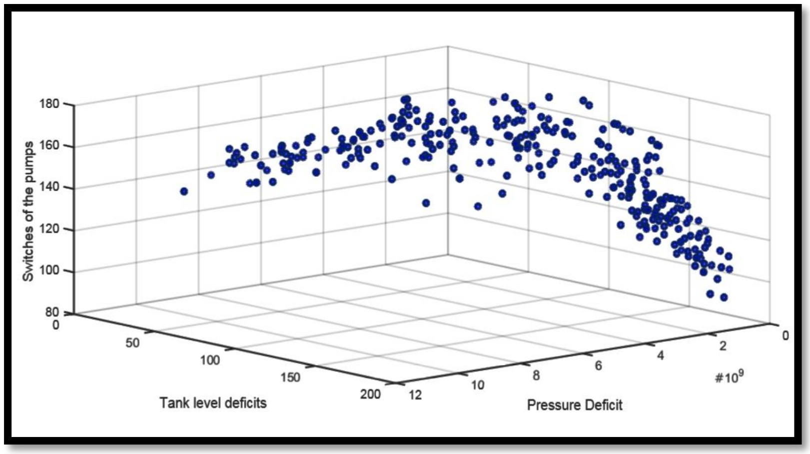


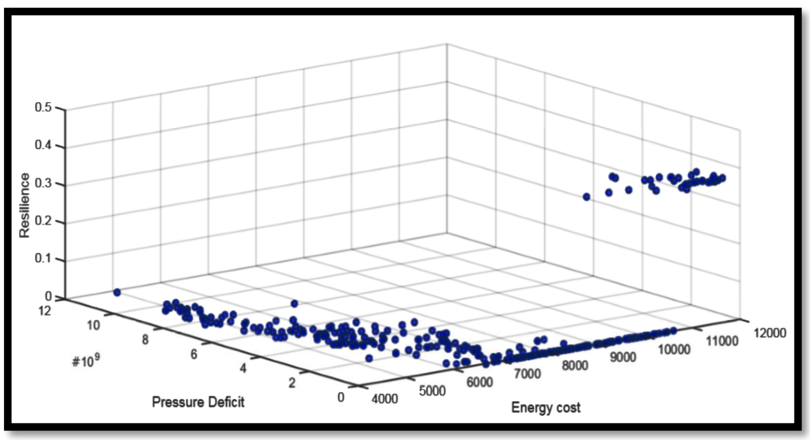
Figure 4.3. 3-D representation of the Pareto front for the optimal pump scheduling

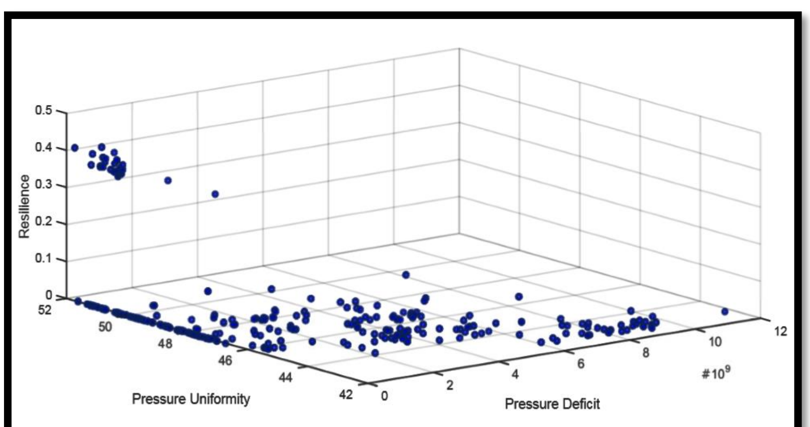
Regarding the MCDM approach, a brief description of the TOPSIS implementation is next provided. This method is basically founded on the concept of distance between each alternative to a positive ideal solution and to a negative ideal solution (nadir). In this regard, the best alternative is that characterized by the shortest distance from the positive ideal solution, and the farthest from the nadir. The choice of the TOPSIS as MCDM method to be integrated with multi-objective optimisation is due to its capability of ranking a wide number of alternatives. This approach can be considered as a driver in implementing the alternative that represents the best trade-off according to the various considered criteria. The process is supported by analysing feedback coming from a team of experts. The TOPSIS method, implemented to rank the set of alternatives, requires a decision matrix as input data, in which the assessment of each alternative under the considered evaluation criteria is given, besides the vector of criteria weights reflecting the perceptions of the involved team of experts concerning the object of analysis. The combined approach for optimal pump scheduling is applied to the D-town network, a benchmark water distribution network presented in (Stokes *et al.*, 2012). This network is formed by 396 nodes, 13 pumps and 4 pressure reducing valves. It has been explored in the literature from the energy and leakage management points of view.

The NSGA-II algorithm implemented in Matlab was run using 900 random solutions to find the Pareto front for the optimal pump scheduling problem. By observing Figure 4.3, it can be highlighted that the more expensive the operation, the lower the deficit of pressure. This relation is clear, since more expensive operations are related to longer use of pumps, thus putting more hydraulic head into the system. The optimal operational cost increases when the number of switches decreases. Larger number of switches allows better pump management, saving more energy, even if this can impair the future behaviour of the pumps. Lastly, tank deficit increases with the operational costs, since the higher the hydraulic head in the network, the higher the volume overflowed from the tanks. Working on the solutions of the Pareto front, the multi-criteria analysis aims to identify what is the most adequate. To such an aim, the following four criteria related to water distribution network management are considered:

- operational cost obtained from the Pareto front;
- operational lack of service, herein considered as pressure deficit at the demand nodes;
- pressure uniformity (PU) parameter, for evaluating pressure compliance. This parameter allows to assess pressure in the system in terms of the differences between the operational and the minimal and average pressures in the system. Less uniform pressure zones, with high pressure difference values, are found in the network corresponding to bigger values of PU;
- the resilience of the network, calculated as proposed in (Todini, 2000).

To identify the correlations of criteria with respect to the solutions in the Pareto front, the following group of figures (4.4) presents a 3D representation of the criteria parameters.





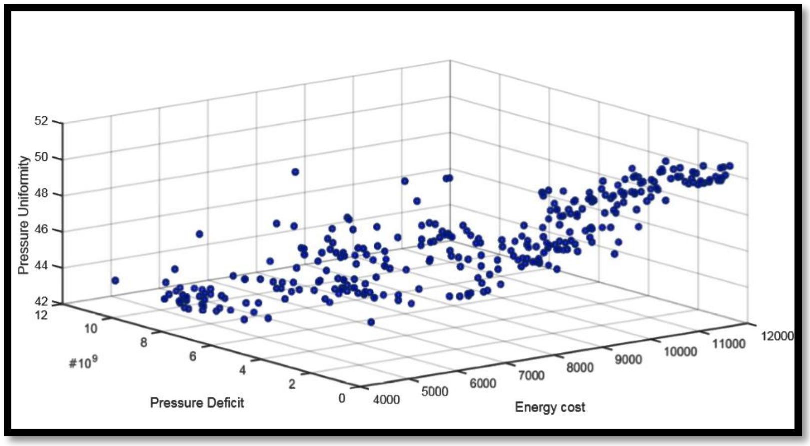


Figure 4.4. 3-D representation of the criteria for each solution of the Pareto front

The TOPSIS methodology was carried out to rank a set of 315 Pareto solutions. Each solution was codified with a code, PS_n, n varying from 1 to 315, and was quantitatively evaluated under the four given criteria, respectively identified as C_1, C_2, C_3 and C_4. The first three criteria have to be minimized whereas the fourth has to be maximized. The ideal and the nadir points for the weighted, normalized decision matrix are, respectively:

$$A^* = (8.60E-03;\ 0.00E+00;\ 1.26E-02, 4.51E-02); \quad (4.5)$$

$$A^- = (2.11E-02;\ 4.23E-02;\ 1.53E-02;\ 0.00E+00). \quad (4.6)$$

Three scenarios were then analysed: the first considers equal assignment of weights; in the second scenario lower importance is given to the economic aspect, whereas the third scenario contemplates the prominence of cost with respect to the other features.

After implementing the TOPSIS and achieving the complete ranking of the 315 alternatives, we provide the set of the ten best solutions, the ones in the first ten positions in the ranking according to the closeness coefficient. Results for the three scenarios and the numerical evaluations of the criteria are given in the following tables (4.11, 4.12, 4.13).

Table 4.11. Final ranking for the vector of criteria weights [0.25, 0.25, 0.25, 0.25]

Ranking position	Solution	C_1	C_2	C_3	C_4	C_i^*
1	PS_{49}	9.32E+03	0.00E+00	5.08E+01	4.36E-01	0.876997066
2	PS_{203}	9.39E+03	0.00E+00	5.04E+01	4.31E-01	0.87518652
3	PS_{218}	8.79E+03	1.83E-01	4.86E+01	3.95E-01	0.87194253
4	PS_{47}	9.25E+03	9.79E-01	4.71E+01	3.95E-01	0.863532413
5	PS_{50}	1.00E+04	0.00E+00	5.03E+01	4.13E-01	0.855952728
6	PS_{196}	1.01E+04	0.00E+00	5.03E+01	4.10E-01	0.852093356
7	PS_{234}	9.66E+03	0.00E+00	5.02E+01	3.93E-01	0.851803008
8	PS_{302}	1.03E+04	0.00E+00	5.17E+01	4.13E-01	0.848682341
9	PS_{175}	1.01E+04	0.00E+00	5.02E+01	3.93E-01	0.842288929
10	PS_1	1.05E+04	0.00E+00	5.08E+01	4.08E-01	0.84207829

Table 4.12. Final ranking for the vector of criteria weights [0.10, 0.30, 0.30, 0.30]

Ranking position	Solution	C_1	C_2	C_3	C_4	C_i^*
1	PS_{49}	9.32E+03	0.00E+00	5.08E+01	4.36E-01	0.943693148
2	PS_{203}	9.39E+03	0.00E+00	5.04E+01	4.31E-01	0.943618474
3	PS_{50}	1.00E+04	0.00E+00	5.03E+01	4.13E-01	0.929216493
4	PS_{196}	1.01E+04	0.00E+00	5.03E+01	4.10E-01	0.925949449
5	PS_{302}	1.03E+04	0.00E+00	5.17E+01	4.13E-01	0.924835185
6	PS_1	1.05E+04	0.00E+00	5.08E+01	4.08E-01	0.920890314
7	PS_{169}	1,06E+04	0,00E+0	5,07E+01	4,07E-01	0.91954257
8	PS_{218}	8.79E+03	1.83E-01	4.86E+01	3.95E-01	0.919142038
9	PS_{47}	9.25E+03	9.79E-01	4.71E+01	3.95E-01	0.919006964
10	PS_{289}	1.11E+04	0.00E+00	5.11E+01	4.10E-01	0.91892125

Table 4.13. Final ranking for the vector of criteria weights [0.40, 0.20, 0.20, 0.20]

Ranking position	Solution	C_1	C_2	C_3	C_4	C_i^*
1	PS_{218}	8.79E+03	1.83E-01	4.86E+01	3.95E-01	0.794510671
2	PS_{49}	9.32E+03	0.00E+00	5.08E+01	4.36E-01	0.787181766
3	PS_{203}	9.39E+03	0.00E+00	5.04E+01	4.31E-01	0.783915221
4	PS_{47}	9.25E+03	9.79E-01	4.71E+01	3.95E-01	0.777539553
5	PS_{234}	9.66E+03	0.00E+00	5.02E+01	3.93E-01	0.760784162
6	PS_{50}	1.00E+04	0.00E+00	5.03E+01	4.13E-01	0.756493058
7	PS_{196}	1.01E+04	0.00E+00	5.03E+01	4.10E-01	0.751291207
8	PS_{302}	1.03E+04	0.00E+00	5.17E+01	4.13E-01	0.745726008
9	PS_{175}	1.01E+04	0.00E+00	5.02E+01	3.93E-01	0.743955655
10	PS_1	1.05E+04	0.00E+00	5.08E+01	4.08E-01	0.736068121

By observing the reported results, it is possible to note some variations in the final rankings. The solution PS_{49} appears to be the more suitable trade-off among the set of optimal alternatives, according to the evaluations of the considered criteria, occupying the first position in two considered scenarios, and the second position in the last scenario.

To evaluate the effects of leakage (Farley and Trow, 2003) in the optimal solution, leaks were added for each pipe. The leakage model (see next equation) considers the following pressure-driven model (Kabaasha *et al.*, 2018):

$$Q_{t,m}^{leak} = \beta \cdot \left(\frac{P_{m,t}^{u}+P_{m,t}^{d}}{2}\right)^{\alpha}; \qquad (4.7)$$

in which the pressure in the leakage orifice of a pipe m is taken as the mean value between the upstream, $P_{m,t}^{u}$, and the downstream, $P_{m,t}^{d}$, pressures. Coefficients β and α depend on the leakage features; the values herein adopted are 10^{-6} and 0.9, respectively (Lambert, 2001).

In terms of the four criteria, solution PS_{49} evaluated under the leakage condition presents an increase of energy consumption and of PU, while resilience decreases. In the leakage scenario, the pumps work out of the optimal operation point, resulting in lower efficiency. As leakage changes the operational point of pumps and the pressure in the network, PU increases, thus pointing to lower pressure uniformity in the system.

The evaluation of leakage is very useful to plan and implement maintenance interventions for the selected configuration of network.

PART II

RELIABILITY ANALYSIS AND MAINTENANCE MONITORING

As previously underlined, maintenance management has to be undertaken on the basis of operational features of systems and of the various objectives to be pursued.

The present Part II of the thesis deals with the main aspects related to the process of maintenance management of complex systems. The topic of reliability analysis is developed in Chapter 5, and the most important parameters to lead such kind of study are discussed. In addition, after having examined reliability configurations of interest for complex systems, a combined approach integrating reliability analysis and MCDM methods is proposed and then applied to a real-world case study.

Chapter 6 treats the theme of human reliability analysis, considered as fundamental in any level and kind of industrial/business activities (Chidambaram, 2016; Hinshaw, 2016). Indeed, human factors are intrinsically involved in processes and may be responsible of several accidents and incidents if not correctly identified and managed (Ergai *et al.*, 2016). In particular, some techniques of human reliability analysis are recalled and interactions of human factors are evaluated, by means of a MCDM approach, with special regard to manufacturing processes in which the role of maintenance is crucial.

Lastly, Chapter 7 is centred on the process of maintenance monitoring. With this regard, the fundamental part played by KPIs is underlined. Particular attention is given to the phase of their selection. Also, the blockchain technology is proposed to optimise predictive maintenance and a proposal of application is presented.

The MCDM methods used in the present Part II are the Fuzzy TOPSIS (FTOPSIS) (Chen, 2000) in Chapter 5, and the Decision-Making Trial and Evaluation Laboratory (DEMATEL) methods in Chapters 6 and 7.

Chapter 5

Reliability analysis

A preliminary reliability analysis of the system under study is essential, representing a fundamental step in supporting strategic actions (Exner *et al*., 2014). This enables to fulfil important objectives in terms of security and, in general, optimise system performance (Catelani *et al*., 2013).

5.1. Choosing the fundamental parameters of study

With relation to a generic complex system, it is possible to define the *reliability function* $R_S(t)$ as the probability of functioning without failing for a given interval of time *t*, at predetermined environmental conditions.

The definition of reliability presumes that a specific criterion for verifying the functioning state of the system has been previously established. In some cases, indeed, it is necessary to fix a threshold beyond which the system is considered as faulty. Moreover, it is necessary to preliminary define the particular working conditions and the interval of time t as observation period, during which the functioning of the system is required.

The reliability function varies with time and the variation depends on the probability law related to failure occurrence in time. Reliability evaluation has to be led on the basis of historical data referring to the behaviour of the system under analysis during its lifespan. In particular, this evaluation has to involve the elements/components in which the system is decomposed.

Not only is reliability fundamental for organising how to conduct processes and organize production, but also for optimising safety and security conditions in industrial workplaces. Increasing reliability means an initial increasing of cost, due to the investment in a better performing system, but it reveals to be strategic, above all in terms of reduction of maintenance interventions.

Broadly speaking, the reliability function of an element i can be expressed with relation to the failure probability, $F_i(t)$ as:

$$R_i(t) = 1 - F_i(t) = 1 - \int_0^t f_i(t)dt = \int_t^\infty f_i(t)dt; \quad (5.1)$$

$f_i(t)$ being the failure density function. Moreover, $\lambda_i(t)$ being the failure rate of the element *i*, it is possible to rewrite the preceding expression as a negative exponential:

$$R_i(t) = e^{-\int_0^t \lambda_i(t)dt}. \quad (5.2)$$

There are situations in which the calculation of reliability is simpler, due to the fact that the failure rate can be assumed as constant or, in other terms, failures occur randomly and the element can be considered as not affected by the occurrence of former failures, something that is referred to as a system *without memory*. This is, for instance, the case of electronic components, which, differently from the mechanical ones, are less affected by the wear phenomenon due to damage accumulation. However, this assumption represents just a way to simplify calculations since, in reality, the failure rate is never constant. Its general trend is commonly known as *bathtub curve* and is shown in Figure 5.1.

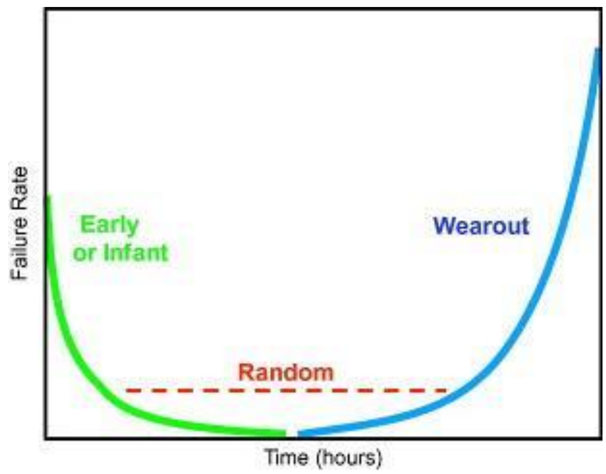


Figure 5.1. A traditional bathtub curve provided by Roesch (2012)

By observing the function failure rate it is possible to isolate three main phases. The first one, in which the function decreases, is characterised by early *infant mortality* failures. The second phase, known as *maturity period*, corresponds to the useful life of the considered system/element, and the failure rate can be approximately considered as constant. Lastly, in the third phase, known as *wearout*, the function increases, and the transition from the normal to the catastrophic wear occurs. Each component can be characterised by one or more tracts of the curve.

The overall reliability of a system R_S depends on the reliabilities R_i of its elements and is achievable by knowing reliability bounds among them. The *Mean Time To Failure* (*MTTF*) is calculated as follows:

$$MTTF = \int_0^\infty R_S(t)dt. \tag{5.3}$$

The *MTTF* is a parameter of fundamental importance for analysing systems, since it provides an estimation of their mean time of functioning.

Real industrial systems are most often repairable, so that another parameter playing a crucial role, besides reliability, is the *availability function*, $A(t)$. It is defined as the probability that a given system is available at time *t*, without taking into consideration the possible occurrence of failures before *t*. For the calculation of the availability, the repair rate μ has to be considered, whose meaning is analogous to the failure rate λ for reliability. By assuming both the repair rate and the failure rate as constant, the availability at time *t* can be calculated by means of the following expression:

$$A(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot e^{-(\lambda+\mu)t}. \qquad (5.4)$$

If we are interested in calculating this parameter in a very long-time interval, we can refer to the *stationary availability* which, for *t* tending to infinity, can be expressed as follows:

$$A_s = \frac{\mu}{\lambda+\mu} = \frac{MTTF}{MTTF+MTTR}; \qquad (5.5)$$

$MTTF$ being the mean time to failure, and $MTTR$ the mean time to repair, respectively, corresponding to the inverse of the failure and the repair rates, provided they are assumed constant. The meaning of the stationary availability is the expected time percentage in which the system is in a functioning state (Certa *et al*., 2013a), that is the percentage of functioning time over the total time with relation to a generic failure-repair cycle.

As highlighted above, reliability and availability are two common measures of complex system performance, their role being essential in product and service quality (Akhavein and Fotuhi Firuzabad, 2011). Therefore, reliability and availability analyses are fundamental to support the analyst in the implementation of actions addressed to the improvement of the technical and economic performance of the system under investigation (Mi *et al*., 2016; Chalabi *et al*., 2016; Alrabghi and Tiwari, 2016). Since reliability and availability analyses are based on the identification of the major system criticalities, reliability relations among components need to be firstly established, and then the set of priority components to be maintained has to be selected.

5.2. Focusing on particularly relevant reliability configurations

Management of maintenance activities aims to optimise the reliability and availability parameters (Alzghoul and Löfstrand, 2011; Choi and Chang, 2016) by specifically taking into account uncertainty affecting data (Wang *et al*., 2016). In this regard, numerous contributions propose the use of mathematical programming. Vasili *et al*. (2011) present a detailed literature review and focus on optimization models for preventive maintenance policies, risk-based optimization models, and models constrained to ensure safety conditions. According to Yssaad and Abene (2015), maintenance optimization can be effectively pursued using a reliability-centred maintenance (RCM) approach (Moubray, 1991). The authors demonstrate the global improvement of reliability and availability parameters of power distribution systems arising from the implementation of an RCM approach; they use a Failure Mode, Effects and Criticality Analysis (FMECA) (Certa *et al*., 2017). Francese *et al*. (2014) and Curcurù *et al*. (2012; 2013; 2016) deal with reliability analyses of complex systems under the presence of epistemic uncertainty affecting input data. Martón *et al*. (2016) propose a model for the simultaneous optimization of testing and maintenance activities on ageing equipment with multiple items. The authors emphasize that the available literature proposes numerous models for assessing Reliability, Availability and Maintainability (RAM) of safety equipment.

Most of such models assess the risk level of technological systems or define appropriate design and/or surveillance and maintenance policies that ensure an optimum level of safety during the plant's operational life. Shariatkhah *et al*. (2015) propose a model that takes into account the dynamic behaviour of an energy conversion system to evaluate its availability. The authors mostly stress the need to consider the dependence of different forms of energy and propose a combined Montecarlo and Markov chain-based approach (Juneja and Shahabuddin, 1992). Sabouhi *et al*. (2016) refer to power plants to present a reliability model aimed at optimizing maintenance strategies and also highlight how system reliability must take into account data related to system critical components. Pang *et al*. (2016) apply a failure mechanism analysis to the main critical components of an aircraft to understand how the reliability of the system could be affected.

Lu and Wu (2014) propose for reliability analysis an approach analogous to that used in project management, by considering the decomposition of the general activity of the investigated system into its various working phases. Specifically, the authors perform a reliability analysis that considers the success of the overall mission and characterise such a state by means of the various system behaviours during each working phase. In such a way, the failure and repair behaviours of each component are characterized. In (Lu *et al*., 2015), the authors use the same approach to analyse the reliability of an aircraft when separately considering climbing, cruising, and landing phases.

As highlighted by Billinton and Allan (1992), relations among system components could often be represented by block diagrams, that are block schemes exhaustively representing components and how they are connected each other.

The simplest configurations are related to series and parallel systems. A system whose components are in series fails when just one of its elements fails, whereas a system with components in parallel functions until all its elements fail, that is to say system functioning is guaranteed when just one component functions. However, it is not always possible to represent components of a system as connected in series or in parallel. Indeed, real industrial systems are more often complex, that is to say, characterised by various functional interdependencies among elements. It is then necessary to apply advanced analysis techniques, influencing the calculation of reliability.

Obviously, configurations with redundant components are commonly designed to increase the overall level of the system reliability/availability. Chambari *et al*. (2012) underline the important role of redundancy in both reliability and cost optimization. They deal with a Redundant Allocation Problem (RAP) to find out the best redundancy strategy that improves the system operating conditions. A further RAP is solved by Yeh and Hsieh (2011), who propose a penalty guided artificial bee colony algorithm to investigate the optimal number of redundant components in design problems. Garg and Sharma (2013) present a fuzzy multi-objective method to undertake the RAP development and make the model more flexible and suitable for decision making.

SureshBabu *et al*. (2012) agree with the need to use redundancy of components to optimize system reliability. However, Sharma *et al*. (2011) emphasize the need to find a trade-off between the maximization of the system reliability and the minimization of resource utilization. Referring to the last point, Swetha *et al*. (2015) notice the general underutilization of resources in redundancy techniques, and so they apply the algorithm Resource Reclaimed Scheme (RRS) to allocate and schedule the critical and non-critical tasks of an avionic mission system. Alebrant Mendes *et al*. (2014) focus on the preventive maintenance of redundant systems and propose a Markov model for determining the time interval between two consecutive maintenance inspections to optimize system availability and maintenance costs. Markov models are also used by Hellmich and Berg (2015) to organize the repair activities of standby safety systems. Huang *et al*. (2015) state that the standby redundancy is a helpful practice. Montoro-Cazorla and Pérez-Ocón (2014) deal with the possibility of including standby units to increase the system operational time. In particular, they illustrate the calculation of availability, reliability, and rate of occurrence of failures when considering a system with one online component and $n-1$ cold standby components.

Moreover, partial redundancy is a significant configuration to improve systems' reliability/availability. Partial redundancy is implemented in a k-out-of-n configuration (Mo *et al*., 2015), for which a system is comprised of n components out of which at least k (with $k \leq n$) have to run simultaneously to assure the functioning state of the system, in other words, if $n - k + 1$ components fail then the system fails.

For partially redundant systems, the available literature presents a wide variety of mathematical programming models where costs, reliability, and availability are commonly considered as objectives and/or constraints. Arulmozhi (2002) focuses on k-out-of-n systems and proposes an equation to calculate the value of the reliability function by means of a recursive algorithm. Lu and Lewis (2008) observe that the k-out-of-n configuration enables safety objectives based on increasing the system reliability level to be met. Kang and Kim (2012) develop a method to quantify the unavailability of a k-out-of-n reactor protection system in a nuclear power plant. The method also enables an investigation into the most dangerous situations related to the entire system. Referring to a k-out-of-n surveillance system, Zhang and Pham (2014) formulate an optimization model where the cost minimization is the objective function, and apply an algorithm to finally select the best maintenance policy. As for the optimal design of k-out-of-n systems, Moghaddass and Zuo (2011) research the need for finding an effective trade-off between the system configuration to be designed and the maintenance strategy to be implemented.

5.3. An exact formula for the stationary availability of k-out-of-n systems

As previously underlined, for industrial systems with reparable components, such as production systems, the most interesting parameter (Ahmed *et al*., 2014) used to drive the maintenance management is the stationary availability A_s, whose maximisation is considered a strategic objective to be pursued. A closed formula has been proposed and validated (Carpitella *et al*., 2018d) to easily calculate the exact stationary availability for a k-out-of-n system.

Let us consider a system S of n identical redundant components each one characterized by constant failure and repair rates λ and μ, respectively. The difficulty to know the trend of the failure rate over the time implies the assumption of being constant. As the repair rate concerns, the main part of reliability studies is grounded upon the assumption of its constancy over time to simplify the computation of the reliability and availability values of systems constituted by reparable components. Without such a hypothesis, several systems, such as the k-out-of-n ones herein analysed, could be investigated only by simulation. Furthermore, electronic components are always characterized by a constant failure rate, whereas mechanical components have a slightly increasing failure rate.

Therefore, the individual stationary availability of a repairable component is computed by the well-known equation:

$$A_s = \frac{\mu}{\lambda+\mu}. \tag{5.6}$$

Let us also consider the following hypotheses regarding the entire system:

- all components are repairable as well as the whole system;

- all components are stochastically independent and identical from a reliability point of view;
- there are no constraints about the maximum availability of maintenance crews.

These hypotheses guarantee the possibility of executing a generic maintenance operation and the possibility of easily aggregating different system states. The following proposition gives an exact formula to calculate the stationary availability of the k-out-of-n system, i.e. $A_{S_{\binom{n}{k}}}$.

Proposition. Under the stated hypotheses, the exact stationary availability $A_{S_{\binom{n}{k}}}$ of a k-out-of- n system is given by the following formula:

$$A_{S_{\binom{n}{k}}} = \frac{\sum_{i=k}^{n}\binom{n}{i}\cdot\mu^{i}\cdot\lambda^{(n-i)}}{\sum_{i=k}^{n}\binom{n}{i}\cdot\mu^{i}\cdot\lambda^{(n-i)}+\binom{n}{k-1}\cdot\mu^{(k-1)}\cdot\lambda^{(n-k+1)}}. \tag{5.7}$$

Proof. In a specific time instant, the possible states of a component are the functioning state (noted by C) and the failure state (noted by its complementary state, $\bar{\mathrm{C}}$). The probability of being in a functioning state coincides with the component availability, whereas the probability of being in a failure state coincides with the component unavailability. Referring to the entire system, the probability of being in a specific state is the probability of the intersection of the states of its components. Under the aforementioned hypotheses, component states are stochastically independent so that the probability of their intersection can be calculated by means of their product.

The stationary availability of the systems is calculated in the proposed formula by computing the ratio between the probability of the functioning states and the probability of all the possible states (both functioning and failure states in which the system S may be in a generic time instant). The ratio is obtained by means of the natural partition of the event space. The numerator represents the probability of the union of the system functioning states. Being the latter mutually exclusive, the probability of the union is precisely the sum of the probabilities of each functioning state. In the denominator of the proposed formula, all the possible states are considered. Indeed, in addition to all the possible functioning states, one has to consider the failure states of the system that may occur when $n - k + 1$ components fail. The number of configurations that imply the system failure is $\binom{n}{k-1}$. Therefore, the denominator of the proposed formula is the probability of the union of events considered in the numerator and the events representing the failure of the system. This finishes the proof.

The formula (5.7) may also be proved to be in agreement with the fundamental theorem of Markov chains. Indeed, the finite number of states (nodes) of the system may be represented by a strongly directed graph. Under the hypotheses previously described, all the transitions (links) between any two Markov states can be straightforwardly derived in terms of λ and μ. As a result, the process is governed by a stochastic regular matrix, which has a unique associated stationary probability, by virtue of the fundamental theorem of Markov chains. Moreover,

- The ratio between each possible functioning state and the denominator of the proposed formula gives back the probability of the analogue state represented in the Markov chain method. As a result, the sum of all those probabilities is the functioning probability or stationary availability of the system.
- Likewise, dividing the term $\binom{n}{k-1} \cdot \mu^{(k-1)} \cdot \lambda^{(n-k+1)}$ by the entire denominator, the system unavailability $U_{S_{\binom{n}{k}}}$ can be obtained, that is $U_{S_{\binom{n}{k}}} = 1 - A_{S_{\binom{n}{k}}}$, which coincides with the probability of the system being in a failure state.

We illustrate this in the following by means of a simple numerical example.

Let us consider a 2-out-of-3 system. By applying the proposed equation, the following stationary availability is obtained:

$$A_{S_{\binom{n}{k}}} = \frac{\mu^2+3\cdot\lambda\cdot\mu}{\mu^2+3\cdot\lambda\cdot\mu+3\cdot\lambda^2}. \qquad (5.8)$$

To validate the proposed equation, let us consider the Markov chain associated with the analysed system. Figure 5.2 represents the associated directed Markov graph where 0 and 1 are the system functioning states (and 2, obviously, the failure state). In particular, 0 is the state where all the three components are available (3C). Since $\bar{C}$ stands for the failure state of a component, the meanings of states 1 and 2 are now evident.

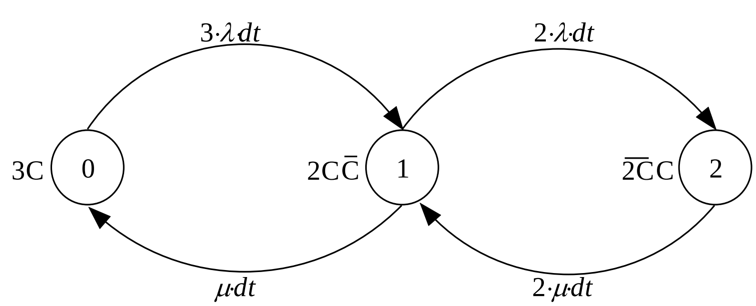


Figure 5.2. Markov graph for a **2**-out-of-**3** system

As mentioned above, λ and μ are the constant failure and repair rates of each component, respectively. The transition probabilities from one state to another, as shown in Figure 5.2, only depend on the current state and not on the preceding sequence of events. Since λ and μ are assumed to be constant over time, the transition probabilities are constant as well, and the relative stochastic process is a homogeneous Markov process.

Considering the Markov graph of Figure 5.2, let $\mathrm{pr}_0(t)$, $\mathrm{pr}_1(t)$ and $\mathrm{pr}_2(t)$ be the probability of the system being at states 0, 1 and 2 at time t. Therefore, the following differential equations hold:

$$\begin{aligned}
\frac{\mathrm{dpr}_0(t)}{\mathrm{d}t} &= -3\cdot\lambda\cdot\mathrm{pr}_0(t)+\mu\cdot\mathrm{pr}_1(t);\\
\frac{\mathrm{dpr}_1(t)}{\mathrm{d}t} &= +3\cdot\lambda\cdot\mathrm{pr}_0(t)-(2\cdot\lambda+\mu)\cdot\mathrm{pr}_1(t)+2\cdot\mu\cdot\mathrm{pr}_2(t);\\
\frac{\mathrm{dpr}_2(t)}{\mathrm{d}t} &= +2\cdot\lambda\cdot\mathrm{pr}_1(t)-2\cdot\mu\cdot\mathrm{pr}_2(t).
\end{aligned} \qquad (5.9)$$

The time-dependent availability value is obtained by the resolution of the previous system of differential equations.

To obtain the value of the parameter of interest, i.e. the stationary availability, the following system of linearly dependent equations needs to be solved. It arises from the previous differential system (5.9) by considering that for *t* tending to infinity the probability of being at each possible state i is constant, and thus $\frac{\mathrm{dpr}_i(\mathrm{t})}{\mathrm{dt}}=0$.

$$\begin{aligned}
&-3\cdot\lambda\cdot P_0+\mu\cdot\mathrm{pr}_1=0;\\
&3\cdot\lambda\cdot\mathrm{pr}_0-(2\cdot\lambda+\mu)\cdot\mathrm{pr}_1+2\cdot\mu\cdot\mathrm{pr}_2=0;\\
&2\cdot\lambda\cdot\mathrm{pr}_1-2\cdot\mu\cdot\mathrm{pr}_2=0.
\end{aligned} \qquad (5.10)$$

To solve this linearly dependent system of equations, a new equation is introduced to replace one equation from the system, accounting for the global probability:

$$\mathrm{pr}_0+\mathrm{pr}_1+\mathrm{pr}_2=1. \qquad (5.11)$$

As a result, the following probability values are obtained:

$$\begin{aligned}
\mathrm{pr}_0 &= \frac{\mu^2}{\mu^2+3\cdot\lambda\cdot\mu+3\cdot\lambda^2};\\
\mathrm{pr}_1 &= \frac{3\cdot\lambda\cdot\mu}{\mu^2+3\cdot\lambda\cdot\mu+3\cdot\lambda^2};\\
\mathrm{pr}_2 &= \frac{3\cdot\lambda^2}{\mu^2+3\cdot\lambda\cdot\mu+3\cdot\lambda^2}.
\end{aligned} \qquad (5.12)$$

In contrast with the calculation of the Perron eigenvector of the system matrix, or with other methods found in the literature based on the resolution of sets of differential equations in the variables pr_i, the results presented here stress the higher computational simplicity of the proposed exact formula. The system steady state availability is thus given by the sum of the probabilities of occurrence of the system functioning states, namely:

$$A_{S_{\binom{n}{k}}} = \mathrm{pr}_0 + \mathrm{pr}_1 = \frac{\mu^2+3\cdot\lambda\cdot\mu}{\mu^2+3\cdot\lambda\cdot\mu+3\cdot\lambda^2}. \quad (5.13)$$

Also, one can observe that the last equation coincides with the stationary availability previously calculated for the 2-out-of-3 system, and it is the sum of two terms, each corresponding to pr_0 and pr_1 respectively. Furthermore, the system unavailability $U_{S_{\binom{n}{k}}}$ is given by the value of pr_2 and it is obviously the complement to one of the right hand side of equation (5.15):

$$U_{S_{\binom{n}{k}}} = \mathrm{pr}_2 = \frac{3\cdot\lambda^2}{\mu^2+3\cdot\lambda\cdot\mu+3\cdot\lambda^2}. \quad (5.14)$$

After having validated the formula (5.7) by means of the classic method of Markov chain (Häggström, 2002), a multi-objective mathematical optimisation model has been proposed to obtain the Pareto front (Deb, 2001; Hwang *et al.*, 1993) and to lead the optimal design of a k-out-of-n system through a MCDM-based approach (Carpitella *et al.*, 2018d).

In conclusion, the analysis of system configuration is mainly aimed at assessing reliability and availability. For systems characterised by a complex configuration and in the presence of many interconnections among components, the advanced techniques presented in the following sections are useful to lead an in-depth evaluation.

5.4. FMEA/FMECA to analyse complex systems in-depth

The phase of reliability analysis, preliminary to the implementation of an effective maintenance process, can be carried out by applying advanced techniques suitable for complex systems such as the FMEA or its evolution FMECA (Carpitella *et al.*, 2018c).

As established by the CEI EN 60812 Standard (2006), these analyses represent a valid support method to semi-quantitatively measure the criticality of system failure modes. These techniques are particularly effective and require the elaboration of a reliability block diagram, which enables to represent the reliability configuration of a given system and the connexion among its components. Knowledge of reliability features of systems from historical series of data related to occurred faults is necessary. In particular, FMEA/FMECA analyses are the main techniques of reliability analysis herein proposed to determine maintenance action priority. The output of the analysis is a list of possible failure modes that could affect the system and the identification of the main criticalities. This is helpful because results derived from reliability analysis can be combined with multi-criteria decision methods to obtain important information for implementing maintenance policies and/or scheduling maintenance activities according to a set of differently weighted criteria.

As defined by the related Standard, the FMEA is a systematic procedure of system analysis aimed at identifying potential failure modes, their causes and effects on system performance. The FMEA is conducted by means of the following phases.

- Developing preliminary considerations to explain the reason why this kind of analysis is undertaken and which are the advantages of the specific operation environment considered.
- Defining the system by means of the following intermediate steps: collection of information; definition of bounds and analysis levels; and representation of system structure.
- Establishing system functions and requirements through an in-depth characterisation of tasks and processes carried out by each component.
- Characterising failure modes, failure causes, and failure effects. To such an aim, all the failure possibilities have to be analysed for each component, by carefully highlighting causes related to each modality of failure and, lastly, by taking into account all the effects on the entire system. In particular, the distinction between local and systemic effects has to be formalised.
- Reporting results in an appropriate worksheet, with the purpose of collecting and synthetizing information. This is a crucial step to eventually manage the phase of risk assessment and the following implementation of reductive/preventive measures.

The FMEA technique has both strengths and drawbacks. On the one hand, effective management of costs can be cited among its application advantages. Indeed, planning of maintenance interventions aimed at minimizing inefficiencies is made easier by an *a priori* identification of possible failures and their causes. Moreover, the FMEA is very helpful in acquiring detailed knowledge of the system under analysis, and hence in increasing company flexibility in its own operational sector. Lastly, this technique effectively supports in pursuing continuous enhancement of quality levels.

On the other hand, the main drawback related to the FMEA is represented by its subjective nature. The analyst evaluates both the overall failure scenario and the parameters determining intervention priority. Indeed, the level of detail in element description and interactions among components strictly depends on the subjective perception of the decision maker.

Despite the mentioned weakness, the FMEA still remains a widely used tool in the literature. Liu *et al*. (2013a) carry out a literature review of 75 papers focused on FMEA application to a wide range of practical cases, with the aim to analyse and compare strengths and weaknesses.

Broadly speaking, it is possible to assert that the FMEA provides a robust support in optimising system reliability through the implementation of preventive and/or corrective interventions. Similarly, the FMECA is mainly implemented on the basis of the previously described phases but aims to quantitatively evaluate the criticality of each failure modes.

Many authors (Koning *et al*., 2009; Aven, 2016b) consider FMECA and the development of risk analyses as an essential part of maintenance management strategies. Vernez and Vuille (2009) emphasize the good adaptability of the FMECA as a tool for analysing complex macro-systems with various hierarchical levels. They support the use of the methodology for reliability optimization and identifying major vulnerabilities.

The identification of critical components, which are components whose functioning of failure state directly impacts on the whole system reliability, is then aimed at preventing "system failure" events. In fact, ranking failures to highlight those requiring an immediate intervention is a helpful practice aimed at identifying critical components.

Naturally, such kind of analysis has to be typically conducted by considering components as dynamically interacting together and not just by their own, as if they were merely single and not connected parts. Moreover, worksheets reporting results make it easier the management of all the identified failure modes to implement the following phase of risk analysis. In this regard, criticality analyses are generally approached by integrating FMEA and FMECA.

Again, on the basis of the CEI EN 60812 Standard (2006) definition, FMECA is an extension to FMEA that enables prioritizing failure modes on the basis of criticality. Specifically, the criticality of each failure mode is computed by combining the risk parameters severity (S), occurrence (O) and detection (D). S is an estimate of how strongly the effects of the failure mode will affect the system; O is the frequency of occurrence of the failure mode in a determined period of time; and D represents the probability of detecting a failure event. For each identified failure mode, the product of parameters O, S and D leads to the RPN (risk priority number):

$$RPN = S \cdot O \cdot D. \tag{5.15}$$

Each risk factor generally takes a discrete value in the range [0, 10].

Analogously to the previously discussed FMEA development, the first step to apply FMECA is the description of the considered system and the construction of a hierarchical structure. To obtain an exhaustive description of the system, it is first necessary to acquire information about the reliability relations among the system components and physically describe them, with their own order and position (defining system boundaries and levels). It is clearly suggested that those components that will neither be evaluated nor taken into consideration in the analysis are excluded from the study. The functional relationships among the components can be finally formalized in a system block diagram. Moreover, it is necessary to define all the possible failure modes for each component, detect the failure causes, and define both the local and the system level failure effects. Also, in this case, all the results must be summarized in worksheets that support the analyst in formalizing the phase of risk evaluation: namely, the computation of the *RPN* related to each failure mode.

5.5. Real-world case study of a street-cleaning vehicle

A real-world application of FMEA/FMECA aimed at optimising maintenance of a complex system is presented in Carpitella *et al*. (2018c). This application was developed using the results of a project in which the University of Palermo was involved as a partner with several enterprises. The project examined the development of a new and innovative vehicle providing a street cleaning service which incorporated a smart telediagnosis system. Figures 5.3 and 5.4, respectively, represent the hierarchical structures of the system and of one of its main sub-components. After having characterised the specific functions carried out by its components and sub-systems, failure modes, causes and effects have been formalised in appropriate worksheets (Table 5.1).

Moreover, the reliability diagrams, built on the basis of the hierarchical structures, are given (Figures 5.5 and 5.6) with the final list of all the highlighted failure modes (Table 5.2), each one progressively tagged by A, B, …, F, according to the code previously attributed to the relative subsystem (1, 2, …, 5). For instance, the ID 5.2.1.A indicates the failure mode "A" (hydraulic engine fault) of component "5.2.1." (rear roller).

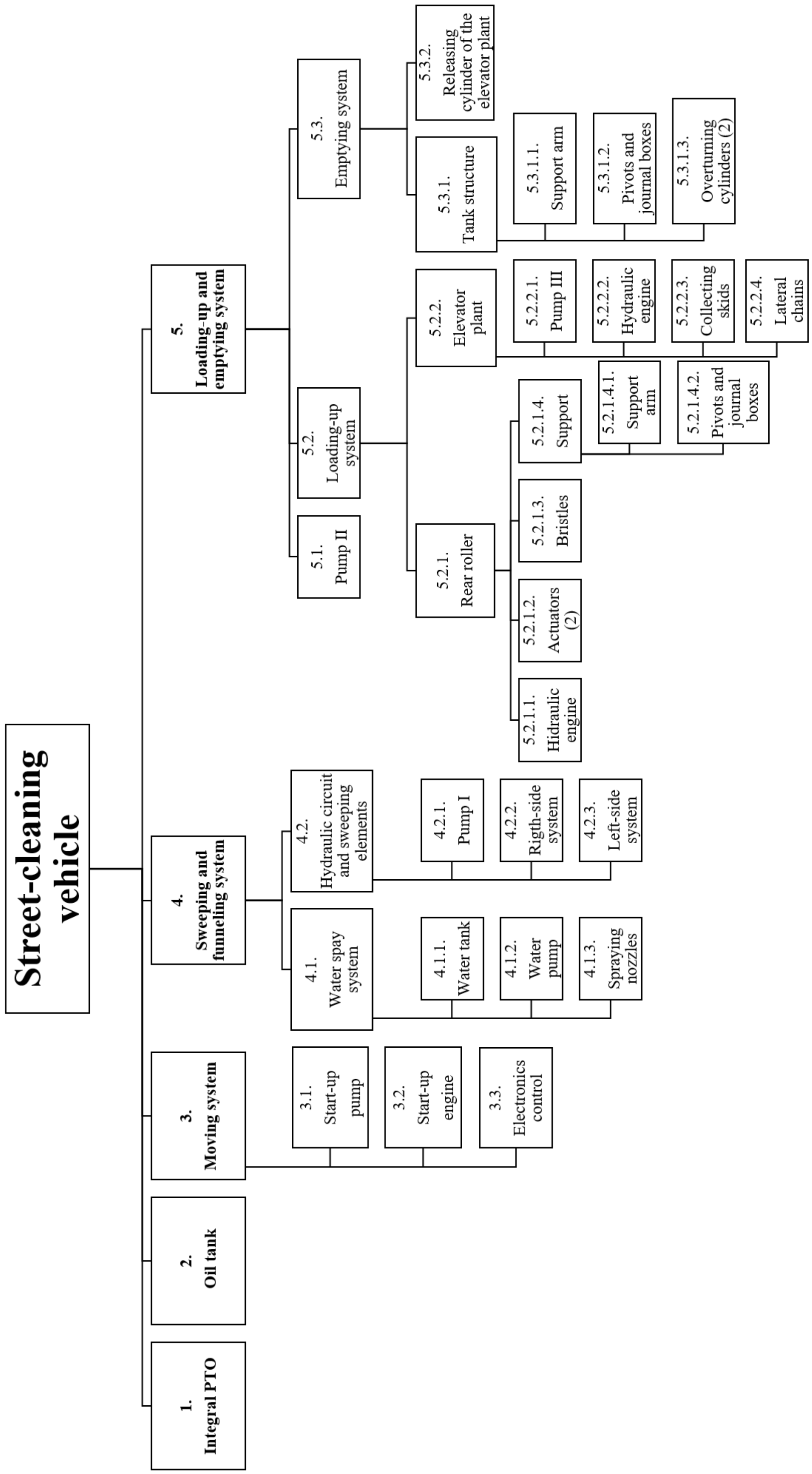


Figure 5.3. Hierarchical structure of the complex system "Street-cleaning vehicle"

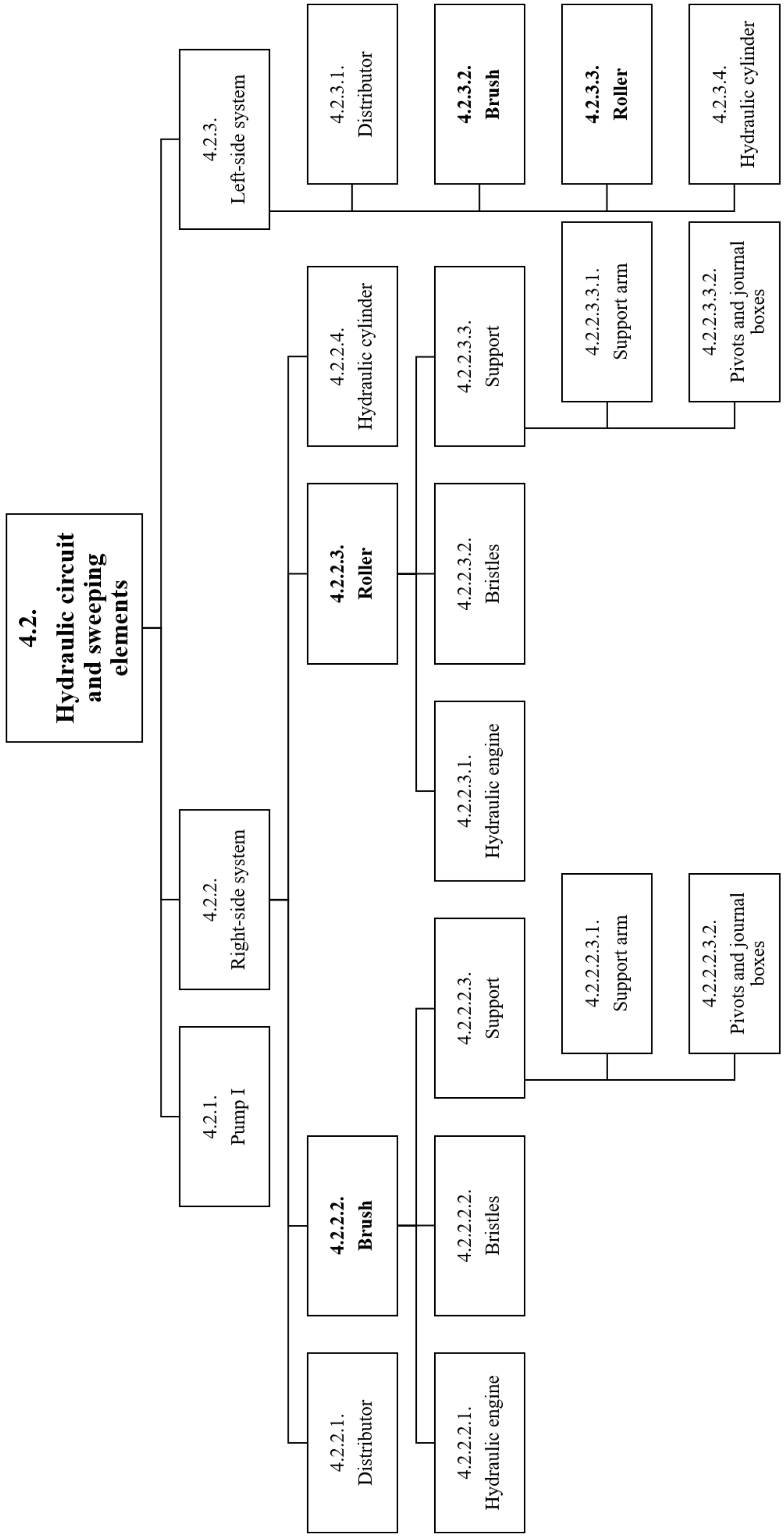


Figure 5.4. Hierarchical structure of the sub-system “Hydraulic circuit and sweeping elements”

Table 5.1. Failure Modes and Effects analysis

ID	Component	Failure Modes	Failure Causes	Failure Effects
1.	Integral power take-off (PTO)	Broken PTO mechanism	Construction defects; lack of lubrication.	• Compromised functionality of hydraulic circuits.
		Worn PTO bearings	High usage time; lack of lubrication.	• Compromised functionality of hydraulic circuits.
		Broken PTO universal joint shafts	Joint break due to overstressing; lack of lubrication.	• Interruption of wheel movement; • machine stop (towing requirement).
		General electrical system fault	Burned fuse; pneumatic solenoid valve fault; control unit fault.	• Failing in switching from mechanical to hydrostatic drive (and vice versa); • failure in performing the sweeper functions.
2.	Oil tank	Overheated oil	Exchanger clogging; lack of water.	• Hydraulic system blocked.
		Insufficient oil level	Maintenance not carried out or leaks.	• Circuit not under pressure.
		Clogged filters	Accumulation of pollutants	• Pressure reduction in the hydraulic circuit.
3.1.	Start-up pump	Fault in distribution system	No power supply; fluid characteristics; failure of valves or other elements.	• Failure of the hydraulic motor to feed the vehicle.
		Mechanical fault	Wear of the elements (bearings, journal boxes, etc.); wear of the sealing elements.	• Failure of the hydraulic motor to feed the vehicle.
3.2.	Start-up engine	Stopped start-up engine	Drive pump failure; overheated oil.	• Stopped sweeper.
		Mechanical fault	Bearing wear.	• Stopped sweeper.
3.3.	Electronics control	Fault in electrical system	Burned fuse; control unit failure; absence of authorization and control signals.	• Non-functioning in hydrostatic transmission; • stopped sweeper.
4.1.1.	Water tank	Empty water tank	Procurement not carried out / failed level switch.	• Fluid delivery failure; • operational delay; • overheated oil.
		Hole in water tank	Ineffective maintenance.	• Fluid delivery failure; • operational delay; • overheated oil.

4.1.2.	Water pump	Fault in distribution system	No power supply; fluid characteristics; failure of valves or other elements.	• Compromised functioning of the water spraying system; • un-compacted powders.
		Mechanical fault	Wear of the elements (bearings, journal boxes, etc.); wear of the sealing elements.	• Compromised functioning of the water spraying system; • un-compacted powders.
4.1.3.	Spraying nozzles	No working nozzles	Pump failure.	• No water jet is dispensed; • un-compacted powders.
		Clogged nozzles	Ineffective maintenance.	• Partial delivery of the water jet; • ineffective compacting of powders.
4.2.1.	Pump I	Fault distribution system	No power supply; fluid characteristics; failure of valves or other elements.	• Compromised functioning of hydraulic circuit and hydraulic actuators; • Work position not taken; • Brush and roller rotation not allowed.
		Mechanical fault	Wear of the elements (bearings, journal boxes, etc.); wear of the sealing elements.	• Compromised functioning of hydraulic circuit and hydraulic actuators; • Work position not taken; • Brush and roller rotation not allowed.
4.2.2.2.1./ 4.2.2.3.1./ 4.2.3.2.1./ 4.2.3.3.1.	Hydraulic engine	Stopped start-up engine	Pump I failure; overheated oil.	• Stopped brushes; • stopped lateral rollers; • waste not conveyed.
		Mechanical fault	Bearing wear.	• Excessive vibration.
4.2.2.4./ 4.2.3.4.	Hydraulic cylinders	Stopped hydraulic cylinders	Pump I and / or pump II failure; excessive friction; hydraulic circuit failure.	• Translation of brushes / rollers not carried out (elements not adherent to the ground when working or not lifted during transportation).
		Mechanical fault	Wear of the sealing elements.	• Irregular translation and loss of oil.
4.2.2.2.3.1. 4.2.2.3.3.1. 4.2.3.2.3.1. 4.2.3.3.3.1.	Support arms	Broken arms	Deformation due to impact with large waste or sidewalks.	• Compromised functionality of brushes and side rollers.
		Stopped arms	Hydraulic system fault.	• Failure in opening / closing side arms; • changes in action range of conveyance system.

4.2.2.2.3.2. 4.2.2.3.3.2. 4.2.3.2.3.2. 4.2.3.3.3.2.	Pivots and journal boxes	Slackened pivots	Incorrect assembly / stress due to vibrations.	• Excessive vibration; • risk of detachment of the brush (s) or roller (s) from the holder.
		Worn journal boxes	Wrong assembly / action of pins inside the journal boxes.	• Incorrect joint between arms and brushes or rollers.
4.2.2.2.2./ 4.2.2.3.2/ 4.2.3.2.2./ 4.2.3.3.2/	Bristles	Damaged brush or roller	Mechanical action of conveyed waste and road surface.	• Inefficiency in waste collection; • low adherence of bristles to the ground.
5.1.	Pump II	Fault distribution system	No power supply; fluid characteristics; failure of valves or other elements.	• Compromised functioning of the loading and unloading system; • work position not taken; • waste not loaded; • tank not emptied.
		Mechanical fault	Wear of the elements (bearings, journal boxes, etc.); wear of the sealing elements.	• Compromised functioning of the loading and unloading system; • Work position not taken; • waste not loaded; • tank not emptied.
5.2.1.1.	Hydraulic engine	Stopped engine	Pump II failure; overheated oil.	• Stopped rear roller; • waste not loaded.
		Mechanical fault	Bearing wear.	• Excessive vibration.
5.2.1.2.	Actuators (2)	Stopped piston	Pump II failure; excessive friction; hydraulic circuit failure.	• Rear roller translation not carried out (system not adherent to the ground when working or not lifted during transportation).
		Mechanical fault	Wear of the sealing elements.	• Irregular translation and loss of oil.
5.2.1.4.1.	Support arms	Broken arms	Deformation due to impact with large waste or sidewalks.	• Compromised functionality of the rear roller.
		Stopped arms	Hydraulic system fault.	• Work position not taken; • waste loading not carried out.
5.2.1.4.2.	Pivots and journal boxes	Slackened pivots	Incorrect assembly / stress due to vibrations.	• Excessive vibration; • risk of detachment of the roller from the support.
		Worn journal boxes	Wrong assembly / action of pins.	• Incorrect joint between arms and roller.

5.2.1.3.	Bristles	Damaged brush or roller	Mechanical action of conveyed waste and road surface.	• Inefficiencies in collecting waste; • low adherence of bristles to the ground.
5.2.2.1.	Pump III	Fault distribution system	No power supply; fluid characteristics; failure of valves or other elements.	• Compromised functionality of the elevator plant; • difficulty in the interaction between the elevator plant and the collection tank; • loading of waste in the tank not carried out; • stopped elevator plant.
		Mechanical fault	Wear of the elements (bearings, journal boxes, etc.); wear of the sealing elements.	• Compromised functionality of the elevator plant; • difficulty in the interaction between the elevator plant and the collection tank; • loading of waste in the tank not carried out; • stopped elevator plant.
5.2.2.2.	Hydraulic engine	Stopped engine	Pump III failure; overheated oil.	• Stopped elevator plant • waste not loaded in the tank.
		Mechanical fault	Bearing wear.	• Excessive vibration.
5.2.2.3.	Collecting skids	Broken skid	Detachment of one or more skids from the support for waste action	• Difficulty in conveying waste; • clogging near the rear roller.
		Worn skid	Crawling between skids and carter;	• Powders dispersion.
5.2.2.4.	Lateral chains	Clogged lateral chains	Presence of small waste;	• Discontinuous motion.
		Stopped lateral chains	Hydraulic system fault.	• Blocked motion; • loading not carried out.
		One or more meshes broken	Impacts or wear.	• Blocked motion; • loading not carried out.
5.3.1.1.	Support arm	Broken arm	Deformation due to impact with large waste or sidewalks;	• Impossibility of tank overturning.
		Stopped arm	Hydraulic system fault.	• Emptying not performed; • operational delay.
5.3.1.2.	Pivots and journal boxes	Slackened pivots or worn journal boxes	Incorrect assembly / stress due to load.	• Excessive vibration; • overturning defects.

5.3.1.3.	Overturning cylinder (2)	Stopped piston	Pump II failure; excessive friction; hydraulic circuit failure.	• Non-functioning emptying system; • delay in starting the next mission.
		Mechanical fault	Wear of the sealing elements.	• Irregular movement and loss of oil.
5.3.2.	Releasing cylinder of the elevator plant	Stopped piston	Pump II failure; excessive friction; hydraulic circuit failure.	• Failure in translating the elevator plant; • waiting in emptying the tank (if full) or in restoring the conveyor system.
		Mechanical fault	Wear of the sealing elements.	• Irregular translation and loss of oil.

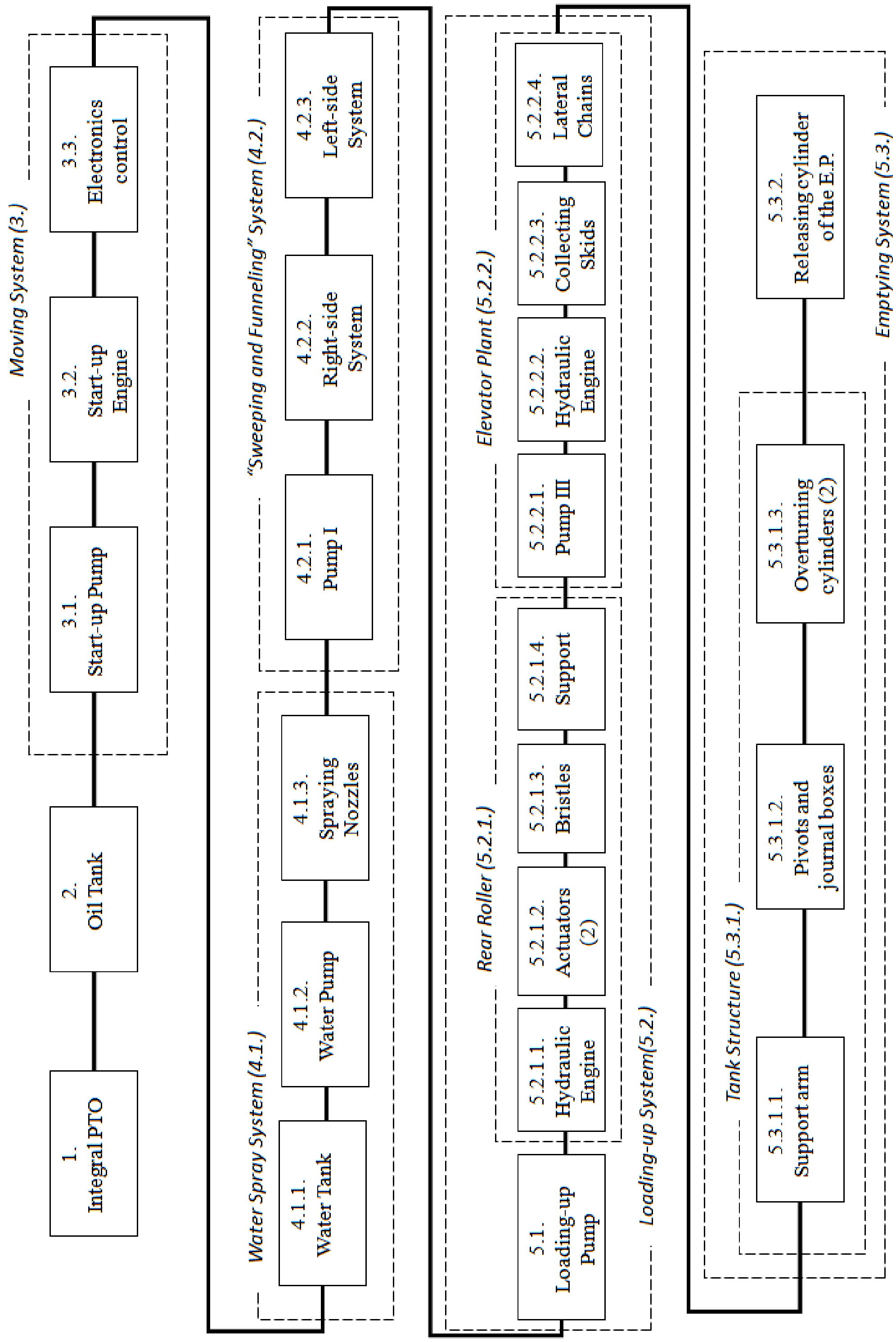


Figure 5.5. Reliability diagram of the system "Street-cleaning vehicle"

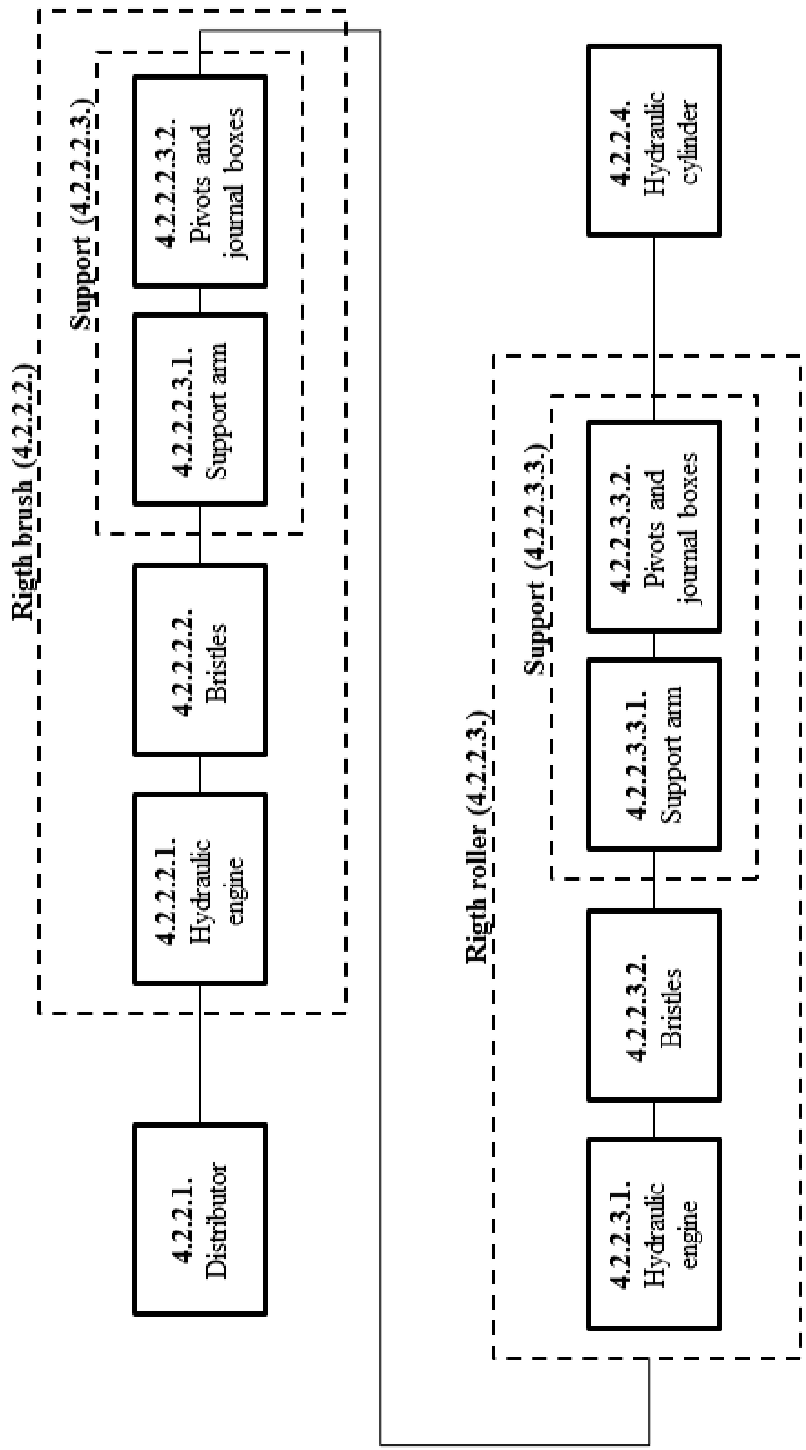


Figure 5.6. Reliability diagram of the sub-system “Right side system”

Table 5.2. List of failure modes and evaluation criteria

FAILURE MODES		ID	C_1	C_2	C_3
1. Integral PTO	Broken PTO mechanism	1A	VH	HI	R
	Worn PTO bearings	1B	VH	HI	O
	Broken PTO universal joint shafts	1C	VH	HI	R
	General electrical system fault	1D	L	MI	O
2. Oil tank	Overheated oil	2A	H	HI	P
	Insufficient oil level	2B	L	MI	P
	Clogged filters	2C	L	MI	P
3.1. Start-up pump	Fault in distribution system (start-up pump)	3.1A	VH	HI	R
	Mechanical fault (start-up pump)	3.1B	VH	HI	R
3.2. Start-up engine	Stopped start-up engine	3.2A	VH	HI	R
	Mechanical fault (start-up engine)	3.2B	VH	HI	R
3.3. Electronics control	Fault in electrical system	3.3A	VH	MI	O
4.1.1. Water tank	Empty water tank	4.1.1A	L	MI	P
	Hole in water tank	4.1.1B	M	HI	R
4.1.2. Water pump	Fault in distribution system of the water pump	4.1.2A	VH	MI	O
	Mechanical fault of the water pump	4.1.2B	VH	MI	R
4.1.3. Spraying nozzles	No working nozzles	4.1.3A	L	MI	P
	Clogged nozzles	4.1.3B	L	MI	P
4.2.1. Pump I	Fault distribution system in Pump I	4.2.1A	H	MI	R
	Mechanical fault in Pump I	4.2.1B	H	MI	R
4.2.2. Right-side system	Damaged brush or roller	4.2.2A	H	MI	P
	Faulty hydraulic cylinders	4.2.2B	L	MI	O
	Fault in electrical system	4.2.2C	M	MI	O

4.2.3. Left-side system	Damaged brush or roller	4.2.3A	H	MI	P
	Faulty hydraulic cylinders	4.2.3B	L	MI	O
	Fault in electrical system	4.2.3C	M	MI	O
5.1. Pump II	Fault in distribution system (loading-pump)	5.1A	L	MI	R
	Mechanical fault (loading-up pump)	5.1B	M	MI	R
5.2.1. Rear roller	Fault in hydraulic engine (rear roller)	5.2.1A	L	MI	R
	Fault in actuator (rear roller)	5.2.1B	M	MI	O
	Worn bristles (rear roller)	5.2.1C	L	MI	P
	Fault in support arm (rear roller)	5.2.1D	H	MI	O
	Slackened pivots or worn journal boxes (rear roller)	5.2.1E	M	MI	P
5.2.2. Elevator plant	Fault in Pump III (elevator plant)	5.2.2A	M	MI	R
	Fault in hydraulic engine (elevator plant)	5.2.2B	L	MI	R
	Broken skid (elevator plant)	5.2.2C	M	HI	P
	Broken chain (elevator plant)	5.2.2D	H	HI	P
5.3.1. Tank structure	Fault in support arm in the tank structure (emptying system)	5.3.1A	H	HI	O
	Slackened pivots or worn journal boxes in the tank structure (emptying system)	5.3.1B	M	HI	P
	Overturning cylinder fault in the tank structure (emptying system)	5.3.1C	M	HI	R
5.3.2. Releasing cylinder of the elevator plant	Broken or stopped releasing cylinder of the elevator plant	5.3.2A	M	HI	R

Regarding the evaluation of failure mode criticality, despite its wide use, the classical *RPN* has been largely criticized for having many drawbacks. As a result, numerous contributions have been made in the literature to enhance the classical FMECA.

Carmignani (2009) suggests the use of a fourth parameter in the *RPN* calculation. The author proposes taking into account the profitability – based on costs and possible profits after minimizing losses due to failures. Bevilacqua *et al*. (2000) propose a modified FMECA where the *RPN* consists of the weighted sum of six parameters (safety, importance of the machine for the process, maintenance costs, failure frequency, downtime length and operating conditions). Furthermore, a sensitivity analysis based on the Monte Carlo simulation to verify the robustness of the final results is performed.

In the presented application, each failure mode has been assessed by considering three evaluation criteria, namely C_1, C_2 and C_3, that differ from those considered by the *RPN* index. The first two criteria refer to the severity parameter whereas the last criterion concerns the frequency of occurrence. In particular, C_1 and C_2 both refer to the execution of maintenance activities related to specific faults and respectively represent the operation time (expressed in hours), and the modality of the maintenance action execution (expressed using a quantitative scale of difficulty values). Specifically, a maintenance action implies a lower level of difficulty if carried out in the same place where the failure occurred, and by an immediately available operator. Similarly, the maintenance action is medium-complex when it is necessary to ask for a specialized maintenance team; and finally, the action is complex if the repair must be made in a repair shop (by also considering the vehicle transport time). In the following section, a MCDM-based approach to rank failure modes is presented as an alternative to the classical *RPN* calculation.

5.6. Alternative approach to the RPN calculation

Once reliability analysis has been accomplished in the most exhaustive way, integrating the related results by means of MCDM methods reveals to be a useful approach to effectively support the process of maintenance management.

Most works in the literature propose the support of MCDM to carry out FMEA and FMECA analyses. Braglia (2000) proposes the AHP (1980, 1994) to pairwise compare the potential causes of failure by assuming as criteria the classical risk factors *O*, *S*, and *D* together with the expected cost due to failures. Braglia and Bevilacqua (2000) also suggest the use of the AHP to support maintenance staff in the identification of failure mode criticality.

In Zammori and Gabbrielli (2011), the FMEA is combined with the ANP technique (Saaty and Ozdemir, 2005) to take into account possible interactions among the principal causes of failure. Indeed, as asserted by Jafari and Fiondella (2016), the possible occurrence of failures could negatively impact on system components, above all in terms of the MTTF. The authors studied how the results of a reliability analysis could be influenced by increasing or decreasing the correlation among the elements of the system under study. Hauge *et al*. (2016) analyse a data set of failures in a system belonging to an oil industry in Norway, aiming at identifying common failure causes and adopting a suitable maintenance policy. In particular, the authors propose an “equipment checklist” for collecting data, making them easily accessible and minimising uncertainty. However, uncertainty generally represents a complex factor to manage and also, according to Berner and Flage (2016), its presence has to be necessarily reflected by reliability analysis.

Emovon *et al*. (2015) prioritize the analysed risk factors with the aim of making a detailed and realistic study of marine machinery systems by means of the VIKOR method (Liu *et al*., 2013b). Braglia *et al*. (2003) develop a fuzzy criticality assessment model which is easy to implement and design. They present a risk function where ‘if-then’ fuzzy rules are automatically generated and the proposed methodology is tested in a real process plant. To take into account the uncertainty that often occurs in the evaluation of parameters *O*, *S*, and *D*, the authors propose the fuzzy-technique for order preference by similarity to ideal solution (FTOPSIS) (Chen, 2000). In particular, FTOPSIS is the fuzzy development of the TOPSIS technique (Hwang and Yoon, 1981). The method is widely used in the literature in various fields (Aiello *et al*., 2009; Rostamzadeh and Sofian, 2011). A combined FTOPSIS and fuzzy-AHP (Chan *et al*., 2008) approach is proposed by Kutlu and Ekmekçioğlu (2012). The fuzzy-AHP method is applied to weight the risk factors that are successively used within the FTOPSIS approach to obtain the final closeness coefficients on the basis of which failure modes are prioritized. Rostamzadeh and Sofian (2011) also suggest combining the methods FAHP and FTOPSIS to increase manufacturing system performance. Broadly speaking, a fuzzy-based approach is widely taken into account in managing the phase of risk assessment, because of its capability to effectively manage uncertainty. Grassi *et al*. (2009) present a multi-attribute fuzzy model to quantitatively calculate the risk shared among different activities with relation to a generic process plant.

Regarding the real-world case study previously analysed, the FTOPSIS is herein proposed (Carpitella *et al*., 2018c) for failure mode prioritization, alternatively to the classical use of the *RPN*, because of its effectiveness in managing data uncertainty thanks to the use of fuzzy numbers. Indeed, human judgments are often vague and uncertain so that, in practical real-life situations, eliciting exact numerical values is difficult. More realistically, experts are better able to express their judgments on criteria weights and/or alternative ratings by means of linguistic variables. With this recognition, the traditional TOPSIS was extended by Chen (2000) under a fuzzy environment where linguistic variables are used to rate alternatives and/or weight criteria.

A brief description of the FTOPSIS methodology is presented next. The first step regards the definition of the fuzzy decision matrix $\tilde{X}$:

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{x}_{m1} & \cdots & \tilde{x}_{mn} \end{bmatrix}; \tag{5.16}$$

where the generic component $\tilde{x}_{ij}$ is the fuzzy number that represents the rating of alternative i under criterion j. The fuzzy numbers used here are either TFN or TrFN (Chen *et al.*, 2006), fully characterized by the following triple and quadruple sets of ordered numbers, respectively (see Chapter 1, Section 1.3):

$$\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij}); \tag{5.17}$$

$$\tilde{x}_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij}). \tag{5.18}$$

Matrix $\tilde{X}$ must be normalized with relation to each criterion to obtain the normalized decision matrix $\tilde{Z}$:

$$\tilde{Z} = \begin{bmatrix} \tilde{z}_{11} & \cdots & \tilde{z}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{z}_{m1} & \cdots & \tilde{z}_{mn} \end{bmatrix}; \tag{5.19}$$

where the components, considering TFNs, are obtained as follows:

$$\tilde{z}_{ij} = \left(\frac{a_{ij}}{c_j^*}, \frac{b_{ij}}{c_j^*}, \frac{c_{ij}}{c_j^*}\right),\ j \in I'; \tag{5.20}$$

$$\tilde{z}_{ij} = \left(\frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right),\ j \in I''; \tag{5.21}$$

I' being the subset of criteria to be maximized, and I'' the subset of criteria to be minimized; c_j^* and a_j^- are:

$$c_j^* = \max_i c_{ij} \text{ if } j \in I'; \tag{5.22}$$

$$a_j^- = \min_i a_{ij} \text{ if } j \in I''. \tag{5.23}$$

Analogously, considering TrFNs, the elements of matrix $\tilde{Z}$ are obtained as follows:

$$\tilde{z}_{ij} = \left(\frac{a_{ij}}{d_j^*}, \frac{b_{ij}}{d_j^*}, \frac{c_{ij}}{d_j^*}, \frac{d_{ij}}{d_j^*}\right),\ j \in J'; \tag{5.24}$$

$$\tilde{z}_{ij} = \left(\frac{a_j^-}{d_{ij}}, \frac{a_j^-}{c_{ij}}, \frac{a_j^-}{b_{ij}}, \frac{a_j^-}{a_{ij}}\right),\ j \in J''; \tag{5.25}$$

J' being the subset of criteria to be maximized and J'' the subset of criteria to be minimized; d_j^* and a_j^- are:

$$d_j^* = \max_i d_{ij} \text{ if } j \in J'; \tag{5.26}$$

$$a_j^- = \min_i a_{ij} \text{ if } j \in J''. \tag{5.27}$$

The weighted normalized matrix $\widetilde{U}$ is then computed to account for the various criteria weights. Thus, the generic component $\tilde{u}_{ij}$ of matrix $\widetilde{U}$ is calculated as:

$$\tilde{u}_{ij} = \tilde{z}_{ij} \cdot w_j; \tag{5.28}$$

w_j being the relative importance weight of criterion j.

Referring to matrix $\widetilde{U}$, the fuzzy positive ideal solution A^* and the fuzzy negative ideal solution A^- are chosen as follows:

$$A^* = (\tilde{u}_1^*, \tilde{u}_2^*, \dots, \tilde{u}_n^*); \tag{5.29}$$

$$A^- = (\tilde{u}_1^-, \tilde{u}_2^-, \dots, \tilde{u}_n^-); \tag{5.30}$$

where, for TFNs, $\tilde{u}_j^* = (0, 0, 0)$ and $\tilde{u}_j^- = (1, 1, 1)$, $j = 1 \dots n$, if the best score for criterion j is the minimal value, and on the contrary if it is the maximum value of the relative scale. Obviously, by considering a minimal value of preference, $\tilde{u}_j^*$ and $\tilde{u}_j^-$ are respectively $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$ when TrFNs are considered.

Then, distances of each alternative to A^* and A^- are computed by means of the vertex method (Chen, 2000). According to this method, the distance $d(\tilde{m}, \tilde{n})$ between two triangular fuzzy numbers $\tilde{m} = (m_1, m_2, m_3)$ and $\tilde{n} = (n_1, n_2, n_3)$ is a crisp value determined as:

$$d(\tilde{m}, \tilde{n}) = \sqrt{\frac{1}{3}[(m_1 - n_1)^2 + (m_2 - n_2)^2 + (m_3 - n_3)^2]}. \tag{5.31}$$

Similarly, the distance between two generic trapezoidal fuzzy numbers (Kahlili-Damghani and Sadi-Nezhad, 2013) $\tilde{t} = (t_1, t_2, t_3, t_4)$ and $\tilde{r} = (r_1, r_2, r_3, r_4)$ is the following crisp value:

$$d(\tilde{t}, \tilde{r}) = \sqrt{\frac{1}{4}[(t_1 - r_1)^2 + (t_2 - r_2)^2 + (t_3 - r_3)^2 + (t_4 - r_4)^2]}. \tag{5.32}$$

Therefore, for each alternative i, aggregating with respect to the whole set of criteria, the related distances from A^* and A^- can be calculated as:

$$d_i^* = \sum_{j=1}^{n} d\left(\tilde{u}_{ij}, \tilde{u}_j^*\right) i = 1 \dots n; \tag{5.33}$$

$$d_i^- = \sum_{j=1}^{n} d\left(\tilde{u}_{ij}, \tilde{u}_j^-\right) i = 1 \dots n. \tag{5.34}$$

To rank alternatives, the closeness coefficient CC_i is finally computed using the following equation:

$$CC_i = \frac{d_i^-}{d_i^- + d_i^*}. \quad (5.35)$$

Thus, referring to the proposed analysis and, according to the value of CC_i, the ranking order of all alternatives can be determined.

Contrarily to the traditional *RPN* calculation, the proposed method permits to take into account the relative importance of the risk parameters.

In particular, the maintenance team has weighted the three considered criteria (C_1, C_2 and C_3) using the AHP, obtaining the following vector of weights: [0.409, 0.197, 0.394]. It means that the criterion considered of utmost importance is the operation time to implement a maintenance intervention. The ratings of failure modes against criteria is carried out by using linguistic variables. Table 5.3 and Figures 5.7 to 5.9, respectively collect and graphically illustrate the corresponding fuzzy scales of evaluation (Chan *et al*., 2008; Wang *et al*., 2015) on the basis of triangular and trapezoidal fuzzy numbers, respectively noted with TFN and TrFN.

Table 5.3. Linguistic terms and associated fuzzy numbers

C_1			**C_2**			**C_3**		
Linguistic term	**Fuzzy number**	**Type of FN**	**Linguistic term**	**Fuzzy number**	**Type of FN**	**Linguistic term**	**Fuzzy number**	**Type of FN**
Very low (VL)	$(0,0,\frac{1}{2},\frac{3}{2})$	**TrFN**	Low impact (LI)	(1,2,3)	**TFN**	Improbable (I)	(0,0,1,3)	**TrFN**
Low (L)	$(\frac{1}{2},\frac{3}{2},\frac{5}{2})$	**TFN**	Medium impact (MI)	(2,3,4)	**TFN**	Remote (R)	(1,3,5)	**TFN**
Medium (M)	$(\frac{3}{2},\frac{5}{2},\frac{7}{2})$	**TFN**	High impact (HI)	(3,4,5)	**TFN**	Occasional (O)	(3,5,7)	**TFN**
High (H)	$(\frac{5}{2},\frac{7}{2},\frac{9}{2})$	**TFN**				Probable (P)	(5,7,9)	**TFN**
Very high (VH)	$(\frac{7}{2},\frac{9}{2},5,5)$	**TrFN**				Frequent (F)	(7,9,10,10)	**TrFN**

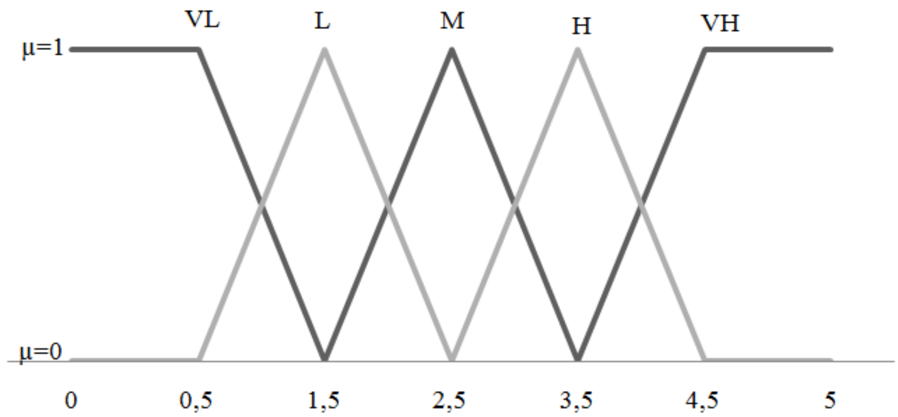


Figure 5.7. Linguistic variables for C_1, time of operation

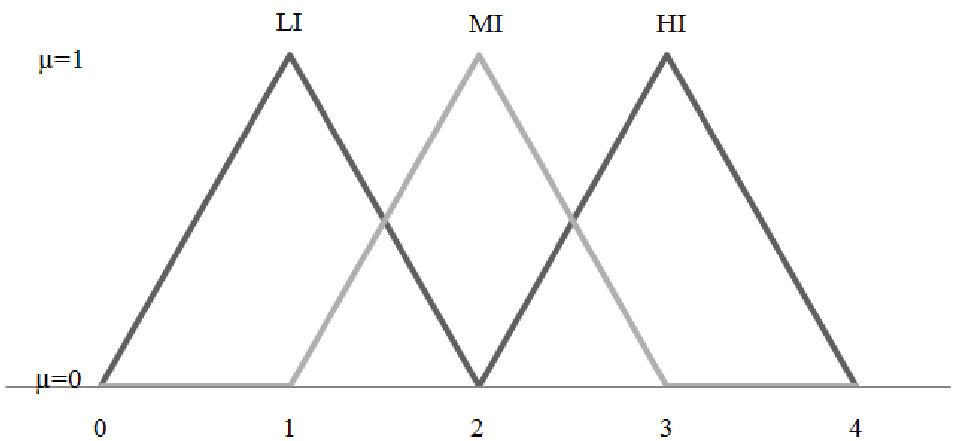


Figure 5.8. Linguistic variables for C_2, modality of execution

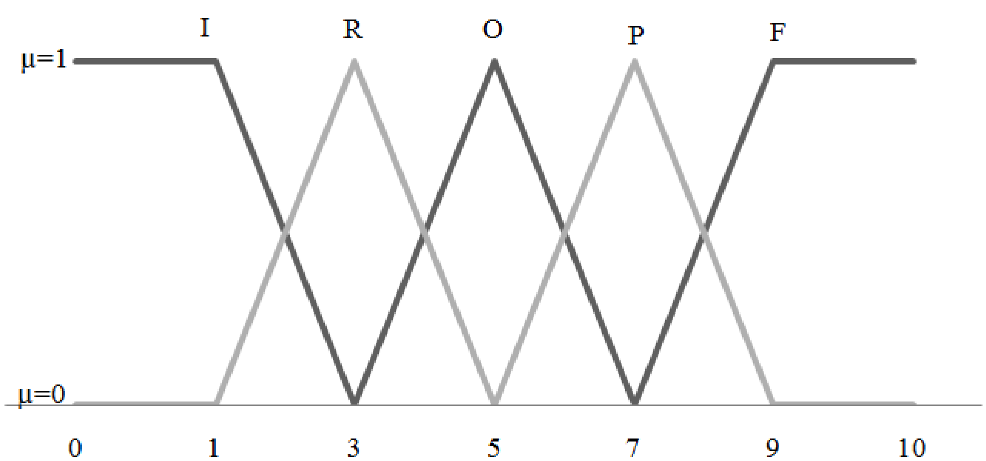


Figure 5.9. Linguistic variables for C_3, frequency of occurrence

Failure modes are then ranked by means of the calculation of the closeness coefficient CC_i considered by the FTOPSIS application (Table 5.4). This coefficient expresses the criticality of failure modes and is computed for each one of them on the basis of the distance d_i^* to an ideal positive fuzzy solution (preferably the shorter) and the distance d_i^- to an ideal negative fuzzy solution (preferably the higher). In this way, failure modes deemed to be the most critical are those characterised by a greater value of CC_i.

A threshold of 90% has lastly been fixed for the closeness coefficient to highlight the main criticalities among the entire set of failure modes related to the system under consideration.

Table 5.4. FTOPSIS results

ID - FM	Failure Mode	d_i^*	d_i^-	CC_i
1A	Broken PTO mechanism	0.340325	2.70939	0.888408
1B	Worn PTO bearings	0.209601	2.798121	0.930313
1C	Broken PTO universal joint shafts	0.340325	2.70939	0.888408
1D	General electrical system fault	0.445381	2.619742	0.854694
2A	Overheated oil	0.194575	2.810765	0.935257
2B	Insufficient oil level	0.414908	2.646928	0.864491
2C	Clogged filters	0.414908	2.646928	0.864491
3.1A	Fault in distribution system (start-up pump)	0.340325	2.70939	0.888408
3.1B	Mechanical fault (start-up pump)	0.340325	2.70939	0.888408
3.2A	Stopped start-up engine	0.340325	2.70939	0.888408
3.2B	Mechanical fault (start-up engine)	0.340325	2.70939	0.888408
3.3A	Fault in electrical system	0.278047	2.73963	0.907861
4.1.1A	Empty water tank	0.414908	2.646928	0.864491
4.1.1B	Hole in water tank	0.387407	2.666641	0.87315
4.1.2A	Fault in distribution system of the water pump	0.262598	2.754173	0.912954
4.1.2B	Mechanical fault of the water pump	0.393322	2.665441	0.871411
4.1.3A	No working nozzles	0.414908	2.646928	0.864491
4.1.3B	Clogged nozzles	0.414908	2.646928	0.864491
4.2.1A	Fault distribution system in Pump I	0.408771	2.650899	0.8664
4.2.1B	Mechanical fault in Pump I	0.408771	2.650899	0.8664
4.2.2A	Damaged brush or roller	0.247573	2.766816	0.91787

4.2.2B	Faulty hydraulic cylinders	0.445381	2.619742	0.854694
4.2.2C	Fault in electrical system	0.30968	2.711423	0.897494
4.2.3A	Damaged brush or roller	0.247573	2.766816	0.91787
4.2.3B	Faulty hydraulic cylinders	0.445381	2.619742	0.854694
4.2.3C	Fault in electrical system	0.30968	2.711423	0.897494
5.1A	Fault in distribution system (loading-pump)	0.576105	2.531011	0.814585
5.1B	Mechanical fault (loading-up pump)	0.440405	2.622692	0.856222
5.2.1A	Fault in hydraulic engine (rear roller)	0.576105	2.531011	0.814585
5.2.1B	Fault in actuator (rear roller)	0.30968	2.711423	0.897494
5.2.1C	Worn bristles (rear roller)	0.414908	2.646928	0.864491
5.2.1D	Fault in support arm (rear roller)	0.278047	2.73963	0.907861
5.2.1E	Slackened pivots or worn journal boxes (rear roller)	0.279207	2.738609	0.90748
5.2.2A	Fault in Pump III (elevator plant)	0.440405	2.622692	0.856222
5.2.2B	Fault in hydraulic engine (elevator plant)	0.576105	2.531011	0.814585
5.2.2C	Broken skid (elevator plant)	0.226209	2.782558	0.924817
5.2.2D	Broken chain (elevator plant)	0.194575	2.810765	0.935257
5.3.1A	Fault in support arm in the tank structure (emptying system)	0.225049	2.783579	0.925199
5.3.1B	Slackened pivots or worn journal boxes in the tank structure (emptying system)	0.226209	2.782558	0.924817
5.3.1C	Overturning cylinder fault in the tank structure (emptying system)	0.387407	2.666641	0.87315
5.3.2A	Broken or stopped releasing cylinder of the elevator plant	0.387407	2.666641	0.87315

Table 5.5 may be used as a driver during the planning of maintenance activities establishing priority of intervention.

Table 5.5. Ranking of the more critical failure modes

Ranking	ID - FM
1^{st}	2A
2^{nd}	5.2.2D
3^{rd}	1B
4^{th}	5.3.1A
5^{th}	5.2.2C
6^{th}	5.3.1B
7^{th}	4.2.2A
8^{th}	4.2.3A
9^{th}	4.1.2A
10^{th}	3.3A
11^{th}	5.2.1D
12^{th}	5.2.1E

Chapter 6

Human reliability analysis

Decision-making is intimately linked to human condition. Decisions usually derive from a combination of descriptive and experiential information (Weiss-Cohen *et al.*, 2016), and the need to make decisions pervades human life at virtually any level, individual, social, entrepreneurial, political, etc., definitely conditioning human behaviour.

In this context, Human Reliability Analysis (HRA) is considered as a useful tool in predicting and quantifying the occurrence of human errors during the execution of a specific task, that may be referred, for example, to manufacturing or project development activities (Carpitella *et al.*, 2017c) for supporting organisational risk assessment. In particular, human reliability is defined as the probability to successfully accomplish a general human activity (Swain, 1990). It is assessed with the aim of supporting the risk evaluation and, in particular, of determining the impact of human contribution to the risk of failure or success. Since processes characterising the systems under investigation involve varied factors, a multidisciplinary approach is necessary to manage human errors. Generally speaking, human errors (Reer, 2008a; 2008b) may be classified into Errors Of Commission (EOC) and Errors Of Omission (EOO). The first category refers to errors made during the identification, interpretation and execution phases of a specific task (*i.e.* errors of sequence, errors of timing, etc.) whereas the second category consists of leaving out a step of the task or the whole task itself due to forgetfulness or inattention.

6.1. The role of human factors in operational environments

Companies are managed following previously designed strategies, and operate according to processes implemented on the basis of the available resources. These strategies and processes are complex systems that integrate workers, plants and environment. Balancing and mutually adapting these elements make it possible, among others, the implementation of actions aimed at preventing the occurrence of accidents and occupational disease within workplaces, and also to identify near misses. Thus, the concept of human management system (HMS) takes relevant importance. In this context, one of the most important organizational objectives to be pursued is to promote a safe and environmentally responsible manner of working (Gholami *et al.*, 2015).

Clerici *et al.* (2016) affirm that an organization is a plurality of "human elements", and risks often depend on organizational criticalities, whose reduction can be undertaken by implementing effective human resource management (HRM). In particular, HRM is defined as a system of structured procedures aimed at optimizing the manpower management in a company (Azadeh and Zarrin, 2016), its workers being the most valuable assets of the organisation (Boatca and Cirjaliu, 2015). As asserted by Cirjaliu and Draghici (2016), nowadays companies seek to continuously improve the well-being and satisfaction of their human resources within their own operational environments.

An important aspect to take into account within this context is integrated by human factors and ergonomics (HF/E), whose optimal management is crucial to achieve central objectives, for instance the transition to sustainable development (Radjiyev *et al.*, 2015; Thatcher and Yeow, 2016). Ergai *et al.* (2016) underline as investigating on these aspects depends on the specific features of the workplace of reference and on the evaluator's background.

The importance of this concept is broadly shared in the literature. Wilson (2014) asserts that any understanding of system ergonomics must be related to the idea of system engineering. Hassall *et al.* (2015) stress that analyses based on human factors and ergonomics are commonly used to improve safety and productivity, particularly in complex systems. Sobhani *et al.* (2017) underline as the improvement of workplace ergonomic conditions gives opportunities to better deal with production variations and optimize the performance of system operation.

Amount and intensity of human interactions with processes generally depend on the field in which the organisation operates. Carpitella *et al.* (2017c) present a literature review in this regard, which is herein extended. Saravia-Pinilla *et al.* (2016) analyse the strong bond existing between environmental and human factors. In particular, the authors highlight a gap in the existing literature about this topic, and propose a model combining human and environmental factors with relation to the processes of product/service design and an *ad hoc* development to potentiate decision-making processes.

A tool particularly effective in conducting human factor-based analyses for reducing accidents and incidents is represented by the Human Factor Analysis and Classification System (HFACS), developed by Wiegmann and Shappell (2003), and applied in a wide variety of contexts, such as, for instance, aviation industry (Omole and Walker, 2015) or maritime safety (Soner *et al.*, 2015). Chen *et al.* (2013) focus on marine casualties and incidents and deal with human factors management with the aim of reducing accidents and avoiding disasters. The authors implement the framework HFACS for maritime accidents (HFACS-MA), a useful support to increase the level of safety and reduce human errors by identifying possible accident causes. Madigan *et al.* (2016) refer to the rail industry and stress the importance of carefully taking into account also latent factors. They propose HFACS by accomplishing a retrospective analysis to examine causes of minor incidents to prevent future and more severe events.

Even in those cases in which a high degree of automation is pursued, such as in manufacturing industries (Choe *et al.*, 2015), it is neither possible nor convenient totally eliminating human contribution.

Industries with high production volumes may consider machines and computers as faster and more reliable than humans in leading automatic operations. In this case, the human contribution given to automated processes would be barely necessary, and this may help reduce possible errors due to psychological and physical factors such as health, stress, age, mood, and so on. Moreover, the more customised the manufacturing process, the more crucial the role of human factors.

Also, the aspect of dependence among various phases of a process has to be considered and managed. This kind of dependence strongly impacts on the reliability level, as asserted by Zio *et al.* (2009). Indeed, considering, for instance, a sequence of two interdependent tasks, a fault on one of them increases the probability of failing on the other. The authors propose a framework based on a fuzzy system for eliciting expert knowledge about those factors mostly influencing dependence between two successive tasks. In particular, relationships between the input factors and the conditional human error probability are represented by means of a set of transparent fuzzy logic rules, and an application, related to two tasks required in response to an accident scenario at a nuclear power plant, is analysed.

Therefore, a current challenge faced by organisations consists in integrating even more machines and workers (Evans and Fendley, 2017), with the aim of creating a systematic operational environment and optimising all the available resources. In this context, human reliability strongly influences organisations' outcomes and plays an important role in evaluating risks related to industrial/business activities. With this recognition, the human factor can be considered a possible trigger event of faults and, thus, modelling human behaviour can be useful to understand the evolution of error probability.

Various levels of behaviour are assumed, which can be summarized as skill-based, rule-based and knowledge-based (Drivalou and Marmaras, 2009). Specific errors are associated to each of these kinds of behaviours (Reason, 1990) and the Human Error

Probability (HEP) increases by transiting from the first to the third. Concerning the skill-based behaviour, the main causes of error can be the lack of concentration and the presence of stressful situations. Regarding the rule-based behaviour, errors are mostly related to wrong use of procedures and rules. Lastly, considering the knowledge-based level, errors are due to incorrect interpretations of a specific situation and to incomplete or incorrect knowledge.

The HRA is centred on the quantification of the HEP. Literature contributions suggest numerous qualitative and quantitative HRA techniques which give particular importance to judgements expressed by experts on the relative application context (Bell and Holroyd, 2009; Paltrinieri and Øien, 2014).

HRA methods are usually split into four categories, namely the first, the second, the third generation and the expert judgement-based category. Table 6.1 gives a brief description of the aforementioned categories (Hollnagel, 1998), whereas Figure 6.1 shows the main relevant HRA methods considered in the literature (Bell and Holroyd, 2009). Four of them may be applied in generic fields, whereas the others are focused on the nuclear sector and, in some cases, on other specific fields such as the rail industry. Moreover, nine methods are publicly available (PA) and eight methods are not (NPA).

Table 6.1. Categories of HRA methods and acronyms' meaning

Category	**Human Reliability Analysis (HRA)**
	Description
First Generation Methods (FGM)	FGMs are mainly focused on the skill and rule-based levels of human behaviour. They are commonly used in quantitative risk assessment to estimate the probability of occurrence of human errors. The Human Error Probability (HEP) is determined by breaking down a task into its basic components and then modifying some factors to explore the effects due to such variation.
Second Generation Methods (SGM)	With respect to FGMs, SGMs are more focused on the operative context and on EOCs in predicting human error. However, they are more scarcely developed and used than FGMs.
Third Generation Methods (TGM)	TGMs are emerging tools proposed as possible development of FGMs, above all of the Human Error Assessment & Reduction Technique (HEART) method.
Expert Judgement Based Method (EJBM)	EJBMs provide experts with structured frameworks to face particularly hazardous contexts. They deal with the determination of error probability in a particular scenario but are not completely validated.
HRA Method	**Acronyms' Meaning**
THERP	Technique for Human Error-Rate Prediction
ASEP	Accident Sequence Evaluation Program
HEART	Human Error Assessment & Reduction Technique
SPAR-H	Standardized Plant Analysis Risk-Human
HRMS	Human reliability management system
JHEDI	Justified Human Error Data Information
INTENT	Not an acronym
ATHEANA	A Technique for Human Error Analysis
CREAM	Cognitive reliability and error analysis method
CAHR	Connectionism Assessment of Human Reliability
CESA	Commission Errors Search and Assessment
CODA	Conclusions from occurrences by descriptions of actions
MERMOS	Assessment method for performance of safety operation
NARA	Nuclear Action Reliability Assessment
APJ	Absolute Probability Judgement
PC	Paired comparisons
SLIM-MAUD	Success likelihood index methodology, Multi-Attribute Utility Decomposition

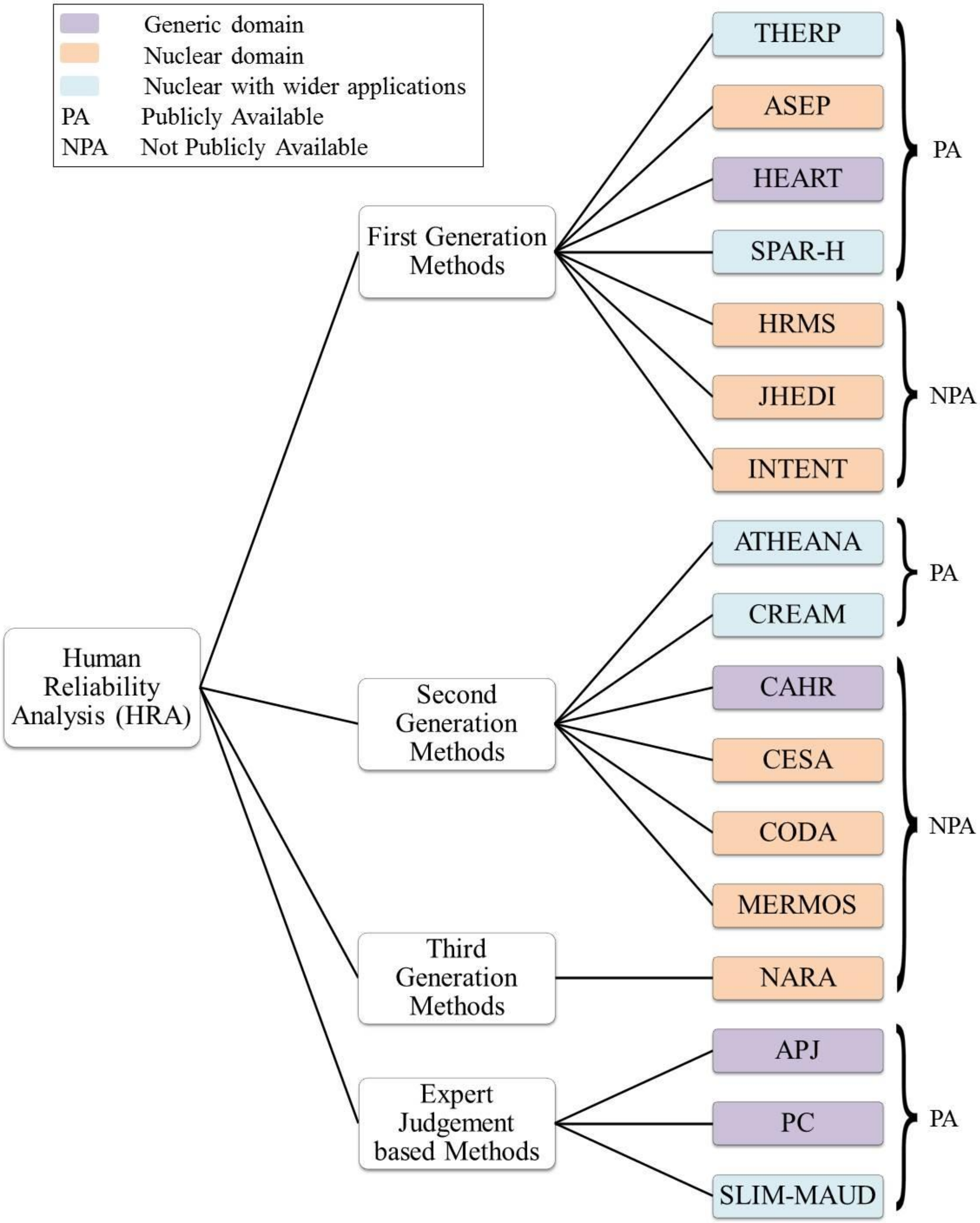


Figure 6.1. HRA methods

6.2. The THERP to calculate the probability of human error

The Technique for Human Error Rate Prediction (THERP) is herein proposed to evaluate the success probability of a software development project, the human factor being fundamental for the success of the whole project. Actually, in the software development field all project phases (i.e. requirement analysis, system design, implementation and testing) need to be performed in series and their success strictly depends on human actions as well as on interactions between subsequent phases. Namely, the way and the time within which a task is performed affect the success or the failure of the next phase. The latter means that the probability of success or failure of the subsequent phase is conditioned to the related probability of success or failure of the preceding one. In this section, such a dependence degree between phases is considered with relation to a software development project.

Developed in the Sandia Laboratories for the US Nuclear Regulatory Commission (Swain and Guttmann, 1983), THERP is nowadays one of the most used HRA methods to evaluate the probability of a human error occurring throughout the completion of a specific task. THERP belongs to the category of FGMs and is broadly used in a range of applications even beyond its original nuclear setting. The THERP-based technique to compute the HEP in accomplishing a specific task comprises various steps and makes use of a large HEPs database developed by Swain and Guttmann, other than historical accident reports, plant data and expert judgments. The THERP handbook also suggests the use of the so-called Performance Shaping Factors (PSFs) for the HEP calculation depending on what extent each factor applies to the task. In addition, THERP suggests a way to compute the probability of success (or failure) of a human task conditioned by the success (or failure) of another preceding human task. The computation of such a Conditioned Human Error Probability (CHEP) is based on the dependence degree existing between tasks. In particular, the dependence degree varies in a continuous way from a level of a complete negative dependence until a complete positive dependence passing through the level of zero dependence.

The negative dependence between two consecutive tasks A and B occurs when the failure of A increases the success probability of the event B or when the success of A increases the failure probability of B. Instead, the positive dependence occurs when the success (or failure) of A increases the success (or failure) of B. The THERP handbook (Swain and Guttmann, 1983) only deals with the positive dependence, which leads to an optimistic result, and suggests the assumption of independence when tasks are characterized by a negative dependence, which, conversely, leads to a conservative result.

In such a way, the variation interval of the dependence degree is limited by the Zero Dependence bound and the Complete Positive Dependence. To simplify the decisional process model and considering that intermediate points do not generate very different values of probabilities, the THERP model suggests five points of dependence, namely Zero Dependence (ZD), Low Dependence (LD), Moderate Dependence (MD), High Dependence (HD), and Complete Dependence (CD). ZD is applied when no dependence relation exists between two consecutive tasks (or events).

Even though rare, it could be chosen with relation to a weak relation. When some uncertainty exists in assuming ZD or LD, the choice of LD is deemed to be more opportune. In fact, such a choice produces precautionary results while the success probability is almost unchanged.

The MD degree is suggested when an evident relation between the two considered events exists. If an event substantially but not completely influences the other event, the high degree of dependence (HD) is chosen. CD implies that the success of an event is totally determined by the success of the preceding one.

Unfortunately, there are no rules to identify the degree of dependence, whose definition actually depends on the analyst perception, expertise and knowledge of the context under investigation. In this regard, the THERP handbook suggests as follows.

1. Assessing the time-space relation between events: the dependence degree increases with relation to events close in time and in space.
2. Evaluating the functional link between two events: if they are linked by a functional link, the dependence degree is stronger.
3. Analysing the effect of stress on the relation among the members of the team: stress affects by increasing the degree of dependence, mostly when individuals are characterized by reduced experience and personality.
4. Evaluating the similarity among the members of the team: homogeneous people in experience, training, status, etc., tend to interact more.

On the basis of the previously described dependence relations, the following table shows the equations proposed by the THERP model to derive the conditioned probability of failure (or success) of the event N given the basic probability of failure (or success) of the preceding event $(N-1)$.

Table 6.2. Equations for the computation of the conditioned probability of human error

Level of Dependence	Failure Equations	Success Equations
Zero	$\text{pr}(F_N\|F_{N-1}\|\text{ZD}) = \text{pr}(F_N)$	$\text{pr}(S_N\|S_{N-1}\|\text{ZD}) = \text{pr}(S_N)$
Low	$\text{pr}(F_N\|F_{N-1}\|\text{LD}) = = \frac{[1 + 19 \cdot \text{pr}(F_N)]}{20}$	$\text{pr}(S_N\|S_{N-1}\|\text{LD}) = = \frac{[1 + 19 \cdot \text{pr}(S_N)]}{20}$
Moderate	$\text{pr}(F_N\|F_{N-1}\|\text{MD}) = = \frac{[1 + 6 \cdot \text{pr}(F_N)]}{7}$	$\text{pr}(S_N\|S_{N-1}\|\text{MD}) = = \frac{[1 + 6 \cdot \text{pr}(S_N)]}{7}$
High	$\text{pr}(F_N\|F_{N-1}\|\text{HD}) = = \frac{[1 + \text{pr}(F_N)]}{2}$	$\text{pr}(S_N\|S_{N-1}\|\text{HD}) = = \frac{[1 + \text{pr}(S_N)]}{2}$
Complete	$\text{pr}(F_N\|F_{N-1}\|\text{CD}) = 1$	$\text{pr}(S_N\|S_{N-1}\|\text{CD}) = 1$

As already asserted, the presented application case deals with the computation of the success probability of a software development project, by means of THERP. Generally speaking, projects addressed to software development comprise four phases (*i.e.* A, B, C and D) to be performed in series. Specifically, A is related to the requirement analysis, B refers to the system design, C is the implementation task and D the testing. A, B, C and D are carried out by experts in the field so that their success strictly depends on human factors. Judgments on the basic HEP related to each phase are elicited from an expert who also classifies tasks into two categories, *i.e.* critical and routine (Table 6.3).

Table 6.3. HEP related to each task

Tasks	Type of Task	HEP
A	Critical	0.05
B	Routine	0.005
C	Routine	0.005
D	Routine	0.005

Each criticality arises from different factors. For example, the criticality of the phase A arises from a not clear understanding of the customer specifications. On the basis of HEPs of Table 6.3 and of the dependence degree between tasks elicited from the expert (Table 6.4), THERP equations of Table 6.2 – reported in the section "application of dependence equations" of Swain and Guttmann (1983) – enable to compute the conditioned probabilities among the various software project development phases.

Table 6.4. Dependence degree between tasks

Tasks	**Dependence Degree**
$B\|A$	High
$D\|A$	Zero
$C\|A$	Low
$C\|B$	Complete
$D\|B$	Moderate
$D\|C$	Moderate

To get the probability of the event of interest, namely the success of the software development project, through the Event Tree of Figure 6.2, the following equation is used:

$$\mathrm{pr(S) = pr(D|C|B|A) \cdot pr(C|B|A) \cdot pr(B|A) \cdot pr(A)} = 0.9957 \cdot 1 \cdot 0.9975 \cdot 0.95 = \\ = 0.9435. \tag{6.1}$$

Regarding the application of the dependence equations, with relation to three generic consecutive phases x, y and k, it is possible to note that if the degree of dependence between y and x (*i.e.* $y|x$) is LD and between k and y (*i.e.* $k|y$) is HD, it follows up that the dependence degree $k|y|x$ is HD.

After having discussed about techniques of human reliability analysis and provided a practical example of THERP application estimating the probability of human error, the following section proposes a MCDM approach aimed at considering the role of human factors in industrial processes. In particular the DEMATEL methodology is applied to evaluate the degree of interdependency among human factors involved in a real industrial process for which maintenance is crucial.

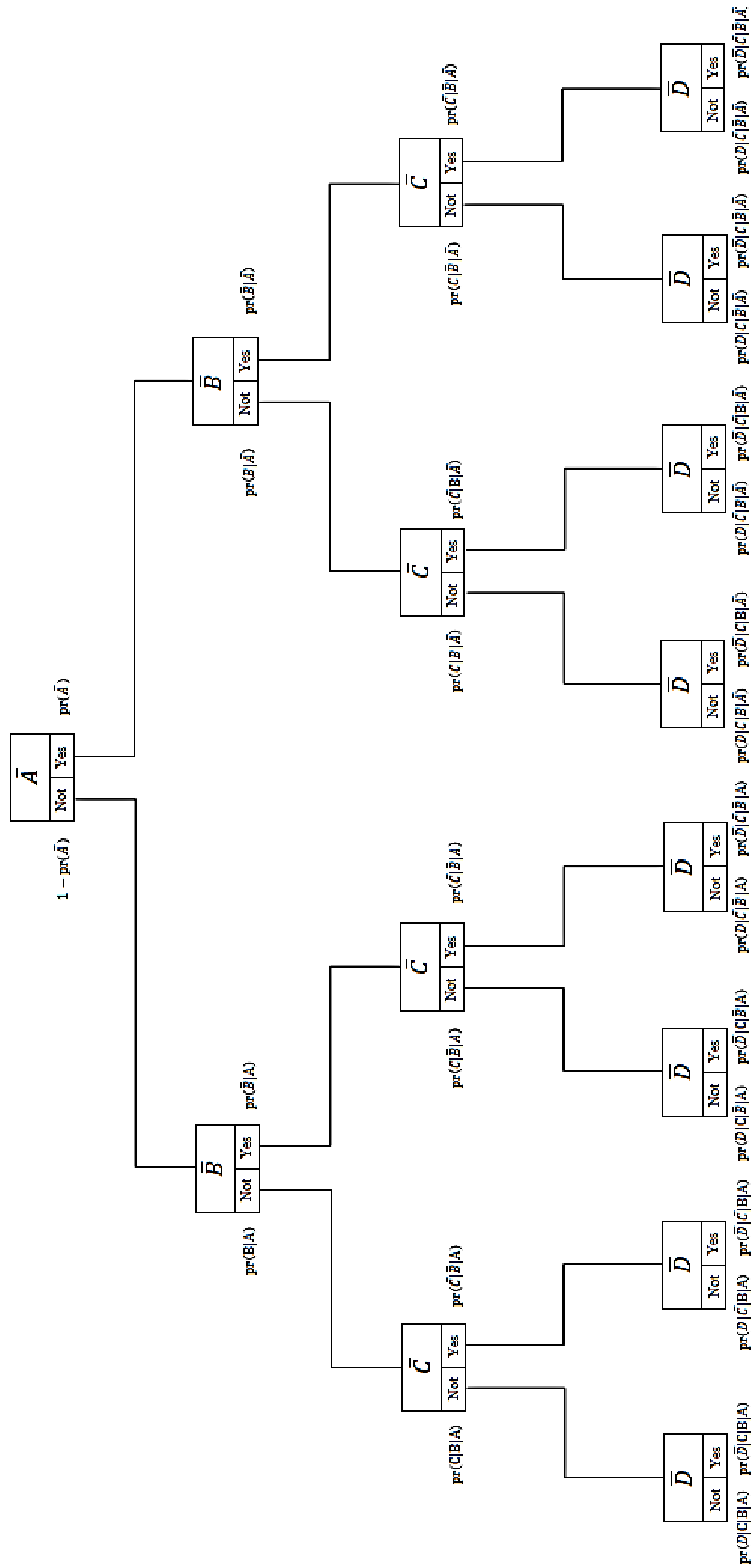


Figure 6.2. Event tree

6.3. The DEMATEL to evaluate interdependencies among human factors

On the basis of all the above, organisational risk assessment in industrial environments is conducted with the aim of evaluating, eliminating or at least minimising risks related to ineffective manners of work, in terms of methods and operation management from humans. Such kind of risks derives from psychological and physical conditions that negatively impact on the quality of work and life.

In particular, when leading organisational risk assessment, the main areas presented in Table 6.5 are analysed with a deep level of detail. The purpose consists in highlighting the presence of possible stressful aspects related to human factors and ergonomics within each area, which could potentially damage the global wellness and health of workers, and then the performance of the whole organisation.

Table 6.5. Description of investigated area related to human factors and ergonomics

ID	Investigated area	Description
A_1	Organizational culture and role	Sharing values upon which the organisation policy is grounded; maintaining relationships among different levels of the same organization; being aware about the own role within the company.
A_2	Career development and job stability	Having clear the possibilities of development in terms of career advances; knowing the path of professional growing; achieving contractual stability.
A_3	Communication, information, consultation and participation of workers	Empowering communication among all the levels of the hierarchy structure of the company; involving workers within decision-making processes to pursue general business objectives.
A_4	Training, awareness and competence	Promoting training paths aimed at increasing specific competencies of workers and at continuously improving the level of safety & security related to industrial processes.
A_5	Operational control: indication of measures and instruments	Defining scheme, minimum contents and work procedures to lead a safe execution of the main tasks; identifying the main criticalities to be monitored; monitoring and controlling processes and outputs; planning and implementing maintenance interventions on the basis of the policies undertaken by the organisation.

ID	Investigated area	Description
A_6	Extraordinary situations and changes management	Defining criteria, methods and responsibilities to identify possible scenarios of extraordinary situations causing exceptional or unusual results; establishing intervention measures; managing changes to implement corrective measures.
A_7	Outsourcing and interference management	Evaluating direct and indirect impacts of the outsourcing process; implementing a framework of cooperation with external companies to optimise safety both of internal workers and third parties.
A_8	Workload and working hours	Examining the entity of workload; balancing responsibilities related to each group of workers; managing and correctly planning the amount of working hours per person; integrating work with life and social contexts of workers.

These factors are present in almost all the working environments. Among all organisational aspects, the European agreement on work-related stress held in Brussels in the year 2004 (European Social Partners, 2008) underlines as managing the problem of stress at work leads to greater efficiency and improvement of health and safety conditions, with consequent economic and social benefits for companies, workers and society. For this reason, the same agreement established to offer models and guidelines for evaluating work-related stress on the basis of two phases, namely preliminary assessment and in-depth evaluation. The first phase is based on the identification of verifiable and quantitative stress indicators. The second phase should be undertaken through surveys, focusing on groups and semi-structured interviews to homogeneous groups of workers.

By analysing the results coming from such evaluations, we propose to focus on the more critical human factors emerged for each target area (Table 6.5). With this aim, the DEMATEL methodology is suggested to select, within the set of highlighted human factors, those most influencing the others. This approach is useful to suggest an order in planning and implementing mitigation measures of organisational risk.

In complex systems, many aspects, factor or criteria are, either directly or indirectly, deeply intertwined (sometimes in a hidden way), and mutual interference affect other elements, thus making it difficult to find priorities for action and eventually hindering decision-making. In many cases, pursuing a specific objective may inadvertently impair several other objectives. So, having a clear vision of the system contributes to the identification of workable solutions. DEMATEL has shown to help confirm interdependence among variables and restrict the relation that reflects the characteristics of a system of management trend (Hori and Shimizu, 1999; Tamura *et al.*, 2002; Chiu *et al.*, 2006). DEMATEL's outcome is a visual representation, through which decision-makers may organize better the actions to take. The purpose of the use of DEMATEL in this thesis (Carpitella *et al.*, 2018b) is to discern the direction and intensity of direct and indirect relationships that flow between a number of well-defined elements. Thus, experts' knowledge is used to contribute to better understand the problem components and the way they interrelate.

The implementation of the DEMATEL methodology can be broadly summarised through four steps (Tafreshi *et al.*, 2016) described next and summarised in Figure 6.3 and then described in detail. The methodology requires a preliminary and clear definition of the problem under analysis. The goal of the problem and the main considered elements must be identified with the help of experts.

I STEP – Building the direct-relation matrix, A

The first step must be implemented after producing as input data the non-negative matrices $X^{(k)}$, $1 \leq k \leq H$, where H is the number of involved experts issuing judgments concerning the mutual influence between pairs of elements. The elements $x_{ij}^{(k)}, i, j = 1, .., n$, where n is the number of compared elements of matrices $X^{(k)}$ are the numerical values encoding the judgments. The numerical value meanings for a typical element $x_{ij}^{(k)}$ are defined as: 0 (no influence), 1 (very low influence), 2 (low influence), 3 (high influence), 4 (very high influence). The main diagonal of each matrix is filled with zeroes.

Lastly, the output of this step is the calculation of the direct-relation matrix A, aimed at incorporating the matrices filled in by the experts. A is a $n \times n$ square matrix whose entries a_{ij} are obtained by:

$$A = \frac{1}{H} \sum_{k=1}^{H} X^{(k)}. \qquad (6.2)$$

II STEP - Obtaining the normalised direct-relation matrix, D

The second step consists in building the normalised direct-relation matrix from the direct-relation matrix obtained as output in the previous step. The normalised matrix is calculated as

$$D = s\,A, \qquad (6.3)$$

where s is a positive number slightly smaller than:

$$\min\left[\frac{1}{\max\limits_{1\le i\le n}\sum_{j=1}^{n}a_{ij}},\frac{1}{\max\limits_{1\le j\le n}\sum_{i=1}^{n}a_{ij}}\right]. \qquad (6.4)$$

Based on matrix D, the initial influence that elements exert on and receive from the others is shown. Then, a continuous decrease of the indirect effects among the considered elements may be obtained along the consecutive powers of matrix D. This enables to obtain the total relation matrix, as explained next.

III STEP - Calculating the total-relation matrix, T

The third step of the procedure is aimed at incorporating direct and indirect effects, through the calculation of the total relation matrix T. This matrix reflects both direct and indirect effects among elements, and is achievable through the sum of the powers of matrix D. Observe that $\lim\limits_{n\to\infty} D^n = 0$, since the spectral radius of D is smaller than 1, since, by (6.4), it is bounded by the maximum row and column sum. As a result, see, for example, Example 7.3.1 in (Meyer, 2001), the power series of D, $I + D + D^2 + ...$, converges to $(I - D)^{-1}$ where I is the identity matrix of size n. Consequently, the total relation matrix may be written as:

$$T = D(I - D)^{-1}. \qquad (6.5)$$

As said, this matrix represents the build-up of mutual direct and indirect effects among elements. Observe that the diagonal entries of matrix D (accounting for the direct effects) are zero; however, the diagonal elements of T collect all the non-direct effects associated to their corresponding factors.

IV STEP - Drawing the impact-relations map

The fourth step aims to obtain an influential relation map by defining $\mathbf{r} = (r_i)$ and $\mathbf{c} = (c_j)$ as $n \times 1$ and $1 \times n$ vectors respectively representing the sum of the rows and the sum of the columns of the total relation matrix T. Particularly, r_i represents both direct and indirect effects of element i on the others, whereas c_i summarizes both direct and indirect effects of the other elements on element i. In such a way, the sum $r_i + c_i$ gives the overall effect (prominence) of element i and the subtraction $r_i - c_i$ helps in dividing the elements into cause and effect groups (relation). Prominence allows to rank factors according to their global influence, while relation enables to group elements into the cause group, if the subtraction is positive, and into the effect group otherwise.

Prominence ranking gives crucial information on the impact associated to the factors. However, a cutoff on the factor list is performed through a suitable threshold, bearing in mind that if the threshold is too high important factors may be excluded and if it is too low, too many factors – some of them irrelevant – may be included, which will turn the solution too complex and thus impractical. In the literature, the threshold value is determined in a variety of ways: by experts through discussions (Tzeng *et al.*, 2007; Lin and Tzeng, 2009) or brainstorming techniques (Azadeh *et al.*, 2015), by following results of literature review, the maximum mean deentropy (Lee and Lin, 2013), the average of all elements in the matrix T (Sara *et al.*, 2015), among others. In this study it is used this last value. Finally, a causal diagram chart is drawn by mapping the dataset of $(r_i + c_i, r_i - c_i)$, which gives a graphical representation of the main interrelations among factors. Typically, only the interrelations among factors considered within the cutoff are drawn, for the sake of clarity.

A summary of these steps can be seen in Figure 6.3.

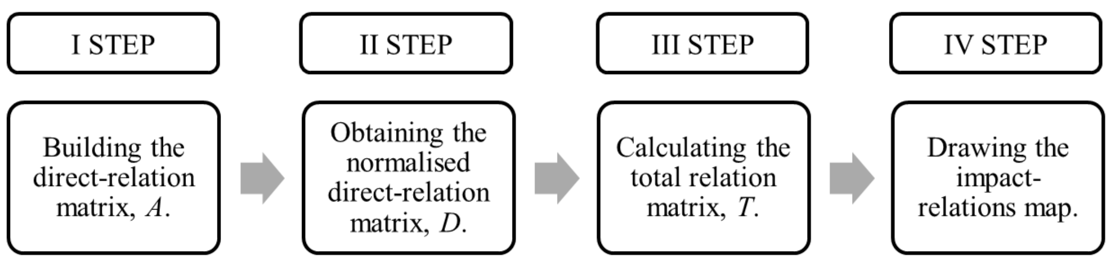


Figure 6.3. Steps for implementing the DEMATEL methodology

The main goal of the DEMATEL application in the present case study consists in identifying key factors based on the causal relationships and the degrees of interrelationship between them, with the aim of providing companies with a structured way of understanding the nature of interdependencies within a set of human factors. As previously asserted, the definition of human factors results from a previous context evaluation carried out in terms of an organizational risk analysis. In other terms, the aim is to identify aspects influencing the others and aspects being influenced by others for pursuing a higher level of safety and security in leading industrial processes. To demonstrate the usefulness of our approach, a real-world case study is developed to evaluate interdependencies among critical human factors analysed in a manufacturing process of a Sicilian firm with the aim of reducing organizational criticalities.

The case study refers to a manufacturing firm, a winery located in Trapani, Sicily (Italy). The wine bottling process carried out in the company is analysed. This process is composed of 13 different phases, provided in Figure 6.4, and takes place in the area dedicated to delivery and production. In the mentioned area, there are three fixed stations and a movable position, respectively occupied by the following operators:

1. W_1, worker dedicated to control that bottles are filled in and plugged;
2. W_2, worker dedicated to control the global quality of bottles;
3. W_3, worker dedicated to wrap final products;
4. W_4, worker dedicated to carry out the following two activities: raw materials (empty bottles, labels and corks) and packaging supply; handling of wrapped final products.

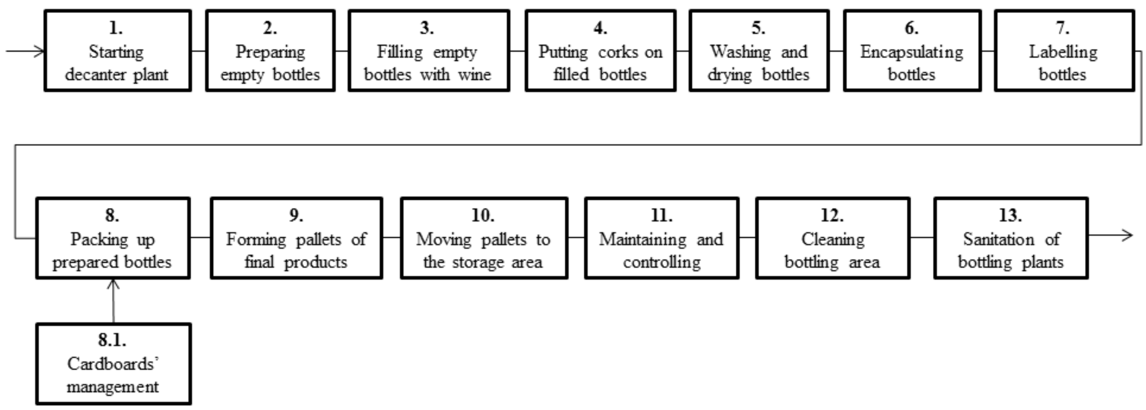


Figure 6.4. Phases of the bottling process

The scheme of the production line representing the bottling process is shown in Figure 6.5. The stations indicated as "DP" and "P", respectively, represent the point in which empty bottles are first taken off from pallets (in which they were originally stocked) for starting the bottling process, and the point in which bottles (after having been filled in, plugged and checked) are finally put in pallets and wrapped to be sent to the storage or final customer areas.

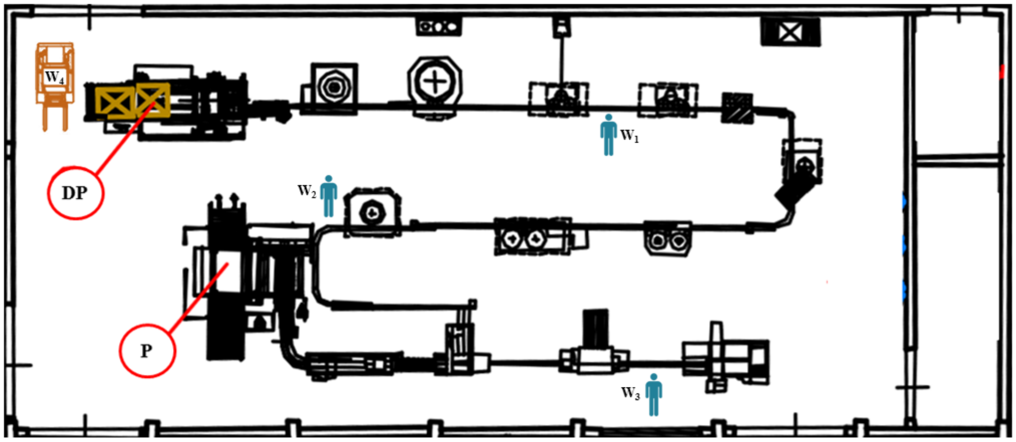


Figure 6.5. Scheme of the production line representing the bottling process

With relation to the described process, the firm recently undertook an organisational risk assessment by focusing on the group of workers distributed in the interested zone. In particular, the work-related stress was evaluated by adopting the guidelines provided in 2011 by the National (Italian) Institute for Insurance against Accidents at Work (INAIL, 2011). Within that evaluation, the areas of Table 6.5 were deeply investigated by means of detailed surveys with the workers. These surveys aimed at highlighting the possible presence of critical human factors for each area, with the final purpose of managing critical aspects and then reducing the organisational risk as much as possible. In particular, the 16 human factors of Table 6.6 (listed with relation to their related area) emerged as possible source of problems. The application of the DEMATEL methodology is suggested for establishing an order for implementing mitigating measures.

Table 6.6. Critical human factors related to each area

ID	Investigated area	Description
A_1	Organisational culture and role	• **HF_1** System of security and safety management not implemented; • **HF_2** Ethical and behavioural code not implemented.
A_2	Career development and job stability	• **HF_3** Criteria for career advancement are not defined; • **HF_4** Reward systems related to the correct management of human resources are not defined for supervisors; • **HF_5** Reward systems related to the achievement of security objectives are not defined.

ID	Investigated area	Description
A_3	Communication, information, consultation and participation of workers	• **HF_6** Work may depend on tasks previously accomplished by others; • **HF_7** Tools involving workers within decisions and strategies are not implemented; • **HF_8** Rigid protocols supervising and controlling workers are implemented.
A_5	Operational control: indication of measures and instruments	• **HF_9** Workers are exposed to noise between the I and the II levels of action; • **HF_{10}** Inadequate ventilation and microclimate; • **HF_{11}** Inadequate lighting; • **HF_{12}** Workers may be exposed to the risk of recurring movements.
A_8	Workload and working hours	• **HF_{13}** Unpredictably variations of workload; • **HF_{14}** Workers cannot regulate machines' rhythm; • **HF_{15}** Workers lead tasks having high level of responsibility for stakeholders, plants and production; • **HF_{16}** Shifts may be not well organised.

The DEMATEL is now applied to evaluate existing interdependencies within the set of $n = 16$ human factors detailed in Table 6.6. Five experts in the field ($H = 5$) were involved to such an aim, whose roles are defined in Table 6.7.

Table 6.7. Roles of the decision makers

Decision maker	Role
H_1	Maintenance responsible
H_2	Quality manager
H_3	Consultant
H_4	Chief of the safety and security system
H_5	Department chief

The experts composing the decision-making group contribute to the process development by playing diverse but complementary roles. Indeed, these subjects have been involved with the aim to guarantee as complete as possible understanding about the problem under analysis.

Each decision-maker was asked to evaluate the direct influence between any two human factors by means of integer scores from 0 to 4. Three non-negative square matrices $X^{(1)}$, $X^{(2)}$, $X^{(3)}$, $X^{(4)}$ and $X^{(5)}$ were collected and then aggregated to obtain the direct-relation matrix A of order 16 (Table 6.8).

Table 6.8. Direct-relation matrix ***A***

A	**HF_1**	**HF_2**	**HF_3**	**HF_4**	**HF_5**	**HF_6**	**HF_7**	**HF_8**	**HF_9**	**HF_{10}**	**HF_{11}**	**HF_{12}**	**HF_{13}**	**HF_{14}**	**HF_{15}**	**HF_{16}**
HF_1	0.000	3.200	1.800	1.600	4.000	2.000	2.400	2.200	4.000	3.600	4.000	4.000	1.800	2.800	4.000	2.200
HF_2	2.200	0.000	2.800	3.000	2.600	0.000	0.000	3.200	1.200	1.600	1.600	1.200	0.000	0.000	2.600	1.400
HF_3	2.200	2.000	0.000	4.000	3.000	3.000	3.000	3.000	0.000	0.000	0.000	0.000	0.000	0.000	1.800	1.600
HF_4	2.000	3.200	3.200	0.000	4.000	1.200	4.000	3.200	0.000	0.000	0.000	0.000	2.400	1.200	3.600	3.600
HF_5	3.600	2.000	3.200	4.000	0.000	2.800	2.800	2.800	2.400	2.800	2.400	2.800	1.200	4.000	4.000	2.400
HF_6	2.400	1.000	2.200	2.400	3.200	0.000	2.000	3.400	1.400	1.400	1.400	1.400	4.000	2.800	3.000	3.800
HF_7	1.400	0.000	2.200	2.600	3.200	3.000	0.000	4.000	1.600	1.600	1.600	1.600	3.200	0.000	2.200	4.000
HF_8	2.600	2.200	3.000	3.000	3.600	2.400	4.000	0.000	1.800	1.800	1.800	1.800	2.400	2.200	3.200	2.000
HF_9	4.000	1.200	0.000	0.000	2.800	1.400	2.000	1.800	0.000	0.200	0.200	0.200	3.000	1.400	3.200	1.800
HF_{10}	4.000	1.200	0.000	0.000	2.800	1.400	2.000	1.800	0.200	0.000	0.200	0.200	2.600	1.000	2.800	1.800
HF_{11}	4.000	1.200	0.000	0.000	2.800	1.400	2.000	1.800	0.200	0.200	0.000	0.200	2.600	1.400	2.200	1.800
HF_{12}	4.000	1.200	0.000	0.000	2.800	1.400	2.000	1.800	0.200	0.200	0.200	0.000	3.000	1.000	3.200	3.000
HF_{13}	1.800	0.000	0.000	2.200	1.400	4.000	3.200	2.000	3.000	3.000	3.000	3.000	0.000	3.200	3.800	3.800
HF_{14}	4.000	0.000	0.000	1.000	3.000	2.800	0.000	2.200	1.000	1.000	1.600	1.000	3.600	0.000	3.200	2.200
HF_{15}	4.000	2.200	1.400	4.000	3.600	3.000	2.200	3.200	2.200	3.200	1.800	1.200	3.400	3.200	0.000	3.200
HF_{16}	3.600	2.000	1.200	4.000	2.000	3.400	3.000	2.000	1.800	2.800	1.800	2.000	3.600	1.800	2.000	0.000

Tables 6.9 and 6.10 show the normalized direct-relation matrix D and the total relation matrix T. Lastly, Table 6.11 shows the values of $r_i + c_i$ and $r_i - c_i$ associated to the various factors, and the ranking of factors, obtained on the basis of their prominence, $r_i + c_i$, which collects the direct and indirect effects related to all the other factors.

Table 6.9. Normalised direct-relation matrix ***D***

D	**HF_1**	**HF_2**	**HF_3**	**HF_4**	**HF_5**	**HF_6**	**HF_7**	**HF_8**	**HF_9**	**HF_{10}**	**HF_{11}**	**HF_{12}**	**HF_{13}**	**HF_{14}**	**HF_{15}**	**HF_{16}**
HF_1	0.000	0.070	0.039	0.035	0.087	0.044	0.052	0.048	0.087	0.079	0.087	0.087	0.039	0.061	0.087	0.048
HF_2	0.048	0.000	0.061	0.066	0.057	0.000	0.000	0.070	0.026	0.035	0.035	0.026	0.000	0.000	0.057	0.031
HF_3	0.048	0.044	0.000	0.087	0.066	0.066	0.066	0.066	0.000	0.000	0.000	0.000	0.000	0.000	0.039	0.035
HF_4	0.044	0.070	0.070	0.000	0.087	0.026	0.087	0.070	0.000	0.000	0.000	0.000	0.052	0.026	0.079	0.079
HF_5	0.079	0.044	0.070	0.087	0.000	0.061	0.061	0.061	0.052	0.061	0.052	0.061	0.026	0.087	0.087	0.052
HF_6	0.052	0.022	0.048	0.052	0.070	0.000	0.044	0.074	0.031	0.031	0.031	0.031	0.087	0.061	0.066	0.083
HF_7	0.031	0.000	0.048	0.057	0.070	0.066	0.000	0.087	0.035	0.035	0.035	0.035	0.070	0.000	0.048	0.087
HF_8	0.057	0.048	0.066	0.066	0.079	0.052	0.087	0.000	0.039	0.039	0.039	0.039	0.052	0.048	0.070	0.044
HF_9	0.087	0.026	0.000	0.000	0.061	0.031	0.044	0.039	0.000	0.004	0.004	0.004	0.066	0.031	0.070	0.039
HF_{10}	0.087	0.026	0.000	0.000	0.061	0.031	0.044	0.039	0.004	0.000	0.004	0.004	0.057	0.022	0.061	0.039
HF_{11}	0.087	0.026	0.000	0.000	0.061	0.031	0.044	0.039	0.004	0.004	0.000	0.004	0.057	0.031	0.048	0.039
HF_{12}	0.087	0.026	0.000	0.000	0.061	0.031	0.044	0.039	0.004	0.004	0.004	0.000	0.066	0.022	0.070	0.066
HF_{13}	0.039	0.000	0.000	0.048	0.031	0.087	0.070	0.044	0.066	0.066	0.066	0.066	0.000	0.070	0.083	0.083
HF_{14}	0.087	0.000	0.000	0.022	0.066	0.061	0.000	0.048	0.022	0.022	0.035	0.022	0.079	0.000	0.070	0.048
HF_{15}	0.087	0.048	0.031	0.087	0.079	0.066	0.048	0.070	0.048	0.070	0.039	0.026	0.074	0.070	0.000	0.070
HF_{16}	0.079	0.044	0.026	0.087	0.044	0.074	0.066	0.044	0.039	0.061	0.039	0.044	0.079	0.039	0.044	0.000

Table 6.10. Total direct-relation matrix $\boldsymbol{T}$

T	**HF1**	**HF2**	**HF3**	**HF4**	**HF5**	**HF6**	**HF7**	**HF8**	**HF9**	**HF10**	**HF11**	**HF12**	**HF13**	**HF14**	**HF15**	**HF16**
HF1	0.209	0.175	0.140	0.186	0.285	0.198	0.211	0.221	0.190	0.195	0.192	0.187	0.209	0.190	0.287	0.224
HF2	0.162	0.068	0.123	0.155	0.172	0.090	0.098	0.167	0.086	0.102	0.095	0.085	0.094	0.076	0.171	0.129
HF3	0.164	0.113	0.075	0.186	0.188	0.157	0.165	0.174	0.067	0.075	0.068	0.066	0.102	0.081	0.162	0.145
HF4	0.199	0.155	0.158	0.137	0.243	0.155	0.214	0.210	0.090	0.102	0.091	0.088	0.181	0.130	0.235	0.217
HF5	0.281	0.154	0.173	0.240	0.208	0.218	0.223	0.237	0.158	0.179	0.160	0.164	0.199	0.214	0.289	0.232
HF6	0.228	0.117	0.139	0.191	0.243	0.144	0.188	0.224	0.128	0.139	0.129	0.126	0.233	0.177	0.242	0.237
HF7	0.191	0.090	0.132	0.183	0.227	0.192	0.137	0.222	0.122	0.132	0.122	0.120	0.203	0.109	0.209	0.227
HF8	0.236	0.145	0.160	0.207	0.258	0.194	0.231	0.162	0.136	0.148	0.138	0.134	0.202	0.164	0.251	0.206
HF9	0.205	0.092	0.066	0.099	0.181	0.129	0.141	0.145	0.074	0.086	0.079	0.076	0.167	0.116	0.193	0.148
HF10	0.199	0.089	0.063	0.095	0.175	0.124	0.136	0.140	0.075	0.078	0.075	0.073	0.153	0.103	0.178	0.142
HF11	0.197	0.088	0.062	0.093	0.173	0.122	0.135	0.138	0.074	0.081	0.070	0.072	0.151	0.110	0.164	0.141
HF12	0.210	0.095	0.068	0.104	0.185	0.133	0.146	0.149	0.081	0.090	0.082	0.075	0.171	0.110	0.197	0.177
HF13	0.223	0.096	0.090	0.180	0.210	0.225	0.211	0.197	0.159	0.171	0.160	0.156	0.161	0.186	0.260	0.240
HF14	0.223	0.077	0.072	0.130	0.201	0.170	0.116	0.165	0.103	0.111	0.116	0.101	0.194	0.099	0.210	0.171
HF15	0.285	0.157	0.137	0.239	0.278	0.221	0.211	0.242	0.157	0.190	0.151	0.134	0.241	0.201	0.208	0.246
HF16	0.252	0.139	0.120	0.219	0.222	0.210	0.208	0.199	0.135	0.166	0.137	0.137	0.226	0.155	0.225	0.162

Table 6.11. Final ranking

	$\boldsymbol{r_i + c_i}$	$\boldsymbol{r_i - c_i}$	**Ranking**	$\boldsymbol{r_i + c_i}$ ↓
HF1	6.762	-0.167	**HF5**	6.779
HF2	3.721	0.024	**HF15**	6.776
HF3	3.765	0.210	**HF1**	6.762
HF4	5.250	-0.037	**HF8**	5.964
HF5	6.779	-0.121	**HF16**	5.954
HF6	5.568	0.202	**HF13**	5.809
HF7	5.390	-0.152	**HF6**	5.568
HF8	5.964	-0.020	**HF7**	5.390
HF9	3.833	0.162	**HF4**	5.250
HF10	3.943	-0.148	**HF14**	4.477
HF11	3.733	0.006	**HF10**	3.943
HF12	3.866	0.276	**HF12**	3.866
HF13	5.809	0.039	**HF9**	3.833
HF14	4.477	0.040	**HF3**	3.765
HF15	6.776	-0.181	**HF11**	3.733
HF16	5.954	-0.133	**HF2**	3.721

Human factors with higher $r_i + c_i$ value give crucial information regarding, in this case, how to reduce organisational risk, since their variations have greater impact on the variations of all the other aspects. As explained before, a threshold has to be established for not taking into account negligible effects. As said, this threshold is here calculated as the average of all the elements in matrix T. In this case the threshold is 0.159. Now, those factors having a value of $T(\mathrm{HF}_i; \mathrm{HF}_i)$ higher than the threshold are selected.

Accordingly, it is suggested that the human factors occupying the first six positions of the ranking need to be more carefully monitored during the process of organisational risk management. They are, in order:

- **HF$_5$:** reward systems related to the achievement of security objectives are not defined;
- **HF$_{15}$**: workers lead tasks having high level of responsibility for stakeholders, plants and production;
- **HF$_1$**: system of security and safety management not implemented;
- **HF$_8$**: rigid protocols supervising and controlling workers are implemented;
- **HF$_{16}$**: shifts may be not well organised;
- **HF$_{13}$**: unpredictably variations of workload.

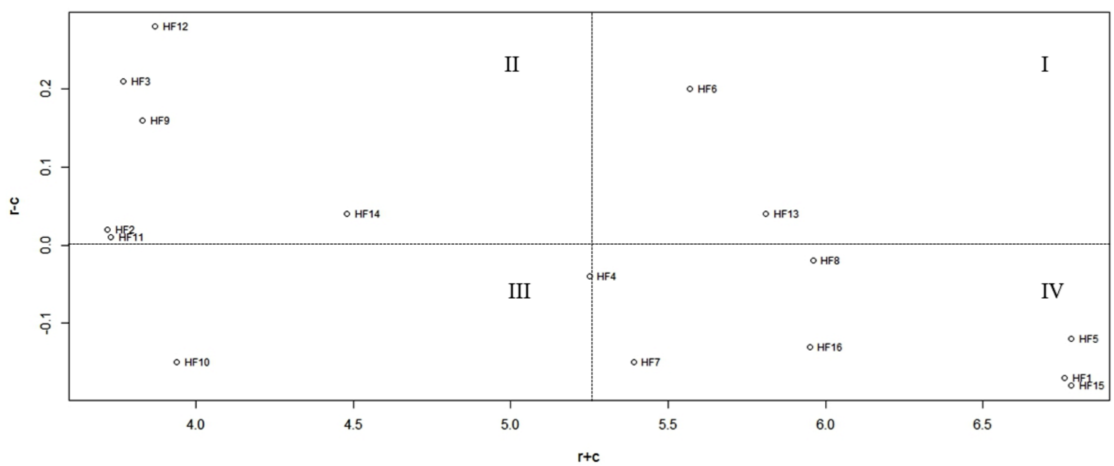


Figure 6.6. DEMATEL chart with HFs spread out into quadrants

Figure 6.6 presents the four quadrants of the chart derived from the DEMATEL application. From this representation, decision makers can visually identify causal relationships among the considered human factors. The rationale for selecting, Si *et al.* (2018), may be summarized as follows:

- factors in quadrant I are identified as core factors or intertwined givers since they have high prominence and relation;
- factors in quadrant II have low prominence but high relation, which are impacted by other factors and cannot be directly improved;

- factors in quadrant III have low prominence and relation and are relatively disconnected from the system;
- factors in quadrant IV are identified as driving factors or autonomous givers because they have high prominence but low relation.

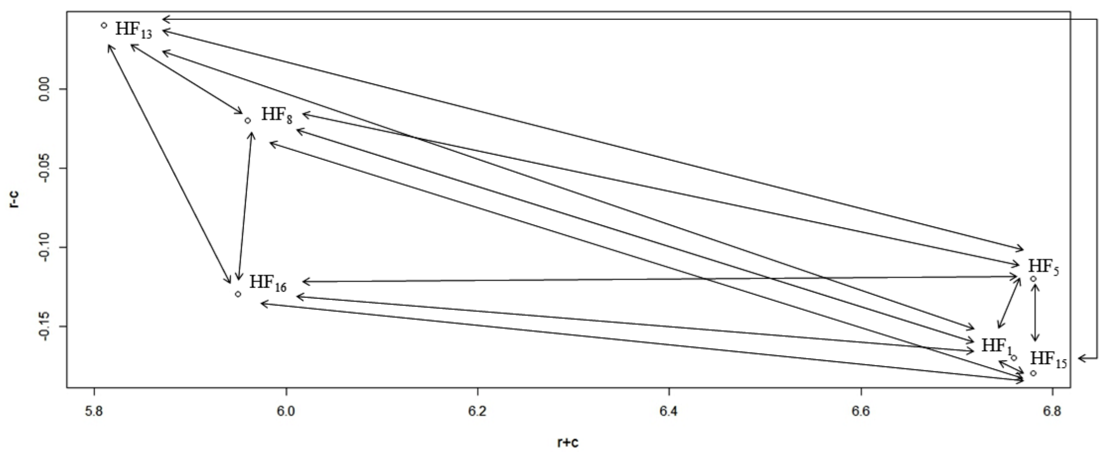


Figure 6.7. Chart representing interdependencies among the six selected HFs

Figure 6.7 shows the interdependencies among the selected HFs, the casual factors. In this methodology, arrows for the factors with values $T(\mathrm{HF}_i; \mathrm{HF}_i)$ lower than the threshold are not customary indicated in the graph, meaning that the corresponding interdependencies can be neglected (Büyüközkan and Güleryüz, 2016). The relations corresponding to the ten unselected HFs are thus not represented for the sake of clarity, despite some relation of interdependence between them and the other factors may exists.

By analysing the six selected human factors, one can observe that human factor HF_5, by occupying the first position of the ranking, reveals the need for better defining reward systems related to the achievement of security objectives. This could be pursued by motivating workers in actively participating to the implementation of a system of security and safety management, as suggested also by human factor HF_1. Thus, this implementation may simultaneously enhance these two factors and can be addressed by starting from a clearer definition of procedures related to the planning and execution of preventive maintenance intervention for the bottling plant. Moreover, three of the selected factors (HF_{15}, HF_{16}, HF_{13}) belong to the area A_8 (see Table 6.6), that is, “Workload and working hours”. It means that, for example, interventions aimed at rearranging aspects related to the entity of workload and the amount of working hours per worker could help improve the entire process under the organisational point of view.

Lastly, among the six selected HFs, the value of the difference $(r_i - c_i)$ is positive just for HF_{13}, what makes this factor a possible cause of bad process organisation and its improvement will produce benefits. The other five factors have associated a negative value of difference $(r_i - c_i)$, so these factors must be interpreted as cause factors of perceived risk.

On the basis of the shown evaluation, the company should consider as primary action the definition of reward systems related to the achievement of security objectives. This may be undertaken by motivating workers in taking part in the implementation of a system of security and safety management starting from a clearer definition of procedures related to the planning and execution of preventive maintenance intervention for the bottling plant. In this way, HF_5 and HF_1 would be simultaneously taken into account.

This aspect will be further investigated, also in terms of management of the monitoring process, throughout the following chapter.

Chapter 7

Maintenance monitoring of complex systems

The present chapter deals with the organisation of maintenance interventions in terms of scheduling and policies. The main focus is on the process of maintenance monitoring. In particular, the innovative blockchain technology is suggested as a management tool to support the implementation of preventive maintenance for complex systems. Moreover, a multi-criteria decision support system (DSS) is proposed to first select a suitable set of maintenance KPIs leading the monitoring process to be then considered as the objective of a mathematical programming model. The chapter is based on a currently led research that treats the present topic (Carpitella *et al.*, 2018, under review).

7.1. Organisation of maintenance actions

Maintenance activities must be integrated and scheduled within the life cycle of the system under investigation, and take into consideration the reliability features required by the system itself. Panteleev *et al.* (2014) emphasize the role of the maintenance and repair organization (MRO) and highlight the importance of implementing and planning periodic maintenance actions on machines to positively impact on their life cycle. The importance of maintenance scheduling is also emphasized by Certa *et al.* (2013a), who deal with the problem of selecting the elements of a repairable and stochastically deteriorating multi-component system to be replaced during each scheduled and periodic system stop within a finite optimization cycle. The simultaneous minimization of both the total expected maintenance cost and system unavailability is ensured by means of a mathematical programming-based approach. In (Certa *et al.*, 2012), the authors formulate a constrained mathematical model to determine both the optimal number of periodic inspections within a finite time frame and the system elements to be replaced during each scheduled inspection. A further mathematical model is proposed by Taghipour *et al.* (2011) to identify the optimal periodicity of the inspection intervals for a repairable system subjected to hidden failures over finite and infinite optimization times. The objective function to be optimized is the total expected cost. Yang *et al.* (2016) underline the importance of optimizing the scheduling of inspection, repair, and replacement activities in preventive maintenance policies. The authors propose a replacement policy to maximize reliability in a mission-based system. On the basis of the same concepts, Poór *et al.* (2014)

consider the key role of machine and plant maintenance in terms of facility management (FM) for manufacturing companies.

As affirmed by Viswanath Dhanisetty *et al.* (2018), the existing literature shows a wide variety of examples making use of MCDM methods as supporting tools to manage maintenance decision making processes. Ruschel *et al.* (2017) lead a systematic review focused on the vast amount of decision-making models adopted in the literature and aimed at enhancing the following areas: maintenance management, maintenance planning, maintenance policy selection, maintenance efficiency analysis, equipment lifecycle management, process monitoring analysis, machine health prognosis, reliability analysis, system and component degradation, maintenance outsourcing, joint optimization, multi-level system integration, multi-state system optimization, risk and consequence analysis, maintenance cost estimation, and inspection and maintenance intervals. The authors also report the percentages of seven types of maintenance policies analysed in the literature, namely: condition-based (40%), preventive (23%), reliability-centred (14%), predictive (10%), others (7%), risk-based (4%), corrective (2%).

Maintenance actions refer to three main groups of maintenance policies, namely corrective (Wang *et al.*, 2014), preventive (Sidibe *et al.*, 2017) and opportunistic (Ba *et al.*, 2017). Their descriptions, strengths and weaknesses are presented in the table below (Carpitella *et al.*, 2017a).

Table 7.1. Main maintenance policies

Policy	Description	Strenghts	Weaknesses
Corrective maintenance	Interventions of corrective maintenance are carried out upon the occurrence of failures. An action of replacement or repair should be accomplished in a minimal time and at a minimal cost.	• Exploitation of the whole useful life of components;	• High risk of plant shutdown; • Negative impact on production, reliability and availability.
Preventive maintenance	Interventions of preventive maintenance are generally realized at constant intervals. Searching for optimal value of interval aims to continuously increase system condition.	• Positive impact on production, reliability and availability; • Control of costs.	• Execution of several interventions, even if not always necessary.

Opportunistic maintenance	This kind of policy considers the possibility of exploiting a period of plant shutdown to conduct maintenance interventions on components for which the planned time of intervention is close but not totally reached.	• Positive impact on production, reliability and availability; • Minimisation of the time to be dedicated to maintenance interventions.	• Execution of several interventions, even if not always necessary.

Within the mentioned policy categories, several maintenance interventions can be planned and implemented. In particular, according to the strategic choices undertaken by the organisation, interventions need to be scheduled, eventually evaluating possibilities of integration among them.

The preventive maintenance policy can be developed and improved by means of diagnostic tools to monitor the degrading state of components (Perelman *et al.*, 2016). Generally, the purpose consists in predicting in a reliable way the instant of time for the execution of maintenance interventions. This kind of action is called predictive maintenance (Jiang *et al.*, 2015) and represents a current challenge faced by organisations.

An exhaustive description of the concept of predictive maintenance is given by Forsthoffer (2017). This author defines this kind of maintenance policy as based on monitoring components to acquire data on important parameters (such as, for instance, temperature and vibration). The collection of this data is aimed at predicting the root cause of change in operational conditions before failures occur and so avoiding unnecessary interventions of preventive maintenance. Indeed, as asserted by the author, the difference between predictive and standard preventive maintenance is that the latter does not make use of sensors or instrumentation to monitor operating units, potentially leading to unnecessary interventions and significant loss of revenue.

Industrial equipment and plants are characterised by ever-increasing complexity, mainly due to the presence of various levels and combinations of interdependencies among elements (Nguyen *et al.*, 2015), and Van Horenbeek and Pintelon (2013) demonstrate the efficacy, also in terms of cost savings, of predictive maintenance in managing interdependencies and in reacting to changes in the deterioration state (especially for critical components). De Benedetti *et al.* (2018) underline the usefulness of techniques aimed at detecting anomalies in a timely manner and affirm that the availability of daily predictive maintenance alerts would perfectly meet the need for promptly reacting and planning maintenance interventions. The last aspect has been considered for formulating an indicator.

Indeed, Lindberg *et al.* (2015) consider the number of alarms over a period of time as a KPI for the maintenance area. The authors also enumerate a list of KPIs that could be used to monitor operational conditions of systems and predict when maintenance will be required. They particularly refer to parameters such as the heat transfer rate in heat exchangers, pump efficiency, equipment wear (based, for instance, on operating hours, speed, load, or start-ups), and vibration amplitude as measured for predicted performance.

7.2. Blockchain technology supporting preventive maintenance

The information society (Beniger, 1989) is mainly characterised by the transition from the ‘industrial economy’ to the emerging ‘network economy’. In other words, we are passing from a situation in which a single enterprise develops physical or intellectual products and, through their ownership, competes in a market of reference – to the creation of new digital frontiers and information infrastructures based on information sharing. This transition is clearly revolutionising the structure of business activities (Malone and Laubacher, 1998) and in this direction, distributed ledger technologies offer a plethora of benefits both to public and private sector organisations. As explained in a report published by the UK Government Office for Science (2015), these types of technologies are highly efficient since modifications implemented to the ledger by any participant with the necessary permission are immediately reported to all users (that is in all copies of the ledger). Moreover, it is extremely difficult to corrupt the ledger because any unauthorised change is rejected.

Blockchain distributed ledger technology is considered as one of the most promising technologies of the new economy (Savelyev, 2018). Swan (2018) defines blockchain as ‘a software protocol for the secure transfer of unique instances of value (*e.g.*, money, property, contracts, and identity credentials) via the internet without requiring a third-party intermediary such as a bank or government’. The major application of blockchain regards the electronic currency bitcoin, whose ecosystem is a network of users communicating with each other using a dedicated protocol via the internet (Vranken, 2017). Chen (2018) stresses the important support given by the blockchain technology to the world of entrepreneurship and innovation, since innovators can create digital tokens to represent and democratise a wide range of assets. With this perspective, the use of the blockchain technology is particularly appropriate for innovative industrial and service systems that must guarantee continuous operation.

This section aims to integrate the blockchain technology with industrial maintenance. In many cases (Madureira *et al.*, 2017; Carpitella *et al.*, 2016), and according to the organisational maintenance policies (Qiu *et al.*, 2017), the prioritisation of failures or anomalies, detected (for example) by using techniques of reliability analysis, is strategic for scheduling maintenance interventions and reducing risks. Bevilacqua and Ciarapica (2018) discuss the case of a refinery risk management system, stressing the fundamental role of human factor-based analyses. Furthermore, authors such as Gopalakrishnan *et al.* (2015) and Guo *et al.* (2013) affirm that prioritising maintenance work-orders is a challenging procedure and has a positive influence on production.

Good maintenance planning is essential to increase the level of competitiveness of enterprises and enhance the process of synchronisation between intervention and production operations (Colledani *et al.*, 2018). Typically, a strategic plan for maintenance interventions requires an effective combination between implementation actions and control of performed interventions. It is necessary to monitor the effects of maintenance actions during the implementation of maintenance policies, and the support given by KPIs is strategic.

Given the helpful contribution of blockchain technology in recording interactions, managing data, and information flow, the present section of the thesis proposes linking the implementation of preventive maintenance policies with prompt action by maintenance crews.

As stressed by Gubbi *et al.* (2013), digital control systems to automate process controls, operator tools, and service information systems to optimise plant safety and security are within the purview of the 'internet of things' (IoT) – which also extends to asset management via predictive maintenance, statistical evaluation, and measurements to maximise reliability. The IoT is the network of physical devices, vehicles, home appliances, and other items embedded with electronics, software, sensors, actuators, and connectivity that enables these objects to connect and exchange data. Khan and Salah (2018) provide a definition of IoT (Atzori *et al.*, 2010) as a network in which various devices provided with sensors are interconnected through a private or a public communication network, and discuss how blockchain can be a key enabler to solve many IoT security problems.

As recently stressed by Rodrigues *et al.* (2018), blockchain technology has evolved very recently beyond the financial markets, gaining more public attention, so that other promising blockchain applications areas are emerging. The authors offer an interesting survey on some of the major issues for IoT and, and they present a review of emerging topics related to IoT security and blockchain. Savelyev (2018) asserts that value-exchange transactions are sequentially grouped into blocks – with each block chained to the previous one and recorded in a peer-to-peer (P2P) network by means of cryptographic trust and assurance mechanisms. In this way, the author highlights that blockchain provides a new paradigm for data storage security without requiring the intermediate action of a central authority. He also stresses the main advantages of this technology: transparency (all recorded data is shared); redundancy (each user owns a copy of the data); immutability (formal consensus has to be formulated and shared by all parts to change records); and disintermediation (no costs associated with the presence of intermediaries need be shouldered).

Backman *et al.* (2017) report the following six-step description for the running process of a P2P network (for instance a Bitcoin network): '1) new transactions are broadcast to all nodes; 2) each node collects new transactions into a block; 3) each node works on finding a proof-of-work for its block; 4) when a node finds a proof-of-work, it broadcasts the block to all nodes; 5) nodes accept the block only if all transactions in it are valid and not already spent; 6) nodes express their acceptance of the block by working on creating the next block in the chain, using the hash of the accepted block as the previous hash'. Undoubtedly, using a reliable and programmable infrastructure for the trustworthy exchange of information, data, and transactions effectively supports interactions between any two parties. Indeed, P2P collaborative networks of companies and individuals support their mutual interests, and this represents the objective to be optimised in complex system management.

The devices in IoT can be remotely controlled to perform the desired functionality and the information shared among the devices then takes place through the network, which employs standard communication protocols. The smart connected devices or 'things' range from simple wearable accessories to large machines and each contains sensor chips.

The use of sensors refers to the predictive maintenance policy which, according to Raza and Ulansky (2017), represents the most promising strategy for technical systems and production lines and can be applied whenever a deteriorating physical parameter such as vibration, pressure, voltage, or current can be quantitatively measured. Civerchia *et al.* (2017) focus on the industrial internet of things (IIoT) and present an advanced monitoring solution based on sensors, designed to support advanced predictive maintenance applications. The results emphasise the potential of wireless sensor devices to monitor equipment status and highlight the usefulness of advanced and pervasive predictive maintenance to contain costs and avoid dangerous situations. Dong *et al.* (2017) develop a predictive maintenance plan for coal equipment based on IoT technology. The authors identify sensors and detection technology as the foundation of IoT, since they are indispensable for capturing thermal, mechanical, optical, electrical, acoustic and displacement signals – and providing processing, transmission, and analysis information, as well as feedback.

7.3. KPIs for maintenance monitoring

When accomplishing the process of maintenance monitoring, the advantages deriving from the use of KPIs are potentiated through the presence of a network of sensors to monitor the state of wear of critical components. Referring to large-scale industrial processes, Zhang *et al.* (2017b) propose a framework for KPI-based process monitoring and fault detection (PM-FD) considering that relationships between KPIs and processes are characterised by the presence of many control loops, sensors, and actuators.

In general, as underlined in (Carpitella *et al.*, 2018e), KPI-based assessment enables critical, synthetic, significant, and key information to be obtained by measuring the main results of the maintenance actions on the overall system. KPI-based assessment also plays a role in the strategic interface between the processes of scheduling and control (Bauer *et al.*, 2016). The related literature (Wireman, 1998; Weber and Thomas, 2006; SMRP press release, 2007) presents a wide number of indicators developed for the maintenance function and aimed at optimising system performance, especially for economic, technical, and organisational aspects. Given the wide variety of indicators, a structured methodology is suggested to select the most representative indicators in (Fangucci *et al.*, 2017). The phase of selection is certainly important to conduct reliability analyses during the process of maintenance management. Moreover, as reported by Stricker *et al.* (2017), relationships linking the considered KPIs should be identified and characterised to exhaustively estimate their effects on system health.

Generally, performance indicators are used in maintenance management to make decisions, plan activities and processes, and acquire a clearer vision of organisational phenomena. Moreover, making a comparison between two values assumed by the same indicators at two distinct time instants helps to discover the likely margins of optimisation. Indicator-based evaluations must also be set within a proper time horizon for producing historical series of values.

The used indicators must effectively respond to the existing needs by managing dynamics within companies, environment mutations, and the possible presence of criticalities. Measurements of indicators can be summarised into three main categories: cost, time, and quality measurements. The table below reports a list of representative maintenance KPIs, referring both to general aspects of maintenance monitoring (Gonzalez *et al.*, 2017) and to predictive maintenance (Lindberg *et al.*, 2015).

Table 7.2. List of maintenance KPIs

N.	ID	KPI	Description
1	**SC**	Schedule compliance (%)	Ratio between scheduled maintenance tasks completed in time and total number of tasks.
2	**CEF**	Component efficiency	Measured by efficiency of specific components (*i.e.* heat transfer rate of heat exhangers, and so on).
3	**NA**	Number of alarms	Number of predictive maintenance alerts in time period.
4	**ER**	Equipment reliability	Ability of equipment to perform given conditions for a given time interval.
5	**TD**	Total downtime	Time the system is down over total monitoring time.
6	**NI**	Number of interventions	Scheduled and unplanned interventions to lead in management strategies.
7	**SW**	System wear	System functioning conditions and mainly based on operating hours, speed, load, vibration amplitude of equipment and so on.
8	**AMC**	Total annual maintenance cost vs annual maintenance budget (%)	Used to assess if expenditure is as anticipated or higher on specific asset maintenance.

7.4. KPIs-based DSS to implement predictive maintenance interventions

On the basis of the reported indicators, a multi-criteria DSS involving experts and decision makers is herein proposed. The DSS integrates DEMATEL methodology (whose application has been previously exposed in section 6.3) and a mathematical programming model to check the efficacy of preventive maintenance policies by using data sensors integrated with blockchain technology.

The DEMATEL method permits to take into consideration interdependencies existing within a set of elements, so that it is used as decision-making support tool to select a representative set of indicators (from among all the KPIs of Table 7.2). These indicators represent drivers when making decisions on the maintenance actions using a mathematical model. Since the state of critical components can be monitored by means of a suitable network of sensors, blockchain technology may support the measurement of selected indicators. Indeed, organisations rely on internal/external maintenance teams to make repairs or substitutions in complex systems, and the incorporation of blockchain technology is helpful in managing the related information flow among maintenance stakeholders.

To demonstrate the usefulness of the approach, it can be applied to subsystems requiring preventive maintenance as analysed within the real-world case study of the cleaning service vehicle previously presented (chapter 5, section 5.5).

First step: DEMATEL application

With relation to the maintenance KPIs of Table 7.2, their ranking has been achieved using DEMATEL and a case study involving three experts ($H = 3$) has been developed. These experts are managers in the technical area and they were asked to fill in the three non-negative matrices (Tables 7.3 to 7.5). Such an approach is useful to translate technical skills acquired by the experts to maintenance management. The direct-relation matrix A aggregating expert judgments and total relation matrix T are respectively reported in Tables 7.6 and 7.7, whereas the final chart of interdependencies among the three selected KPIs (considering a threshold fair to 0.802) is shown in Figure 7.1. Relations with and among the other KPIs, despite existing, have been omitted for the sake of graphical clarity.

Table 7.3. Non-negative matrix filled in by expert $\boldsymbol{H_1}$

$\boldsymbol{H_1}$	SC	CEF	NA	ER	TD	NI	SW	AMC
SC	0	4	3	3	3	4	4	4
CEF	2	0	2	3	4	3	3	3
NA	4	3	0	4	4	4	4	4
ER	3	3	3	0	3	2	4	4
TD	4	4	3	3	0	4	4	4
NI	3	3	2	3	4	0	2	4
SW	1	3	3	3	3	2	0	3
AMC	4	3	3	3	3	2	2	0

Table 7.4. Non-negative matrix filled in by expert $\boldsymbol{H_2}$

$\boldsymbol{H_2}$	SC	CEF	NA	ER	TD	NI	SW	AMC
SC	0	3	3	3	3	3	3	3
CEF	3	0	3	2	4	4	4	3
NA	3	4	0	4	4	4	3	4
ER	2	3	2	0	3	3	3	3
TD	4	3	3	3	0	4	3	4
NI	3	4	3	2	4	0	2	4
SW	4	4	4	3	4	4	0	3
AMC	4	4	4	3	4	4	3	0

Table 7.5. Non-negative matrix filled in by expert $\boldsymbol{H_3}$

$\boldsymbol{H_3}$	SC	CEF	NA	ER	TD	NI	SW	AMC
SC	0	3	3	3	4	3	4	4
CEF	3	0	2	3	3	4	3	3
NA	4	4	0	4	4	4	3	4
ER	4	3	2	0	3	3	3	3
TD	4	3	4	3	0	4	4	4
NI	2	4	4	4	3	0	2	4
SW	3	3	3	4	4	3	0	3
AMC	4	3	3	4	4	3	2	0

Table 7.6. Direct-relation matrix $\boldsymbol{A}$

$\boldsymbol{A}$	SC	CEF	NA	ER	TD	NI	SW	AMC
SC	0.00	3.33	3.00	3.00	3.33	3.33	3.67	3.67
CEF	2.67	0.00	2.33	2.67	3.67	3.67	3.33	3.00
NA	3.67	3.67	0.00	4.00	4.00	4.00	3.33	4.00
ER	3.00	3.00	2.33	0.00	3.00	2.67	3.33	3.33
TD	4.00	3.33	3.33	3.00	0.00	4.00	3.67	4.00
NI	2.67	3.67	3.00	3.00	3.67	0.00	2.00	4.00
SW	2.67	3.33	3.33	3.33	3.67	3.00	0.00	3.00
AMC	4.00	3.33	3.33	3.33	3.67	3.00	2.33	0.00

Table 7.7. Total direct-relation matrix ***T*** and final ranking

T	**SC**	**CEF**	**NA**	**ER**	**TD**	**NI**	**SW**	**AMC**	$r_i + c_i$	$r_i - c_i$	**Ranking**
SC	0.70	0.84	0.75	0.79	0.88	0.84	0.79	0.89	12.809	0.142	TD
CEF	0.74	0.67	0.68	0.73	0.83	0.79	0.73	0.81	12.532	-0.588	AMC
NA	0.91	0.94	0.73	0.91	0.99	0.95	0.87	0.99	13.114	1.470	NA
ER	0.73	0.75	0.66	0.62	0.79	0.74	0.71	0.80	12.006	5.795	SC
TD	0.89	0.90	0.81	0.84	0.82	0.92	0.84	0.96	13.860	0.089	NI
NI	0.76	0.81	0.72	0.76	0.85	0.69	0.71	0.86	12.720	-0.409	CEF
SW	0.77	0.81	0.73	0.78	0.86	0.80	0.65	0.84	12.290	0.191	SW
AMC	0.83	0.83	0.75	0.79	0.88	0.82	0.75	0.76	13.308	-0.480	ER

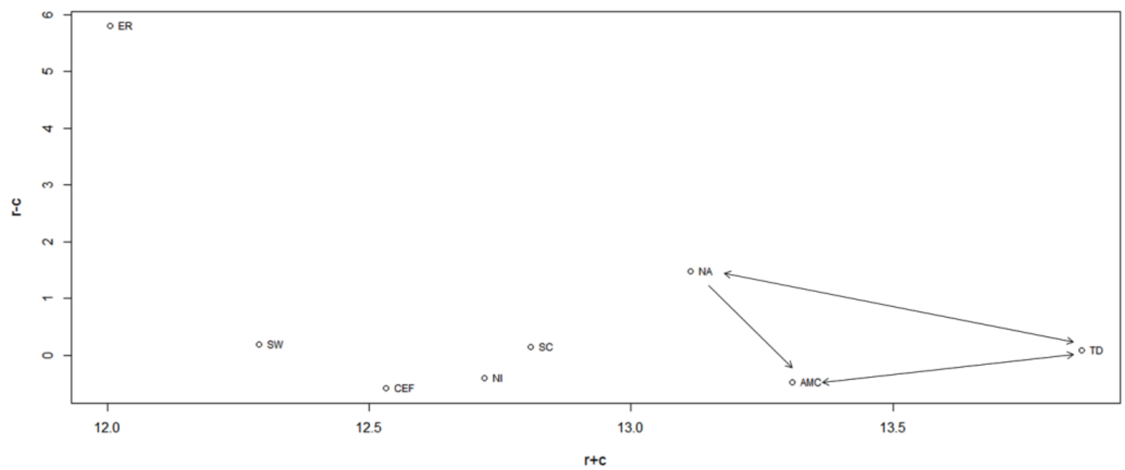


Figure 7.1. Impact-relations map

By observing the final ranking of KPIs, it is possible to note that the indicators TD, AMC, and NA have a higher value associated with relation to the sum $r_i + c_i$, and this means that their variations can correspond to variations of all the other aspects. Moreover, by considering the values of the subtraction $r_i - c_i$, the indicators TD and NA belong to the cause group, whereas the indicator AMC belongs to the effect group.

The following preventive maintenance mathematical model has been formulated on the basis of the selected KPIs (all referred to an annual basis) with the purpose of providing a support during the phases of scheduling and implementation of interventions.

Second step: predictive maintenance mathematical model

Let us consider a set of critical elements related to a generic complex system, identified using preliminary reliability analysis. Considering that these elements are monitored by sensors measuring parameters directly correlated to their wear state y, we indicate with m_k the value of this parameter acquired at time $t_k = k\cdot\Delta t$. By hypothesizing a linear bound between m and y (Curcurù *et al.*, 2010), we have that:

$$m_k = a + by_k + \delta_k; \tag{7.1}$$

in which a and b are the coefficients of linear transformation and δ_k represents the error due to imprecision of the sensor measuring the trend of the wear state y.

Let us fix a programmed and constant interval of time T_{pj} for executing preventive maintenance interventions on generic critical component belonging to the set $i = 1, \cdots, I$. Since the state of critical components is monitored by sensors, on the basis of the acquired information it is necessary, at each temporal instant t_k, to decide whether to execute maintenance intervention at $t_k + \Delta t_k$ or not. If the decision about executing the intervention is not taken at time t_k, the decision will be postponed to the next observation, without excluding the possibility of executing the intervention at the end of the programmed interval of time T_{pj}. In particular, a scheduled intervention of preventive maintenance (whose duration will be fair to T_{pS_i}) will be executed as programmed if, during the various observations, the number of alerts (S_i) given by sensors related to a specific component *i* is lower than a fixed threshold S_i^*. Instead, if S_i is higher or equal to S_i^*, then an intervention of predictive maintenance will be executed before the programmed time and the related duration will be fair to T_{pD_i}.

This condition is verified if the value of monitored parameter m_k given by (7.1) belongs to the range (m^*, m^{**}), where m^* is the value of parameter corresponding to an accepted value of failure probability $F(t_{k,i})$, called $Risk^*$, at the time k for the component *i*, whereas at m^{**} the component fails.

The objective function of the proposed model is expressed as a minimisation of the unavailability U of the analysed system. The formulation of the just cited objective function has been guided by the KPI occupying the first position in the ranking obtained by the DEMATEL method – that is the total downtime (TD):

$$U = TD = 1 - \left[\frac{\sum_{j=1}^{N}(T_{p_j} - \sum_{i\in I}(T_{pD_i}\cdot X_{pD_i}) + \sum_{i\in I}(T_{pS_i}\cdot X_{pS_i} + \sum_{i\in I}(T_{M_i}\cdot X_{S_i}))}{number\ of\ working\ hours\ per\ year}\right]; \tag{7.2}$$

where:

- $j = 1, \cdots, N$ is the index defining the interval in which the programmed intervention of preventive maintenance is executed;
- $i = 1, \cdots, I$ is the index representing the generic critical element belonging to the system to be monitored;
- X_{pD_i} is the Boolean variable assuming value fair to 1 if the intervention of predictive maintenance is executed in advance with respect to the programmed time, 0 otherwise;
- X_{pS_i} is the Boolean variable assuming value fair to 1 if the intervention of preventive maintenance is executed at the programmed time, 0 otherwise;
- X_{S_i} is the Boolean variable assuming value fair to 1 if the condition $\mathrm{S}_i \geq \mathrm{S}_i^*$ is verified within the reference interval, 0 otherwise.
- T_{M_i} is the time needed to organise the activities for monitoring and controlling the wear state of components belonging to the set I. These activities will be implemented only when the number of alerts is higher than or equal to the fixed threshold, and it implies that the time T_{M_i} will be $\neq 0$. This condition is assured by means of the constraint (7.4) and, in such a case, the cost C_{S_i} (considered in the calculation of the total cost CT in the formula (7.6)) will be computed.

The following relation exists among the various components of time considered within the model: $T_{M_i} > T_{pS_i} > T_{pD_i}$.

Moreover, the optimisation problem is subjected to the following constraints.

$$X_{pD_i} + X_{pS_i} = 1 \quad \forall i \in I; \tag{7.3}$$

The constraint (7.3) expresses that the intervention can be executed in advance with respect to the programmed time or at the programmed time.

$$\begin{cases} X_{S_i} \geq \frac{S_i - S_i^*}{S_i^*} \\ X_{S_i} < \frac{S_i}{S_i^*} \end{cases} \quad \forall i \in I; \tag{7.4}$$

The constraint (7.4) ensures that the Boolean variable X_{S_i} is equal to 1 if the number of alerts S_i given by sensors is higher than or equal to a fixed threshold S_i^*, 0 otherwise. Moreover, the number of S_i alerts over the programmed interval of time T_{pj}, corresponds to the KPI occupying the third position in the ranking obtained by the DEMATEL method.

$$\begin{cases} X_{pD_i} \geq F(t_{k,i}) - Risk^* \\ X_{pD_i} < \frac{F(t_{k,i})}{Risk^*} \end{cases} \quad \forall k, \forall i \in I; \tag{7.5}$$

The constraint (7.5) guarantees that the predictive maintenance intervention is executed if the accepted level of failure probability is overcome.

$$AMC = \frac{CT}{Annual\ budget} = \frac{\sum_{i \epsilon I}(C_{pD_i} + C_{pS_i} + C_{S_i})}{Annual\ budget} \leq 1. \quad (7.6)$$

The last constraint (7.6) ensures that the annual cost budget constraint, related to the execution of the maintenance actions, is respected. The AMC is the indicator in the second position of the DEMATEL ranking.

7.5. Description of the analysed complex service system

The present section proposes the application of the presented DSS to the real-world complex system previously analysed in chapter 5 (section 5.5), that is the innovative street cleaning vehicle endowed with a smart remote diagnosis (telediagnosis) system. The blockchain-supported preventive maintenance is implemented for the mentioned system. In detail, the cleaning service activities were grouped into three main phases: namely, vehicle handling, waste collection, and tank emptying.

The vehicle starts its service by moving from the starting point to the destination point at high speed, and then reduces speed to about 7 km/h during waste collection.

In chapter 5 (section 5.6), a combined multi-criteria decision-making approach was applied to rank failure modes resulting from the related FMECA. The obtained ranking of failure modes highlights the major criticalities.

Moreover, a sensitivity analysis was made (Carpitella *et al.*, 2018c) to test the influence of the three considered criteria, by changing the relative assigned priorities, on the ranking results. For each vector of weights, failure modes deemed to be the most critical ones are those characterized by a CC_i value smaller than or equal to 10% (Table 7.8).

Table 7.8. Sensitivity analysis results

w_{C1}	w_{C2}	w_{C3}	Ranking	ID - FM
0.2	0.4	0.4	1st 2nd 3rd 4th 5th 6th	2A 5.2.2D 5.2.2C 5.3.1B 1B 5.3.1A
0.4	0.4	0.2	1st 2nd 3rd 4th 5th 6th	2A 5.2.2D 1B 5.3.1A 5.2.2C 5.3.1B
0.6	0.1	0.3	1st 2nd 3rd 4th 5th 6th 7th 8th 9th 10th 11th 12th 13th 14th 15th 16th 17th 18th 19th 20th 21st 22nd	2A 5.2.2D 1B 5.3.1A 4.2.2A 4.2.3A 4.1.2A 5.2.2C 5.3.1B 3.3A 5.2.1D 5.2.1E 4.2.2C 4.2.3C 5.2.1B 1A 1C 3.1A 3.1B 3.2A 3.2B 4.1.2B
0.1	0.6	0.3	1st 2nd 3rd 4th 5th	2A 5.2.2D 5.2.2C 5.3.1B 1B

0.3 0.1 0.6	1^{st}	2A
	2^{nd}	5.2.2D
	3^{rd}	5.2.2C
	4^{th}	5.3.1B
	5^{th}	4.2.2A
	6^{th}	4.2.3A
	7^{th}	1B
	8^{th}	5.3.1A
	9^{th}	5.2.1E
	10^{th}	4.1.2A
	11^{th}	3.3A
	12^{th}	5.2.1D
	13^{th}	4.2.2C
	14^{th}	4.2.3C
	15^{th}	5.2.1B

By observing results of the sensitivity analysis, it is possible to note that the final rankings vary with varying criteria weights, but five failure modes appear in all the different scenarios considered by the sensitivity analysis as the most critical. These failure modes are summarised in the following Table 7.9 with their failure causes, effects and related involved component.

Table 7.9. List of most critical failure modes

ID	Failure Modes	Failure Causes	Failure Effects	Component
2A	Overheated oil	• Exchanger clogging • lack of water	• Hydraulic system blocked	Oil tank
5.2.2D	Broken chain	• Hydraulic system fault • impacts or wear	• Blocked motion • Loading not carried out	Lateral chains (elevator plant)
5.2.2C	Broken skid	• Detachment of one or more skids from the waste action support	• Difficulty in conveying waste • Clogging near the rear roller	Collecting skids (elevator plant)
5.3.1B	Slackened pivots or worn journal boxes	• Incorrect assembly / stress due to load	• Excessive vibration • Overturning defects	Tank structure (emptying system)
1B	Worn PTO bearings	• High usage time; lack of lubrication	• Compromised functionality of hydraulic circuits	Integral PTO

By analysing the components mainly involved in failure modes and related failure causes, it is clear that the functioning of the various sweeping elements directly depends on the state of the hydraulic system. This system is influenced by the state of the related hydraulic pumps, whose moving parts and supports are subjected to progressive wear causing increased vibrations.

The pump degradation state can then be correlated to a parameter associated with vibration and can be monitored by suitable and widely available vibration sensors.

With relation to the proposed DSS, the KPIs selected using the DEMATEL method (namely, total downtime (TD), number of alarms (NA) and total annual maintenance cost vs annual maintenance budget (AMC)) are used to monitor the efficiency of preventive maintenance on analysed components – that is, actions implemented by means of sensors installed on the three main hydraulic pumps. This means that the set I of elements to be monitored consists of pump I (component ID: 4.2.1.), pump II (component ID: 5.1.) and pump III (component ID: 5.2.2.1.). Acceleration is the parameter correlated to the wear state y of pumps to be measured by sensors.

The execution of maintenance activities related to pump faults are characterised by an operational time of between two and four hours. The modality of the maintenance action execution also implies a medium-complex level of difficulty for pumps since the related intervention needs a specialized maintenance team. Indeed, it is not possible to carry out the intervention in the same place where the failure occurs and this must take place in the repair shop (meaning that vehicle transport time must be taken into account).

The role of blockchain technology consists in collecting data from sensors installed on the mentioned components and transmitting them as fast as possible to the maintenance crew by assuring a secure information exchange.

CONCLUSIONS AND FUTURE DEVELOPMENTS

This doctoral thesis has been developed under an agreement of co-tutelle between two university institutions, namely the University of Palermo (Università degli Studi di Palermo, UNIPA) and the Polytechnic University of Valencia (Universitat Politècnica de València, UPV).

The main focus of the work regards the usage of multi-criteria decision-making (MCDM) methods, robustly supported by in-deep mathematical introspection, to optimise such a crucial field in industry as maintenance management of complex systems. In the course of the doctoral development, three international traineeships (respectively in Spain, UK and Germany) were accomplished to complement this thesis with topics related to mechanisms of judgments' consistency improvement, computer programming and website implementation proposing a MCDM method for worldwide companies use.

Conclusions

The main chapters of the thesis develop the following issues: decision-support models for complex system management, reliability analysis, and maintenance management of complex systems.

Regarding the first issue, various MCDM methods are applied to solve a number of real-world cases. First of all, the AHP (analytic hierarchy process) is approached from different mathematical perspectives. After briefly presenting the steps to apply this method, mainly focusing on maintenance management, the fundamental role of experts in this field is underlined. The availability of decision makers with a well-recognised experience in the treated field enables to collect reliable opinions supporting the solving process of decision-making problems. However, due to the natural limits of human thinking, an important point to be highlighted within AHP application regards the process of checking consistency of judgments. This aspect refers to the judgment attributed by experts as pairwise comparisons between pairs of elements.

To deal with the topic of consistency improvement, the linearization scheme is adopted. It is a rigorous framework providing a mechanism capable of manipulating inconsistent pairwise comparison matrices for achieving their closest consistent matrix. One cannot forget that the synthetic consistent results thus achieved may not reflect the effective opinions initially expressed by the decision makers. For this reason, to bridge the gap between theory and practice and adequately combine traditional theory-driven science objectivity and human behaviour subjectivity, the establishment of a feedback-based relationship with the experts is compulsory. It is fundamental that the experts eventually agree with the final consistent results. If this trade-off is not reached, calculations could lead to wrong decisions with their associated wastes. In this case, consequently, judgments should be elicited again.

The most common difficulty is represented by a condition of uncertainty, in which decision-makers may be immersed in the tasks of attributing their evaluations and of making suitable selections when facing various factors or criteria. This situation has been approached through various mathematical theoretical lines.

The fuzzy extension of the AHP (FAHP) is applied to manage vagueness of human judgments in determining the final vector of crisp weights with relation to a given set of elements. In particular, experts are asked to attribute judgments about the importance between pairs of elements in terms of linguistic evaluations associated to fuzzy numbers.

Moreover, the graph theory is proposed to treat incomplete comparison matrices of pairwise comparison judgments, representing situations in which experts are not fully sure about one or more factors and may prefer not to express any preference.

A probabilistic approach has also been considered as a good support when an expert or a group of decision-makers have doubts in assigning crisp values to their judgments. They could, instead, provide probabilistic values and, in this case, pairwise comparison matrices of AHP are treated as random reciprocal matrices with one or more random entries, which are random positive variables capturing expert uncertainty. The necessary theory to handle AHP-based decisions under the umbrella of the probability theory is developed, and lower bounds of probability in terms of confidence intervals for the various variables involved are estimated.

Lastly, still within the AHP, it is discussed the situation in which the number of elements to be considered in a decision-making process is huge. A consistent clustering of the entries of an AHP comparison matrix is addressed so that, if approved by the experts, *a posteriori* gathering of various elements (criteria or alternatives) becomes possible. In particular, a mechanism for reducing the size of a large comparison matrix is designed to consistently compress it, and eventually group some of the original elements into clusters. Naturally, such a reduction must be performed so that consistency is preserved and, in any case, the experts must eventually validate any manipulation of their original comparisons.

Beyond the AHP, other MCDM methods are applied, sometimes in a combined strategy, and compared to optimise management maintenance problems. Among them, methods belonging to the ELECTRE family, particularly the ELECTRE I and III, are used as support tools to strategically organise suitable maintenance interventions. The TOPSIS method and its fuzzy version (FTOPSIS) are proposed to manage situations in which ranking many decision alternatives is useful. Also, the TOPSIS has been integrated with a multi-objective optimisation perspective, to select the solution representing the best trade-off (among the non-dominated solutions belonging to a Pareto front) under the evaluation of various criteria.

The following chapter of the thesis was developed during two international traineeships and regards the creation of a new website aimed at proposing the use of the AHP method to worldwide companies and professionals. The objective consists in providing a support to deal with various kinds of decision-making problems in an interactive way. Fundamental importance is given to the process of feedbacks exchange with the decision makers involved, with the purpose to achieve a final solution representing a good trade-off between experts' opinions and consistency maintenance.

After having demonstrated the efficacy of MCDM methods as support tools in the maintenance field, another chapter of the thesis is devoted to the implementation of reliability analyses on complex systems (also in terms of human reliability analysis), on the basis of which any maintenance activity should be implemented. In particular, the most important functions involved in such kinds of analyses are explained and a new formula for calculating the stationary availability of systems characterised by a k-out-of-n reliability configuration has been proposed and validated. Indeed, the stationary availability is one of the most relevant parameters which the management and planning of maintenance activities is based on. The proposed novel formula represents an effective alternative to the classical method based on Markov chains, which requires greater computational effort.

Advanced techniques for reliability analyses such as FMEA and FMECA are illustrated and applied to real complex systems, supported again by a MCDM perspective. Specifically, an alternative approach to the classical RPN (risk priority number) for assessing system failure modes is proposed by means of the application of the fuzzy TOPSIS (FTOPSIS) method and by considering the relative importance of the involved risk parameters. The proposed approach takes into account data uncertainty and consists in ranking all the possible failure modes resulting from a FMECA application. Results are derived through the support of a maintenance team of experts and may be used as a driver during the planning of maintenance activities in terms of priorities. Indeed, the main critical failure modes related to the system under consideration are highlighted.

Moreover, the issue of human reliability analysis has been developed with relation to the role of human factors in practical operational contexts. In this regard, the THERP (technique for human error rate prediction) technique has been proposed to compute the success probability of projects highly depending on human factors. Lastly, the DEMATEL methodology is applied to evaluate the interdependencies existing among criteria considered important within processes for which the role of maintenance is crucial.

The last chapter of the thesis deals with maintenance management, above all in terms of monitoring processes and technological innovation pursued through effective scheduling of maintenance interventions. These aspects have been developed by supporting decision-making processes about the implementation of suitable maintenance policies and by seeking to optimise costs and production. Various types of maintenance policies are analysed and a comparison among them is made. Special attention is given to predictive maintenance policies, implemented by means of surveillance systems (typically composed of sensors) to monitor wear on critical components.

Moreover, the chapter emphasises the need to monitor the performance of maintenance activities and evaluate their effectiveness through suitable KPIs (key performance indicators), mainly relevant to cost, time, and quality. The thesis proposes integrating maintenance management with the innovative technology blockchain to optimise the process of control of system states, and useful KPIs in the maintenance field are analysed. In this regard, among the plethora of indicators existing in the literature, a MCDM approach is proposed to carry out a suitable selection.

Throughout the doctoral thesis, several maintenance applications are proposed to show the practical usefulness of the accomplished research. In particular, a wide range of complex systems has been object of analysis, and the following practical problems have been, in order, sorted out: deciding a leakage control policy in water supply; location of shelves for materials handling; selecting the best data processing technique for water networks; scheduling maintenance activities for industrial water distribution systems; optimal pump scheduling in water distribution networks; prioritizing risk in manufacturing processes; evaluating interdependencies among human factors involved in manufacturing processes; ranking of criticalities for a complex service system; selection of a suitable set of maintenance KPIs; and elaborating a decision support system supporting predictive maintenance of a service system.

Future developments

With relation to the themes analysed in this thesis, the undertaken research may be expanded through various lines in terms of future developments.

- From a methodological point of view, several improvements of various MCDM methods should be devised. Just to provide an example, still within the AHP, consistency improvement using the linearization scheme should be further extended to the case in which the expert does not want to see some of his/her judgments changed. This idea opens a new window to consistency improvement since other types of constraints (not only to have a fixed pairwise comparison) may be easily implemented by using the classical Riesz representation Theorem (Riesz, 1909; Rudin, 1986).
- The topic of human reliability analysis could be further investigated in the treated field, by applying also techniques belonging to the categories of second and third generation, and expert-judgment based methods integrated with multi-criteria decision-making methods. In particular, this theme could be explored with the aim to give value to human resources involved in carrying out maintenance actions for complex systems. Also, with relation to their specific skills and by means of data acquisition and elaboration, the development of a platform aimed at assigning, qualifying and promoting maintenance workers may be undertaken.
- The topic of blockchain technology, proposed as management tool to support preventive maintenance of complex systems, may be further expanded in terms of practical implementation for other production/service systems, different from the particular case herein analysed.
- The website developed in collaboration with the German company IngeniousWare GmbH, proposing a MCDM method to worldwide enterprises, could be further expanded. The main idea would follow the line of programming other MCDM methods bundled within a recommender expert system, so that the main decision-maker involved in a decision problem would be advised with the most appropriate support for his/her problem of interest. This selection would be based on the objective of the problem, *i.e.* ranking various alternatives (even if the set is huge), selecting the best option among different possibilities, or clustering solutions by grouping them on the basis of their common characteristics. To such an aim, the website could contemplate a starting section in which the chief of the project would be prompted to provide specific key information about the problem. This information would then be handled by a suitable expert system that would recommend the method better fitting his/her needs. In this way, subjects located in various geographic areas could easily keep in

touch to carry out group-decision-making processes. The related application would follow the procedure contemplated by the method to be used, which would, obviously, need specific software development.

APPENDIXES

APPENDIX A

1. Problem setting

In these appendices, the symbols $\mathscr{M}_{n,m}$, $\mathscr{M}_{n,m}^+$, $\mathscr{M}_n$, $\mathscr{M}_n^+$ will denote the sets of $n\times m$ real matrices, $n\times m$ positive matrices, $n\times n$ real matrices, and $n\times n$ positive matrices, respectively.

The following problem was solved in (Benítez *et al.*, 2015): given an incomplete reciprocal matrix $A\in\mathscr{M}_n^+$, find a reciprocal completion of A, say X, such that

$$\mathrm{d}(X,\mathscr{C}_n)\le \mathrm{d}(X',\mathscr{C}_n)$$

for any $X'\in\mathscr{M}_n^+$ reciprocal completion of A, where $\mathscr{C}_n$ denotes the subset of $\mathscr{M}_n$ composed of consistent matrices. Here $\mathrm{d}(\cdot,\cdot)$ is the following distance defined in $\mathscr{M}_n^+$:

$$\mathrm{d}(X,Y)=\|\mathrm{LOG}(X)-\mathrm{LOG}(Y)\|_F,$$

where $\mathrm{LOG}:\mathscr{M}_n^+\to\mathscr{M}_n$ is such that if a_{ij} is the (i,j)-entry of A, then the (i,j)-entry of LOG(A) is $\log(a_{ij})$. Furthermore, $\|\cdot\|_F$ is the Frobenius norm (i.e., $\|A\|_F^2=\mathrm{tr}(A^TA)$). Let us observe that A is reciprocal if and only if LOG(A) is skew-symmetric. Observe that the rule $\langle A,B\rangle=\mathrm{tr}(A^TB)$ defines an inner product in $\mathscr{M}_{n,n}$ and that the aforementioned Frobenius norm is induced by this inner product.

To be more precise, the stated problem can be formulated as follows.

Problem 1. Let $A\in\mathscr{M}_n$ be an incomplete reciprocal matrix. Let $(i_1,j_1),\dots,(i_k,j_k)$ the unknown entries of A above the main diagonal of A. Let $X(\lambda_1,\dots,\lambda_k)\in\mathscr{M}_n$ be a completion of A and such that $X_{i_r,j_r}=\exp(\lambda_r)$, $X_{j_r,i_r}=\exp(-\lambda_r)$ for $r=1,\dots,k$. Find $\lambda_1,\cdots,\lambda_k$ such that

$$\mathrm{d}(X(\lambda_1,\dots,\lambda_k),\mathscr{C}_n)\le \mathrm{d}(X(\lambda_1',\dots,\lambda_k'),\mathscr{C}_n)$$

for any $\lambda_1',\dots,\lambda_k'\in\mathbb{R}$.

The solution of Problem 1 was given in the next result, see Theorem 4 in (Benítez *et al.*, 2015). From now on, any vector of $\mathbb{R}^n$ will be considered as a column and denoted as $\mathbf{1}_n=[1\ \cdots\ 1]^T\in\mathscr{M}_{n,1}$. The standard basis of $\mathbb{R}^n$ will be denoted by $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$.

Theorem 1. *Let $A\in\mathscr{M}_n^+$ be an incomplete reciprocal matrix and $(i_1,j_1),\dots,(i_k,j_k)$ its unknown entries above its main diagonal. Any solution of Problem 1 satisfies*

$$\boldsymbol{\lambda}=S\mathbf{m},\qquad \left(D-\frac{1}{n}S^TS\right)\mathbf{m}=\mathbf{b}, \tag{A.1}$$

where $\boldsymbol{\lambda}=[\lambda_1\ \cdots\ \lambda_k]^T$, $\mathbf{m}=[\mu_1\ \cdots\ \mu_{n-1}]^T$, S is the $k\times(n-1)$ matrix whose (r,s)-entry is $\mathbf{d}_{i_rj_r}^T\mathbf{y}_s$, D is the diagonal $(n-1)\times(n-1)$ matrix whose (s,s)-entry is $\|\mathbf{y}_s\|^2$, and

$$\mathbf{b}=[\mathbf{w}^T\mathbf{y}_1\ \cdots\ \mathbf{w}^T\mathbf{y}_{n-1}]^T,$$

being $\mathbf{w}=\frac{1}{n}\sum_{i<j}c_{ij}\mathbf{d}_{ij}$. Here

$$c_{ij}=\begin{cases}\log a_{ij} & \text{if we know the } (i,j)\text{-entry of } A,\\ 0 & \text{if we do not know the } (i,j)\text{-entry of } A,\end{cases} \tag{A.2}$$

$\{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}\}$ is an orthogonal basis of $(\mathrm{span}\{\mathbf{1}_n\})^\perp$ and $\mathbf{d}_{ij}=\mathbf{e}_i-\mathbf{e}_j$.

The purpose is to study system (A.1) in terms of certain graph related to the incomplete matrix A. In particular, the solution of Problem 1 is proved to be unique if, and only if, this graph is connected.

The meaning of the values $\lambda_1,\dots,\lambda_k$ in the above Theorem 1 is clear: the missing entry (i_r, j_r) of A must be filled with $\exp(\lambda_r)$. One can see $\mu_1,\dots,\mu_{n-1}$ as auxiliary values useful to find $\boldsymbol{\lambda}$. The meaning of $\boldsymbol{\mu}$ is herein given.

If A is an incomplete reciprocal matrix, then

$$\mathscr{A} = \{\mathrm{LOG}(X) : X \text{ is a reciprocal completion of } A\}$$

is a linear manifold because if X is any reciprocal completion of A, then

$$\mathrm{LOG}(X) = \mathrm{LOG}(X_0) + \sum_{r=1}^{k} \lambda_r (\mathbf{e}_{i_r}\mathbf{e}_{j_r}^T - \mathbf{e}_{j_r}\mathbf{e}_{i_r}^T), \tag{A.3}$$

where in this last equality, X_0 is the reciprocal completion of A with 1s on its missing entries.

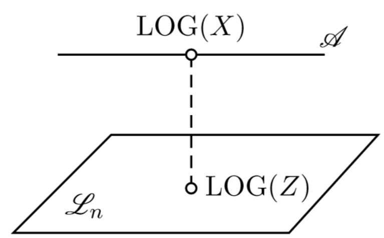


Figure A.1: The matrices LOG(X) and LOG(Z) minimize the distance between $\mathscr{A}$ and $\mathscr{L}_n$

Also,

$$\mathscr{L}_n = \{\mathrm{LOG}(Z) : Z \in \mathscr{M}_n, Z \text{ is consistent}\}$$

is a linear subspace of $\mathscr{M}_n$. In fact, it can be proved (see Theorems 2.2 and 2.4 (Benítez *et al.*, 2011a) that if we define the linear mapping $\phi_n : \mathbb{R}^n \to \mathscr{M}_n$ by $\phi_n(\mathbf{v}) = \mathbf{v}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{v}^T$, then $\mathrm{im}\,\phi_n = \mathscr{L}_n$, $\ker\phi_n = \mathrm{span}\{\mathbf{1}_n\}$, and a basis of $\mathscr{L}_n$ is $\{\phi_n(\mathbf{y}_1),\dots,\phi_n(\mathbf{y}_{n-1})\}$. Here, as in Theorem 1, $\{\mathbf{y}_s\}_{s=1}^{n-1}$ is an orthogonal basis of $(\mathrm{span}\{\mathbf{1}_n\})^{\perp}$.

With these preparatives, if $\mathrm{LOG}(X) \in \mathscr{A}$ and $\mathrm{LOG}(Z) \in \mathscr{L}_n$ are the matrices such that minimize $\mathrm{d}(X', Z') = \|\mathrm{LOG}(X') - \mathrm{LOG}(Z')\|_F$ for $\mathrm{LOG}(X') \in \mathscr{A}$ and $\mathrm{LOG}(Z') \in \mathscr{L}_n$, then

$$\mathrm{LOG}(Z) = \mu_1\phi_n(\mathbf{y}_1) + \cdots + \mu_{n-1}\phi_n(\mathbf{y}_{n-1}) = \phi_n(\mu_1\mathbf{y}_1 + \cdots + \mu_{n-1}\mathbf{y}_{n-1}).$$

See Theorem 4 in (Benítez *et al.*, 2015) for a deeper explanation. Therefore, Theorem 1 also gives the consistent matrix closest to the best completion of A. Furthermore, by defining $Y = [\mathbf{y}_1 \cdots \mathbf{y}_{n-1}] \in \mathscr{M}_{n,n-1}$ and $\boldsymbol{\theta} = Y\mathbf{m}$, then $\mathrm{LOG}(Z) = \phi_n(\boldsymbol{\theta})$. In other words, vector $\boldsymbol{\theta}$ gives the consistent matrix closest to the best completion of A.

The following theorem is important to fill matrix A, because we can forget the scalars $\lambda_1,\dots,\lambda_r$ and fix our attention to $\boldsymbol{\theta}$.

Theorem 2. *Let $A \in \mathscr{M}_n^+$ be an incomplete reciprocal matrix and $(i_1, j_1),\dots,(i_k, j_k)$ its unknown entries above its main diagonal. Let X be a reciprocal completion of A and Y be a consistent matrix of order n such that $\mathrm{d}(X,Z) \leq \mathrm{d}(X', Z')$ for all X' reciprocal consistent completion of A and Z' a consistent matrix. Then for $r = 1,\dots,k$, the entry (i_r, j_r) of X equals to the entry (i_r, j_r) of Z.*

Proof. Let us denote $B_{i,j}=\mathbf{e}_i\mathbf{e}_j^T-\mathbf{e}_j\mathbf{e}_i^T$. If $M=(m_{ij})\in\mathscr{M}_n$, then by using that $\mathrm{tr}(PQ)=\mathrm{tr}(QP)$ holds for any pair of matrices P,Q such that PQ and QP are meaningful,

$$\langle B_{i,j},M\rangle=\mathrm{tr}(B_{i,j}^T M)=\mathrm{tr}(\mathbf{e}_j\mathbf{e}_i^T M)-\mathrm{tr}(\mathbf{e}_i\mathbf{e}_j^T M)=\mathrm{tr}(\mathbf{e}_i^T M\mathbf{e}_j)-\mathrm{tr}(\mathbf{e}_j^T M\mathbf{e}_i)=m_{ij}-m_{ji}. \quad (A.4)$$

By (A.3), the support subspace of $\mathscr{A}$ is the subspace spanned by $B_{i_1,j_1},\ldots,B_{i_r,j_r}$ Since $\mathrm{LOG}(X)-\mathrm{LOG}(Z)$ is orthogonal to the support subspace of $\mathscr{A}$, by using (A.4) for $M=\mathrm{LOG}(X)-\mathrm{LOG}(Z)$, one has that the (i_r,j_r) entry of $L(X)$ equals to the (i_r,j_r) entry of $L(Z)$ for $r=1,\ldots,k$. □

2. Some review of graph theory

Some basic facts of graph theory are going to be reviewed in the following. The reader is encouraged to consult (Bapat, 2011) for a further insight. In the forthcoming it will be assumed that any graph has no loops.

The concepts of the Laplacian matrix and the incidence matrix of a graph G with vertices $\{1,2,\ldots,n\}$, edges $\{e_1,e_2,\ldots,e_m\}$ and no loops are presented. The *Laplacian matrix* of G is the $n\times n$ matrix, denoted by $L(G)$, defined as follows: if $i\neq j$, then the (i,j)-entry of $L(G)$ is 0 if vertices i and j are not adjacent, and it is -1 if i and j are adjacent. The (i,i)-entry of $L(G)$ is the degree of vertex i (i.e., the number of edges incident to vertex i).

Suppose that each edge of G has assigned an orientation, which is arbitrary but fixed. The *incidence matrix* of G, denoted by $Q(G)$, is the $n\times m$ matrix defined as follows: the rows and the columns of $Q(G)$ are indexed by vertices and edges, respectively. The (i,j)-entry of $Q(G)$ is 0 if vertex i and edge e_j are not incident, and otherwise it is 1 or -1 depending if e_j begins or finishes at i, respectively. For a graph G one has the following equalities:

$$L(G)=Q(G)Q(G)^T, \qquad \mathbf{1}_n^T Q(G)=\mathbf{0}. \quad (A.5)$$

A basic property of the Laplacian and incidence matrices is that

$$\mathrm{rk}(L(G))=\mathrm{rk}(Q(G))=n-p,$$

where p is the number of connected components of G and n is the number of vertices of G.

If G is a graph with vertices $\{1,\ldots,n\}$, then the complement of G, denoted by $\overline{G}$, is the graph with the same vertices and the edges are defined by the following rule: i and j are adjacent in $\overline{G}$ if and only if i and j are not adjacent in G. It is easy to see that

$$L(G)+L(\overline{G})=nI_n-\mathbf{1}_n\mathbf{1}_n^T. \quad (A.6)$$

The proof is simple: if $i\neq j$, then only one of the two following possibilities can occur: "i and j are adjacent" or "i and j are not adjacent", hence $L(G)_{ij}+L(\overline{G})_{ij}=-1$, which equals the (i,j)-entry of $nI_n-\mathbf{1}_n\mathbf{1}_n^T$. Since vertex i can be adjacent to the $n-1$ remaining vertices, then $L(G)_{ii}+L(\overline{G})_{ii}=n-1$, which again equals the (i,i)-entry of $nI_n-\mathbf{1}_n\mathbf{1}_n^T$.

3. Main results

Next, the system (A.1) appearing in Theorem 1 will be studied. To this end, an incomplete reciprocal matrix $A=(a_{ij})\in\mathscr{M}_n^+$ is associated to a directed graph in the following way. We have $i\to j$ when $i<j$ and the entries a_{ij} and a_{ji} are known. This graph will be denoted G_A. Recall that the Laplacians of G_A and $\overline{G_A}$ are independent on the orientation of the edges. However, the incidence matrices of G_A and $\overline{G_A}$ depend on the chosen orientation and thus, the edges need to be ordered. To such an aim, the lexicographical order is used, $(i_1\to j_1)\prec(i_2\to j_2)$ when $i_1<i_2$ or $(i_1=j_1)\&(j_1<j_2)$. An example is presented in Figure A.2.

To understand the third item of the next theorem, let us observe that by (A.3) and Theorem 1, the values $\lambda_1,\ldots,\lambda_k$ provide the set of solutions of Problem 1.

$$A = \begin{bmatrix} 1 & a & * \\ a^{-1} & 1 & b \\ * & b^{-1} & 1 \end{bmatrix}$$

$$Q(G_A) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \quad L(G_A) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Figure A.2: Example of an incomplete reciprocal matrix, its associated directed graph, the incidence matrix, and the Laplacian

Theorem 3. *Let $A \in \mathcal{M}_n^+$ be an incomplete reciprocal matrix and G_A its associate graph. Let p be the number of connected components of G_A. Under the notation of Theorem 1, one has*

(i) *The rank of $nD - S^T S$ is $n - p$.*

(ii) *The solutions $[\boldsymbol{\lambda}^T \ \mathbf{m}^T]^T$ of system (A.1) is a linear manifold whose dimension is $p-1$.*

(iii) *The set*

$$\mathscr{S} = \left\{ S\mathbf{m} : \left(D - \frac{1}{n} S^T S \right) \mathbf{m} = \mathbf{b} \right\}$$

is a linear manifold whose dimension is $p-1$.

Proof. We express matrices D and S in another way. Define $Y = [\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}] \in \mathcal{M}_{n,n-1}$, where the meaning of the vectors $\mathbf{y}_i$ is written in Theorem 1: they form an orthogonal basis of $(\operatorname{span}\{\mathbf{1}_n\})^\perp$. Since $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal system, we have

$$D = \begin{bmatrix} \|\mathbf{y}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\mathbf{y}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\mathbf{y}_{n-1}\|^2 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1^T \\ \mathbf{y}_2^T \\ \vdots \\ \mathbf{y}_{n-1}^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \cdots & \mathbf{y}_{n-1} \end{bmatrix} = Y^T Y. \tag{A.7}$$

Observe that the matrix $[\mathbf{d}_{i_1 j_1} \ \cdots \ \mathbf{d}_{i_k j_k}] \in \mathcal{M}_{n,k}$ is the incidence matrix of the graph $\overline{G_A}$. Therefore,

$$S = \begin{bmatrix} \mathbf{d}_{i_i j_1}^T \mathbf{y}_1 & \cdots & \mathbf{d}_{i_1 j_1}^T \mathbf{y}_{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{d}_{i_k j_k}^T \mathbf{y}_1 & \cdots & \mathbf{d}_{i_k j_k}^T \mathbf{y}_{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{i_i j_1}^T \\ \vdots \\ \mathbf{d}_{i_k j_k}^T \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_{n-1} \end{bmatrix} = Q(\overline{G_A})^T Y. \tag{A.8}$$

Hence,

$$D - \frac{1}{n} S^T S = \frac{1}{n} \left[n Y^T Y - Y^T Q(\overline{G_A}) Q(\overline{G_A})^T Y \right] = \frac{1}{n} Y^T \left[n I_n - L(\overline{G_A}) \right] Y. \tag{A.9}$$

Another useful equality is

$$\mathbf{1}_n^T Y = 0, \tag{A.10}$$

because the columns of Y are orthogonal to $\mathbf{1}_n$. Therefore, we obtain by (A.6), (A.9), and (A.10)

$$D - \frac{1}{n} S^T S = \frac{1}{n} Y^T \left(L(G_A) + \mathbf{1}_n \mathbf{1}_n^T \right) Y = \frac{1}{n} Y^T L(G_A) Y. \tag{A.11}$$

Let us define $Z=[Y\ \mathbf{1}_n]\in\mathscr{M}_n$. Obviously, Z is a nonsingular matrix because the $n-1$ first columns of Z form an orthogonal basis of $(\text{span}\{\mathbf{1}_n\})^{\perp}$. Observe that from (A.5) we obtain

$$\begin{aligned} Z^T L(G_A) Z &= \begin{bmatrix} Y^T \\ \mathbf{1}_n^T \end{bmatrix} L(G_A) \begin{bmatrix} Y & \mathbf{1}_n \end{bmatrix} \\ &= \begin{bmatrix} Y^T L(G_A) Y & Y^T L(G_A)\mathbf{1}_n \\ \mathbf{1}_n^T L(G_A) Y & \mathbf{1}_n^T L(G_A)\mathbf{1}_n \end{bmatrix} = \begin{bmatrix} Y^T L(G_A) Y & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}. \end{aligned}$$

Since Z is nonsingular, by (A.11) and the previous computation,

$$\text{rk}(nD-S^TS)=\text{rk}(Y^TL(G_A)Y)=\text{rk}(Z^TL(G_A)Z)=\text{rk}(L(G_A))=n-p, \tag{A.12}$$

where p is the number of connected components of G_A. This proves (i).

If d is the dimension of the manifold $\{[\boldsymbol{\lambda}^T\mathbf{m}^T]^T : \boldsymbol{\lambda},\mathbf{m} \text{ satisfy } (A.1)\}$, then d is the dimension of the null space of the matrix

$$\begin{bmatrix} I_k & -S \\ 0 & D-\frac{1}{n}S^TS \end{bmatrix}\in\mathscr{M}_{k+n-1}.$$

Thus, by the previous item (i)

$$d=k+n-1-\text{rk}\begin{bmatrix} I_k & -S \\ 0 & D-\frac{1}{n}S^TS \end{bmatrix}=k+n-1-(k+\text{rk}(nD-S^TS))=p-1.$$

This proves (ii).

Let us prove (iii). The dimension of $\mathscr{S}$ equals $\dim\mathscr{S}_1$, where $\mathscr{S}_1=\{S\mathbf{m}:(nD-S^TS)\mathbf{m}=\mathbf{0}\}$. But $\mathscr{S}_1$ is the image of the linear mapping $\Phi:\mathscr{N}\to\mathbb{R}^k$, where $\mathscr{N}$ is the null space of $nD-S^TS$ and $\Phi(\mathbf{v})=S\mathbf{v}$. Thus,

$$\dim\mathscr{S}_1=\dim\text{im}\,\Phi=\dim\mathscr{N}-\dim\ker\Phi.$$

Since $nD-S^TS$ is a square $(n-1)\times(n-1)$ matrix, by using item (i), one obtains

$$\dim\mathscr{N}=n-1-\text{rk}(nD-S^TS)=n-1-(n-p)=p-1.$$

Thus, to finish the proof, we must prove $\ker\Phi=\{\mathbf{0}\}$. Let $\mathbf{x}\in\mathbb{R}^{n-1}$ such that $\Phi(\mathbf{x})=\mathbf{0}$, *i.e.*, $S\mathbf{x}=\mathbf{0}$ and $(nD-S^TS)\mathbf{x}=\mathbf{0}$. Hence $D\mathbf{x}=\mathbf{0}$. The nonsingularity of D (as one can easily see from (A.7)), leads to $\mathbf{x}=\mathbf{0}$. □

We get the following two corollaries:

Corollary 1. *There exists at least one solution to Problem 1.*

Corollary 2. *Under the notation of Theorem 3, the following three conditions are equivalent:*

(i) *G_A is connected.*

(ii) *The matrix $nD-S^TS$ is nonsingular.*

(iii) *The solution of Problem 1 is unique.*

The equivalence of statements (i) and (iii) of Corollary 2 was proven in (Bozóki *et al.*, 2010). Observe that Theorem 3 also characterizes the degree of freedom of the set of solutions.

Next, the system (A.1) is going to be expressed in a simpler way, making more explicit the role of the graph G_A. The following lemma is used.

Lemma 1. *Let G be a graph with n vertices and m edges. Let* $\{\mathbf{y}_1,\dots,\mathbf{y}_{n-1}\}$ *be any basis of* $(\mathrm{span}\{\mathbf{1}_n\})^{\perp}$ *and* $Y=[\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}]$. *If* $\mathbf{v}\in\mathbb{R}^m$, *then* $Y^T Q(G)\mathbf{v}=\mathbf{0}\Leftrightarrow Q(G)\mathbf{v}=\mathbf{0}$.

Proof. The '⇐' part is trivial. It will be proved the '⇒' part: the vector $Q(G)\mathbf{v}$ is orthogonal to $\mathbf{y}_1,\dots,\mathbf{y}_{n-1}$. By the second equality of (A.5), also $Q(G)\mathbf{v}$ is orthogonal to $\mathbf{1}_n$. Hence $Q(G)\mathbf{v}\in\mathbb{R}^n$ is orthogonal to a basis of $\mathbb{R}^n$, and thus, $Q(G)\mathbf{v}=\mathbf{0}$. □

From now on, m will denote the number of edges of the graph G_A. Therefore, the incidence matrix of the graph G_A, namely $Q(G_A)$, is an $n\times m$ matrix.

Theorem 4. *Let* $A\in\mathscr{M}_n^+$ *be an incomplete reciprocal matrix and* $(i_1,j_1),\dots,(i_k,j_k)$ *its unknown entries. Any solution* $\boldsymbol{\lambda}=[\lambda_1 \ \cdots \ \lambda_k]^T$ *of Problem 1 satisfies*

$$\boldsymbol{\lambda}=Q(\overline{G_A})^T\boldsymbol{\theta},\quad L(G_A)\boldsymbol{\theta}=Q(G_A)\boldsymbol{\rho},\tag{A.13}$$

where $\boldsymbol{\rho}=[\log(a_{i_1,j_1}) \ \cdots \ \log(a_{i_m,j_m})]^T$.

Proof. The notation of Theorem 1 is used. Also, it is denoted $Y=[\mathbf{y}_1 \ \cdots \ \mathbf{y}_{n-1}]\in\mathscr{M}_{n,n-1}$, and $\boldsymbol{\theta}=Y\mathbf{m}$. By (A.8), the first equality of (A.1) reduces to $\boldsymbol{\lambda}=Q(\overline{G_A})^T\boldsymbol{\theta}$. Let us prove the second equality of (A.13).

$$\mathbf{b}=\begin{bmatrix}\mathbf{y}_1^T\mathbf{w}\\ \vdots\\ \mathbf{y}_{n-1}^T\mathbf{w}\end{bmatrix}=\begin{bmatrix}\mathbf{y}_1^T\\ \vdots\\ \mathbf{y}_{n-1}^T\end{bmatrix}\mathbf{w}=Y^T\mathbf{w}=\frac{1}{n}Y^T\sum_{i<j}c_{ij}\mathbf{d}_{ij}.\tag{A.14}$$

Observe that by the definition of the numbers c_{ij} (see (A.2)), in the summation appearing in (A.14), the indices can be restricted with no problem to the edges of the graph G_A. Thus, we have

$$\sum_{i<j}c_{ij}\mathbf{d}_{ij}=Q(G_A)\boldsymbol{\rho}.$$

Therefore, $\mathbf{b}=\frac{1}{n}Y^TQ(G_A)\boldsymbol{\rho}$, and the second equality of (A.1) becomes

$$(nD-S^TS)\mathbf{m}=Y^TQ(G_A)\boldsymbol{\rho}.$$

Now, it is enough to recall expression (A.11) to get $Y^TL(G_A)\boldsymbol{\theta}=Y^TQ(G_A)\boldsymbol{\rho}$. From here and the first equality of (A.5), we get $Y^TQ(G_A)(Q(G_A)^T\boldsymbol{\theta}-\boldsymbol{\rho})=\mathbf{0}$. From Lemma 1, we get $Q(G_A)(Q(G_A)^T\boldsymbol{\theta}-\boldsymbol{\rho})=\mathbf{0}$. Therefore, the second equality of (A.13) has been proven. □

A drawback associated to the second equality of system (A.13) is that matrix $L(G_A)$ is always nonsingular since $L(G_A)$ is an $n\times n$ matrix and $\mathrm{rk}(L(G_A))=n-p$, where p is the number of connected components of G_A.

In (Benítez *et al.*, 2014b) it was characterised when an incomplete, positive, and reciprocal matrix can be completed to become a consistent matrix. Concretely, it was stated in Theorems 7 and 10 of (Benítez *et al.*, 2014b) that, under the notation of Theorem 1 of this paper, A can be completed to be consistent if and only if there exists $\mathbf{x}\in\mathbb{R}^n$ such that $Q(G_A)^T\mathbf{x}=\boldsymbol{\rho}$, and in this case, we have $\boldsymbol{\lambda}=Q(\overline{G_A})^T\mathbf{x}$. It is possible to observe that, precisely, the second system in (A.13) corresponds to the least squares system related to $Q(G_A)^T\mathbf{x}=\boldsymbol{\rho}$.

Next, the system (A.13) is analysed by decomposing it in simpler systems.

For the sake of readability, Table A.1 indicates the notation for some parameters of the graph G_A.

Rearranging the points of G_A, the matrix $Q(G_A)$ has the following structure

$$Q(G_A)=\begin{bmatrix}0_{s,m}\\ Q_1\end{bmatrix}\in\mathscr{M}_{n,m},$$

Table A.1: Used notation for the parameters of a graph

n	No. of points
p	No. of connected components
m	No. of edges
s	No. of isolated points
$G_1,\ldots,G_q$	Connected components of G_A with more than 2 points
n_i	No. of points of the connected component G_i
m_i	No. of edges of the connected component G_i

where

$$Q_1 = Q(G_1) \oplus \cdots \oplus Q(G_q) \in \mathscr{M}_{n-s,m}, \qquad Q(G_i) \in \mathscr{M}_{n_i,m_i},$$

$G_1,\ldots,G_q$ being the connected components of G composed of more than two points. The ideas to study system (A.13) are: a) “forgetting” the isolated points and b) studying each connected component separatedly.

Observe that the number of isolated points plus q equals the number of connected components of G_A, i.e., $s+q=p$. Since n_i is the number of points of G_i for $i=1,\ldots,q$, evidently, we have

$$s+n_1+\cdots+n_q = n.$$

Also, observe that $\mathrm{rk}(Q(G_i)) = n_i - 1$ because G_i is connected. This is in full agreement with the fact that $n-p = \mathrm{rk}(Q(G_A)) = \mathrm{rk}(Q_1) = \mathrm{rk}(Q(G_1)) + \cdots + \mathrm{rk}(Q(G_q))$.

Also, the Laplacian of G_A has a block structure:

$$L(G_A) = Q(G_A)Q(G_A)^T = \begin{bmatrix} 0_{s,m} \\ Q_1 \end{bmatrix} \begin{bmatrix} 0_{m,s} & Q_1^T \end{bmatrix} = \begin{bmatrix} 0_{s,s} & 0 \\ 0 & Q_1 Q_1^T \end{bmatrix} \tag{A.15}$$

and

$$Q_1 Q_1^T = L(G_1) \oplus \cdots \oplus L(G_q). \tag{A.16}$$

Let us study system (A.13). First, with the notation of Theorem 4, we shall simplify $Q(G_A)\boldsymbol{\rho}$.

$$Q(G_A)\boldsymbol{\rho} = \begin{bmatrix} 0_{s,m} \\ Q_1 \end{bmatrix} \boldsymbol{\rho} = \begin{bmatrix} \mathbf{0}_s \\ Q_1 \boldsymbol{\rho} \end{bmatrix}.$$

Recall that we have denoted by m the number of edges of G_A and by m_i the number of edges of G_i for $i=1,\cdots,q$. Let us note $m_1+\cdots+m_q = m$. We partition $\boldsymbol{\rho} \in \mathscr{M}_{m,1}$ as follows:

$$\boldsymbol{\rho}^T = \begin{bmatrix} \boldsymbol{\rho}_1^T & \cdots & \boldsymbol{\rho}_q^T \end{bmatrix}, \qquad \boldsymbol{\rho}_i \in \mathscr{M}_{m_i,1}.$$

Therefore

$$Q_1 \boldsymbol{\rho} = \begin{bmatrix} Q(G_1)\boldsymbol{\rho}_1 \\ \vdots \\ Q(Q_q)\boldsymbol{\rho}_q \end{bmatrix}.$$

Now, let us recall that $\boldsymbol{\theta} \in \mathscr{M}_{n,1}$. We decompose

$$\boldsymbol{\theta}^T = \begin{bmatrix} \boldsymbol{\theta}_0^T & \boldsymbol{\theta}_1^T & \cdots & \boldsymbol{\theta}_q^T \end{bmatrix},$$

where $\boldsymbol{\theta}_0 \in \mathscr{M}_{s,1}$ and $\boldsymbol{\theta}_i \in \mathscr{M}_{n_i,1}$ for $i = 1, \cdots, q$. Now, (A.13), (A.15), and (A.16) lead to

$$L(G_i)\boldsymbol{\theta}_i = Q(G_i)\boldsymbol{\rho}_i, \quad i = 1, \cdots, q. \tag{A.17}$$

To solve system (A.13), we must think on the connected components of G_A. However, let us note that the systems (A.17) are always singular since the Laplacian of any graph is always a singular matrix.

So, what is the general solution of (A.17)? First, the systems (A.17) are solvable because these systems are the least square systems of $Q(G_i)^T \boldsymbol{\theta}_i = \boldsymbol{\rho}_i$. Let $\widehat{\boldsymbol{\theta}}_i$ be a solution of (A.17). We know that the general solution of (A.17) is $\widehat{\boldsymbol{\theta}}_i + \mathscr{N}(L(G_i))$, where $\mathscr{N}(\cdot)$ stands for the null space of a matrix. Since $\mathrm{rk}(L(G_i)) = n_i - 1$ and $L(G_i) \in \mathscr{M}_{n_i}$ (recall that G_i is a *connected* component of the graph G_A), then

$$\dim \mathscr{N}(L(G_i)) = n_i - \mathrm{rk}(L(G_i)) = 1.$$

Thus, to find $\mathscr{N}(L(G_i))$, it is enough to find a nonzero vector in $\mathscr{N}(L(G_i))$. But from (A.5) one gets $L(G_i)\mathbf{1}_{n_i} = 0$. Hence

$$\mathscr{N}(L(G_i)) = \{\alpha \mathbf{1}_{n_i} : \alpha \in \mathbb{R}\}.$$

Therefore, the general solution of (A.17) is

$$\widehat{\boldsymbol{\theta}}_i + \alpha \mathbf{1}_{n_i}, \qquad \alpha \in \mathbb{R},$$

where $\widehat{\boldsymbol{\theta}}_i$ is a particular solution of (A.17).

Now, it will be illustrated how to find a particular solution of (A.17). Let Y_i be a matrix in $\mathscr{M}_{n_i, n_i - 1}$ whose $n_i - 1$ columns form a basis of $(\mathrm{span}\{\mathbf{1}_{n_i}\})^\perp$ and let $\widehat{\mathbf{m}}_i$ be the unique solution of the linear system

$$Y_i^T L(G_i) Y_i \widehat{\mathbf{m}}_i = Y_i^T Q(G_i) \boldsymbol{\rho}_i. \tag{A.18}$$

This system has a unique solution because $Y_i^T L(G_i) Y_i \in \mathscr{M}_{n_i-1, n_i-1}$, (A.11), and (A.12) imply that $Y_i^T L(G_i) Y_i$ is nonsingular. Lemma 1 leads to $Y_i \widehat{\mathbf{m}}_i$ is a solution of (A.17). Hence the general solution of (A.17) is

$$Y_i \widehat{\mathbf{m}}_i + \alpha \mathbf{1}_{n_i}, \quad \alpha_i \in \mathbb{R}.$$

Hence, we can solve the right system in (A.13). Since $\boldsymbol{\theta}_0 \in \mathbb{R}^s$ is arbitrary, then if $\boldsymbol{\theta}$ is any solution of the right linear system in (A.13), then

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_0 \\ Y_1 \widehat{\mathbf{m}}_1 + \alpha_1 \mathbf{1}_{n_1} \\ \vdots \\ Y_q \widehat{\mathbf{m}}_q + \alpha_q \mathbf{1}_{n_q} \end{bmatrix}, \qquad \boldsymbol{\theta}_0 \in \mathbb{R}^s, \alpha_1, \cdots, \alpha_q \in \mathbb{R} \text{ are arbitrary.} \tag{A.19}$$

We have arrived to the following theorem. Recall that the mapping $\phi_n : \mathbb{R}^n \to \mathscr{M}_n$ is defined by $\phi_n(\mathbf{v}) = \mathbf{v}\mathbf{1}_n^T - \mathbf{1}\mathbf{v}^T$. Also, it is useful to recall Theorem 2.

Theorem 5. *Let $A \in \mathscr{M}_n^+$ be an incomplete reciprocal matrix whose unspecified entries above its main diagonal are $(i_1, j_1), \ldots, (i_k, j_k)$. Let G_A be its associate graph whose parameters are specified in Table A.1. Let $Y_i \in \mathscr{M}_{n_i, n_i - 1}$ a matrix whose $n_i - 1$ columns form a basis of $(\mathrm{span}\{\mathbf{1}_{n_i}\})^\perp$, let $\widehat{\mathbf{m}}_i$ be the unique vector satisfying (A.18), and let $\boldsymbol{\theta}$ be any vector of $\mathbb{R}^n$ given by (A.19). If X is a reciprocal completion of A such that $\mathrm{d}(X, \mathscr{C}_n) \leq \mathrm{d}(X', \mathscr{C}_n)$ for any reciprocal completion X' of A, then the (i_r, j_r) entry of X is the (i_r, j_r) entry of Y, where $\mathrm{LOG}(Y) = \phi_n(\boldsymbol{\theta})$.*

4. Synthetic example

Let A be the following incomplete reciprocal matrix:

$$A = \left[\begin{array}{cccc|cc} 1 & 2 & 4 & * & * & * \\ 1/2 & 1 & 5 & * & * & * \\ 1/4 & 1/5 & 1 & 2 & * & * \\ * & * & 1/2 & 1 & * & * \\ \hline * & * & * & * & 1 & 3 \\ * & * & * & * & 1/3 & 1 \end{array}\right].$$

Let us observe that by deleting the 4th, 5th, and 6th rows and columns of matrix A, we get a nonsingular matrix. Hence, $\mathrm{rk}(A) \geq 3$ and, in view of Theorem 3 of (Benítez *et al.*, 2012a), A cannot be completed to be consistent. It is easy to check that the associated graph G_A has two connected components, $G_1 = \{1,2,3,4\}$ and $G_2 = \{5,6\}$. Since G_A is not connected, by Corollary 2, the solution of problem 1 is not unique (in fact, the solutions of system (A.1) constitute a one-dimensional linear manifold, in view of Theorem 3).

Let us find $\widehat{\mathbf{m}}_1$: since the number of points of G_1 is $n_1 = 4$ and Y_1 is a matrix whose $n_1 - 1$ columns are a basis of $(\mathrm{span}\{\mathbf{1}_{n_1}\})^{\perp}$, then we can pick

$$Y_1 = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

Furthermore, one can easily see that the Laplacian of G_1 is the following matrix:

$$L(G_1) = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

To construct $Q(G_A)$ and $\boldsymbol{\rho}_1$ we employ the lexicographical order.

$$Q(G_1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \boldsymbol{\rho}_1 = \begin{bmatrix} \log a_{12} \\ \log a_{13} \\ \log a_{23} \\ \log a_{34} \end{bmatrix} = \begin{bmatrix} \log 2 \\ \log 4 \\ \log 5 \\ \log 2 \end{bmatrix}.$$

The solution of the system $Y_1^T L(G_1) Y_1 \widehat{\mathbf{m}}_1 = Y_1^T Q(G_1) \boldsymbol{\rho}_1$ is $\widehat{\mathbf{m}}_1 \simeq [0.194, 0.499, 0.423]^T$. Now, $Y_1 \widehat{\mathbf{m}}_1 \simeq [1.116, 0.728, -0.576, -1.269]^T$. Let us now find $\widehat{\mathbf{m}}_2$ and $Y_2 \widehat{\mathbf{m}}_2$. Since $n_2 = 2$ is the number of points of G_2,

$$Y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad L(G_2) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad Q(G_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{\rho}_2 = \begin{bmatrix} \log a_{35} \end{bmatrix} = \begin{bmatrix} \log 3 \end{bmatrix}.$$

The system $Y_2^T L(G_2) Y_2 \widehat{\mathbf{m}}_2 = Y_2^T Q(G_2) \boldsymbol{\rho}_2$ is $4\widehat{\mathbf{m}}_2 = 2 \log 3$. Hence $\widehat{\mathbf{m}}_2 = (\log 3)/2$ and

$$Y_2 \widehat{\mathbf{m}}_2 = \begin{bmatrix} (\log 3)/2 \\ -(\log 3)/2 \end{bmatrix} \simeq \begin{bmatrix} 0.549 \\ -0.549 \end{bmatrix}.$$

Therefore, by (A.19),

$$\boldsymbol{\theta} \simeq [1.116+\alpha_1, 0.727+\alpha_1, -0.5756+\alpha_1, -1.2669+\alpha_1, 0.5493+\alpha_2, -0.5493+\alpha_2]^T.$$

Since $\mathrm{LOG}(Y) = \phi_n(\boldsymbol{\theta})$ and Theorem 5, the (i_r, j_r) entry of the optimal completion of A is the (i_r, j_r) entry of Y, which is $\exp(\theta_{i_r})/\exp(\theta_{j_r}) = \exp(\theta_{i_r} - \theta_{i_j})$. Thus, if X_{i_r,j_r} is the (i_r, j_r) entry of the optimal completion of A, then $a_{14} = \exp(\theta_1 - \theta_4) \simeq 10.858$, $a_{15} = \exp(\theta_1 - \theta_5) \simeq 1.763 \exp(\alpha_1 - \alpha_2)$, and so on. Finally, we get (we denote $K = \exp(\alpha_1 - \alpha_2)$) that the optimal completion of A is

$$\left[\begin{array}{cccc|cc}
1 & 2 & 4 & 10.858 & K1.763 & K5.288 \\
1/2 & 1 & 5 & 7.368 & K1.196 & K3.588 \\
1/4 & 1/5 & 1 & 2 & K0.325 & K0.974 \\
0.092 & 0.136 & 1/2 & 1 & K0.162 & K0.487 \\
\hline
K^{-1}0.567 & K^{-1}0.836 & K^{-1}3.080 & K^{-1}6.160 & 1 & 3 \\
K^{-1}0.189 & K^{-1}0.279 & K^{-1}1.027 & K^{-1}2.053 & 1/3 & 1
\end{array}\right].$$

5. Comparison with other methods

In this subsection we compare our approach with two well-know PCM completion methods, namely, Van Uden's rule (Van Uden, 2002) and Harker's method (Harker, 1987).

Let A be an incomplete reciprocal $n \times n$ matrix ($n > 2$). If only one entry a_{ik} above the diagonal is missing, Van Uden proposes the following equality for calculating the missing element

$$a_{ik} = \sqrt[n-2]{X/Y}, \qquad X = \prod_{j \neq k} a_{ij}, \; Y = \prod_{j \neq i} a_{kj}. \tag{A.20}$$

The intuitive idea for this proposal is the following: if we consider just the fixed indices i, k, and a third index j (varying in $\{1, \ldots, n\} \setminus \{i, k\}$), we get an incomplete 3×3 submatrix and to achieve the consistency of this submatrix, we should set $a_{ik} = a_{ij} a_{jk} = a_{ij}/a_{kj}$. Since index j can take $n-2$ possible values, then we have $n-2$ possible values of a_{ik}. It is natural to consider the geometric mean of these values. We shall see that Theorem 5 includes Van Uden's rule. The notation $\mathscr{R}(\cdot)$ is introduced to indicate the range space of a matrix.

Rearranging the indices, it is possible to assume a_{12} and a_{21} as missing entries. Observe that the associate graph G_A is connected, and thus, the solution of Problem 1 is unique (Corollary 2). To find this solution, in view of Theorem 5 and (A.18), the system $Y^T L(G_A)\boldsymbol{\theta} = Y^T Q(G_A)\boldsymbol{\rho}$ has to be studied, where Y is an $n \times (n-1)$ matrix whose columns form an orthogonal basis of $\mathrm{span}\{\mathbf{1}_n\}^\perp$, $\boldsymbol{\theta} \in \mathbb{R}^n$,

$$\boldsymbol{\rho} = [\log a_{13} \cdots \log a_{1n} \; \log a_{23} \cdots \log a_{2n} \; l_1 \cdots l_r]^T,$$

and any l_m is of the form $\log a_{i_m j_m}$ with $3 \leq i_m < j_m$. In view of Lemma 1, the equation $Y^T L(G_A)\boldsymbol{\theta} = Y^T Q(G_A)\boldsymbol{\rho}$ is equivalent to $L(G_A)\boldsymbol{\theta} = Q(G_A)\boldsymbol{\rho}$. It is evident, by the definition of the Laplacian matrix of the graph G_A, that

$$L(G_A) = \begin{bmatrix} (n-2)I_{n-2} & -U_{2,n-2} \\ -U_{n-2,2} & nI_{n-2} - U_{n-2} \end{bmatrix}.$$

Since G_A is the complete graph of order n without the edge connecting vertices 1 and 2, by denoting with $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the standard basis of $\mathbb{R}^n$, then it is possible to write

$$Q(G_A) = [\, \mathbf{e}_1 - \mathbf{e}_3 \mid \cdots \mid \mathbf{e}_1 - \mathbf{e}_n \mid \mathbf{e}_2 - \mathbf{e}_3 \mid \cdots \mid \mathbf{e}_2 - \mathbf{e}_n \mid \mathbf{f}_1 \mid \cdots \mid \mathbf{f}_r \,],$$

where the vectors $\mathbf{f}_1,\dots,\mathbf{f}_r$ have the form $\mathbf{e}_i - \mathbf{e}_j$, where $3 \le i < j$, since in the graph G_A, if $i, j \in \{3,\dots,n\}$, then i and j are connected. By defining $s_1 = \sum_{j=3}^{n} \log a_{1j}$ and $s_2 = \sum_{j=3}^{n} \log a_{2j}$, then

$$\begin{aligned} Q(G_A)\boldsymbol{\rho} &= \sum_{j=3}^{n} \log a_{1j}(\mathbf{e}_1 - \mathbf{e}_j) + \sum_{j=3}^{n} \log a_{2j}(\mathbf{e}_2 - \mathbf{e}_j) + \sum_{j=1}^{r} l_j \mathbf{f}_j \\ &= s_1\mathbf{e}_1 + s_2\mathbf{e}_2 - \sum_{j=3}^{n}\left(\log a_{1j} + \log a_{2j}\right)\mathbf{e}_j + \sum_{j=1}^{r} l_j \mathbf{f}_j. \end{aligned}$$

Observe that $\mathbf{f}_1,\dots,\mathbf{f}_r \in \operatorname{span}\{\mathbf{e}_3,\dots,\mathbf{e}_n\}$. Thus, exists $\mathbf{v} \in \mathbb{R}^{n-2}$ such that

$$Q(G_A)\boldsymbol{\rho} = \begin{bmatrix} s_1 \\ s_2 \\ \mathbf{v} \end{bmatrix}.$$

Since $Q(G_A)\boldsymbol{\rho} \in \mathscr{R}[Q(G_A)] = \mathscr{R}[Q(G_A)Q(G_A)^T] = \mathscr{R}[L(G_A)]$, there exists $\boldsymbol{\theta} \in \mathbb{R}^n$ such that $L(G_A)\boldsymbol{\theta} = Q(G_A)\boldsymbol{\rho}$. Hence, denoting $\mathbf{s} = [s_1\ s_2]^T$ and decomposing $\boldsymbol{\theta}^T = [\boldsymbol{\theta}_1^T\ \boldsymbol{\theta}_2^T]^T$, $\boldsymbol{\theta}_1 \in \mathbb{R}^2$ and $\boldsymbol{\theta}_2 \in \mathbb{R}^{n-2}$, we have

$$\begin{bmatrix} (n-2)I_2 & -U_{2,n-2} \\ -U_{n-2,2} & nI_{n-2} - U_{n-2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{v} \end{bmatrix}.$$

Therefore, $(n-2)\boldsymbol{\theta}_1 - U_{2,n-2}\boldsymbol{\theta}_2 = \mathbf{s}$. If $\boldsymbol{\theta}_1 = [\xi_1, \xi_2]^T$ and $\boldsymbol{\theta}_2 = [\xi_3,\dots,\xi_n]^T$, then

$$(n-2)\xi_1 - (\xi_3 + \cdots + \xi_n) = s_1 \qquad \text{and} \qquad (n-2)\xi_2 - (\xi_3 + \cdots + \xi_n) = s_2.$$

By subtracting these two equalities, $(n-2)(\xi_1 - \xi_2) = s_1 - s_2$. Now, since $s_1 - s_2 = \sum_{j=3}^{n}(\log a_{1j} - \log a2j) = \log(\prod_{j=3}^{n} a_{1j}/a_{2j})$, we get

$$a_{12} = \mathrm{e}^{\xi_1 - \xi_2} = \mathrm{e}^{(s_1 - s_2)/(n-2)} = \sqrt[n-2]{\mathrm{e}^{s_1 - s_2}} = \sqrt[n-2]{\prod_{j=3}^{n} a_{1j}/a_{2j}},$$

which is Van Uden's rule (A.20) for $i = 1$ and $k = 2$.

There are other methods to deal with an incomplete reciprocal matrix when just one entry above the main diagonal is missing. It is possible to cite the one proposed by Shiraishi *et al.* (1998) and the heuristic approach given by Harker (1987). The foundation of the method proposed by Shiraishi *et al.* (1998) is based on the following theorem: let A be a reciprocal $n \times n$ matrix ($n > 2$), if $p_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \cdots + c_n$, then $c_1 = -n$, $c_2 = 0$, and $c_3 \le 0$. Furthermore, $c_3 = 0$ if and only if A is consistent. So, it is natural to maximize c_3 in this kind of problems. As one can see in section 3 in (Shiraishi *et al.*, 1998), the Van Uden's rule follows a different approach.

To better show the performance and validity of the method proposed by this thesis, a final empirical comparison is made with Harker's method. Let A be the following reciprocal matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1/2 & 1 & 4 & 2 \\ 1/3 & 1/4 & 1 & 1/2 \\ 1 & 1/2 & 2 & 1 \end{bmatrix},$$

with priorities given by the eigenvector $(0.361, 0.318, 0.097, 0.224)^T$. By using Theorem 3 in (Benítez *et al.*, 2014a), the consistent matrix closest to A is

$$X_A \simeq \begin{bmatrix} 1 & 1.107 & 3.464 & 1.565 \\ 0.9036 & 1 & 3.130 & 1.414 \\ 0.2887 & 0.3194 & 1 & 0.4518 \\ 0.6389 & 0.7071 & 2.213 & 1 \end{bmatrix}.$$

Let us proceed to delete some entries of A obtaining, as an example,

$$\widehat{A} = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 4 & \star \\ 1/3 & 1/4 & 1 & 1/2 \\ \star & \star & 2 & 1 \end{bmatrix}.$$

Note that the rank of the 3×3 upper left block of $\widehat{A}$ is 3; hence, the rank of $\widehat{A}$ is greater than 1 and, as a result, $\widehat{A}$ cannot be completed to be a consistent matrix.

The missing data are estimated by means of the Harker's rule. To this end, the derived reciprocal matrix is built:

$$\widetilde{A} = \begin{bmatrix} 2 & 2 & 3 & 0 \\ 1/2 & 2 & 4 & 0 \\ 1/3 & 1/4 & 1 & 1/2 \\ 0 & 0 & 2 & 3 \end{bmatrix}$$

By using Octave, the largest normalised eigenvalue is calculated as $\lambda_{\max} \simeq 4.083$, with associated eigenvector $\mathbf{v} \simeq (0.4243, 0.2927, 0.09939, 0.1836)^T$, the priority vector found by the Harker's method. With this vector one can get the matrix $H = (H_{ij})$, where $H_{ij} = \mathbf{v}_i/\mathbf{v}_j$. In this example,

$$H \simeq \begin{bmatrix} 1 & 1.449 & 4.269 & 2.311 \\ 0.6900 & 1 & 2.945 & 1.595 \\ 0.2343 & 0.3400 & 1 & 0.5414 \\ 0.4327 & 0.6272 & 1.847 & 1 \end{bmatrix},$$

which, obviously, is not a completion of $\widehat{A}$.

Let us now use the method proposed in the thesis (the complete set of details is omitted). First of all, since the associated graph is connected, the optimal completion is unique. Let

$$X(a,b) = \begin{bmatrix} 1 & 2 & 3 & a \\ 1/2 & 1 & 4 & b \\ 1/3 & 1/4 & 1 & 1/2 \\ 1/a & 1/b & 2 & 1 \end{bmatrix}$$

be this solution. By Theorem 2, the entries (1,4) and (2,4) of $X(a,b)$ (and their respective symmetrical entries) coincide with the corresponding entries of Z, where Z is the consistent matrix such that $\mathrm{d}(X(a,b), Z) = \mathrm{d}(X(a,b), \mathscr{C}_4)$, and $\mathscr{C}_4$ is the set of 4×4 consistent matrices (recall that $\mathrm{d}(\cdot,\cdot)$ is the distance defined as $\mathrm{d}(M,N) = \|\mathrm{LOG}(M) - \mathrm{LOG}(N)\|_F$). By the previous consideration of Theorem 2, one has $Z = E(\phi_4(\boldsymbol{\theta}))$. This vector $\boldsymbol{\theta}$ can be obtained by equalities (A.18) and (A.19) getting $\boldsymbol{\theta} \simeq (0.6310, 0.2648, -0.7945, -0.1014)^T$, and thus,

$$Z = E(\phi_4(\boldsymbol{\theta})) \simeq \begin{bmatrix} 1 & 1.443 & 4.160 & 2.080 \\ 0.6933 & 1 & 2.884 & 1.443 \\ 0.2404 & 0.3467 & 1 & 0.5000 \\ 0.4808 & 0.6933 & 2.000 & 1 \end{bmatrix}.$$

Accordingly, the optimal completion of $\widehat{A}$ is

$$X(Z_{14}, Z_{24}) = \begin{bmatrix} 1 & 2 & 3 & Z_{14} \\ 1/2 & 1 & 4 & Z_{24} \\ 1/3 & 1/4 & 1 & 1/2 \\ Z_{41} & Z_{42} & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2.080 \\ 1/2 & 1 & 4 & 1.443 \\ 1/3 & 1/4 & 1 & 1/2 \\ 0.4808 & 0.6933 & 2 & 1 \end{bmatrix}.$$

It is possible to note that matrices Z and H are similar. We can also check that $\mathrm{d}(X_A, Z) = 0.6355 < 0.7988 = \mathrm{d}(X_A, H)$, which shows that, in this example, the matrix Z obtained by the proposed method is closer to X_A than the matrix H obtained by the Harker's rule.

Observe that the proposed method gives the optimal completion of matrix $\widehat{A}$ (evidently, $X(Z_{14}, Z_{24})$ is a completion of $\widehat{A}$), while the Harker's rule gives just a priority vector $\mathbf{v}$, and the matrix H such that $H_{ij} = \mathbf{v}_i / \mathbf{v}_j$, is not, in general, a completion of $\widehat{A}$.

Additionally, it can be checked (by using Octave, for example) that the largest eigenvalue of $X(Z_{14}, Z_{24})$ is $\lambda_{\max} \simeq 4.081$, from which we easily find the consistency index of $X(Z_{14}, Z_{24})$, which equals $CI = (\lambda_{\max} - 4)/(4-1) \simeq 0.02714$. Finally, since $CR = CI/RI = 0.03050 < 0.1 = 10\%$, according to Saaty's criterion, the consistency of $X(Z_{14}, Z_{24})$ is acceptable and a priority vector is the normalised eigenvector of $X(Z_{14}, Z_{24})$ associated to $\lambda_{\max}$, which is $\mathbf{w} \simeq (0.4164, 0.2890, 0.1001, 0.1949)^T$.

This example shows that given an incomplete matrix which cannot be completed to be consistent, we can get a completion whose consistency is acceptable.

APPENDIX B

1. The definition of a random reciprocal matrix

When an expert has doubts in assigning a specific value to an entry in a reciprocal matrix, then the idea of using random instead of constant entries is suggested. Consequently, we will consider matrices $A=(a_{ij})$ whose components can be random variables. Another use of random variables in AHP can be the following: imagine that two experts express their judgements and thus form two reciprocal matrices, say $A=(a_{ij})$ and $B=(b_{ij})$. If there exists $i \neq j$ with $a_{ij} \neq b_{ij}$, then one can consider a discrete random variable X such that $\text{pr}(X=a_{ij})=w_A$ and $\text{pr}(X=b_{ij})=w_B$, where w_A, w_B are the respective weights given to the experts (of course, $0 \leq w_A, w_B \leq 1$, $w_A+w_B=1$).

A *random reciprocal matrix* is an $n \times n$ matrix $A=(a_{ij})$ whose entries are positive random variables whose expectation and variances are finite and $a_{ij}a_{ji}=1$, see Vargas (1982).

Let $B=(b_{ij})$ be the closest consistent matrix to A. What can be said about b_{ij}? And about the priority vector? These questions will be dealt with in this section.

The *expectation* and *variance* of a random variable X will be denoted by $\text{E}(X)$ and $\text{Var}(X)$, respectively. The *covariance* of the random variables X and Y will be denoted by $\text{Cov}(X,Y)$. Throughout this article, when we write E, Var, or Cov we will assume that these numbers are finite. To deal with random reciprocal matrices, it is plausible that the geometric mean is more natural than the arithmetic mean. Another reason is the following: if $A=(a_{ij})$ is a positive random matrix, since $a_{ij}=1/a_{ji}$, then it is natural that "mean of $(a_{ij})=1/$mean of (a_{ji})" holds. However, this property does not hold when the mean is the expectation E. Since the function $x \mapsto 1/x$ is convex, then, by Jensen's inequality, one has $\text{E}(X)^{-1} \leq \text{E}(X^{-1})$, and the equality holds if, and only if $\text{Var}(X)=0$. Therefore, another kind of expectation is going to be defined and used.

2. The geometric expectation and AHP

Given a positive random variable X, we define the *geometric expectation* by

$$\text{G}(X)=\exp(\text{E}(\log X)).$$

Equivalently, $\log[\text{G}(X)]=\text{E}[\log(X)]$. This expectation has found several applications in economics, see, e.g., Bean (2012) and Paolella (2006). From the very well-known properties of the expectation, one can give the following result.

Theorem 1. *Let X, Y be positive random variables. Then*

(i) $\text{G}(aX^b)=a\,\text{G}(X)^b$, *for constants $a>0$ and $b \in \mathbb{R}$.*

(ii) $\text{G}(XY)=\text{G}(X)\text{G}(Y)$.

In particular, if X is positive, then $\text{G}(X^{-1})=\text{G}(X)^{-1}$. By Jensen's inequality, since $x \mapsto \log x$ is a concave function, we have $\log[\text{E}(X)] \geq \text{E}[\log X]=\log[\text{G}(X)]$, i.e., $\text{G}(X) \leq \text{E}(X)$, and the inequality becomes an equality if and only if there exists $c \in \mathbb{R}$ such that $\text{pr}(X=c)=1$.

Theorem 2. *Let $A=(a_{ij})$ be an $n\times n$ reciprocal random matrix. Let $B=(b_{ij})$ the closest consistent matrix. If $\mathbf{x}=[x_1,\ldots,x_n]^T$ is a random priority vector of the matrix B, then*

$$\mathrm{G}(b_{ij})=\frac{\mathrm{G}(x_i)}{\mathrm{G}(x_j)}$$

and there exists $C\in\mathbb{R}$ such that

$$\mathrm{G}(x_i)=C\sqrt[n]{\mathrm{G}(a_{i1})\cdots\mathrm{G}(a_{in})}.$$

Proof: The expression for $\mathrm{G}(x_i)$ follows from Theorem 1. The expression for $\mathrm{G}(b_{ij})$ follows from $B=\mathbf{x}J(\mathbf{x})^T$ and Theorem 1. □

Observe that in the above theorem there is no need to assume that the judgements in matrix A have to be independent.

3. The geometric variance, the geometric covariance and AHP

Measures of deviation from the geometric expected value $\mathrm{G}(X)$ analogous to the variance of X can be defined. For a given positive random variable X, we define the *geometric variance* as follows:

$$\mathrm{Var}_g(X)=\mathrm{Var}(\log X). \tag{B.1}$$

In some textbooks, the expression $\exp(\mathrm{Var}(\log X))$ can be found as the definition for the geometric variance; however, (B.1) is easier to handle. Obviously, $\mathrm{Var}_g(X)\geq 0$ and $\mathrm{Var}_g(X)=0$ if and only if there exists $c>0$ such that $\mathrm{pr}(X=c)=1$.

We shall give two examples to show why we will not use the "usual" variance and why we suggest using the geometric variance.

1. Let us consider the following two situations:

   (i) a_{12} is the discrete random variable such that $\mathrm{pr}(a_{12}=1)=\mathrm{pr}(a_{12}=2)=1/2$.

   (ii) b_{12} is the discrete random variable such that $\mathrm{pr}(b_{12}=8)=\mathrm{pr}(b_{12}=9)=1/2$.

   In the first situation, the expert has doubts between "equal importance" and "weak importance" (in (Saaty, 2008c) one can find the fundamental scale in AHP proposed by Saaty). In the second situation, the expert's doubts are much smaller (his/her doubts vary between "major importance" and "extreme importance").

   However, $\mathrm{Var}(a_{12})=\mathrm{Var}(b_{12})$ —as one can trivially deduce from the expression $\mathrm{Var}(X+k)=\mathrm{Var}(X)$, where X is a random variable and $k\in\mathbb{R}$ is a constant. This fact is not intuitive since the expert's doubts in the first situation are greater than in the second situation. In contrast, one has $\mathrm{Var}_g(a_{12})=0.12011$ and $\mathrm{Var}_g(b_{12})=0.00347$.

2. In AHP theory, if $A=(a_{ij})$ is a reciprocal matrix, then $a_{ij}=1/a_{ji}$. Therefore, it must be intuitive that "variance of $1/X$ = variance of X". However, the "usual variance" does not satisfy this property (a trivial example is the random variable X such that $\mathrm{pr}(X=1)=\mathrm{pr}(X=2)=1/2$). Instead, we will see that the geometric variance does satisfy this property (see item (i) of Theorem 3).

The following is a step further in the same line of definitions. Given two positive random variables X and Y, the *geometric covariance* of X and Y is defined as

$$\mathrm{Cov}_g(X,Y)=\mathrm{Cov}(\log X,\log Y).$$

We next prove several properties of the geometric variance and geometric covariance.

Theorem 3. *Let X and Y be positive random variables.*

(i) $\mathrm{Var}_g(X^r) = r^2 \mathrm{Var}_g(X)$, *where $r \in \mathbb{R}$ is a constant.*

(ii) $\mathrm{Var}_g(XY) = \mathrm{Var}_g(X) + \mathrm{Var}_g(Y) + 2\mathrm{Cov}_g(X, Y)$.

(iii) *If X and Y are independent,* $\mathrm{Var}_g(XY) = \mathrm{Var}_g(X) + \mathrm{Var}_g(Y)$.

(iv) *If $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are positive random variables, and $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ are real constants, then* $\mathrm{Cov}_g(\prod_{i=1}^n X_i^{a_i}, \prod_{j=1}^m Y_j^{b_j}) = \sum_{i,j} a_i b_j \mathrm{Cov}_g(X_i, Y_j)$.

(v) *If A is a positive constant, then* $\mathrm{Var}_g(AX) = \mathrm{Var}_g(X)$ *and* $\mathrm{Cov}_g(A, X) = 0$.

Proof: (i): We use that if Z is a random variable and $a \in \mathbb{R}$, then $\mathrm{Var}(aZ) = a^2 \mathrm{Var}(Z)$.

$$\mathrm{Var}_g(X^r) = \mathrm{Var}(\log X^r) = \mathrm{Var}(r \log X) = r^2 \mathrm{Var}(\log X) = r^2 \mathrm{Var}_g(X).$$

(ii): By the previous definitions and known properties of the variance, we have

$$\begin{aligned} \mathrm{Var}_g(XY) &= \mathrm{Var}[\log(XY)] \\ &= \mathrm{Var}(\log X + \log Y) \\ &= \mathrm{Var}(\log X) + \mathrm{Var}(\log Y) + 2\mathrm{Cov}(\log X, \log Y) \\ &= \mathrm{Var}_g(X) + \mathrm{Var}_g(Y) + 2\mathrm{Cov}_g(X, Y). \end{aligned}$$

(iii): Since X and Y are independent, $\log X$ and $\log Y$ are also independent, hence the covariance of $\log X$ and $\log Y$ are zero. The conclusion follows from the computation made in the proof of (ii).

(iv): It follows from the definition of the geometric covariance and the property

$$\mathrm{Cov}\left(\sum_i a_i X_i, \sum_j b_j Y_j\right) = \sum_{i,j} a_i b_j \mathrm{Cov}(X_i, Y_j),$$

which is valid for arbitrary random variables X_i, Y_j and constants a_i, b_j.

(v): Since A is a constant, using the properties of the expectation,

$$\begin{aligned} \mathrm{Cov}_g(A, X) &= \mathrm{Cov}(\log A, \log X) \\ &= \mathrm{E}(\log A \log X) - \mathrm{E}(\log A)\mathrm{E}(\log X) = \big(\log A\big)\mathrm{E}(\log X) - \big(\log A\big)\mathrm{E}(\log X) = 0. \end{aligned}$$

The theorem is proved. □

Property (ii) above can be generalized by applying the formula of the variance of the sum of n random variables. If $X_1, \ldots, X_n$ are positive random variables, then

$$\mathrm{Var}_g(X_1 \cdots X_n) = \sum_{i=1}^n \mathrm{Var}_g(X_i) + 2\sum_{i<j} \mathrm{Cov}_g(X_i, X_j) \tag{B.2}$$

and if $X_1, \ldots, X_n$ are pairwise independent, then

$$\mathrm{Var}_g(X_1 \cdots X_n) = \sum_{i=1}^n \mathrm{Var}_g(X_i).$$

Now we give the geometric variance of the closest consistent matrix to a given random reciprocal matrix.

Theorem 4. *Let $A=(a_{ij})$ be an $n\times n$ reciprocal random matrix. Let $B=(b_{ij})$ be the closest consistent matrix. If $\mathbf{x}=[x_1,\dots,x_n]^T$ is a random vector which is a priority vector of the matrix B, then*

$$\mathrm{Var}_g(b_{ij})=\mathrm{Var}_g(x_i)+\mathrm{Var}_g(x_j)-2\mathrm{Cov}_g(x_i,x_j),$$

$$\mathrm{Cov}_g(b_{ij},b_{rs})=\mathrm{Cov}_g(x_i,x_r)-\mathrm{Cov}_g(x_i,x_s)-\mathrm{Cov}_g(x_j,x_r)+\mathrm{Cov}_g(x_j,x_s),$$

$$\mathrm{Var}_g(x_i)=\frac{1}{n^2}\left[\sum_{j=1}^{n}\mathrm{Var}_g(a_{ij})+2\sum_{j<k}\mathrm{Cov}_g(a_{ij},a_{ik})\right],$$

and

$$\mathrm{Cov}_g(x_i,x_j)=\frac{1}{n^2}\sum_{r,s}\mathrm{Cov}_g(a_{ir},a_{js}). \tag{B.3}$$

Proof: Since $b_{ij}=x_i/x_j$, it follows from Theorem 3 that

$$\begin{aligned}\mathrm{Var}_g(b_{ij})&=\mathrm{Var}_g(x_ix_j^{-1})\\&=\mathrm{Var}_g(x_i)+\mathrm{Var}_g(x_j^{-1})+2\mathrm{Cov}_g(x_i,x_j^{-1})=\mathrm{Var}_g(x_i)+\mathrm{Var}_g(x_j)-2\mathrm{Cov}_g(x_i,x_j).\end{aligned}$$

In an analogous way, we can prove the expression of $\mathrm{Cov}_g(b_{ij},b_{rs})$. If, in addition, we use (B.2), the remaining expressions can be similarly proved. □

Given a random reciprocal matrix $A=(a_{ij})$, it is reasonable to assume that a_{ij} are independent for $1\le i<j\le n$ (see Rosenbloom (1996)).

Corollary 1. *Under the notation of Theorem 4, if a_{ij} are pairwise independent for $1\le i<j\le n$, then*

$$\mathrm{Var}_g(x_i)=\frac{1}{n^2}\sum_{j=1}^{n}\mathrm{Var}_g(a_{ij}),\quad i=1,\dots,n,$$

and

$$\mathrm{Cov}_g(x_i,x_j)=-\frac{1}{n^2}\mathrm{Var}_g(a_{ij}),\quad i,j=1,\dots,n,\ i\neq j.$$

Proof: By the independence hypothesis, if $\mathrm{Cov}_g(a_{ir},a_{js})\neq 0$, then $(i,r)=(j,s)$ or $(i,r)=(s,j)$. The expression for the geometric variance follows from Theorem 4. To complete the proof, if $i\neq j$, then the unique non vanishing term on the right hand side of (B.3) corresponds to $(i,r)=(s,j)$, which is $\mathrm{Cov}_g(a_{ir},a_{js})=\mathrm{Cov}_g(a_{ij},a_{ji})=\mathrm{Cov}_g(a_{ij},a_{ij}^{-1})=-\mathrm{Cov}_g(a_{ij},a_{ij})=-\mathrm{Var}_g(a_{ij})$. □

If $\mathbf{x}=[x_1,\dots,x_n]^T$ is a vector of random variables, we define the matrix whose (i,j)-entry is $\mathrm{Cov}_g(x_i,x_j)$. This matrix will be named as the *geometric variance-covariance matrix* of $\mathbf{x}$ and denoted from now on by $\Sigma_g(\mathbf{x})$. Notice that $\mathrm{Cov}_g(x_i,x_i)=\mathrm{Var}_g(x_i)$. Observe that the geometric variance of b_{ij} can be computed by using the geometric variance-covariance matrix and Theorem 4. If $\mathbf{d}_{ij}$ denotes the column vector of $\mathbb{R}^n$ whose ith component is 1 and whose jth component is -1, and its remaining components are 0, then $\mathrm{Cov}_g(b_{ij},b_{rs})=\mathbf{d}_{ij}^T\Sigma_g(\mathbf{x})\mathbf{d}_{rs}$.

The importance of the random variables b_{ij} comes from the fact that these random variables are useful to rank the priorities. Recall that if a priority vector of the consistent matrix $B=(b_{ij})$ is $\mathbf{x}=[x_1,\dots,x_n]^T$, then $b_{ij}=x_i/x_j$. Hence, $b_{ij}>1$ if and only if $x_i>x_j$ and, thus, $\mathrm{pr}(b_{ij}>1)$ is the probability of the ith alternative being preferred to the jth alternative. Also, the random variables b_{ij} are useful to rank a complete order of preferences: for example, $x_i>x_j>x_k \Longleftrightarrow b_{ij}>1$ and $b_{jk}>1$; thus, that rank order can be evaluated by finding $\mathrm{pr}(b_{ij}>1 \text{ and } b_{jk}>1)$.

4. Chebyshev's inequalities and their applications in AHP

There are basic inequalities in probability theory used to give bounds for certain probabilities. These inequalities are important because they provide useful information about *arbitrary* random variables. Chebyshev's inequality says that the probability that a random variable X is outside the interval $[\mathrm{E}(X)-\varepsilon, \mathrm{E}(X)+\varepsilon]$ is negligible if $\mathrm{Var}(X)/\varepsilon^2$ is small enough. Precisely, we have that for any $\varepsilon > 0$,

$$\mathrm{pr}(|X-\mathrm{E}(X)| \geq \varepsilon) \leq \frac{\mathrm{Var}(X)}{\varepsilon^2}.$$

We give now a similar inequality concerning the geometric expectation and variance.

Theorem 5. *Let X be a positive random variable. For any $u > 0$ one has*

$$\mathrm{pr}(\mathrm{e}^{-u} < X/\mathrm{G}(X) < \mathrm{e}^{u}) \geq 1 - \frac{\mathrm{Var}_g(X)}{u^2}.$$

Proof: Since log is an increasing function,

$$\begin{aligned}\mathrm{pr}(\mathrm{e}^{-u} < X/\mathrm{G}(X) < \mathrm{e}^{u}) &= \mathrm{pr}(\mathrm{e}^{-u}\,\mathrm{G}(X) < X < \mathrm{e}^{u}\,\mathrm{G}(X)) \\ &= \mathrm{pr}(-u + \log \mathrm{G}(X) < \log X < u + \log \mathrm{G}(X)) \\ &= \mathrm{pr}\left(\left|\log X - \log \mathrm{G}(X)\right| < u\right) \\ &= \mathrm{pr}\left(\left|\log X - \mathrm{E}(\log X)\right| < u\right) = 1 - \mathrm{pr}\left(\left|\log X - \mathrm{E}(\log X)\right| \geq u\right).\end{aligned}$$

From Chebyshev's inequality, one has

$$\mathrm{pr}\left(\left|\log X - \mathrm{E}(\log X)\right| \geq u\right) \leq \frac{\mathrm{Var}(\log X)}{u^2} = \frac{\mathrm{Var}_g(X)}{u^2}.$$

Therefore, the conclusion of the theorem follows. □

In Alirio *et al.*, (2012) it is proven the following two dimensional version of Chebyshev's inequality.

Theorem 6. *Let X and Y be two random variables and $\varepsilon > 0$. Then*

$$\mathrm{pr}(|X-\mu_x| \geq \varepsilon\sigma_x \ \text{ or } \ |Y-\mu_y| \geq \varepsilon\sigma_y) \leq \frac{1+\sqrt{1-\rho^2}}{\varepsilon^2},$$

where $\mu_x = \mathrm{E}(X)$, $\mu_y = \mathrm{E}(Y)$, $\sigma_x^2 = \mathrm{Var}(X)$, $\sigma_y^2 = \mathrm{Var}(Y)$, and ρ is the correlation between X and Y, i.e., $\rho = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$.

The following theorem gives bounds for some probabilities.

Theorem 7. *Let X e Y be positive random variables. If $\varepsilon > 0$, then*

$$\mathrm{pr}\left(\mathrm{e}^{-\varepsilon \mathrm{Var}_g(X)} < \frac{X}{\mathrm{G}(X)} < \mathrm{e}^{\varepsilon \mathrm{Var}_g(X)} \ \text{ and } \ \mathrm{e}^{-\varepsilon \mathrm{Var}_g(Y)} < \frac{Y}{\mathrm{G}(Y)} < \mathrm{e}^{\varepsilon \mathrm{Var}_g(Y)}\right) \geq 1 - \frac{1+\sqrt{1-\rho^2}}{\varepsilon^2},$$

where ρ is the correlation between $\log X$ *and* $\log Y$.

Proof: Let $\omega_x = \mathrm{Var}_g(X)$ and $\omega_y = \mathrm{Var}_g(Y)$. Since $x \mapsto \log x$ is a non decreasing function, then

$$\begin{aligned}\mathrm{e}^{-\varepsilon\omega_x} < X/\mathrm{G}(X) < \mathrm{e}^{\varepsilon\omega_x} &\iff -\varepsilon\omega_x + \mathrm{E}(\log X) < \log X < \varepsilon\omega_x + \mathrm{E}(\log X) \\ &\iff |\log X - \mathrm{E}(\log X)| < \varepsilon\omega_x\end{aligned}$$

and, similarly for $Y/\mathrm{G}(Y)$. Therefore,

$$\begin{aligned}&\mathrm{pr}\left(\mathrm{e}^{-\varepsilon\omega_x}<\frac{X}{\mathrm{G}(X)}<\mathrm{e}^{\varepsilon\omega_x}\ \text{ and }\ \mathrm{e}^{-\varepsilon\omega_y}<\frac{Y}{\mathrm{G}(Y)}<\mathrm{e}^{\varepsilon\omega_y}\right)\\&\quad=\mathrm{pr}\big(|\log X-\mathrm{E}(\log X)|<\varepsilon\omega_x\ \text{ and }\ |\log Y-\mathrm{E}(\log Y)|<\varepsilon\omega_y\big)\\&\quad=1-\mathrm{pr}\big(|\log X-\mathrm{E}(\log X)|\geq\varepsilon\omega_x\ \text{ or }\ |\log Y-\mathrm{E}(\log Y)|\geq\varepsilon\omega_y\big).\end{aligned}$$

Recall that $\omega_x=\mathrm{Var}_g(X)=\mathrm{Var}(\log X)$ and $\omega_y=\mathrm{Var}(\log Y)$; hence the conclusion of the theorem follows from Theorem 6. □

Example. Let us consider the following random reciprocal matrix

$$A=\begin{bmatrix}1 & a_{12} & 2\\ a_{12}^{-1} & 1 & 3\\ 1/2 & 1/3 & 1\end{bmatrix},$$

where a_{12} is a positive random variable. For the sake of conciseness, we denote $\gamma=\mathrm{G}(a_{12})$ and $\omega=\mathrm{Var}_g(a_{12})$. Note that by Theorem 1, one has that $\mathrm{G}(a_{12}^{-1})=1/\gamma$. Let $\mathbf{x}=[x_1\ x_2\ x_3]^T$ be the priority vector of the closest consistent matrix to A. By Theorem 2, exists $C>0$ such that

$$\mathrm{G}(x_1)=C\sqrt[3]{2\gamma},\qquad \mathrm{G}(x_2)=C\sqrt[3]{3/\gamma},\qquad \mathrm{G}(x_3)=C\sqrt[3]{1/6}. \tag{B.4}$$

Let $B=\mathbf{x}J(\mathbf{x})^T=(b_{ij})$ be the closest consistent matrix to A. If G denotes the 3×3 matrix whose (i,j) entry is $\mathrm{G}(b_{ij})$, then by Theorem 2, $\mathrm{G}(b_{ij})=\mathrm{G}(x_i)/\mathrm{G}(x_j)$, hence

$$G=\begin{bmatrix}1 & \sqrt[3]{2\gamma^2/3} & \sqrt[3]{12\gamma}\\ \sqrt[3]{3/2\gamma^2} & 1 & \sqrt[3]{18/\gamma}\\ \sqrt[3]{1/12\gamma} & \sqrt[3]{\gamma/18} & 1\end{bmatrix}. \tag{B.5}$$

By Theorems 3 and 4, one has that

$$\mathrm{Var}_g(x_1)=\frac{1}{9}\mathrm{Var}_g(a_{12})=\frac{\omega}{9},\quad \mathrm{Var}_g(x_2)=\frac{1}{9}\mathrm{Var}_g(a_{12}^{-1})=\frac{1}{9}\mathrm{Var}_g(a_{12})=\frac{\omega}{9},\quad \mathrm{Var}_g(x_3)=0.$$

Now we write the variance-covariance geometric matrix of the random vector $\mathbf{x}$, denoted by $\Sigma_g(\mathbf{x})$. From the previous computations we know the entries of the main diagonal of $\Sigma_g(\mathbf{x})$ because $\mathrm{Cov}_g(x_i,x_i)=\mathrm{Var}_g(x_i)$. By property (v) of Theorem 3, the unique non vanishing term in the left hand side of $\mathrm{Cov}_g(x_1,x_2)=n^{-2}\sum_{r,s}\mathrm{Cov}_g(a_{1r},a_{2s})$ is $\mathrm{Cov}_g(a_{12},a_{21})$. But

$$\mathrm{Cov}_g(a_{12},a_{21})=\mathrm{Cov}_g(a_{12},a_{12}^{-1})=-\mathrm{Cov}_g(a_{12},a_{12})=-\mathrm{Var}_g(a_{12})=-\omega.$$

Since $\Sigma_g(\mathbf{x})$ is symmetric, $\mathrm{Cov}_g(a_{21},a_{12})=-\omega$. Finally, since the third row of A is composed of constants, then the third row and the third column of $\Sigma_g(\mathbf{x})$ must be filled with zeroes, because from item (v) of Theorem 3 and Theorem 4,

$$\mathrm{Cov}_g(x_3,x_i)=\frac{1}{3^2}\sum_{r,s}\mathrm{Cov}_g(a_{3r},a_{is})=0.$$

Thus,

$$\Sigma_g(\mathbf{x})=\frac{1}{9}\begin{bmatrix}\omega & -\omega & 0\\ -\omega & \omega & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{B.6}$$

If V is the 3×3 matrix whose (i,j) entry is $\mathrm{Var}_g(b_{ij})$, again by Theorem 4, we have

$$V_{12}=\mathrm{Var}_g(b_{12})=\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\Sigma_g(\mathbf{x})\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}=\frac{4\omega}{9}.$$

The remaining entries of V can be similarly computed and we can obtain

$$V=\frac{\omega}{9}\begin{bmatrix} 0 & 4 & 1 \\ 4 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{B.7}$$

Finally, we will find $\mathrm{Cov}_g(b_{ij},b_{rs})$ for $1\le i<j\le n$, $1\le r<s\le n$, and $(i,j)\neq(r,s)$. By Theorem 4 and (B.6),

$$\mathrm{Cov}_g(b_{12},b_{13})=\mathbf{d}_{12}^T\Sigma_g(\mathbf{x})\mathbf{d}_{13}=\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\Sigma_g(\mathbf{x})\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}=\frac{2\omega}{9}.$$

Similarly, we obtain $\mathrm{Cov}_g(b_{12},b_{23})=-2\omega/9$ and $\mathrm{Cov}_g(b_{13},b_{23})=-\omega/9$. Observe that there is no need to compute more covariances because $\mathrm{Cov}_g(X,X)=\mathrm{Var}_g(X)$, $\mathrm{Cov}_g(X,Y^{-1})=-\mathrm{Cov}_g(X,Y)$, and $\mathrm{Cov}_g(k,X)=0$ when X,Y are positive random variables and $k\in\mathbb{R}$ is a constant.

We will use Theorem 5 to study the random variable b_{12} (recall that this random variable is the $(1,2)$ entry of B, which is the closest consistent matrix to the given reciprocal matrix A). Let $u>0$. We know that

$$\mathrm{pr}\left(\mathrm{e}^{-u}\cdot\sqrt[3]{2\gamma^2/3}<b_{12}<\mathrm{e}^{u}\cdot\sqrt[3]{2\gamma^2/3}\right)\ge1-\frac{4\omega/9}{u^2}. \tag{B.8}$$

To fix ideas, let us assume that the expert has no preference between $a_{12}=5$ or $a_{12}=6$. Thus, it is natural to say that a_{12} is a random variable such that $\mathrm{pr}(5\le a_{12}\le6)=1$ and $\mathrm{G}(a_{12})$ is the geometric mean of 5 and 6, i.e., $\gamma=\mathrm{G}(a_{12})=\sqrt{5\cdot6}=\sqrt{30}\simeq5.477$.

To give a value to $\mathrm{Var}_g(a_{12})$, let us consider that the larger the variance of a random variable, the worse the behaviour of X. Moreover, since $\mathrm{pr}(\log5\le\log a_{12}\le\log6)=1$, then $\mathrm{Var}_g(a_{12})=\mathrm{Var}(\log a_{12})\le(\log6-\log5)^2/4\simeq0.00831$ (see Bhatia and Davis (2000)). We will assume the worst situation: $\omega=\mathrm{Var}_g(a_{12})=0.00831$.

We use (B.4) to get that the random (non normalised) vector $\mathbf{x}$ of priorities satisfies

$$C[\mathrm{G}(x_1)\ \mathrm{G}(x_2)\ \mathrm{G}(x_3)]\simeq C[2.221\ \ 0.8182\ \ 0.5503].$$

The geometric variance-covariance matrix of $\mathbf{x}$ is given in (B.6). If B is the nearest consistent matrix to A, then the geometric mean of the entries of B is given by (B.5); especifically, in this example we have

$$G\simeq\begin{bmatrix} 1 & 2.714 & 4.036 \\ 0.3684 & 1 & 1.487 \\ 0.2478 & 0.6727 & 1 \end{bmatrix},$$

and the matrix of the variances $(\mathrm{Var}_g(b_{ij}))$ is given in (B.7).

We will use Theorem 5 to exemplify about the preference order between the first and the second alternative (the remaining orders can be dealt with analogously). From (B.8) we have for any $u>0$ that

$$\mathrm{pr}(2.714\cdot\mathrm{e}^{-u}\le b_{12}\le2.714\cdot\mathrm{e}^{u})\ge1-\frac{0.003693}{u^2}.$$

We list some concrete values of u to see the goodness of these the bounds.

Value of u	Interval of b_{12}	Lower bound of the probability
0.7	[1.347, 5.466]	0.99246
0.3	[2.011, 3.664]	0.95896
0.15	[2.336, 3.153]	0.83585

We can see that $\mathrm{pr}(x_1 < x_2) = \mathrm{pr}(x_1 x_2^{-1} < 1) = \mathrm{pr}(b_{12} < 1) < \mathrm{pr}(b_{12} \notin [1.347, 5,466])$, hence $\mathrm{pr}(x_1 < x_2)$ is very small. What is more, $\mathrm{pr}(x_1 < 2x_2) = \mathrm{pr}(b_{12} < 2) < \mathrm{pr}(b_{12} \notin [2.011, 3.644]) < 1 - 0.95896 \simeq 0.041$, almost negligible.

Now we study the probability of certain preference order, for example, $x_1 < x_2 < x_3$. Observe that $x_1 < x_2 < x_3$ if and only if $x_1 x_2^{-1} < 1$ and $x_2 x_3^{-1} < 1$, i.e., $b_{12} < 1$ and $b_{23} < 1$. By Theorem 7 we have that for all $\varepsilon > 0$ one has

$$\mathrm{pr}\left(\mathrm{e}^{-\varepsilon\omega_{12}} < \frac{b_{12}}{\mathrm{G}(b_{12})} < \mathrm{e}^{\varepsilon\omega_{12}} \text{ and } \mathrm{e}^{-\varepsilon\omega_{23}} < \frac{b_{23}}{\mathrm{G}(b_{23})} < \mathrm{e}^{\varepsilon\omega_{23}}\right) \geq 1 - \frac{1+\sqrt{1-\rho^2}}{\varepsilon^2}, \tag{B.9}$$

where $\omega_{12} = \mathrm{Var}_g(b_{12})$, $\omega_{23} = \mathrm{Var}_g(b_{23})$, and ρ is the correlation between $\log(b_{12})$ and $\log(b_{23})$. Since

$$\rho = \frac{\mathrm{Cov}_g(b_{12}, b_{23})}{\sqrt{\mathrm{Var}_g(b_{12})}\sqrt{\mathrm{Var}_g(b_{23})}} = \frac{-2\omega/9}{\sqrt{4\omega/9}\sqrt{\omega/9}} = -1,$$

then we obtain from (B.9) the following table for several values of ε.

Value of ε	Interval of b_{12}	Interval of b_{23}	Lower bound of the probability
$\varepsilon = 1.5$	[2.699, 2.729]	[1.485, 1.489]	0.56
$\varepsilon = 2$	[2.694, 2.734]	[1.484, 1.490]	0.75
$\varepsilon = 3$	[2.685, 2.745]	[1.483, 1.491]	0.89
$\varepsilon = 5$	[2.665, 2.765]	[1.480, 1.493]	0.96
$\varepsilon = 10$	[2.616, 2.817]	[1.473, 1.501]	0.99

As we can see, we get good bounds for these probabilities.

5. The log-normal distribution and AHP

We say that the random variable X follows a *log-normal* distribution with parameters μ and σ (denoted as $X \sim \log\mathcal{N}(\mu, \sigma)$) if X is positive and $\log X$ follows a normal distribution such that $\mathrm{E}(\log X) = \mu$ and $\mathrm{Var}(\log X) = \sigma^2$. Evidently,

$$\mathrm{G}(X) = \exp(\mathrm{E}(\log X)) = \mathrm{e}^{\mu}, \qquad \mathrm{Var}_g(X) = \mathrm{Var}(\log X) = \sigma^2$$

The importance in AHP of this distribution lies in the following fact: if $X \sim \log\mathcal{N}(\mu, \sigma)$, then $1/X$ also follows a log-normal distribution. More concretely, $1/X \sim \log\mathcal{N}(-\mu, \sigma)$.

We will use the following two results, which can be found in any textbook dealing with multivariate normal distributions.

Theorem 8. *The random vector $\mathbf{x} \in \mathbb{R}^k$ is multivariate normal if and only if $\mathbf{a}^T\mathbf{x}$ is univariate normal for all $\mathbf{a} \in \mathbb{R}^k$.*

Theorem 9. *If the random variables $X_1, \ldots, X_m$ are independent and if X_i has a normal distribution ($i = 1, \ldots, m$), then $a_1 X_1 + \cdots a_m X_m$ has a normal distribution for arbitrary constants $a_1, \ldots, a_m \in \mathbb{R}$.*

When the judgements are independent and follow a log-normal distribution, we can give the following theorem.

Theorem 10. *Let $A=(a_{ij})\in\mathcal{M}_n^+$ be a reciprocal random matrix. Assume that a_{ij} are independent for $1\le i<j\le n$ and $a_{ij}\sim\log\mathcal{N}(\mu_{ij},\sigma_{ij})$. Let $B=(b_{ij})$ be the closest consistent matrix to A and $\mathbf{x}=[x_1,\dots,x_n]^T$ be a priority vector of B. Then the random vectors $\mathbf{y}=[\log x_1,\dots,\log x_n]^T$ and*

$$\mathbf{b}=[\log b_{12},\dots,\log b_{1n},\log b_{23},\dots,\log b_{2n},\dots,\log b_{n-1,n}]^T$$

follow a multivariate normal distribution.

Proof: We use Theorem 8 to prove that $\mathbf{y}$ has a multivariate normal distribution. Let $\mathbf{a}=[\xi_1,\dots,\xi_n]^T\in\mathbb{R}^n$. From $x_i=C\sqrt[n]{a_{i1}\cdots a_{in}}$ for some fixed constant $C>0$, if we denote $l_{ij}=\log(a_{ij})$ for all indices i,j, then

$$\mathbf{a}^T\mathbf{y}=\sum_{i=1}^{n}\xi_i\log x_i=\frac{C}{n}\sum_{i=1}^{n}\xi_i(l_{i1}+\dots+l_{in}). \tag{B.10}$$

Since a_{ij} are independent for $1\le i<j\le n$, from Theorems 8 and 9, the vector

$$\mathbf{l}=[l_{12},\dots,l_{1n},l_{23},\dots,l_{2n},\dots,l_{n-1,n}]^T\in\mathbb{R}^p$$

(here $p=n(n-1)/2$) has a multivariate normal distribution. In addition, using $l_{ij}=-l_{ji}$, $l_{ii}=0$, and (B.10), we can see that there exists $\mathbf{c}\in\mathbb{R}^p$ such that $\mathbf{a}^T\mathbf{y}=\mathbf{c}^T\mathbf{l}$. By Theorem 8, $\mathbf{a}^T\mathbf{y}$ has a univariate normal distribution. Since $\mathbf{a}$ is arbitrary, again by Theorem 8, the random vector $\mathbf{y}$ has a multivariate normal distribution.

Let $\mathbf{d}=[d_{12},\dots,d_{n-1,n}]^T\in\mathbb{R}^p$. By using $b_{ij}=x_i/x_j$ we have

$$\mathbf{d}^T\mathbf{b}=\sum_{i<j}d_{ij}\log b_{ij}=\sum_{i<j}d_{ij}(\log x_i-\log x_j)=\frac{C}{n}\sum_{i<j}\sum_{k=1}^{n}d_{ij}(l_{ik}-l_{jk})$$

Using again $l_{rs}=-l_{sr}$ and $l_{rr}=0$, there exists a vector $\mathbf{e}\in\mathbb{R}^p$ such $\mathbf{d}^T\mathbf{b}=\mathbf{e}^T\mathbf{l}$. A similar argument as before can be used to prove that $\mathbf{b}$ follows a multivariate normal distribution. □

We do not specifiy the parameters of the multivariate distributions of the foregoing theorem as they can be easily found in Theorem 2, Theorem 4, and Corollary 1.

Example. Let us consider the following reciprocal random matrix

$$A=\begin{bmatrix}1 & a_{12} & a_{13}\\ 1/a_{12} & 1 & 2\\ 1/a_{13} & 1/2 & 1\end{bmatrix}.$$

The expert considers $3\le a_{12}\le 4$ and $4\le a_{13}\le 5$. Therefore, it is natural to set $\mathrm{G}(a_{12})=\sqrt{12}$ and $\mathrm{G}(a_{13})=\sqrt{20}$. The expert assumes that a_{12} and a_{13} follow a log-normal distribution and these variables are independent. To set the geometric variance of a_{12}, several random samples from the log-normal distribution with $\mathrm{G}(a_{12})=\sqrt{12}$ and $\mathrm{Var}_g(a_{12})=0.5^2$ were generated. In Octave, ten samples can be easily obtained by executing

```
exp(normrnd(log(sqrt(12)),0.5,10,1)).
```

By performing this, we can observe that there are samples outside [3,4], which is not admissible by the expert, and therefore, we must decrease the variance. After several tries, the expert says that the value of $\mathrm{Var}_g(a_{12})=0.05^2$ is adequate. In a similar way, $\mathrm{G}(a_{13})=\sqrt{20}$ and $\mathrm{Var}_g(a_{13})=0.05^2$ will be considered.

We denote $\gamma_{12} = \mathrm{G}(a_{12})$, $\gamma_{13} = \mathrm{G}(a_{13})$, and $\omega = \mathrm{Var}_g(a_{12}) = \mathrm{Var}_g(a_{13})$. Let $B = (b_{ij})$ the consistent matrix closest to A and let $[x_1\ x_2\ x_3]^T$ be a priority vector of B. By Theorem 2, there exists $C > 0$ such that

$$\mathrm{G}(x_1) = C\sqrt[3]{\gamma_{12}\gamma_{13}}, \quad \mathrm{G}(x_2) = C\sqrt[3]{2/\gamma_{12}}, \quad \mathrm{G}(x_3) = C\sqrt[3]{1/(2\gamma_{13})}, \quad \mathrm{G}(b_{ij}) = \mathrm{G}(x_i)/\mathrm{G}(x_j).$$

As an example we shall find $\mathrm{pr}(x_1 < 2x_2)$ and $\mathrm{pr}(x_1 < 3x_2\ \&\ x_1 < 5x_3)$. Observe first that

$$\mathrm{pr}(x_1 < x_2) = \mathrm{pr}(x_1/x_2 < 2) = \mathrm{pr}(b_{12} < 2) = \mathrm{pr}(\log b_{12} < \log 2).$$

By Theorem 10, $\log b_{12}$ follows a normal distribution. To find its parameters, we apply Theorem 2:

$$\mathrm{E}(\log b_{12}) = \log(\mathrm{G}(b_{12})) = \log(\mathrm{G}(x_1)/\mathrm{G}(x_2)) = \frac{1}{3}\left[2\log\gamma_{12} + \log\gamma_{13} - \log 2\right] \simeq 1.097. \quad \text{(B.11)}$$

By Theorem 4, one gets $\mathrm{Var}(\log b_{12}) = \mathrm{Var}_g(b_{12}) = \mathrm{Var}_g(x_1) + \mathrm{Var}_g(x_2) - 2\mathrm{Cov}_g(x_1, x_2)$. But Corollary 1 leads to

$$\mathrm{Var}_g(x_1) = \frac{1}{9}\left(\mathrm{Var}_g(a_{11}) + \mathrm{Var}_g(a_{12}) + \mathrm{Var}_g(a_{13})\right) = \frac{2\omega}{9},$$

$$\mathrm{Var}_g(x_2) = \frac{1}{9}\left(\mathrm{Var}_g(a_{21}) + \mathrm{Var}_g(a_{22}) + \mathrm{Var}_g(a_{23})\right) = \frac{\omega}{9},$$

and

$$\mathrm{Cov}_g(x_1, x_2) = -\frac{1}{9}\mathrm{Var}_g(a_{12}) = -\frac{\omega}{9}.$$

Therefore, $\mathrm{Var}(\log b_{12}) = 5\omega/9 \simeq 0.00139$. Now, it is simple to compute $\mathrm{pr}(\log b_{12} < \log 2)$, obtaining that this probability is approximately 0.

To find $\mathrm{pr}(x_1 < 3x_2\ \&\ x_1 < x_3) = \mathrm{pr}(b_{12} < 3\ \&\ b_{13} < 1)$, we need to know the parameters of the joint distribution of (b_{12}, b_{13}). By Theorems 8 and 10, $(\log b_{12}, \log b_{13})$ follows a bivariate normal distribution. The mean of $\log b_{12}$ was computed in (B.11). Similarly, we have

$$\mathrm{E}(\log b_{13}) = \log(\mathrm{G}(b_{13})) = \log(\mathrm{G}(x_1)/\mathrm{G}(x_3)) = \frac{1}{3}\left[\log 2 + \log\gamma_{12} + 2\log\gamma_{13}\right] \simeq 1.644.$$

The covariance matrix of $(\log b_{12}, \log b_{13})$ is

$$\Sigma = \begin{bmatrix} \mathrm{Var}(\log b_{12}) & \mathrm{Cov}(\log b_{12}, \log b_{13}) \\ \mathrm{Cov}(\log b_{12}, \log b_{13}) & \mathrm{Var}(\log b_{13}) \end{bmatrix},$$

which can be computed by using Theorem 4 and Corollary 1. Observe that $\mathrm{Var}(\log b_{12})$ was computed before. Since

$$\begin{aligned}\mathrm{Var}(\log b_{13}) &= \mathrm{Var}_g(b_{13}) = \mathrm{Var}_g(x_1) + \mathrm{Var}_g(x_3) - 2\mathrm{Cov}_g(x_1, x_3)\\ &= \frac{1}{9}\Big[\left(\mathrm{Var}_g(a_{11}) + \mathrm{Var}_g(a_{12}) + \mathrm{Var}_g(a_{13})\right) + \\ &\quad + \left(\mathrm{Var}_g(a_{31}) + \mathrm{Var}_g(a_{32}) + \mathrm{Var}_g(a_{33})\right) + 2\mathrm{Var}_g(a_{13})\Big] = \frac{5\omega}{9}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Cov}(\log b_{12}, \log b_{13}) &= \mathrm{Cov}_g(b_{12}, b_{13})\\ &= \mathrm{Cov}_g(x_1, x_1) - \mathrm{Cov}_g(x_1, x_3) - \mathrm{Cov}_g(x_2, x_1) + \mathrm{Cov}_g(x_2, x_3)\\ &= \mathrm{Var}_g(x_1) + \frac{1}{9}\mathrm{Var}_g(a_{13}) + \frac{1}{9}\mathrm{Var}_g(a_{21}) - \frac{1}{9}\mathrm{Var}_g(a_{23}) = \frac{4\omega}{9},\end{aligned}$$

we get

$$\Sigma = \frac{\omega}{9}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

Observe that in this example, matrix Σ is not singular.

If Σ were singular, then there would exist constants $\alpha, \beta \in \mathbb{R}$ such that $\log b_{12} = \alpha \log b_{13} + \beta$, being $b_{ij} = x_i / x_j$. In this case (which recall it is not satisfied by the example), one can find $\mathrm{pr}(b_{12} < 3 \ \& \ b_{13} < 5)$.

Finally, to find $\mathrm{pr}(b_{12} < 3 \ \& \ b_{13} < 5) = \mathrm{pr}(\log b_{12} < \log 3 \ \& \ \log b_{13} < 0)$, we will use the Octave program. By executing

```
g12=sqrt(12); g13=sqrt(20);
e1=(2*log(g12)+log(g13)-log(2))/3; % Mean of log(b12)
e2=(2*log(g13)+log(g12)+log(2))/3; % Mean of log(b13)
mu = [e1 e2];
om=0.05^2; % Omega
Sigma= [5 4; 4 5]*om/9; % Covariance matrix of (log b12,log b13)
mvncdf([log(3) log(5)],mu,Sigma)
% pr(log b12 < log 3 & log b13 < log 5)
```

we obtain $\mathrm{pr}(b_{12} < 3 \ \& \ b_{13} < 5) \simeq 0.172$.

APPENDIX C

1. Clustering of entries in reciprocal matrices

In many practical situations, it may be useful to collapse several opinions or judgements into a single one, while trying to maintain the 'non-collapsed' judgements as faithful to the original as possible. This problem is herein addressed after presenting some preliminaries.

If a matrix $A \in \mathscr{M}_n^+$ is reciprocal, to find its closest consistent approximation, one must projected LOG(A) onto $\mathscr{L}_n$. As LOG(A) is skew-symmetric, in the following reasoning, only skew-symmetric matrices are involved instead of reciprocal matrices.

Let $M \in \mathscr{M}_{n+m}$ be a skew-symmetric matrix. Let us partition M as follows

$$M = \begin{bmatrix} M_1 & -M_2 \\ M_2^T & M_3 \end{bmatrix} \in \mathscr{M}_{n+m}, \qquad M_1 \in \mathscr{M}_n,\ M_3 \in \mathscr{M}_m, \tag{C.1}$$

M_1 and M_3 being skew-symmetric (this is because M is skew-symmetric). The relations between the last m judgments of M are reflected in the block M_3, and the relations between the first n judgments and the last m judgments are reflected in the block M_2 (let us note that M_2 can be a non-square matrix —in case $n \neq m$ holds).

If we want to collapse the i-th judgments ($i = n+1, \ldots, n+m$) of M to a single one, then we are forced to consider the following $n+1$ square block matrix:

$$N = \begin{bmatrix} M_1 & -\mathbf{v} \\ \mathbf{v}^T & 0 \end{bmatrix} \in \mathscr{M}_{n+1}, \qquad \mathbf{v} \in \mathbb{R}^n. \tag{C.2}$$

Observe that the 'north-west' blocks of M and N must be equal if we want that the preservation of the judgments in this collapsed matrix N to be as faithful possible. Our purpose is: *how to find vector* $\mathbf{v}$?

If this 'collapse' is coherent, then the information concerning the $1, \ldots, n$ judgments must not be changed. In other words, since the orthogonal projection onto $\mathscr{L}_{n+m}$ and $\mathscr{L}_{n+1}$ provide the best approximations, if

$$p_{n+m}(M) = \begin{bmatrix} X & -Y \\ Y^T & Z \end{bmatrix}, \qquad p_{n+1}(N) = \begin{bmatrix} X' & -\mathbf{y} \\ \mathbf{y}^T & 0 \end{bmatrix}, \qquad X, X' \in \mathscr{M}_n, \tag{C.3}$$

then one must have $X = X'$.

Once the vector $\mathbf{v}$ in matrix N written in (C.2) is found, the expert(s) that filled the matrix M must be asked (in a feedback process) if this vector $\mathbf{v}$ (which reflects the relations between the $1, \ldots, n$ judgements and the collapsed one) is adequate.

The next auxiliary lemma will be useful to prove the main results of this paper. From now on, $U_{n,m}$ will denote the $n \times m$ matrix all of whose entries are 1, which is equivalent to $U_{n,m} = \mathbf{1}_n \mathbf{1}_m^T$.

Lemma 1. *If* $M \in \mathscr{M}_n$ *is skew-symmetric, then* $U_{m,n} M \mathbf{1}_n = \mathbf{0}$.

Proof. Let $\{\mathbf{e}_1,\dots,\mathbf{e}_m\}$ be the standard basis of $\mathbb{R}^m$. The lemma will be proven if we demonstrate $\mathbf{e}_i^T U_{m,n} M \mathbf{1}_n = 0$ for $i = 1,\dots,m$. Since $\mathbf{e}_i^T U_{m,n} = \mathbf{1}_n^T$ we obtain $\mathbf{e}_i^T U_{m,n} M \mathbf{1}_n = \mathbf{1}_n^T M \mathbf{1}_n$. Let us bear in mind that $\mathbf{1}_n^T M \mathbf{1}_n$ is a scalar, and so, coincides with its transpose. Since $M = -M^T$ we have $\mathbf{1}_n^T M \mathbf{1}_n = (\mathbf{1}_n^T M \mathbf{1}_n)^T = \mathbf{1}_n^T M^T \mathbf{1}_n = -\mathbf{1}_n^T M \mathbf{1}_n$. Hence $\mathbf{1}_n^T M \mathbf{1}_n = 0$ and the proof of the lemma is ended. □

Also, the next result (Theorem 4 of Benítez *et al.*, (2014a)) will play an essential role in the sequel, and we include it for the sake of readability. Let us recall that $p_n : \mathscr{M}_n \to \mathscr{L}_n$ denotes the orthogonal projection onto $\mathscr{L}_n$.

Theorem 1. *Let $M \in \mathscr{M}_n$ be skew-Hermitian and $\mathbf{v} \in \mathbb{R}^n$. Then $\phi_n(\mathbf{v}) = p_n(M)$ if and only if there exists $\alpha \in \mathbb{R}$ such that $\mathbf{v} = \frac{1}{n} M \mathbf{1}_n + \alpha \mathbf{1}_n$.*

The main results follow below.

Theorem 2. *Let $M \in \mathscr{M}_{n+m}$ and $N \in \mathscr{M}_{n+1}$ be skew-symmetric matrices decomposed as in (C.1) and (C.2), respectively. Let $p_{n+m}(M)$ and $p_{n+1}(N)$ be decomposed as in (C.3). Then the following conditions are equivalent:*

(i) $X = X'$.

(ii) *There exists $\alpha \in \mathbb{R}$ such that*

$$\mathbf{v} = \frac{n+1}{n+m} M_2 \mathbf{1}_m + \frac{m-1}{n+m} M_1 \mathbf{1}_n + \alpha \mathbf{1}_n. \tag{C.4}$$

Under this equivalence, one has

$$\mathbf{y} = \frac{1}{n+m}[-M_1 \mathbf{1}_n + (I_n + U_n) M_2 \mathbf{1}_m] + \alpha \mathbf{1}_n.$$

Proof. Let us recall that

$$p_{n+m}(M) = \frac{1}{n+m}\left[(M U_{n+m}) - (M U_{n+m})^T\right]. \tag{C.5}$$

Since

$$M U_{n+m} = \begin{bmatrix} M_1 & -M_2 \\ M_2^T & M_3 \end{bmatrix} \begin{bmatrix} U_n & U_{n,m} \\ U_{m,n} & U_m \end{bmatrix} = \begin{bmatrix} M_1 U_n - M_2 U_{m,n} & M_1 U_{n,m} - M_2 U_m \\ M_2^T U_n + M_3 U_{m,n} & M_2^T U_{n,m} + M_3 U_m \end{bmatrix}. \tag{C.6}$$

Therefore, we obtain

$$X = \frac{1}{n+m}\left[(M_1 U_n - M_2 U_{m,n}) - (M_1 U_n - M_2 U_{m,n})^T\right].$$

We have

$$\begin{aligned} M_2 U_{m,n} - (M_2 U_{m,n})^T &= M_2(\mathbf{1}_m \mathbf{1}_n^T) - [M_2(\mathbf{1}_m \mathbf{1}_n^T)]^T \\ &= (M_2 \mathbf{1}_m)\mathbf{1}_n^T - [(M_2 \mathbf{1}_m)\mathbf{1}_n^T]^T \\ &= (M_2 \mathbf{1}_m)\mathbf{1}_n^T - \mathbf{1}_n (M_2 \mathbf{1}_m)^T \\ &= \phi_n(M_2 \mathbf{1}_m). \end{aligned}$$

Hence,

$$X = \frac{1}{n+m}\left[(M_1 U_n) - (M_1 U_n)^T - \phi_n(M_2 \mathbf{1}_m)\right] = \frac{1}{n+m}\left[n p_n(M_1) - \phi_n(M_2 \mathbf{1}_m)\right]. \tag{C.7}$$

Analogously we obtain

$$X' = \frac{1}{n+1}\big[(M_1 U_n - \mathbf{v}\mathbf{1}_n^T) - (M_1 U_n - \mathbf{v}\mathbf{1}_n^T)^T\big] = \frac{1}{n+1}\big[n p_n(M_1) - \phi_n(\mathbf{v})\big]. \tag{C.8}$$

(i) ⇒ (ii). From $X = X'$, (C.7), and (C.8) we obtain

$$\frac{1}{n+m}\big[n p_n(M_1) - \phi_n(M_2 \mathbf{1}_m)\big] = \frac{1}{n+1}\big[n p_n(M_1) - \phi_n(\mathbf{v})\big].$$

Therefore, by recalling that ϕ_n and p_n are linear mappings

$$\phi_n\left(\frac{1}{n+1}\mathbf{v} - \frac{1}{n+m}M_2\mathbf{1}_m\right) = p_n\left(\left(\frac{n}{n+1} - \frac{n}{n+m}\right)M_1\right).$$

Let us recall that M_1 is skew-symmetric. From Theorem 1, there exists $\alpha \in \mathbb{R}$, such that

$$\frac{1}{n+1}\mathbf{v} - \frac{1}{n+m}M_2\mathbf{1}_m = \frac{1}{n}\left(\frac{n}{n+1} - \frac{n}{n+m}\right)M_1\mathbf{1}_n + \alpha\mathbf{1}_n.$$

By performing some easy computations and renaming $(n+1)\alpha$ to α, we obtain the expression of $\mathbf{v}$ given in the statement of the theorem.

(ii) ⇒ (i): We know that exists $\alpha \in \mathbb{R}$, such that

$$\mathbf{v} - \frac{n+1}{n+m}M_2\mathbf{1}_m = \frac{1}{n}\,\frac{n(m-1)}{n+m}M_1\mathbf{1}_n + \alpha\mathbf{1}_n.$$

By Theorem 1 and since $\frac{n(m-1)}{n+m}M_1$ is skew-symmetric, we obtain

$$\phi_n\left(\mathbf{v} - \frac{n+1}{n+m}M_2\mathbf{1}_m\right) = p_n\left(\frac{n(m-1)}{n+m}M_1\right).$$

Now, the linearity of p_n and ϕ_n, together with (C.7) and (C.8) lead to

$$\begin{aligned} X' &= \frac{1}{n+1}\big[n p_n(M_1) - \phi_n(\mathbf{v})\big] \\ &= \frac{1}{n+1}\left[n p_n(M_1) - \left(\phi_n\left(\frac{n+1}{n+m}M_2\mathbf{1}_m\right) + p_n\left(\frac{n(m-1)}{n+m}M_1\right)\right)\right] \\ &= \frac{n}{n+m}p_n(M_1) - \frac{1}{n+m}\phi_n(M_2\mathbf{1}_m) \\ &= X. \end{aligned}$$

This proves the first part of the theorem.

To find $\mathbf{y}$, which appears in (C.3), we use $p_{n+1}(N) = [N U_{n+1} - (N U_{n+1})^T]/(n+1)$. Since (we have marked with $\star$ the entries that we are not interested in)

$$N U_{n+1} = \begin{bmatrix} M_1 & -\mathbf{v} \\ \mathbf{v}^T & 0 \end{bmatrix}\begin{bmatrix} U_n & \mathbf{1}_n \\ \mathbf{1}_n^T & 1 \end{bmatrix} = \begin{bmatrix} \star & M_1\mathbf{1}_n - \mathbf{v} \\ \mathbf{v}^T U_n & \star \end{bmatrix},$$

we obtain (recall that U_n is symmetric)

$$p_{n+1}(N) = \frac{1}{n+1}\begin{bmatrix} \star & M_1\mathbf{1}_n - \mathbf{v} - U_n\mathbf{v} \\ \star & \star \end{bmatrix}.$$

Hence, by using $U_n\mathbf{1}_n = n\mathbf{1}_n$, the first part of the theorem, (C.3), and Lemma 1,

$$\begin{aligned}(n+1)\mathbf{y} &= -M_1\mathbf{1}_n + \mathbf{v} + U_n\mathbf{v}\\ &= -M_1\mathbf{1}_n + \frac{n+1}{n+m}M_2\mathbf{1}_m + \frac{m-1}{n+m}M_1\mathbf{1}_n + \alpha\mathbf{1}_n +\\ &\quad + U_n\left(\frac{n+1}{n+m}M_2\mathbf{1}_m + \frac{m-1}{n+m}M_1\mathbf{1}_n + \alpha\mathbf{1}_n\right)\\ &= -\frac{n+1}{n+m}M_1\mathbf{1}_n + \frac{n+1}{n+m}M_2\mathbf{1}_m + (n+1)\alpha\mathbf{1}_n + \frac{n+1}{n+m}U_nM_2\mathbf{1}_m.\end{aligned}$$

The proof is finished. □

It is noteworthy that the vectors **v** and **y** in Theorem 2 are independent in block M_3. In other words, to collapse the criteria $n+1,\ldots,n+m$ the pairwise comparisons among these criteria can be ignored.

However, the arbitrariness of the scalar α appearing in Theorem 2 leads us to impose another condition. Let us motivate it with the following example. Let $M_1 \in \mathscr{M}_n$ be skew-symmetric and $\mathbf{v} \in \mathbb{R}^n$. Set $M = N = \left[\begin{smallmatrix} M_1 & -\mathbf{v} \\ \mathbf{v}^T & 0 \end{smallmatrix}\right]$. By using Theorem 2 (observe that $m = 1$) we obtain that the presence of α is awkward. If we look at (C.3), we can think on **y** as a 'mixture' of Y. In fact, we shall impose that **y** is the arithmetic mean of Y.

Theorem 3. *Let $M \in \mathscr{M}_{n+m}$ and $N \in \mathscr{M}_{n+1}$ be skew-symmetric matrices decomposed as in (C.1) and (C.2), respectively. Let $p_{n+m}(M)$ and $p_{n+1}(N)$ be decomposed as in (C.3) with $X = X'$. If*

$$\frac{1}{m}Y\mathbf{1}_m = \mathbf{y},$$

then the scalar α appearing in Theorem 2 is

$$\alpha = S\frac{1-m}{(n+m)m}, \tag{C.9}$$

where S is the sum of all the components of M_2.

Proof. We shall use the notation of Theorem 2 and its proof. By (C.5) and (C.6),

$$\begin{aligned}-Y\mathbf{1}_m &= \left[\text{Block (1,2) of } p_{n+m}(M)\right]\mathbf{1}_m\\ &= \frac{1}{n+m}\left[M_1U_{n,m} - M_2U_m - (M_2^TU_n + M_3U_{m,n})^T\right]\mathbf{1}_m\\ &= \frac{1}{n+m}\left[M_1U_{n,m}\mathbf{1}_m - M_2U_m\mathbf{1}_m - U_nM_2\mathbf{1}_m - U_{n,m}M_3^T\mathbf{1}_m\right].\end{aligned}$$

Let us observe that $U_{n,m}\mathbf{1}_m = m\mathbf{1}_n$, $U_m\mathbf{1}_m = m\mathbf{1}_m$ and $U_{n,m}M_3^T\mathbf{1}_m = 0$ (because M_3 is skew-symmetric and using Lemma 1). Therefore,

$$\begin{aligned}-\frac{1}{m}Y\mathbf{1}_m &= \frac{1}{n+m}\left[M_1\mathbf{1}_n - M_2\mathbf{1}_m - \frac{1}{m}U_nM_2\mathbf{1}_m\right]\\ &= \frac{1}{n+m}[M_1\mathbf{1}_n - M_2\mathbf{1}_m - U_nM_2\mathbf{1}_m] + \frac{1}{n+m}U_nM_2\mathbf{1}_m\\ &\quad -\alpha\mathbf{1}_n + \alpha\mathbf{1}_n - \frac{1}{m(n+m)}U_nM_2\mathbf{1}_m\\ &= -\mathbf{y} + \frac{1}{n+m}\left(1-\frac{1}{m}\right)U_nM_2\mathbf{1}_m + \alpha\mathbf{1}_n.\end{aligned}$$

But, as one can easily see, $U_n M_2 \mathbf{1}_m = S\mathbf{1}_n$, where S is the sum of all of components of M_2. Therefore,

$$-\frac{1}{m} Y\mathbf{1}_m = -\mathbf{y} + \left(\frac{m-1}{m(n+m)} S + \alpha\right)\mathbf{1}_n.$$

The proof is finished. □

Definition 1. *Let $M \in \mathcal{M}_{n+m}$ be a skew-symmetric matrix decomposed as in (C.1). If $\alpha \in \mathbb{R}$ is given by (C.9), $\mathbf{v} \in \mathbb{R}^n$ is given by (C.4), and $N \in \mathcal{M}_{n+1}$ is given by (C.2), then we say that the* collapse of the last m judgements of M produces N.

Example 1. We close this section with a rather artificial example. Let A be the following reciprocal matrix

$$A = \left[\begin{array}{cc|cc} 1 & 3 & 2 & 4 \\ 1/3 & 1 & 2 & 3 \\ \hline 1/2 & 1/2 & 1 & 2 \\ 1/4 & 1/3 & 1/2 & 1 \end{array}\right].$$

If we want to collapse the third and fourth judgements, we use Theorems 2 and 3 for $n = m = 2$.

$$M = \mathrm{LOG}(A) = \left[\begin{array}{cc|cc} 0 & 1.0986 & 0.6932 & 1.3863 \\ -1.0986 & 0 & 0.6932 & 1.0986 \\ \hline -0.6932 & -0.6932 & 0 & 0.6932 \\ -1.3863 & -1.0986 & -0.6932 & 0 \end{array}\right] = \left[\begin{array}{cc} M_1 & -M_2 \\ M_2^T & M_3 \end{array}\right].$$

By Theorem 3, we get $\alpha = 0.4839$. By Theorem 2, we get

$$\mathbf{v} = [-1.3774 \ \ -1.1617]^T.$$

Therefore, the collapsed matrix N given by Definition 1 is

$$N = \left[\begin{array}{cc} M_1 & -\mathbf{v} \\ \mathbf{v}^T & 0 \end{array}\right] = \left[\begin{array}{ccc} 0 & 1.0986 & 1.3774 \\ -1.0986 & 0 & 1.1617 \\ -1.3774 & -1.1617 & 0 \end{array}\right].$$

Coming back to the comparison matrices, we obtain that the collapse of the 3rd and 4th judgements produces

$$E(N) = \left[\begin{array}{ccc} 1 & 3 & 3.9647 \\ 1/3 & 1 & 3.1952 \\ 0.25223 & 0.31297 & 1 \end{array}\right],$$

where $E : \mathcal{M}_{n,m} \to \mathcal{M}_{n,m}^{+}$ is the entry-wise exponential mapping (i.e., the (i, j)-entry of $E(X)$ is $\mathrm{e}^{X_{i,j}}$). Observe that LOG and E are inverse mappings of each other.

Now is the time for the expert to decide if he/she agrees with this new comparison matrix.

2. Clustering of entries in consistent matrices

Let us recall that if $A = [a_{ij}] \in \mathcal{M}_n$ is a consistent matrix, then exists $\mathbf{v} = [v_1 \cdots v_n]^T \in \mathbb{R}^n$ such that $a_{ij} = v_i v_j^{-1}$ for all $1 \le i, j \le n$. This vector $\mathbf{v}$ is the priority vector of the matrix A, and it is easily checked that $\mathbf{v}$ is an eigenvector of A associated to the eigenvalue n. This eigenvalue n is the Perron eigenvalue of the positive matrix A.

Before studying how to collapse several judgments in a consistent matrix, let us see a general useful fact: Let $A \in \mathscr{M}_n$ be a consistent matrix. If $\mathbf{z} \in \mathbb{R}^n$ is the priority vector of A, then $\mathrm{LOG}(A) = \phi_n(\mathrm{LOG}(\mathbf{z}))$. In fact: since $a_{ij} = z_i z_j^{-1}$ we have $\log(a_{ij}) = \log(z_i) - \log(z_j)$ for all $1 \le i, j \le n$, and therefore, $\mathrm{LOG}(A) = \phi_n(\mathrm{LOG}(\mathbf{z}))$.

Let $A \in \mathscr{M}^+_{n+m}$ be a consistent matrix and let us partition A as follows:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad A_1 \in \mathscr{M}_n,\ A_4 \in \mathscr{M}_m. \tag{C.10}$$

It is evident that A_1 is consistent (it is the comparison matrix of the $1, \ldots, n$ judgements). Also, A_4 is the comparison matrix of the $n+1, \ldots, n+m$ judgements, which is also consistent. Let $\mathbf{z} \in \mathbb{R}^{n+m}$ be the priority vector of A. Let us decompose $\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}$, where $\mathbf{z}_1 \in \mathbb{R}^n$. Now, one has $\mathrm{LOG}(A) = \phi_{n+m}(\mathrm{LOG}(\mathbf{z}))$. Hence

$$\begin{aligned}
\mathrm{LOG}(A) = \begin{bmatrix} \mathrm{LOG}(A_1) & \mathrm{LOG}(A_2) \\ \mathrm{LOG}(A_3) & \mathrm{LOG}(A_4) \end{bmatrix} &= \phi_{n+m}(\mathrm{LOG}(\mathbf{z})) = \mathrm{LOG}(\mathbf{z})\mathbf{1}_{n+m}^T - \mathbf{1}_{n+m}\mathrm{LOG}(\mathbf{z})^T \\
&= \begin{bmatrix} \mathrm{LOG}(\mathbf{z}_1) \\ \mathrm{LOG}(\mathbf{z}_2) \end{bmatrix} \begin{bmatrix} \mathbf{1}_n^T & \mathbf{1}_m^T \end{bmatrix} - \begin{bmatrix} \mathbf{1}_n \\ \mathbf{1}_m \end{bmatrix} \begin{bmatrix} \mathrm{LOG}(\mathbf{z}_1)^T & \mathrm{LOG}(\mathbf{z}_2)^T \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{LOG}(\mathbf{z}_1)\mathbf{1}_n^T - \mathbf{1}_n \mathrm{LOG}(\mathbf{z}_1)^T & \mathrm{LOG}(\mathbf{z}_1)\mathbf{1}_m^T - \mathbf{1}_n \mathrm{LOG}(\mathbf{z}_2)^T \\ \mathrm{LOG}(\mathbf{z}_2)\mathbf{1}_n^T - \mathbf{1}_m \mathrm{LOG}(\mathbf{z}_1)^T & \mathrm{LOG}(\mathbf{z}_2)\mathbf{1}_m^T - \mathbf{1}_m \mathrm{LOG}(\mathbf{z}_2)^T \end{bmatrix} \qquad (C.11) \\
&= \begin{bmatrix} \phi_n(\mathrm{LOG}(\mathbf{z}_1)) & \mathrm{LOG}(\mathbf{z}_1)\mathbf{1}_m^T - \mathbf{1}_n \mathrm{LOG}(\mathbf{z}_2)^T \\ \mathrm{LOG}(\mathbf{z}_2)\mathbf{1}_n^T - \mathbf{1}_m \mathrm{LOG}(\mathbf{z}_1)^T & \phi_m(\mathrm{LOG}(\mathbf{z}_2)) \end{bmatrix}.
\end{aligned}$$

Therefore, $\mathbf{z}_1$ is the priority vector of A_1 and $\mathbf{z}_2$ is the priority vector of A_4.

Theorem 4. *Let $A \in \mathscr{M}_{n+m}$ be a consistent matrix decomposed as in (C.10) whose priority vector is $[z_1\ \ z_2\ \ \cdots\ \ z_{n+m}]^T$ and $M = \mathrm{LOG}(A)$ be decomposed as in (C.1). Let N be produced by the collapse of the last m judgements of M, and finally, let us denote $\mathbf{w}_1 = [\log z_1\ \ \cdots\ \ \log z_n]^T$ and $s_2 = \log z_{n+1} + \cdots + \log z_{n+m}$. Then*

(i) $N = \begin{bmatrix} M_1 & -\mathbf{v} \\ \mathbf{v}^T & 0 \end{bmatrix}$, *where* $\mathbf{v} = -\mathbf{w}_1 + \dfrac{s_2}{m}\mathbf{1}_n$.

(ii) $p_{n+1}(N) = N$. *We obtain that $N \in \mathscr{L}_{n+1}$, or equivalently, $E(N)$ is a consistent matrix.*

(iii) *The priority vector for $E(N)$ is* $[z_1\ \ \cdots\ \ z_n\ \ \sqrt[m]{z_{n+1}\cdots z_{n+m}}]^T$.

Proof. Let $\mathbf{z}$ be the priority vector of A decomposed as in the previous paragraph. We shall denote $\mathbf{w}_1 = \mathrm{LOG}(\mathbf{z}_1)$ and $\mathbf{w}_2 = \mathrm{LOG}(\mathbf{z}_2)$. We also denote $s_1 = \mathbf{w}_1^T\mathbf{1}_n$ and $s_2 = \mathbf{w}_2^T\mathbf{1}_m$ (observe that s_i is the sum of the components of $\mathbf{w}_i$ for $i = 1, 2$).

Theorem 3 and Theorem 2 are applied to obtain vector $\mathbf{v}$. From decompositions (C.1) and (C.11) we obtain

$$M_1 = \mathbf{w}_1\mathbf{1}_n^T - \mathbf{1}_n\mathbf{w}_1^T \qquad \text{and} \qquad -M_2 = \mathbf{w}_1\mathbf{1}_m^T - \mathbf{1}_n\mathbf{w}_2^T.$$

Now, observe that $\mathbf{1}_n^T\mathbf{1}_n = n$ and $\mathbf{w}_1^T\mathbf{1}_n$ is an scalar (which commutes with any matrix). So,

$$M_1\mathbf{1}_n = \left(\mathbf{w}_1\mathbf{1}_n^T - \mathbf{1}_n\mathbf{w}_1^T\right)\mathbf{1}_n = \mathbf{w}_1\mathbf{1}_n^T\mathbf{1}_n - \mathbf{1}_n\mathbf{w}_1^T\mathbf{1}_n = n\mathbf{w}_1 - s_1\mathbf{1}_n$$

and, analogously,

$$-M_2\mathbf{1}_m=\left(\mathbf{w}_1\mathbf{1}_m^T-\mathbf{1}_n\mathbf{w}_2^T\right)\mathbf{1}_m=\mathbf{w}_1\mathbf{1}_m^T\mathbf{1}_m-\mathbf{1}_n\mathbf{w}_2^T\mathbf{1}_m=m\mathbf{w}_1-s_2\mathbf{1}_n.$$

Firstly, we obtain the value of α in Theorem 3. It is easy to see that the sum of the entries of M_2 is

$$S=\mathbf{1}_n^T M_2\mathbf{1}_m=\mathbf{1}_n^T(-m\mathbf{w}_1+s_2\mathbf{1}_n)=-ms_1+ns_2.$$

Therefore,

$$\alpha=(ms_1-ns_2)\frac{m-1}{m(n+m)}.$$

Now, we find an expression for vector $\mathbf{v}$ appearing in Theorem 2:

$$\begin{aligned}\mathbf{v}&=\frac{n+1}{n+m}M_2\mathbf{1}_m+\frac{m-1}{n+m}M_1\mathbf{1}_n+\alpha\mathbf{1}_n\\&=\frac{n+1}{n+m}(-m\mathbf{w}_1+s_2\mathbf{1}_n)+\frac{m-1}{n+m}(n\mathbf{w}_1-s_1\mathbf{1}_n)+(ms_1-ns_2)\frac{m-1}{m(n+m)}\mathbf{1}_n\\&=-\mathbf{w}_1+\frac{1}{n+m}\left[s_2(n+1)-s_1(m-1)+(ms_1-ns_2)\frac{m-1}{m}\right]\mathbf{1}_n\\&=-\mathbf{w}_1+\frac{s_2}{m}\mathbf{1}_n.\end{aligned}$$

Item (i) is proven. To prove item (ii), we seek for a simplified expression of the vector $\mathbf{y}$ appearing in Theorem 2. Before doing this, we simplify $U_nM_2\mathbf{1}_m$:

$$\begin{aligned}-U_nM_2\mathbf{1}_m&=U_n(m\mathbf{w}_1-s_2\mathbf{1}_n)\\&=m\mathbf{1}_n\mathbf{1}_n^T\mathbf{w}_1-s_2\mathbf{1}_n\mathbf{1}_n^T\mathbf{1}_n=m\mathbf{1}_ns_1-s_2\mathbf{1}_nn=(ms_1-ns_2)\mathbf{1}_n.\end{aligned}$$

Now,

$$\begin{aligned}-\mathbf{y}&=\frac{1}{n+m}[M_1\mathbf{1}_n-(I_n+U_n)M_2\mathbf{1}_m]-\alpha\mathbf{1}_n\\&=\frac{1}{n+m}[n\mathbf{w}_1-s_1\mathbf{1}_n+m\mathbf{w}_1-s_2\mathbf{1}_n+(ms_1-ns_2)\mathbf{1}_n]-\alpha\mathbf{1}_n\\&=\mathbf{w}_1+\left(\frac{ms_1-ns_2}{n+m}-\frac{s_1+s_2}{n+m}-\alpha\right)\mathbf{1}_n\\&=\mathbf{w}_1+\left(\frac{ms_1-ns_2}{n+m}-\frac{s_1+s_2}{n+m}-(ms_1-ns_2)\frac{m-1}{m(n+m)}\right)\mathbf{1}_n\\&=\mathbf{w}_1+\frac{ms_1-ns_2-m(s_1+s_2)}{m(n+m)}\mathbf{1}_n\\&=\mathbf{w}_1-\frac{s_2}{m}\mathbf{1}_n.\end{aligned}$$

Observe that since A is consistent, we have $M=\mathrm{LOG}(A)\in\mathscr{L}_{n+m}$. Therefore, $p_{n+m}(M)=M$. From (C.1) and (C.3) we have $M_1=X$. From Theorem 2 we have $X=X'$. From the above computations we obtain $\mathbf{v}=\mathbf{y}$. Hence, the expressions for N and $p_{n+1}(N)$ given in (C.2) and (C.3) prove item (ii).

To prove item (iii), we recall that the priority vector of a consistent matrix is just a scalar multiple of any of its columns, and therefore, the priority vector for $E(N)$ can be regarded as a multiple of its last column, which is

$$\begin{bmatrix}E(-\mathbf{v})\\1\end{bmatrix}=\begin{bmatrix}E\left(\mathbf{w}_1-\frac{s_2}{m}\mathbf{1}_n\right)\\1\end{bmatrix}.$$

Observe that the ith component of $\mathbf{w}_1 - \frac{s_2}{m}\mathbf{1}_m$ is given by

$$\log z_i - \frac{\log z_{n+1} + \cdots + \log z_{n+m}}{m} = \log\left(\frac{z_i}{\sqrt[m]{z_{n+1}\cdots z_{n+m}}}\right).$$

Thus, if we denote $K = \sqrt[m]{z_{n+1}\cdots z_{n+m}}$, then the last column of $E(N)$ is $[z_1/K \;\; \cdots \;\; z_n/K \;\; 1]^T$. The proof of the third item is finished. □

Note that the clustered matrix, N, and its priority vector are obtained by using the formulas in (i) and (iii), respectively. Both formulas are really straightforward since they involve exclusively simple (linear) vector operations to build the last column (row) of N, and replacing the last m components of the priority vector by the m-th root of all of them.

According to this theorem there are no limitations regarding the size, $n+m$, of the matrix, nor with respect to the number, m, of items to be collapsed. Moreover, if the initial PCM has acceptable consistency, then this theorem guarantees consistency for the clustered structure; while if the initial PCM do not exhibit acceptable consistency, then it would be absurd to use this theorem to derive a clustered structure and claim consistency. Specifically, if $A \in \mathscr{M}_{n+m}$ is a consistent matrix, then item (ii) of Theorem 4 implies that, by collapsing the last m judgements according to Definition 1, thus obtaining a skew-symmetric matrix $N \in \mathscr{M}_{n+1}$, matrix $E(N)$ is consistent. In other words, consistency is preserved by collapsing judgements. Furthermore, since all the involved operations are continuous, if matrix A is close to being consistent (e.g., its consistency is acceptable according to Saaty's criterion), then one can apply Theorem 4 (approximately) to obtain the collapsed matrix and its priority vector.

REFERENCES AND SCIENTIFIC PRODUCTION

REFERENCES

Abedi, M., Torabi, S.A., Norouzi, G.H., Hamzeh, M. (2012). ELECTRE III: A knowledge-driven method for integration of geophysical data with geological and geochemical data in mineral prospectivity mapping. Journal of Applied Geophysics, 87, 9–18.

Ahmed, Q., Khan, F., Ahmed, S. (2014). Improving safety and availability of complex systems using a risk-based failure assessment approach. Journal of Loss Prevention in the Process Industries, 32, 218–229.

Aiello, G., Enea, M., Galante, G., La Scalia, G. (2009). Clean Agent Selection Approached by Fuzzy TOPSIS Decision-Making Method. Fire Technology, 45, 405–418.

Akhavein, A., Fotuhi Firuzabad, M. (2011). A heuristic-based approach for reliability importance assessment of energy producers. Energy Policy, 39 (3), 1562–1568.

Alebrant Mendes, A., Coit, D.W., Ribeiro, J.L.D. (2014). Establishment of the optimal time interval between periodic inspections for redundant systems. Reliability Engineering & System Safety, 131, 148–165.

Alirio Valencia, E., Poveda, Y.A., Escudero Salcedo, C.A. (2012). Desigualdad de Chebyshev bidimensional. Scientia Et Technica, Universidad Tecnológica de Pereira, Pereira, Colombia, 51, 242–246.

Alrabghi, A., Tiwari, A. (2016). A novel approach for modelling complex maintenance systems using discrete event simulation. Reliability Engineering & System Safety, 154, 160–170.

Alzghoul, A., Löfstrand, M. (2011). Increasing availability of industrial systems through data stream mining. Computers & Industrial Engineering, 60 (2), 195–205.

Antonovsky, A., Pollock, C., Straker, L. (2016). System reliability as perceived by maintenance personnel on petroleum production facilities. Reliability Engineering and System Safety, 152, 58–65.

Arulmozhi, G. (2002). Exact equation and an algorithm for reliability evaluation of K-out-of-N:G system. Reliability Engineering and System Safety, 78, 87–91.

Aşchilean, I., Badea, G., Giurca, I., Naghiu, G.S., Iloaie, F.G. (2017). Choosing the Optimal Technology to Rehabilitate the Pipes in Water Distribution Systems Using the AHP Method. Energy Procedia, 112, 19–26.

Atzori, L., Iera, A., Morabito, G. (2010). The internet of things: A survey. Computer Networks, 54 (15), 2787–2805.

Aven, T. (2016a). Supplementing quantitative risk assessments with a stage addressing the risk understanding of the decision maker. Reliability Engineering and System Safety, 152, 51–57.

Aven T. (2016b). Risk assessment and risk management: Review of recent advances on their foundation. European Journal of Operational Research, 253 (1), 1–13.

Ayala-Cabrera, D., Herrera, M., Izquierdo, J., Pérez-García, R. (2011). Location of buried plastic pipes using-multi-agent support base don GPR images. Journal of Applied Geophysics, 75, 679–686.

Ayala-Cabrera, D., Ocaña-Levario, S.J., Pérez-García, R., Izquierdo, J., Herrera, M. (2013a). Visualización de fugas de agua en sistemas de abastecimiento de agua mediante imágenes de GPR. Proceedings of XII Simposio Iberoamericano sobre Abastecimiento de Agua y Drenaje Urbano (SEREA), Buenos Aires, Argentina.

Ayala-Cabrera, D., Izquierdo, J., Montalvo, I., Pérez-García, R. (2013b). Water supply system component evaluation from GPR radargrams using a multi-agent approach. Mathematical and Computer Modeling, 57 (7) 1927–1932.

Ayala-Cabrera, D., Izquierdo, J., Ocaña-Levario, S.J., Pérez-García, R. (2014). 3D model construction of water supply system pipes based on GPR images. Proceedings of the 7th International Congress on Environmental Modeling and Software (iEMSs), CA, USA.

Azadeh, A., Zarrin, M. (2016). An intelligent framework for productivity assessment and analysis of human resource from resilience engineering, motivational factors, HSE and ergonomics perspectives. Safety Science, 89, 55–71.

Azadeh, A., Zarrin, M., Abdollahi, M., Noury, S., Farahmand, S. (2015). Leanness assessment and optimization by fuzzy cognitive map and multivariate analysis," Expert Systems with Applications, 42 (15-16), 6050–6064.

Aznar, J., Guijarro, J. (2008). Nuevos métodos de valoración, Modelos Multicriterio. Universidad Politécnica de Valencia, Spain.

Ba, H.T., Cholette, M.E., Borghesani, P., Zhou, Y., Ma, L. (2017). Opportunistic maintenance considering non-homogenous opportunity arrivals and stochastic opportunity durations. Reliability Engineering & System Safety, 160, 151–161.

Backman, J., Yrjola, S., Valtanen, K., Mämmelä, O. (2017). Blockchain network slice broker in 5G: Slice leasing in factory of the future use case. In Proceedings of Internet of things Business Models, Users, and Networks Conference.

Baddeley, A. (2007). Working Memory, Thought, and Action. Oxford University Press, Oxford.

Bapat, R.B. (2011). Graphs and Matrices. Springer Universitext, New Delhi, India.

Bhatia, R., Davis, C. (2000). A Better Bound on the Variance. The American Mathematical Monthly, 107 (4), 353–357.

Bauer, M., Lucke, M., Johnsson, C., Harjunkoski, I., Schlake, J.C. (2016). KPIs as the interface between scheduling and control. IFAC-PapersOnLine, 49 (7), Pages 687–692.

Bean, M. (2012). Probability: The Science of Uncertainty with Applications to Investments, Insurance, and Engineering. Orient Blackswan.

Bell, J., Holroyd, J. (2009). Review of human reliability assessment methods. Health and Safety Executive.

Beniger, J.R. (1989). The Control Revolution. Harvard University Press.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R. (2011a). Achieving matrix consistency in AHP through linearization. Applied Mathematical Modelling, 35, 4449–4457.

Benítez, J., Delgado-Galván, X., Gutiérrez, J.A., Izquierdo, J. (2011b). Balancing Consistency and Expert Judgment in AHP. Mathematical and Computer Modeling, 54, 1785–1790.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R. (2012a). Improving consistency in AHP decision-making processes. Applied Mathematics and Computation, 219, 2432–2441.stewart

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R. (2012b). An approach to AHP decision in a dynamic context. Decision Support Systems, 53, 499–506.

Benítez, J., Izquierdo, J., Pérez-García, R., Ramos-Martínez, E. (2013). A simple formula to find the closest consistent matrix to a reciprocal matrix. Applied Mathematical Modelling, 38 (15–16), 3968–3974.

Benítez, J., Izquierdo, J., Pérez-García, R., Ramos-Martínez, E. (2014a). A simple formula to find the closest consistent matrix to a reciprocal matrix. Applied Mathematical Modelling, 38, 3968–3974.

Benítez, J., Carrión, L., Izquierdo, J., Pérez-García, R. (2014b). Characterization of consistent completion of reciprocal comparison matrices. Abstract and Applied Analysis, Article ID 349729, 12 pages.

Benítez, J., Delgado-Galván, X., Izquierdo, J., Pérez-García, R. (2015). Consistent completion of incomplete judgments in decision making using AHP. Journal of Computational and Applied Mathematics, 290, 412–422.

Benítez, J., Carpitella, S., Certa, A., Ilaya-Ayza, A.E., Izquierdo, J. (2017). Some awkward issues on pairwise comparison matrices. Proceedings of the 19th International Conference on Mathematical Modeling in Engineering & Human Behaviour, Valencia, Spain, July 17–19, pages 43–48.

Benítez, J., Carpitella, S., Certa, A., Izquierdo, J. (2018a). Characterisation of the consistent completion of AHP comparison matrices using graph theory. Journal of Multi-Criteria Decision Analysis, https://doi.org/10.1002/mcda.1652.

Benítez J., Carpitella S., Certa A., Ilaya-Ayza A.E., Izquierdo J. (2018b). Consistent clustering of entries in large pairwise comparison matrices. Journal of Computational and Applied Mathematics, 343, 98–112.

Benítez, J., Carpitella, S., Certa, A., Izquierdo, J. (2018, under review). Management of uncertain pairwise comparisons in AHP through probabilistic concepts. Submitted to Applied Soft Computing on March 28th, 2018 and currently under review.

Berner, C., Flage, R. (2016). Strengthening quantitative risk assessments by systematic treatment of uncertain assumptions. Reliability Engineering & System Safety, 151, 46–59.

Berrittella, M., Certa, A., Enea, M., Zito, P. (2008). Transport policy and climate change: How to decide when experts disagree. Environmental Science & Policy, 11 (4), 307–314.

Bertolini, M., Bevilacqua, M., Massini, R. (2006). FMECA approach to product traceability in the food industry. Food control, 17 (2), 137–145.

Bevilacqua, M., Braglia, M., Gabbrielli, R. (2000). Monte Carlo simulation approach for a modified FMECA in a power plant. Quality and Reliability Engineering International, 16 (14), 313–324.

Bevilacqua, M., Ciarapica, F.E. (2018). Human factor risk management in the process industry: A case study. Reliability Engineering and System Safety, 169, 149–159.

Beynon, M.J., Curry, B., Morgan, P. (2000). The Dempster–Shafer theory of evidence: An alternative approach to multicriteria decision modelling. Omega, 28 (1), 37–50.

Biedermann, A., Bozza, S., Taroni, F., Aitken. C. (2017). The consequences of understanding expert probability reporting as a decision. Science & Justice, 57 (1), 80–85.

Billinton, R., Allan, R.N. (1992). Reliability evaluation of engineering systems: concepts and techniques. New York, second ed., Plenum.

Bimpas, M., Amditis, A., Uzunoglu, N. (2010). Detection of water leaks in supply pipes using continuous wave sensor operating at 2.45 GHz. Journal of Applied Geophysics, 70, 226–236.

Blagojevic, B., Srdjevic, B., Srdjevic, Z., Zoranovic, T. (2016). Heuristic aggregation of individual judgments in AHP group decision making using simulated annealing algorithm. Information Sciences, 330, 260–273.

Boatca, M.E., Cirjaliu, B. (2015). A Proposed Approach for an Efficient Ergonomics Intervention in Organizations. Procedia Economics and Finance, 23, 54–62.

Bosse, S., Splieth, M., Turowski, K. (2016). Multi-objective optimization of IT service availability and costs. Reliability Engineering and System Safety, 147, 142–155.

Bozóki, S., Fülöp, J., Rónyai, L. (2010). On optimal completions of incomplete pairwise comparison matrices. Mathematical and Computer Modelling, 52, 318–333.

Bozóki, S., Csató, L., Temesi, J. (2016). An application of incomplete pairwise comparison matrices for ranking top tennis players. European Journal of Operational Research, 248 (1), 211–218.

Braglia, M. (2000). MAFMA: multi-attribute failure mode analysis. International Journal of Quality and Reliability Management, 17, 1017–1033.

Braglia, M., Bevilacqua, M. (2000). Fuzzy modeling and analytical hierarchy processing as a means of quantifying risk levels associated with failure modes in production systems. Technology, Law and Insurance, 5, 125–134.

Braglia, M., Frosolini, M., Montanari, R. (2003). Fuzzy TOPSIS Approach for Failure Mode, Effects and Criticality Analysis. Quality and Reliability Engineering International, 19, 425–443.

Brentan, B.M., Luvizotto, E. (2014). Refining PSO applied to electric energy cost reduction in water pumping. Water Resources Management, 4, 19–30.

Bulut, E., Duru, O., Keçeci, T., Yoshida, S. (2012). Use of consistency index, expert prioritization and direct numerical inputs for generic fuzzy-AHP modeling: A process model for shipping asset management. Expert Systems with Applications, 39 (2), 1911–1923.

Büyüközkan, G., Çifçi, G., Güleryüz, S. (2011). Strategic analysis of healthcare service quality using fuzzy AHP methodology. Expert Systems with Applications, 38 (8) 9407–9424.

Büyüközkan, G., Çifçi, G. (2012). A combined fuzzy AHP and fuzzy TOPSIS based strategic analysis of electronic service quality in healthcare industry. Expert Systems with Applications, 39 (3), 2341–2354.

Büyüközkan, G., Güleryüz, S. (2016). An integrated DEMATEL-ANP approach for renewable energy resources selection in Turkey. International Journal of Production Economics, 182, 435–448.

Büyüközkan, G., Güleryüz, S. (2016). An integrated DEMATEL-ANP approach for renewable energy resources selection in Turkey. International Journal of Production Economics, 182, 435–448.

Carmignani, G. (2009). An integrated structural framework to cost-based FMECA: The priority-cost FMECA. Reliability Engineering & System Safety, 94 (4), 861–871.

Carpitella, S., Certa, A., Galante, G., Izquierdo, J., La Fata, C.M. (2016). The FTOPSIS method to support FMECA analyses. Proceedings of 22th ISSAT International Conference on Reliability and Quality in Design, Los Angeles, California, USA.

Carpitella, S., Carpitella, F., Benítez, J., Certa, A., Izquierdo, J. (2017a). Prioritization of maintenance actions in water distribution systems. Proceedings of the Congreso de Métodos Numéricos en Ingeniería CMN, Valencia, Spain, July 3-5.

Carpitella, S., Ocaña-Levario, S.J., Benítez, J., Certa, A., Enea, M., Izquierdo, J. (2017b). The ELECTRE I method to optimise GPR data processing analyses for maintenance of water networks. Proceedings of the 23th ISSAT International Conference on Reliability and Quality in Design, Chicago, Illinois, USA, August 4–6, pages 6–10.

Carpitella, S., Certa, A., Enea, M., Galante, G.M., Izquierdo, J., La Fata, C.M. (2017c). Human Reliability Analysis to support the development of a software project. Proceedings of the 23th ISSAT International Conference on Reliability and Quality in Design, Chicago, Illinois, USA, August 4–6, pages 214–218.

Carpitella, S., Ocaña-Levario, S.J., Benítez, J., Certa, A., Izquierdo, J. (2018a). A hybrid multi-criteria approach to GPR image mining applied to water supply system maintenance. Journal of Applied Geophysics, 159, 754–764.

Carpitella, S., Carpitella, F., Certa, A., Benítez, J., Izquierdo, J. (2018b). Managing human factors to reduce organisational risk in industry. Mathematical and Computational Applications (, 23 (4), 67, DOI: 10.3390/mca23040067.

Carpitella, S., Certa, A., Izquierdo, J., La Fata, C.M. (2018c). A combined multi-criteria approach to support FMECA analyses: a real-world case. Reliability Engineering & System Safety, 169, 394–402.

Carpitella, S., Certa, A., Izquierdo, J., La Fata, CM. (2018d). k-out-of-n systems: an exact formula for stationary availability and multi-objective configuration design based on mathematical programming and TOPSIS. Journal of Computational and Applied Mathematics, 330, 1007–1015.

Carpitella, S., Certa, A., Izquierdo, J. (2018e). DEMATEL-based consensual selection of suitable maintenance KPIs. Proceedings of the 24th ISSAT International Conference on Reliability and Quality in Design, Toronto, Canada, August 2-6.

Carpitella, S., Certa, A., Benítez, J., Enea, M., Izquierdo, J. (2018, under review). Blockchain technology-supported preventive maintenance using a multi-criteria decision support system. Submitted to Computers & Industrial Engineering on July 15th, 2018 and currently under review.

Catelani, M., Ciani, L., Cristaldi, L., Faifer, M., Lazzaroni, M. (2013). Electrical performances optimization of Photovoltaic Modules with FMECA approach. Measurement, 46 (10), 3898–3909.

Certa, A., Enea, M., Galante, G., La Fata, C.M. (2009). Multi-objective human resources allocation in R&D projects planning. International Journal of Production Research, 47 (13) 3503–3523.

Certa, A., Enea, M., Galante, G., Lupo, T. (2012). A Multi-Objective Approach to Optimize a Periodic Maintenance Policy, International Journal of Reliability. Quality and Safety Engineering, 19 (6), 1240002.

Certa, A., Enea, M., Lupo, T. (2013a). ELECTRE III to dynamically support the decision maker about the periodic replacements configurations for a multi-component system. Decision Support Systems, 55 (1), 126–134.

Certa, A., Enea M., Galante G., Lupo T. (2013b). A multi-decision makers approach to select the maintenance plan for a multi-component system. Proceeding of 19th ISSAT International Conference on Reliability and Quality in Design, Honolulu, Hawaii, USA.

Certa, A., Enea, M., Galante, G.M., La Fata, C.M. (2013c). A Multistep Methodology for the Evaluation of Human Resources Using the Evidence Theory. International Journal of Intelligent Systems, 28 (11), 1072–1088.

Certa, A., Enea, M., Hopps, F. (2015). A multi-criteria approach for the group assessment of an academic course: A case study. Studies in Educational Evaluation, 44, 16–22.

Certa, A., Hopps, F., Inghilleri, R., and La Fata, C.M. (2017). A Dempster Shafer Theory-based approach to the Failure Mode, Effects and Criticality Analysis (FMECA) under epistemic uncertainty: application to the propulsion system of a fishing vessel. Reliability Engineering and System Safety, 159, 69–79.

Chalabi, N., Dahane, M., Beldjilali, B., Neki, A. (2016). Optimisation of preventive maintenance grouping strategy for multi-component series systems: Particle swarm based approach. Computers & Industrial Engineering, 102, 440–451.

Chambari, A., Rahmati, S.H.A., Najafi, A.A., Karimi, A. (2012). A bi-objective model to optimize reliability and cost of system with a choice of redundancy strategies. Computers & Industrial Engineering, 63 (1), 109–119.

Chan, F.T.S., Kumar, N., Tiwari, M.K., Lau, H.C.W., Choy, K.L. (2008). Global supplier selection: a fuzzy-AHP approach. International Journal of Production Research, 46 (14), 3825–3857.

Chang, D.-Y. (1996). Application of the extent analysis method on fuzzy AHP. European Journal of Operational Research, 95, 649–655.

Chen, C.T. (2000). Extensions of the TOPSIS for group decision-making under fuzzy environment. Fuzzy Sets and Systems, 114 (1), 1–9.

Chen, C.T., Lin, C.T., Huang, S.F. (2006). A fuzzy approach for supplier evaluation and selection in supply chain management. International Journal of production economics, 102 (2), 289–301.

Chen, S.-T., Wall, A., Davies, P., Yang, Z., Wang, J., Chou, Y.-H. (2013). A Human and Organisational Factors (HOFs) analysis method for marine casualties using HFACS-Maritime Accidents (HFACS-MA). Safety Science, 60, 105–114.

Chen, S.-M., Tsai, W.-H. (2016). Multiple attribute decision making based on novel interval-valued intuitionistic fuzzy geometric averaging operators. Information Sciences, 367–368, 1045–1065.

Cheng, Z., Xiong, Y., Xu, Y. (2016). An opinion diffusion model with decision-making groups: The influence of the opinion's acceptability. Physica A, 461, 429–438.

Chen, Y. (2018). Blockchain tokens and the potential democratization of entrepreneurship and innovation. Business Horizons, 61 (4), 567–575.

Chidambaram, P. (2016). Perspectives on human factors in a shifting operational environment. Journal of Loss Prevention in the Process Industries, 44, 112–118.

Chiu, Y. J., Chen, H. C., Tzeng, G. H., Shyu, J. Z. (2006). Marketing strategy based on customer behavior for the LCD-TV. International Journal of Management and Decision Making, 7(2/3), 143–165.

Choe, P., Tew, J.D., Tong, S. (2015). Effect of cognitive automation in a material handling system on manufacturing flexibility. International Journal of Production Economics, 170 Part C, 891–899.

Choi, I.W., Chang, D. (2016). Reliability and availability assessment of seabed storage tanks using fault tree analysis. Ocean Engineering, 120, 1–14.

Cid-López, A., Hornos, M.J., Carrasco, R.A., Herrera-Viedma, E. (2016). Applying a linguistic multi-criteria decision-making model to the analysis of ICT suppliers' offers. Expert Systems with Applications, 57, 127–138.

Cirjaliu, B., Draghici, A. (2016). Ergonomic Issues in Lean Manufacturing. Procedia - Social and Behavioral Sciences, 221, 105–110.

Civerchia, F., Bocchino, S., Salvadori, C., Rossi, E., Maggiani, L., Petracca, M. (2017). Industrial Internet of Things monitoring solution for advanced predictive maintenance applications. Journal of Industrial Information Integration, 7, 4–12.

Clerici, P., Guercio, A., Quaranta, L. (2016). Human Management System for Occupational Health and Safety. Tipolitografia INAIL, Milano.

Colledani, M., Magnanini, M.C., Tolio, T. (2018). Impact of opportunistic maintenance on manufacturing system performance. CIRP Annals, 67 (1), 499–502.

Costello, F., Watts, P. (2016). People's conditional probability judgments follow probability theory (plus noise). Cognitive Psychology, 89, 106–133.

Crawford, G., Williams, C. (1985). A note on the analysis of subjective judgment matrices. Journal of Mathematical Psychology, 29 (4), 387–405.

Crocco, L., Soldovieri, F., Millington, T., Cassidy, N.J. (2010). Bistatic Tomographic GPR imaging for incipient pipeline leakage evaluation. Progress in Electromagnetics Research, 101, 307–321.

Curcurù, G., Galante, G., Lombardo, A. (2010). A predictive maintenance policy with imperfect monitoring. Reliability Engineering & System Safety, 95 (9), 989–997.

Curcurù, G., Galante, G.M., La Fata C.M. (2012). A bottom-up procedure to calculate the top event probability in presence of epistemic uncertainty. In: Proceedings of the 11th international probabilistic safety assessment and management conference and the annual European safety and reliability conference, PSAM11 ESREL, Helsinki.

Curcurù, G., Galante G.M., La Fata, C.M. (2013). An imprecise Fault Tree Analysis for the estimation of the Rate of Occurrence of Failure (ROCOF). Journal of Loss Prevention in the Process Industries, 26, 1285–1292.

Curcurù, G., Galante, G.M., La Fata, C.M. (2016). A Dempster-Shafer Theory-Based Approach to Compute the Birnbaum Importance Measure under Epistemic Uncertainty. International Journal of Applied Engineering Research, 11 (21), 10574–10585.

De Benedetti, M., Leonardi, F., Messina, F., Santoro, C., Vasilakos, A. (2018). Anomaly detection and predictive maintenance for photovoltaic systems. Neurocomputing, 310, 59–68.

De Tombe, D.J. (2001). Methodology for handling complex societal problems. European Journal of Operational Research, 128, 227–230.

Deb, K. (2001). Multi-objective optimisation using evolutionary algorithms. John Wiley & Sons, New York.

Deb, K., Pratap, A., Agarwal, S., Meyarivan, T.A.M.T. (2002). A fast and elitist multiobjective genetic algorithm: NSGA-II. IEEE Xplore: IEEE Transactions on Evolutionary Computation, 6,182–197.

Delgado-Galván, X., Pérez-García, R., Izquierdo, J., Mora-Rodríguez, J. (2010). Analytic hierarchy process for assessing externalities in water leakage management. Mathematical and Computer Modelling, 52, 1194–1202.

Delgado-Galván, X., Izquierdo, J., Benítez, J., Pérez-García, R. (2014). Joint Stakeholder Decision-Making on the Management of the Silao-Romita Aquifer Using AHP. Environmental Modelling & Software, 51, 310–322.

Demirci, S., Yigit, E., Eskidemir, I.H., Ozdemir, C. (2012). Ground penetrating radar imaging of water leaks from buried pipes based on back-projection method. NDT&E International, 47, 35–42.

Dittrich, Y. (2016). What does it mean to use a method? Towards a practice theory for software engineering. Information and Software Technology, 70, 220–231.

Dong, L., Carnalla, S., Shinozuka, M. (2011). Experimental and analytical study of detecting leakage of water pipes using ground-penetrating radar. Proceedings of the 9th International workshop on remote sensing for disaster response, Standford, California, USA.

Dong, M., Li, S., Zhang, H. (2015). Approaches to group decision making with incomplete information based on power geometric operators and triangular fuzzy AHP. Expert Systems with Applications 42 (21) 7846–7857.

Dong, L., Mingyue, R., Guoying, M. (2017). Application of Internet of Things Technology on Predictive Maintenance System of Coal Equipment. Procedia Engineering, 174, 885–889.

Drivalou, S., Marmaras, N. (2009). Supporting skill-, rule-, and knowledge-based behaviour through an ecological interface: An industry-scale application. International Journal of Industrial Ergonomics, 39 (6), 947–965.

Durán, O., Aguiló, J. (2006). Selección de máquinas de control numérico usando Fuzzy AHP. Espacios, 27 (1), http://www2.scielo.org.ve/scielo.php?script=sci_arttext&pid=S0798-10152006000100006&lng=es&nrm=iso.

Ee, A.W.L., Shaik, S.M., Khoo, H.H. (2015). Development and application of a Combined Approach for Inherent Safety and Environmental (CAISEN) assessment. Process Safety and Environmental Protection, 96, 138–148.

Ekel, P., Kokshenev, I., Parreiras, R., Pedrycz, W., Pereira Jr., J. (2016). Multiobjective and multiattribute decision making in a fuzzy environment and their power engineering applications. Information Sciences, 361–362, 100–119.

Emovon, I., Norman, R.A., Murphy, A.J., Pazouki, K. (2015). An integrated multicriteria decision making methodology using compromise solution methods for prioritizing risk of marine machinery systems. Ocean Engineering, 105, 92–103.

EN 60812:2006 Standard, Analysis techniques for system reliability – Procedure for failure mode and effects analysis (FMEA).

Ergai, A., Cohen, T., Sharp, J., Wiegmann, D., Gramopadhye, A., Shappell, S. (2016). Assessment of the Human Factors Analysis and Classification System (HFACS): Intra-rater and inter-rater reliability. Safety Science, 82, 393–398.

Ergu, D., Kou, G., Peng, Y., Li, F., Shi, Y. (2012). Data consistency in emergency management. International Journal of Computers, Communications & Control, 7 (3), 450–458.

Ergu, D., Kou, G., Peng, Y., Zhang, M. (2016). Estimating the missing values for the incomplete decision matrix and consistency optimization in emergency management. Applied Mathematical Modelling, 40 (1), 254–267.

European Social Partners (BUSINESS EUROPE, CEEP, ETUC, UEAPME). 2008. Implementation of the European autonomous framework agreement on work-related stress. Brussels (Belgium).

Evans, D.C., Fendley, M. (2017). A multi-measure approach for connecting cognitive workload and automation. International Journal of Human-Computer Studies, 97, 182–189.

Exner, K., Lindow, K., Buchholz, C., Stark, R. (2014). Validation of Product-Service Systems – A Prototyping Approach. Procedia CIRP, 16, 68–73.

Falatoonitoosi, E., Leman, Z., Sorooshian, S., Salimi, M. (2013). Decision-Making Trial and Evaluation Laboratory. Research Journal of Applied Sciences, Engineering and Technology, 5 (13), 3476–3480.

Farley, M., Trow, S. (2003). Losses in Water Distribution Networks. IWA Publishing, London.

Farmani, R., Ingeduld, P., Savic, D., Walters, G., Svitak, Z., Berka, J. (2007). Real-time modelling of a major water supply system. Proceedings of the Institution of Civil Engineers-Water Management, 160, 103–108.

Fangucci, A., Galante, G.M., Inghilleri, R., La Fata, C.M. (2017). Structured methodology for selection of maintenance key performance indicators: Application to an oil refinery plant. International Journal of Operations and Quantitative Management, 23 (2), 89–113.

Figueira, J., Mousseau, V., Roy, B. (2005). Electre Methods. In Greco, Salvatore (Ed.), Multiple criteria decision analysis: state of the art surveys (pages 133–153), International Series in Operations Research & Management Science, New York, Springer.

Figueira, J.R., Greco, S., Roy, B., Slowinski, R. (2013). An overview of ELECTRE methods and their recent extensions. Journal of Multi-Criteria Decision Analysis, 20, 61–85.

Finan, J.S., Hurley, W.J. (1997). The analytic hierarchy process: does adjusting a pairwise comparison matrix to improve the consistency ratio help? Computers and Operations Research, 24, 749–755.

Floricel, S., Michela, J.L., Piperca, S. (2016). Complexity, uncertainty-reduction strategies, and project performance. International Journal of Project Management, 34, 1360–1383.

Fontela, E., Gabus, A. (1974). DEMATEL, innovative methods, Technical report no. 2, Structural analysis of the world problematique. Battelle Geneva Research Institute.

Fontela, E., Gabus, A. (1976). The DEMATEL Observe. Battelle Institute, Geneva Research Center.

Forman, E., Peniwati, K. (1998). Aggregating Individual Judgments and Priorities with the Analytic Hierarchy Process. European Journal of Operation Research, 108, 165–169.

Forsthoffer, M.S. (2017). Chapter 11: Predictive and Preventive Maintenance. Forsthoffer's More Best Practices for Rotating Equipment. 501–546.

Forte, E., Pipan, M. (2017). Review of multi-offset GPR applications: Data acquisition, processing and analysis. Signal Processing, 132, 210–220.

Francese, M., Galante, G.M., La Fata, C.M., Passannanti, G. (2014). Handling epistemic uncertainty in the Fault Tree Analysis using interval-valued expert information, Safety and reliability: methodology and applications. In: Proceedings of the European safety and reliability conference-ESREL, Wroclaw, 1683–1690.

Franek, J., Kresta, A. (2014). Judgment Scales and Consistency Measure in AHP. Procedia Economics and Finance, 12, 164–173.

Garg, H., Sharma, S.P. (2013). Multi-objective reliability-redundancy allocation problem using particle swarm optimization. Computers & Industrial Engineering, 64 (1), 247–255.

Gholami, P.S., Nassiri, P., Yarahmadi, R., Hamidi, A., Mirkazemi, R. (2015). Assessment of Health Safety and Environment Management System function in contracting companies of one of the petro-chemistry industries in Iran, a case study. Safety Science, 77, 42–47.

Giang, P.H. (2015). Decision making under uncertainty comprising complete ignorance and probability. International Journal of Approximate Reasoning, 62, 27–45.

Gölcük, İ., Baykasoğlu, A. (2016). An analysis of DEMATEL approaches for criteria interaction handling within ANP. Expert Systems with Applications, 46, 346–366.

Gonzalez, E., Nanos, E.M., Seyr, H., Valldecabres, L., Yürüşen, N.Y., Smolka, U., Muskulus, M., Melero, J.J. (2017). Key Performance Indicators for Wind Farm Operation and Maintenance. Energy Procedia, 137, 559–570.

Gopalakrishnan, M., Bokrantz, J., Ylipää, T., Skoogh, A. (2015). Planning of Maintenance Activities – A Current State Mapping in Industry. Procedia CIRP, 30, 480–485.

Government Office for Science (2015). Distributed Ledger Technology: beyond block chain. A report by the UK Government Chief Scientific Adviser. www.gov.uk/go-science.

Gubbi, J., Buyya, R., Marusic, S., Palaniswami, M. (2013). Internet of Things (IoT): A vision, architectural elements, and future directions. Future Generation Computer Systems, 29 (7), 1645–1660.

Gopalakrishnan, M., Bokrantz, J., Ylipää, T., Skoogh, A. (2015). Planning of Maintenance Activities – A Current State Mapping in Industry. Procedia CIRP, 30, 480–485.

Govindan, K., Jepsen, M.B. (2016). ELECTRE: A comprehensive literature review on methodologies and applications. European Journal of Operational Research, 250 (1), 1–29.

Grassi, A., Gamberini, R., Mora, C., Rimini, B. (2009). A fuzzy multi-attribute model for risk evaluation in workplaces. Safety Science, 47 (5), 707–716.

Guitouni, A., Martel, J.M. (1998). Tentative Guidelines to Help Choosing the Appropriate MCDA Method. European Journal of Operation Research, 109, 501–521.

Guo, W., Jin, J., Hu, S.J. (2013). Allocation of maintenance resources in mixed model assembly systems. Journal of Manufacturing Systems, 32, 473–479.

Gupta, M., Mohanty, B.K. (2016). An algorithmic approach to group decision making problems under fuzzy and dynamic environment. Expert Systems with Applications, 55, 118–132.

Gurbuz, A.C., McClellan, J.H., Scott, W.R. (2012). Compressive sensing of underground structures using GPR. Digital Signal Processing, 22 (1), 66–73.

Häggström, O. (2002). Finite Markov Chains and Algorithmic Applications. Cambridge University Press.

Hao, T., Rogers, C.D.F., Metje, N., Chapman, D.N., Muggleton, J.M., Foo, K.Y., Wang, P., Pennock, S.R., Atkins, P.R., Swingler S.G., Parker, J., Costello, S.B., Burrow, M.P.N., Anspach, J.H., Armitage, R.J, Cohn, A.G., Goddard, K., Lewin, P.L., Orlando, G., Redfern, M.A., Royal, A.C.D., Saul, A.J. (2012). Condition assessment of the buried utility service infrastructure. Tunnelling and Underground Space Technology, 28, 331–344.

Hassall, M., Xiao, T., Sanderson, P., Neal, A. (2015). Human Factors and Ergonomics. International Encyclopaedia of the Social & Behavioral Sciences (Second Edition), 297–305.

Harker, P.T. (1987). Alternative modes of questioning in the analytic hierarchy process. Mathematical Modelling, 9, 353–360.

Hatami-Marbini, A., Tavana, M. (2011). An extension of the Electre I method for group decision-making under a fuzzy environment. Omega, 39 (4) 373–386.

Hauge, S., Hokstad, P., Håbrekke, S., Lundteigen, M.A. (2016). Common cause failures in safety-instrumented systems: Using field experience from the petroleum industry. Reliability Engineering & System Safety, 151, 34–45.

Hellmichn, M., Berg, H.P. (2015). Markov analysis of redundant standby safety systems under periodic surveillance testing. Reliability Engineering and System Safety, 133, 48–58.

Hennissen, P., Beckers, H., Moerkerke, G. (2017). Linking practice to theory in teacher education: A growth in cognitive structures. Teaching and Teacher Education, 63, 314–325.

Hillerman, T., Souza, J.C.F., Reis, A.C.B., Carvalho, R.N. (2017). Applying clustering and AHP methods for evaluating suspect healthcare claims. Journal of Computational Science, 19, 97–111.

Hinshaw, K. (2016). Human factors in obstetrics and gynaecology. Obstetrics, Gynaecology & Reproductive Medicine, 26 (12), 368–370.

Ho, W. (2008). Integrated analytic hierarchy process and its application – A literature review. European Journal of Operational Research, 186, 211–228.

Hoarau, Q., Ginolhac, G., Atto, A.M., Nicolas, J.M. (2017). Robust adaptive detection of buried pipes using GPR. Signal Processing, 132, 293–305.

Hokkanen, J., Salminen, P., Rossi, E., Ettala, M. (1995). The choice of a solid waste management system using the Electre II decision-aid method. Waste Management & Research, 13 (2), 175–193.

Hollnagel, E. (1998). Cognitive Reliability and Error Analysis Method (CREAM). Oxford, Elsevier Science.

Homenda, W., Jastrzebska, A., Pedrycz, W. (2016). Multicriteria decision making inspired by human cognitive processes. Applied Mathematics and Computation 290, 392–411.

Hori, S., Shimizu, Y. (1999). Designing methods of human interface for supervisory control systems. Control Engineering Practice, 7, 1413–1419.

Hsu, W.-K.K., Huang, S.-H.S., Tseng, W.-J. (2016). Evaluating the risk of operational safety for dangerous goods in airfreights – A revised risk matrix based on fuzzy AHP. Transportation Research Part D: Transport and Environment, 48, 235–247.

Hua, Z., Gong, B., Xu, X. (2008). A DS–AHP approach for multi-attribute decision making problem with incomplete information. Expert Systems with Applications, 34 (3), 2221–2227.

Huang, C.-L., Yung, K. (1981). Multiple-Attribute Decision Making. Methods and Applications. A state-of-the-Art Survey. Springer-Verlag, Berlin, Heidelberg, New York.

Huang, C., Chu, P., Chiang, Y. (2008). A fuzzy AHP application in government-sponsored R&D project selection. Omega, 36 (6), 1038–1052.

Huang, W., Loman, J., Song, T. (2015). A reliability model of a warm standby configuration with two identical sets of units. Reliability Engineering and System Safety, 133, 237–245.

Hughes, W.R. (2009). A statistical framework for strategic decision making with AHP: Probability assessment and Bayesian revision. Omega, 37 (2), 463–470.

Hunaidi, O., Giamou, P. (1998). Ground-penetrating radar for detection of leaks in buried plastic water distribution pipes. Proceedings of the seventh International Conference on Ground-Penetrating Radar, Lawrence, Kansas, USA.

Hwang, C.L., Yoon, K. (1981). Multiple Attributes Decision Making Methods and Applications. Springer, Berlin Heidelberg.

Hwang, C.-L., Lai, Y.J., Liu, T.Y. (1993). A new approach for multiple objective decision making. Computers and Operational Research, 20, 889–899.

Ilaya-Ayza, A.E. (2016). Propuesta para la transición de un sistema con suministro de agua intermitente a suministro continuo. Doctoral Dissertation. Universitat Politècnica de València. Valencia, Spain.

Ilaya-Ayza, A.E., Campbell, E., Pérez-García, R., Izquierdo, J. (2016). Network Capacity Assessment and Increase in Systems with Intermittent Water Supply. Water, 8 (4), 126.

Ilaya-Ayza, A.E., Benítez, J., Izquierdo, J., Pérez-García, R. (2017). Multi-criteria optimization of supply schedules in intermittent water supply systems. Journal of Computational and Applied Mathematics, 309, 695–703.

INAIL (2011). Evaluation and management of work-related stress risk, Manual for use by companies in implementation of Legislative Decree 81/08 and subsequent amendments. https://www.inail.it/cs/internet/docs/alg-valutazione-gestione-rischio-stress-lavoro-correlato-ita.pdf.

Ishizaka, A. (2008). A multicriteria approach with AHP and clusters for supplier selection. In Proceedings of the 15th International Annual Euroma Conference, Groningen.

Ishizaka, A. (2012). Clusters and pivots for evaluating a large number of alternatives in AHP. Brazilian Operations Research Society, 32 (1), 87–101.

Islam, R., Biswal, M., Alam, S. (1997). Clusterization of alternatives in the Analytic Hierarchy Process. Military Operations Research, 3, 69–78.

Islam, R., Abdullah, N.A. (2006). Management Decision-Making by the Analytic Hierarchy Process: A proposed modification for large-scale problems. Journal for International Business and Entrepreneurship Development, 3, 18–40.

ISO 22000:2005. Food safety management systems - Requirements for any organization in the food chain. Technical Committee ISO/TC 34 of the International Organization for Standardization.

ISO 55000:2014. Asset management Overview, principles and terminology. International Standard Organization, Switzerland.

Izhakian, Y. (2017). Expected utility with uncertain probabilities theory. Journal of Mathematical Economics, 69, 91–103.

Jabbarova, A.I. (2016). Solution for the Investment Decision Making Problem through Interval Probabilities. Procedia Computer Science, 102, 465–468.

Jafary, B., Fiondella, L. (2016). A universal generating function-based multi-state system performance model subject to correlated failures. Reliability Engineering & System Safety, 152, 16–27.

Jalao, E.R., Wua, T., Shunk, D. (2014). An intelligent decomposition of pairwise comparison matrices for large-scale decisions. European Journal of Operational Research, 238, 270–280.

Jiang, X., Duan, F., Tian, H., Wei, X. (2015). Optimization of reliability centered predictive maintenance scheme for inertial navigation system. Reliability Engineering & System Safety, 140, 208–217.

Jin, F., Ni, Z., Chen, H., Li, Y. (2016). Approaches to decision making with linguistic preference relations based on additive consistency. Applied Soft Computing, 49, 71–80.

Johnson, S.G.B., Rajeev-Kumar, G., Keil, F.C. (2016). Sense-making under ignorance. Cognitive Psychology, 89, 39–70.

Jowitt, P.W., Xu, C. (1990). Optimal valve control in water-distribution networks. Journal of Water Resources Planning and Management, 118, 455–472.

Jowitt, P.W., Germanopoulos, G. (1992). Optimal pump scheduling in water-supply networks. Journal of Water Resources Planning and Management, 118, 406–422.

Jun, D., Tian-tian, F., Yi-sheng, Y., Yu, M. (2014). Macro-site selection of wind/solar hybrid power station based on ELECTRE-II. Renewable and Sustainable Energy Reviews, 35, 194–204.

Jun, L., Huibin, X. (2012). Reliability Analysis of Aircraft Equipment Based on FMECA Method. Physics Procedia, 25, 1816–1822.

Juneja, S., Shahabuddin, P. (1992). Fast simulation of Markovian Reliability/Availability Models with General Repair Policies. In: Proceedings of the Fault-Tolerant Computing, Boston, MA, July 8-10. IEEE 150–159.

Ka, B. (2011). Application of Fuzzy AHP and ELECTRE to China Dry Port Location Selection. The Asian Journal of Shipping and Logistics, 27 (2), 331–353.

Kabaasha, A.M., Piller, O., Van Zyl, J.E. (2018). Incorporating the modified orifice equation into pipe network solvers for more realistic leakage modeling. Journal of Hydraulic Engineering, 144 (2), 04017064.

Kahlili-Damghani, K., Sadi-Nezhad, S. (2013). A hybrid fuzzy multiple criteria group decision making approach for sustainable project selection. Applied Soft Computing, 13 (1), 339–352.

Kang, H.G., Kim, H.E. (2012). Unavailability and spurious operation probability of k-out-of-n reactor protection systems in consideration of CCF. Annals of Nuclear Energy, 49, 102–108.

Karanik, M., Wanderer, L., Gomez-Ruiz, J.A., Pelaez, J.I. (2016). Reconstruction methods for AHP pairwise matrices: How reliable are they? Applied Mathematics and Computation, 279, 103–124.

Kaufman, B.E. (1999). Emotional arousal as a source of bounded rationality. Journal of Economics Behaviour & Organization, 38, 135–144.

Kaya, T., Kahraman, C. (2011). An integrated fuzzy AHP–ELECTRE methodology for environmental impact assessment. Expert Systems with Applications, 38 (7), 8553–8562.

Keeney, R. (1992). Value focused thinking. Harvard University Press.

Khan, M.A., Salah, K. (2018). IoT security: Review, blockchain solutions, and open challenges. Future Generation Computer Systems, 82, 395–411.

Klir, G.J., Yuan, B. (1995). Fuzzy sets and fuzzy logic – Theory and applications. Prentice Hall PTR, Upper Saddle River, New Jersey.

Kolahi, H., Jahangiri, M., Ghaem, H., Rostamabadi, A., Aghabeigi, M., Farhadi, P., Kamalinia, M. (2018). Evaluation of Respiratory Protection Program in Petrochemical Industries; Application of Analytic Hierarchy Process (AHP). Safety and Health at Work, 9, (1), 95–100.

Koning, J., Jaspers, R., Doornink, J., Ouwehand, B., Klinkhamer, F., Snijders, B., Sadakov, S., Heemskerk, C. (2009). Maintenance implications of critical components in ITER CXRS upper port plug design. Fusion Engineering and Design, 84 (7–11), 1091–1094.

Kubler, S., Robert, J., Derigent, W., Voisin, A., Le Traon, Y. (2016). A state-of the-art survey & testbed of fuzzy AHP (FAHP) applications. Expert Systems with Applications, 65, 398–422.

Kumar, A., Sah, B., Singh, A.R., Deng, Y., He, X., Kumar, P., Bansal, R.C. (2017). A review of multi criteria decision making (MC\DM) towards sustainable renewable energy development. Renewable and Sustainable Energy Reviews, 69, 596–609.

Kunz, J. (2015). Objectivity and subjectivity in performance evaluation and autonomous motivation: An exploratory study. Management Accounting Research, 27, 27–46.

Kutlu, A.C., Ekmekçioğlu, M. (2012). Fuzzy failure modes and effects analysis by using fuzzy TOPSIS-based fuzzy AHP. Expert Systems with Applications, 39 (1), 61–67.

Lambert, A. (2001). What do we know about pressure: Leakage relationship in distribution system? In Proceedings of IWA System Approach to Leakage Control and Water Distribution Systems Management, London, UK.

La Rocca, A., Hoholm, T., Mørk, B.E. (2017). Practice theory and the study of interaction in business relationships: Some methodological implications. Industrial Marketing Management, 60, 187–195.

La Scalia, G., Micale, R., Certa, A., Enea, M. (2015). Ranking of shelf life models based on smart logistic unit using the ELECTRE III method. International Journal of Applied Engineering Research, 10 (17), 38009–38015.

Lee, P. T.-W., Lin, C.-W. (2013). The cognition map of financial ratios of shipping companies using DEMATEL and MMDE, Maritime Policy & Management, 40(2), 133–145.

Lee, H.-S., Tzeng, G.-H., Yeih, W., Wang, Y.-J., Yang, S.-C. (2013). Revised DEMATEL: Resolving the Infeasibility of DEMATEL. Applied Mathematical Modelling, 37 (10–11), 6746-6757.

Lin, C., Tzeng, G. (2009). A value-created system of science (technology) park by using DEMATEL, Expert Systems with Applications, 36 (6), 9683–9697.

Lindberg, C.-F., Tan, S.T., Yan, J.Y., Starfelt, F. (2015). Key Performance Indicators Improve Industrial Performance. Energy Procedia, 75, 1785–1790.

Liu, P., Jin, F., Zhang, X., Su, Y., Wang, M. (2011). Research on the multi-attribute decision-making under risk with interval probability based on prospect theory and the uncertain linguistic variables. Knowledge-Based Systems, 24 (4), 554–561.

Liu, H.C., Liu, L., Liu, N. (2013a). Risk evaluation approaches in failure mode and effects analysis: A literature review. Expert Systems with Applications, 40, 828–838.

Liu, H.C., Mao, L.X., Zhang, Z.Y., Li, P. (2013b). Induced aggregation operators in the VIKOR method and its application in material selection. Applied Mathematical Modelling, 37 (9), 6325–6338.

Liu, Z., Kleiner, Y. (2013). State of the art review of inspection technologies for condition assessment of water pipes. Measurement, 46 (1), 1–15.

Liu, X., Wang, W., Peng, R. (2015). An integrated production, inventory and preventive maintenance model for a multi-product production system. Reliability Engineering and System Safety, 137, 76–86.

Liu, S., Chan, F.T.S., Ran, W. (2016). Decision making for the selection of cloud vendor: An improved approach under group decision-making with integrated weights and objective/subjective attributes. Expert Systems with Applications, 55, 37–47.

Løken, E. (2007). Use of multicriteria decision analysis methods for energy planning problems. Renewable and Sustainable Energy Reviews, 11 (7), 1584–1595.

Lolli, F., Ishizaka, A., Gamberini, R., Rimini, B. (2017). A multicriteria framework for inventory classification and control with application to intermittent demand. Journal of Multi-Criteria Decision Analysis, 24 (5–6), 275–285.

Lopes, I., Senra, P., Vilarinho, S., Sá, V., Teixeira, C., Lopes, J., Alves, A., Oliveira, J.A., Figueiredo, M. (2016). Requirements Specification of a Computerized Maintenance Management System – A Case Study. Procedia CIRP, 52, 268–273.

Lu, L., Lewis, G. (2008). Configuration determination for k-out-of-n partially redundant systems. Reliability Engineering & System Safety, 93 (11), 1594–1604.

Lu, J.M., Wu, X.Y. (2014). Reliability evaluation of generalized phased-mission systems with repairable components. Reliability Engineering & System Safety, 121, 136–145.

Lu, J.M., Wu, X.Y., Liu, Y., Lundteigen, M.A. (2015). Reliability analysis of large phased-mission systems with repairable components based on success-state sampling. Reliability Engineering & System Safety, 142, 123–133.

Madigan, R., Golightly, D., Madders, R. (2016). Application of Human Factors Analysis and Classification System (HFACS) to UK rail safety of the line incidents. Accident Analysis & Prevention, 97, 122–131.

Madureira, S., Flores-Colen, I., De Brito, J., Pereira, C. (2017). Maintenance planning of facades in current buildings. Construction and Building Materials, 147, 790–802.

Magyar, Z., Garbai, L., Jasper, A. (2016). Risk-based determination of heat demand for central and district heating by a probability theory approach. Energy and Buildings, 110, 387–395.

Malone, T.W., Laubacher, R.J. (1998). The Dawn of the E-Lance Economy. Harvard Business Review.

Marnell, G. (2014). Chunking the Information Presented to Readers. A critique of Information Mapping. The Australian and New Zealand Journal of Technical Communication, 33, http://www.abelard.com.au/information%20mapping.pdf.

Martón, I., Martorell, P., Mullor, R., Sánchez, A.I., Martorell, S. (2016). Optimization of test and maintenance of ageing components consisting of multiple items and addressing effectiveness. Reliability Engineering and System Safety, 153, 151–158.

Massanet, S., Riera, J.V., Torrens, J., Herrera-Viedma, E. (2016). A model based on subjective linguistic preference relations for group decision making problems. Information Sciences, 355–356, 249–264.

Matos, M.A. (2007). Decision under risk as a multicriteria problem. European Journal of Operational Research, 181 (3), 1516–1529.

Melon, M.G., Beltran, P.A., Cruz, M.C.G. (2008). An AHP-based evaluation procedure for innovative educational projects: a face-to-face vs. computer-mediated case study. Omega, 36 (5), 754–765.

Meyer, C. (2000). Matrix Analysis and Applied Linear Algebra, SIAM, Philadelphia.

Mi, J., Li, Y.F., Yang, Y.J., Peng, W., Huang, H.Z. (2016). Reliability assessment of complex electromechanical systems under epistemic uncertainty. Reliability Engineering & System Safety, 152, 1–15.

Miller, G.A. (1956). The Magic Number Seven, Plus or Minus Two: Some Limits on Our Capacity for Processing Information. The Psychological Review, 63, 81–97.

Mirmousa, S., Dehnavi, H.D. (2016). Development of Criteria of Selecting the Supplier by Using the Fuzzy DEMATEL Method. Procedia - Social and Behavioral Sciences, 230, 281–289.

Mo, Y., Xing, L., Amari, S.V., Dugan, J.B. (2015). Efficient analysis of multi-state k-out-of-n systems. Reliability Engineering and System Safety, 133, 95–105.

Moghaddass, R., Zuo, M.J. (2011). Optimal design of a repairable k-out-of-n system considering maintenance. In: Proceedings of the Reliability and Maintainability Symposium (RAMS), Lake Buena Vista, FL, January 24–27, IEEE 1–6.

Montoro-Cazorla, D., Pérez-Ocón, R. (2014). A redundant n-system under shocks and repairs following Markovian arrival processes. Reliability Engineering and System Safety, 130, 69–75.

Moubray, J. (1991). Reliability-centered maintenance. Butterworth-Heinemann.

Mousavi-Nasab, S.H., Sotoudeh-Anvari, A. (2017). A comprehensive MCDM-based approach using TOPSIS, COPRAS and DEA as an auxiliary tool for material selection problems. Materials & Design, 121, 237–253.

Muhammad, M.N., Cavus, N. (2017). Fuzzy DEMATEL method for identifying LMS evaluation criteria. Procedia Computer Science, 120, 742–749.

Mulliner, E., Malys, N., Maliene, V. (2016). Comparative analysis of MCDM methods for the assessment of sustainable housing affordability. Omega, 59, 146–156.

Narens, L. (2016). An introduction to lattice based probability theories. Journal of Mathematical Psychology, 74, 66–81.

Nguyen, K.-A., Do, P., Grall, A. (2015). Multi-level predictive maintenance for multi-component systems. Reliability Engineering & System Safety, 144, 83–94.

Nikas, A., Doukas, H., López, L.M. (2018). A group decision making tool for assessing climate policy risks against multiple criteria. Heliyon, 4 (3), https://doi.org/10.1016/j.heliyon.2018.e00588.

Ocaña-Levario, S.J. (2014). Pattern recognition for the identification of pipelines buried in water supply systems from GPR images. Master Thesis, Polytechnic University of Valencia, Valencia, Spain.

Ocaña-Levario, S.J., Carreño-Alvarado, E.P., Ayala-Cabrera, D., Izquierdo, J. (2018). GPR image analysis to locate water leaks from buried pipes by applying variance filters. Journal of Applied Geophysics, 152, 236–247.

Omole, H., Walker, G. (2015). Offshore Transport Accident Analysis Using HFACS. Procedia Manufacturing, 3, 1264–1272.

Ortiz-Barrios, M.A., Aleman-Romero, B.A., Rebolledo-Rudas, J., Maldonado-Mestre, H., MontesVilla, L., De Felice, F., Petrillo, A. (2017). The analytic decision-making preference model to evaluate the disaster readiness in emergency departments: The A.D.T. model. Journal of Multi-Criteria Decision Analysis, 24 (5–6), 204–226.

Ossadnik, W., Schinke, S., Kaspar, R.H. (2016). Group Aggregation Techniques for Analytic Hierarchy Process and Analytic Network Process: A Comparative Analysis. Group Decision and Negotiation, 25, 421–457.

Ozdemir, M.S. (2005). Validity and inconsistency in the analytic hierarchy process. Applied Mathematics and Computation, 161, 707–720.

Paltrinieri, N., Øien, K. (2014). Expert Judgment in Human Reliability Analysis: Development of User Guidelines. In: Probabilistic Safety Assessment and Management PSAM, Honolulu, Hawaii.

Pandeya, A., Kumar, A. (2016). A note on "Applying fuzzy linguistic preference relations to the improvement of consistency of fuzzy AHP". Information Sciences, 346–347, 1–5.

Pang, H., Yu, T., Song, B. (2016). Failure mechanism analysis and reliability assessment of an aircraft slat. Engineering Failure Analysis, 60, 261–279.

Panteleev, V.V., Kamaev, V.A., Kizim, A.V. (2014). Developing a Model of Equipment Maintenance and Repair Process at Service Repair Company Using Agent-Based Approach. Procedia Technology, 16, 1072–1079.

Paolella, M. (2006). Fundamental Probability: A Computational Approach. Wiley.

Partovi, F.Y. (2006). An analytical model for locating facilities strategically. Omega, 34, 41–55.

Pereira Jr, J.G., Ekel, P.Y., Palhares, R.M., Parreiras, R.O. (2015). On multicriteria decision making under conditions of uncertainty. Information Sciences, 324, 44–59.

Perelman, L.S., Abbas, W., Koutsoukos, X., Amin, S. (2016). Sensor placement for fault location identification in water networks: A minimum test cover approach. Automatica, 72, 166–176.

Petruni, A., Giagloglou, E., Douglas, E., Geng, J., Leva, M.C., Demichela, M. (2017). Applying Analytic Hierarchy Process (AHP) to choose a human factors technique: Choosing the suitable Human Reliability Analysis technique for the automotive industry. Safety Science, In Press, https://doi.org/10.1016/j.ssci.2017.05.007.

Poór, P., Kuchtová, N., Šimon, M. (2014). Machinery Maintenance as Part of Facility Management. Procedia Engineering, 69, 1276–1280.

Proulx, F., Rodriguez, M.J., Sérodes, J., Bouchard, C. (2007). A methodology for identifying vulnerable locations to taste and odour problems in a drinking water system. Water Science and Technology, 55 (5), 177–183.

Qazi, A., Quigley, J., Dickson, A., Kirytopoulos, K. (2016). Project Complexity and Risk Management (ProCRiM) Towards modelling project complexity driven risk paths in construction projects. International Journal of Project Management, 34 (7), 1183–1198.

Qiu, Q., Cui, L., Shen, J., Yang, L. (2017). Optimal maintenance policy considering maintenance errors for systems operating under performance-based contracts. Computers & Industrial Engineering, 112, 147–155.

Rad, T.G., Sadeghi-Niaraki, A., Abbasi, A., Choi, S.-M. (2018). A methodological framework for assessment of ubiquitous cities using ANP and DEMATEL methods. Sustainable Cities and Society, 37, 608–618.

Radjiyev, A., Qiu, H., Xiong, S., Nam, K. (2015). Ergonomics and sustainable development in the past two decades (1992–2011): Research trends and how ergonomics can contribute to sustainable development. Applied Ergonomics, 46 (Part A), 67–75.

Rao Venkata, R. (2013). Decision Making in the Manufacturing Environment: Using Graph Theory and Fuzzy Multiple Attribute Decision Making. Springer.

Raza, A., Ulansky, V. (2017). Modelling of Predictive Maintenance for a Periodically Inspected System. Procedia CIRP, 59, 95–101.

Reason, J. (1990). Human error. Cambridge University Press.

Reer, B. (2008a). Review of advances in human reliability analysis of errors of commission, Part 1: EOC identification. Reliability Engineering & System Safety, 93 (8), 1091–1104.

Reer, B. (2008b). Review of advances in human reliability analysis of errors of commission—Part 2: EOC quantification. Reliability Engineering & System Safety, 93 (8), 1105–1122.

Riesz, F. (1909). Sur les opérations fonctionnelles linéaires. C.R. Acad. Sci., Paris, 149, 974–977.

Rodrigues, B., Bocek, T., Stiller, T. (2018). The Use of Blockchains: Application-Driven Analysis of Applicability. Advances in Computers, In press, https://doi.org/10.1016/bs.adcom.2018.03.011.

Roesch, W.J. (2012). Using a new bathtub curve to correlate quality and reliability. Microelectronics Reliability, 52 (12), 2864–2869.

Rogers, M.G., Bruen, M., Maystre, L-Y. (2013). Electre and decision support. Methods and Applications in Engineering and Infrastructure Investment.

Rosenbloom, E.S. (1996). A probabilistic interpretation of the final rankings in AHP. European Journal of Operational Research, 96, 371–378.

Rostamzadeh, R., Sofian, S. (2011). Prioritizing effective 7Ms to improve production systems performance using fuzzy AHP and fuzzy TOPSIS (case study). Expert Systems with Applications, 38 (5), 5166–5177.

Roy, B. (1968). Classement et choix en presence de points de vue multiples (La methode ELECTRE). Revue Francaise D Informatique de Recherche Operationnelle, 2 (8), 57–75.

Roy, B. (1991). The outranking approach and the foundations of electre methods. Theory and Decision, 31 (1), 49–73.

Rudin, W. (1986). Real and Complex Analysis, McGraw-Hill.

Ruschel, E., Santos, E.A.P., De Freitas, E., Loures, R. (2017). Industrial maintenance decision-making: A systematic literature review. Journal of Manufacturing Systems, 45, Pages 180–194.

Saaty, T.L. (1977). A scaling method for priorities in hierarchical structures. Journal of Mathematical Psychology, 15 (3) 234–281.

Saaty, T.L. (1980). The Analytic Hierarchy Process: Planning, Priority Setting and Resource Allocation, McGraw-Hill, New York.

Saaty, T.L. (1994). The Analytic Hierarchy Process. McGraw Hill Company, New York.

Saaty, T.L., Vargas, L.G. (1994). Decision Making in Economic, Political, Social and Technological Environments with the Analytic Hierarchy Process. The Analytic Hierarchy Process Series, 7, 1st Edition.

Saaty, T.L. (2000). Fundamentals of Decision Making and Priority Theory with the Analytic Hierarchy Process. RWS Publication, Pittsburg.

Saaty, T.L. (2003). Decision-making with the AHP: why is the principal eigenvector necessary. European Journal of Operations Research, 145, 85–91.

Saaty, T.L., Ozdemir, M.S. (2003). Why the Magic Number Seven Plus or Minus Two. Mathematical and Computer Modelling, 38, 233–244.

Saaty, T.L., Ozdemir, M.S. (2005). The encyclicon: a dictionary of decisions with dependence and feedback based on Analytic Network Process. Pittsburgh, RWS Publications.

Saaty, T.L. (2008a). Fundamentals of decision making and priority theory with the analytic hierarchy process. RWS Publication, Pittsburgh.

Saaty, T.L. (2008b). Decision making with the analytic hierarchy process. International Journal of Services Sciences, 1, 83–98.

Saaty, T.L. (2008c). Relative measurement and its generalization in decision making. Why pairwise comparisons are central in mathematics for the measurement of intangible factors. The analytic hierarchy/network process. Revista Real Academia Ciencias Serie A: Matemáticas, 102 (2), 251–318.

Sabouhi, H., Abbaspour, A., Fotuhi-Firuzabad, M., Dehghanian, P. (2016). Reliability modeling and availability analysis of combined cycle power plants. Electrical Power and Energy Systems, 79, 108–119.

Sara, J., Stikkelman, R.M., Herder, P.M. (2015). Assessing relative importance and mutual influence of barriers for CCS deployment of the ROAD project using AHP and DEMATEL methods, International Journal of Greenhouse Gas Control, 41, 336–357.

Saravia-Pinilla, M.H., Daza-Beltrán, C., García-Acosta, G. (2016). A comprehensive approach to environmental and human factors into product/service design and development. A review from an ergoecological perspective. Applied Ergonomics, 57, 62–71.

Savelyev, A. (2018). Copyright in the blockchain era: Promises and challenges. Computer Law & Security Review, 34 (3), 2018, 550–561.

Seiti, H., Tagipour, R., Hafezalkotob, A., Asgari, F. (2017). Maintenance strategy selection with risky evaluations using RAHP. Journal of Multi-Criteria Decision Analysis, 24 (5–6), 257–274.

Shafer, G. (1976). A mathematical theory of evidence, Princeton University Press, Princeton.

Shah, P., Harris, A.J.L., Bird, G., Catmur, C., Hahn U. (2016). A pessimistic view of optimistic belief updating. Cognitive Psychology, 90, 71–127.

Shariatkhah, M.H., Haghifam, M.R., Parsa-Moghaddam, M., Siano, P. (2015). Modeling the reliability of multi-carrier energy systems considering dynamic behavior of thermal loads. Energy and Buildings, 103, 375–383.

Sharma, V.K., Agarwal, M., Sen, K. (2011). Reliability evaluation and optimal design in heterogeneous multi-state series-parallel systems. Information Sciences, 181, 362–378.

Shen, Y., Hoerl, A., McConnell, W. (1992). An incomplete design in the Analytic Hierarchy Process. Mathematical and Computer Modelling, 16, 121–129.

Shieh, J.-I., Wu, H.-H., Huang, K.-K. (2010). A DEMATEL method in identifying key success factors of hospital service quality. Knowledge-Based Systems, 23 (3), 277–282.

Shiraishi, S., Obata, T., Daigo, M. (1998). Properties of a positive reciprocal matrix and their application to AHP. Journal of the Operation Research Society of Japan, 41, 404–414.

Shoham, K., Leyton-Brown, K. (2009). Multiagent Systems: Algorithmic, Game-Theoretic and Logical Foundations. Cambridge University Press, Cambridge, MA, USA.

Si, S.-L., You, X.-Y., Liu, H.-C., Zhang, P. (2018). DEMATEL Technique: A Systematic Review of the State-of-the-Art Literature on Methodologies and Applications. Mathematical Problems in Engineering, https://doi.org/10.1155/2018/3696457.

Sidibé, I.B., Khatab, A., Diallo, C., Adjallah, K.H. (2016). Kernel estimator of maintenance optimization model for a stochastically degrading system under different operating environments. Reliability Engineering and System Safety, 147, 109–116.

Sidibe, I.B., Khatab, A., Diallo, C., Kassambara, A. (2017). Preventive maintenance optimization for a stochastically degrading system with a random initial age. Reliability Engineering & System Safety, 159, 255–263.

SMRP Press Release. (2007). Global Indicators for Maintenance and Availability Performance. Society for Maintenance and Reliability Professionals, McLean, Virginia, USA.

Sobhani, A., Wahab, M.I.M., Neumann, W.P. (2017). Incorporating human factors-related performance variation in optimizing a serial system. European Journal of Operational Research, 257 (1), 69–83.

Söderbaum, P. (1999). Values, ideology and politics in ecological economics. Ecological Economics, 28, 161–170.

Soner, O., Asan, U., Celik, M. (2015). Use of HFACS–FCM in fire prevention modelling on board ships. Safety Science, 77, 25–41.

Soroudi, A., Rabiee, A., Keane, A. (2017). Information gap decision theory approach to deal with wind power uncertainty in unit commitment. Electric Power Systems Research, 145, 137–148.

Srdjevic, B., Srdjevic, Z., Blagojevic, B. (2014). First-Level Transitivity Rule Method for Filling in Incomplete Pair-Wise Comparison Matrices in the Analytic Hierarchy Process. Applied Mathematics & Information Sciences, 8 (2), 459–467.

Stewart, G.W. (2001). Matrix Algorithms, vol. II, SIAM.

Stokes, C., Wu, W., Dandy, G. (2012). Battle of the water networks II: Combining engineering judgement with genetic algorithm optimisation. In Proceedings of the 14th Water Distribution Systems Analysis Conference, South Australia, Adelaide.

Stricker, N., Micali, M., Dornfeld, D., Lanza, G. (2017). Considering Interdependencies of KPIs – Possible Resource Efficiency and Effectiveness Improvements. Procedia Manufacturing, 8, 300–307.

Supeekit, T., Somboonwiwat, T., Kritchanchai, D. (2016). DEMATEL-modified ANP to evaluate internal hospital supply chain performance. Computers & Industrial Engineering, 102, 318–330.

SureshBabu, S.V., Maheswar, D., Ranganath, G. (2012). Reliability Optimization Model for Redundant Systems with Multiple Constraints. Procedia Engineering, 38, 7–14.

Swan, M. (2018). Blockchain for Business: Next-Generation Enterprise Artificial Intelligence Systems. Advances in Computers, In press, https://doi.org/10.1016/bs.adcom.2018.03.013.

Swain, A. D. (1990). Human Reliability Analysis: Needs, Status, Trends and Limitations. Reliability Engineering and System Safety, 29 (3), 301–313.

Swain, A.D., Guttmann, H.E. (1983). Handbook of Human Reliability Analysis with Emphasis on Nuclear Power Plant Applications. NUREG/CR-1278, USNRC.

Swetha, K., Sreekumar, A., Nath, N.N., Pillay, V.R. (2015). Optimizing Resources in m-Redundant Avionic Systems - Comparative Evaluation. Procedia Computer Science, 57, 1376–1383, 3rd International Conference on Recent Trends in Computing (ICRTC).

Szulecka, J., Monges Zalazar, E. (2017). Forest plantations in Paraguay: Historical developments and a critical diagnosis in a SWOT-AHP framework. Land Use Policy, 60, 384–394.

Tafreshi, P.F., Aghdaie, M.H., Behzadian, M., Abadi, M.G. (2016). Developing a Group Decision Support System for Advertising Media Evaluation: A Case in the Middle East. Group Decision and Negotiation, 25 (5), 1021–1048.

Taghipour, S., Banjevic, D. (2011). Periodic inspection optimization models for a reparable system subjected to hidden failure. IEEE Transactions on Reliability, 60 (1), 275–285.

Tamura, M., Nagata, H., Akazawa, K. (2002). Extraction and systems analysis of factors that prevent safety and security by structural models. In 41st SICE annual conference, Osaka, Japan.

Thatcher, A., Yeow, P.H.P. (2016). Human factors for a sustainable future. Applied Ergonomics, 57, 1–7.

Thomson, J., Wang, L., Royer, M. (2009). State of technology review report on condition assessment of ferrous water transmission and distribution systems. EPA United States Environmental Protection Agency, EPA/600/R-09/055.

Todini, E. (2000). Looped water distribution networks design using a resilience index based heuristic approach. Urban Water, 2 (2), 115–122.

Triantaphyllou, E. (1995). Linear programming based decomposition approach in evaluating priorities from pairwise comparisons and error analysis, Journal of Optimization Theory and Applications, 84, 207–234.

Triantaphyllou, E. (2000). Multi-Criteria Decision Making Methods: A Comparative Study. Vol. 44, Springer.

Tzeng, G.H., Chiang C.H., Li C.W. (2007) Evaluating intertwined effects in e-learning programs: a novel hybrid MCDM model based on factor analysis and DEMATEL, Expert Systems with Applications, 32 (4), 1028–1044.

Vaidya, O.S., Kumar, S. (2006). Analytic hierarchy process: An overview of applications. European Journal of Operational Research, 169 (1), 1–29.

Van Horenbeek, A., Pintelon, L. (2013). A dynamic predictive maintenance policy for complex multi-component systems. Reliability Engineering & System Safety, 120, 39–50.

Van Laarhoven, P., Pedrycz, W. (1983). A fuzzy extension of Saaty's priority theory. Fuzzy Sets and Systems, 11 (1), 199–227.

Van Uden, E. (2002). Estimating missing data in pairwise comparison matrices, Texts in Operational and Systems Research in the Face to Challenge the XXI Century, Methods and Techniques in Information Analysis and Decision Making, Academic Printing House, Warsaw.

Vargas, L.G. (1982). Reciprocal matrices with random coefficient. Mathematical Modelling, 3, 69–81.

Vargas, L.G., De Felice, F., Petrillo, A. (2017). Editorial journal of multicriteria decision analysis special issue on “Industrial and Manufacturing Engineering: Theory and Application using AHP/ANP”. Journal of Multi-Criteria Decision Analysis, 24 (5–6), 201–202.

Vasili, M., Tang, S.H., Ismail, N., Vasili, M.R. (2011). A Maintenance optimization models: a review and analysis. In: Proceedings of the International Conference on Industrial Engineering and Operations Management, Kuala Lumpur, Malaysia, January 22–24.

Vernez, D., Vuille, F. (2009). Method to assess and optimise dependability of complex macro-systems: Application to a railway signalling system. Safety Science, 47 (3), 382–394.

Vincke, P. (1992). Multicriteria Decision-Aid. Wiley.

Viswanath Dhanisetty, V.S., Verhagen, W.J.C., Curran, R. (2018). Multi-criteria weighted decision making for operational maintenance processes. Journal of Air Transport Management, 68, 152–164.

Vranken, H. (2017). Sustainability of bitcoin and blockchains. Current Opinion in Environmental Sustainability, 28, 1–9.

Wang, T.-C., Chen, Y.-H. (2008). Applying fuzzy linguistic preference relations to the improvement of consistency of fuzzy AHP. Information Sciences, 178 (19), 3755–3765.

Wang, Y., Deng, C., Wu, J., Wang, Y., Xiong, Y. (2014). A corrective maintenance scheme for engineering equipment. Engineering Failure Analysis, 36, 269–283.

Wang, X., Yang, F., Wei, H., Zhang, L. (2015). A new ranking method based on TOPSIS and possibility theory for multi-attribute decision making problem. Optik - International Journal for Light and Electron Optics, 126 (24), 4852–4860.

Wang, W., Xiong, J., Xie, M. (2016a). A study of interval analysis for cold-standby system reliability optimization under parameter uncertainty. Computers & Industrial Engineering, 97, 93–100.

Wang, P., Zhouquan Zhu, Z., Wang, Y. (2016b). A novel hybrid MCDM model combining the SAW, TOPSIS and GRA methods based on experimental design. Information Sciences, 345, 27–45.

Wang, Z.-J., Tong, X. (2016). Consistency analysis and group decision making based on triangular fuzzy additive reciprocal preference relations. Information Sciences, 361–362, 29–47.

Wang, H., Xu, Z. (2016). Interactive algorithms for improving incomplete linguistic preference relations based on consistency measures. Applied Soft Computing, 42, 66–79.

Weber, A., Thomas, R. (2006). Key Performance Indicators: Measuring & Managing the Maintenance Function. Ivara Corporation.

Weiss-Cohen, L., Konstantinidis, E., Speekenbrink, M., Harvey, N. (2016). Incorporating conflicting descriptions into decisions from experience. Organizational Behavior and Human Decision Processes, 135, 55–69.

Wiegmann, D.A., Shappell, S.A. (2003). A Human Error Approach to Aviation Accident Analysis: The Human Factors Analysis and Classification System. Ashgate, Aldershot.

Wilson, J.R. (2014). Fundamentals of systems ergonomics/human factors. Applied Ergonomics, 45 (1), 5–13.

Wireman, T. (1998). Developing Performance Indicators for Managing Maintenance. Industrial Press, Inc.

Wright, G. (1985). Behavioral Decision Making. Plenum Press.

Xu, Y., Chen, L., Rodríguez, R.M., Herrera, F., Wang, H. (2016). Deriving the priority weights from incomplete hesitant fuzzy preference relations in group decision making. Knowledge-Based Systems, 99, 71–78.

Yager, R.R., Kreinovich, V. (1999). Decision making under interval probabilities. International Journal of Approximate Reasoning, 22 (3), 195–215.

Yan, H.-B., Ma, T., Huynh, V.-N. (2017). On qualitative multi-attribute group decision making and its consensus measure: A probability based perspective. Omega, 70, 94–117.

Yang, J., Huang, H.Z., He, L.P., Zhu, S.P., Wen, D. (2011). Risk evaluation in failure mode and effects analysis of aircraft turbine rotor blades using Dempster–Shafer evidence theory under uncertainty. Engineering Failure Analysis, 18 (8), 2084–2092.

Yang, L., Ma, X., Zhai, Q., Zhao, Y. (2016). A delay time model for a mission-based system subject to periodic and random inspection and postponed replacement. Reliability Engineering & System Safety, 150, 96–104.

Yeh, W.C., Hsieh, T.J. (2011). Solving reliability redundancy allocation problems using an artificial bee colony algorithm. Computers & Operations Research, 38 (11), 1465–1473.

Yssaad, B., Abene, A. (2015). Rational Reliability Centered Maintenance Optimization for power distribution systems. International Journal of Electrical Power & Energy Systems, 73, 350–360.

Yu, Z.-W., Mao, J.-F. (2017). Probability analysis of train-track-bridge interactions using a random wheel/rail contact model. Engineering Structures, 144, 120–138.

Zadeh, L. (1965). Fuzzy sets. Information and Control, 8, 338–353.

Zammori, F., Gabbrielli, R. (2011). ANP/RPN: A multi criteria evaluation of the risk priority number. Quality and Reliability Engineering International, 28, 85–104.

Zanakis, S.H., Solomon, A., Wishart, N., Dublish, S. (1998). Multi-attribute decision making: A simulation comparison of select methods. European Journal of Operational Research, 107 (3), 507–529.

Zhang, Y., Pham, H. (2014). A Cost Model of an Opportunistic Maintenance Policy on k-out-of-n Surveillance Systems Considering Two Stochastic Processes. In Proceedings of the 20th ISSAT International Conference on Reliability and Quality in Design, Seattle, Washington, USA, August 7–9.

Zhang, H. (2016a). Group decision making based on multiplicative consistent reciprocal preference relations. Fuzzy Sets and Systems, 282, 31–46.

Zhang, H. (2016b). Group decision making based on incomplete multiplicative and fuzzy preference relations. Applied Soft Computing, 48, 735–744.

Zhang, H., Ha, L., Li, Q., Beer, M. (2017a). Imprecise probability analysis of steel structures subject to atmospheric corrosion. Structural Safety, 67, 62–69.

Zhang, K., Shardt, Y.A.W., Chen, Z., Yang, X. Ding, S.X., Peng, K. (2017b). A KPI-based process monitoring and fault detection framework for large-scale processes. ISA Transactions, 68, 276–286.

Zimmermann, H.J. (1985). Fuzzy set theory and its applications. Kluwer, Boston, MA.

Zio, E., Baraldi, P., Librizzi, M., Podofillini, L., Dang, V.N. (2009). A fuzzy set-based approach for modeling dependence among human errors. Fuzzy Sets and Systems, 160 (13), 1947–1964.

SCIENTIFIC PRODUCTION

Journals (published)

Carpitella, S., Ocaña-Levario, S.J., Benítez, J., Certa, A., Izquierdo, J. (2018). A hybrid multi-criteria approach to GPR image mining applied to water supply system maintenance. Journal of Applied Geophysics (SJR IF:0.49, Q3; JCR IF: 1.646, 62.50%), 159, 754–764.

Carpitella, S., Carpitella, F., Certa, A., Benítez, J., Izquierdo, J. (2018). Managing human factors to reduce organisational risk in industry. Mathematical and Computational Applications (SJR IF: 0.2, Q3), 23 (4), 67, DOI: 10.3390/mca23040067.

Benítez, J., Carpitella, S., Certa, A., Izquierdo, J. (2018). Characterisation of the consistent completion of AHP comparison matrices using graph theory. Journal of Multi-Criteria Decision Analysis (SJR IF: 0.35, Q2), https://doi.org/10.1002/mcda.1652.

Benítez, J., Carpitella, S., Certa, A., Ilaya-Ayza, A.E., Izquierdo, J. (2018). Consistent clustering of entries in large pairwise comparison matrices. Journal of Computational and Applied Mathematics (SJR IF: 0.94, Q2; JCR IF: 1.632, 80.75%), 343, 98–112.

Carpitella, S., Certa, A., Izquierdo, J., La Fata, CM. (2018). k-out-of-n systems: an exact formula for stationary availability and multi-objective configuration design based on mathematical programming and TOPSIS. Journal of Computational and Applied Mathematics (SJR IF: 0.94, Q2; JCR IF: 1.632, 80.75%), 330, 1007–1015.

Carpitella, S., Certa, A., Izquierdo, J., La Fata, C.M. (2018). A combined multi-criteria approach to support FMECA analyses: a real-world case. Reliability Engineering & System Safety (SJR IF: 1.66, Q1; JCR IF: 4.139, 93.373%), 169, 394–402.

Journals (under review)

Carpitella, S., Certa, A., Benítez, J., Enea, M., Izquierdo, J. Blockchain technology-supported preventive maintenance using a multi-criteria decision support system. Submitted to Computers & Industrial Engineering (SHR IF:1,46, Q1; JCR IF: 3.195, 81.91%) on July 15th, 2018 and currently under review.

Benítez, J., Carpitella, S., Certa, A., Izquierdo, J. Management of uncertain pairwise comparisons in AHP through probabilistic concepts. Submitted to Applied Soft Computing (SJR IF: 1.2, Q1; JCR IF: 3.907, 90%) on March 28th, 2018 and currently under review.

International Conferences and oral presentations

Carpitella, S., Certa, A., Izquierdo, J. (2018). DEMATEL-based consensual selection of suitable maintenance KPIs. Proceedings of the 24th ISSAT International Conference on Reliability and Quality in Design, Toronto, Canada, August 2–6, pages 270–274.

Carpitella, S., Carpitella, F., Certa, A., Benítez, J., Izquierdo, J. (2018). Assessing organizational risk in industry by evaluating interdependencies among human factors through the DEMATEL methodology. Proceedings of the 20th International Conference on Mathematical Modeling in Engineering & Human Behaviour, Valencia, Spain, July 16–18. (**Oral presentation at the conference**).

Carpitella, S., Brentan, B., Montalvo, I., Izquierdo, J., Certa, A. (2018). Multi-objective and multi-criteria analysis for optimal pump scheduling in water systems. Proceedings of the 13th International Hydroinformatics Conference HIC, Palermo, Italy, July 1–6, 3, 364-371. (**Oral presentation at the conference**).

Carpitella, S., Certa, A., Enea, M. (2018). The ELECTRE I method to support the FMECA. Proceedings of the 16th IFAC Symposium on Information Control Problems in Manufacturing INCOM, Bergamo, Italy, June 11–13, 51 (11), pages 459–464. (**Oral presentation at the conference**).

Carpitella, S., Certa, A., Enea, M., Galante, G., Izquierdo, J., La Fata, C.M., Vella, F. (2017). Combined HACCP and TOPSIS-based approach to prioritize risks in the salmon manufacturing process: an applicative case. Proceedings of the 22th Summer School "Francesco Turco", Palermo, Italy, September 13–15, pages 150–156.

Benítez, J., Carpitella, S., Certa, A., Izquierdo, J., La Fata, C.M. (2017). Some consistency issues in multi-criteria decision making. Proceedings of the 22th Summer School "Francesco Turco", Palermo, Italy, September 13–15, pages 411–418. (**Oral presentation at the conference**).

Carpitella, S., Certa, A., Enea, M., Galante, G.M., Izquierdo, J., La Fata, C.M. (2017). Human Reliability Analysis to support the development of a software project. Proceedings of the 23th ISSAT International Conference on Reliability and Quality in Design, Chicago, Illinois, USA, August 4–6, pages 214–218.

Carpitella, S., Ocana-Levario, S.J., Benítez, J., Certa, A., Enea, M., Izquierdo, J. (2017). The ELECTRE I method to optimise GPR data processing analyses for maintenance of water networks. Proceedings of the 23th ISSAT International Conference on Reliability and Quality in Design, Chicago, Illinois, USA, August 4–6, pages 6–10.

Benítez, J., Carpitella, S., Certa, A., Ilaya-Ayza, A.E., Izquierdo, J. (2017). Some awkward issues on pairwise comparison matrices. Proceedings of the 19th International Conference on Mathematical Modeling in Engineering & Human Behaviour, Valencia, Spain, July 17–19, pages 43–48. (**Oral presentation at the conference**).

Carpitella, S., Carpitella, F., Benítez, J., Certa, A., Izquierdo, J. (2017). Prioritization of maintenance actions in water distribution systems. Proceedings of the Congreso de Métodos Numéricos en Ingeniería CMN, Valencia, Spain, July 3–5. (**Oral presentation at the conference**).

Certa, A., Carpitella, S., Enea, M., Micale, R. (2016). A multi criteria decision making approach to support the risk management: a case study. Proceedings of the 21th Summer School "Francesco Turco", Naples, Italy, September 13–15, pages 242–246. (**Oral presentation at the conference**).

Carpitella, S., Certa, A., Izquierdo, J., La Fata, C.M., Pérez-Garcia, R. (2016) Optimal design of k-out-of-n systems. Proceedings of the XII Seminario Euro Latinoamericano de Sistemas de Ingeniería, Lleida, Spain, September 5–7.

Carpitella, S., Certa, A., Galante, G., Izquierdo, J., La Fata, C.M. (2016). The FTOPSIS method to support FMECA analyses. Proceedings of the 22th ISSAT International Conference on Reliability and Quality in Design, Los Angeles, California, USA, August 4–6, pages 398–402.

Carpitella, S., Certa, A., Galante, G., Izquierdo, J. (2016). An equation to calculate the availability for the k-out-of-n system. Proceedings of 18th International Conference on Mathematical Modeling in Engineering & Human Behaviour, Valencia, Spain, July 18–20, pages 34–39. (**Oral presentation at the conference**).