The Flexibility Method

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1 Summary of key ideas

This article explains in an accessible and simple way what the Method of Flexibility consists of, as well as the procedure to be followed to solve statically hyperstatic framed structures, taking redundant forces as main unknowns. We will illustrate it with an example.

2 Introduction

The basic methods of structural analysis applicable to framed structures are the Flexibility Method and the Stiffness Method.

The Flexibility method is also called force method or compatibility method because the main unknowns are forces and the resultant equations to be solved are compatibility equations. The Stiffness method is the counterpart of the flexibility method, being called also displacement method or equilibrium method, because the main unknowns are the displacements and the resultant equations to be solved are equilibrium equations.

In the Flexibility Method, the degree of static indeterminacy (DSI) will determine the number of unknowns (redundant forces) as well as the decomposition of the hyperstatic structure by applying the principle of superposition. The hyperstatic structure will be decomposed into the sum of a primary isostatic structure (obtained from the original one in which the redundant force or forces have been removed but the load remains the same) and as many unit load states as the DSI multiplied by the associated redundant.

To restore the kinematic conditions to the original structure, compatibility equations are formulated, the unknowns being the redundant forces. From them we obtain the value of all the static unknowns, solving statically the hyperstatic structure.

3 Objectives

After reading this document, the student will be able to:

- Determine and select the primary static unknowns and split the statically indeterminate structure into a primary isostatic structure and unit load structures according to the principle of superposition.

- Formulate the compatibility equations of the Flexibility Method, obtaining the movements of the primary isostatic structure and of the unit load structures (called flexibility coefficients) involved in the compatibility equations

- Solve statically the hyperstatic structure.

4 The Flexibility Method

The Flexibility Method is a method to solve statically hyperstatic structures, talking as primary unknowns the redundant forces. This method is based on the principle of superposition, so the following requirements must be fulfilled: material linearity (linear material behaviour) and geometric linearity (small movements).

The given indeterminate structure is first made statically determinate by introducing suitable number of releases (removing the corresponding redundant forces). The number of releases required is equal to the DSI. The introduction of these releases results in displacement discontinuities at these points and directions under the externally applied loads.

Unknown redundants (forces and moments) are then applied at these releases to restore the continuity or compatibility of the structure. The computation of these unknown forces involves solution of linear simultaneous compatibility equations, whose number is equal to the DSI.

After the unknown redundants are computed all the internal forces can be computed in the entire structure using equations of equilibrium and free bodies diagrams of the members or using the principle of superposition.

The general expression of the compatibility equations is as follows:

$$\Delta_i = \Delta_i^0 + \sum_{i=1, j=1}^{n} c_{ij} \cdot R_j \quad i = 1 \ldots n \tag{1}$$

being:

  n = DSI (degree of static indeterminacy) = number of redundant forces

  Δ_i = movements in the given hyperstatic structure (in the direction and point of application of redundant i)

  Δ_i^0 = movements in the primary isostatic structure (in the direction and point of application of redundant i)

  c_{ij} = flexibility coefficients, movements in i produced by a unit load in j
    - first subscript (i) is the location of the movement
    - second subscript (j) is the location of the unit load
    - the movement (displacement/rotation) at a point P in a structure due to a unit load/moment at a point Q is equal to the movement (displacement/rotation) at point Q in a structure due to a unit load/moment at point P, thus $c_{ij} = c_{ji}$

  R_j = redundant forces

4.1 Application of the Flexibility Method.

The calculation process to be followed to solve statically an hyperstatic structure with the Flexibility method is:

1. DETERMINATION OF THE NUMBER OF UNKNOWNS
   The unknowns of the Flexibility method are the redundant forces; therefore, the first step will be to calculate the Degree of Static Indeterminacy (DSI) of the structure which will give us the number of redundant forces.

2. SELECTION OF THE REDUNDANTS
   The second step will be the selection of the redundant forces.

3. SPLIT THE STATICALLY INDETERMINATE STRUCTURE INTO:
   a) A statically determinate structure referred to as the PRIMARY ISOSTATIC STRUCTURE obtained from the original hyperstatic structure after removing the selected redundant forces. This structure will be loaded with all the applied loads.

   b) Statically determinate structures acted on by the redundant forces (one at a time). The number of these isostatic structures will be the number of redundants (if the structure is hyperstatic to the second degree we will have

2 statically determinate structures acted on by one redundant each). To simplify the process, each structure will be loaded by a unit load in the direction and point of application of the corresponding redundant. Therefore, these unit load structures will be multiplied by their corresponding redundant.

4. EQUILIBRIUM IN THE PRIMARY ISOSTATIC STRUCTURE:
The external and internal equilibrium conditions must be satisfied. With the equilibrium equations we obtain the reactions, the member-end internal forces and the internal forces functions, necessary for the subsequent calculation of the movements (Δ^0) involved in the compatibility equations.

5. EQUILIBRIUM IN THE UNIT LOAD STRUCTURES:
The external and internal equilibrium conditions must be satisfied. With the equilibrium equations we obtain the reactions, the member-end internal forces and the internal forces functions, necessary for the subsequent calculation of the movements (c_{ij}, flexibility coefficients) involved in the compatibility equations.

6. ENFORCE COMPATIBILITY IN THE POINTS OF THE STATICALLY INDETERMINATE STRUCTURE AT WHICH THE SELECTED REDUNDANTS ACT formulating the corresponding compatibility equations.
The number of compatibility equations will be equal to the number of redundants.

7. CALCULATE THE MOVEMENTS IN THE PRIMARY ISOSTATIC STRUCTURE (Δ^0).
The movements of the primary isostatic structure at the point of application and direction of the redundant forces should be obtained. Different methods can be used, namely the classical methods (using the displacement functions and the compatibility conditions) or the Principle of Virtual Work (Unit Load Method).

8. CALCULATE THE MOVEMENTS IN EACH UNIT LOAD STRUCTURE (c_{ij}, flexibility coefficients)
The movements of each unit load structure at the point of application and direction of the corresponding redundant force should be obtained. Different methods can be used, namely the classical methods (using the displacement functions and the compatibility conditions) or the Principle of Virtual Work (Unit Load Method). Only the movements cii can also be obtained by the Principle of Conservation of Energy but not the cij.

9. CALCULATE THE REDUNDANT FORCES.
Solve the compatibility equations to get the redundant forces.

10. SOLVE STATICALLY THE HYPERSTATIC STRUCTURE.
The hyperstatic structure is solved statically (reactions, member-end internal forces and internal forces functions) from the redundant forces, by using the equilibrium equations or by means of the principle of superposition.

4.2 Worked example

We will solve statically the structure shown in figure 1, considering and elastic and linear behavior.

<u>Data</u>: Beam (1): A= 53.8 cm2   I= 8360 cm4   E = 210000 N/mm2
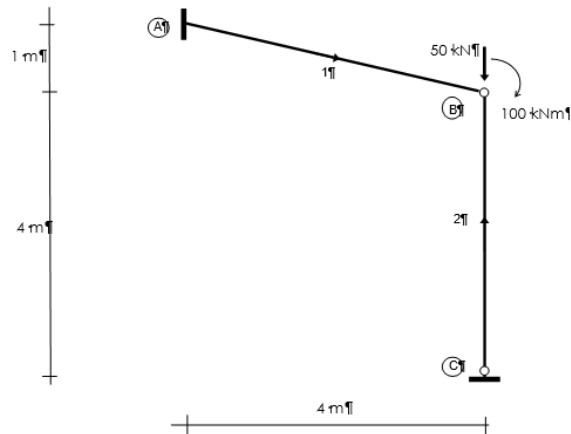   Column (2): A= 14.92 cm2   I= 168 cm4   E = 210000 N/mm2

Figure 1: Hyperstatic structure

1. Degree of Static indeterminacy: DSI = 1, therefore the number of redundant forces is 1

2. We select as redundant force Ry_A. The vertical reaction in C and the Moment reaction in A could also have been chosen. The horizontal reactions in A and C are 0.

3. Splitting the hyperstatic structure into the primary isostatic structure and a unit load structure multiplied by the redundant force (figure 2)
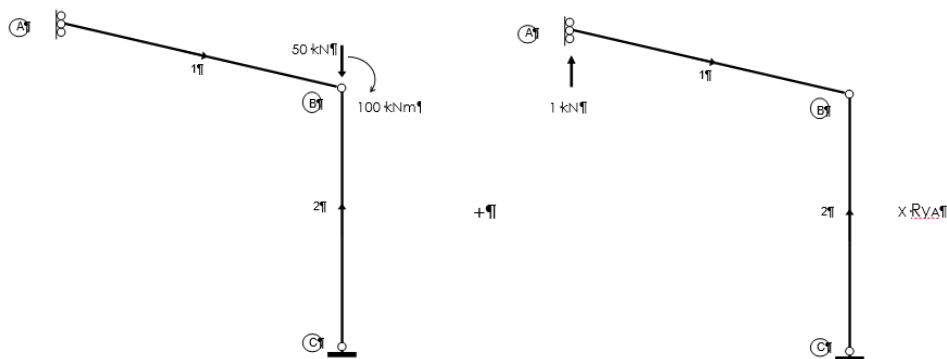

Figure 2: Primary isostatic structure (left) and Unit load structure (right)

4. Equilibrium in the primary isostatic structure (figure 3): we obtain the reactions, the member-end internal forces and the internal forces functions.
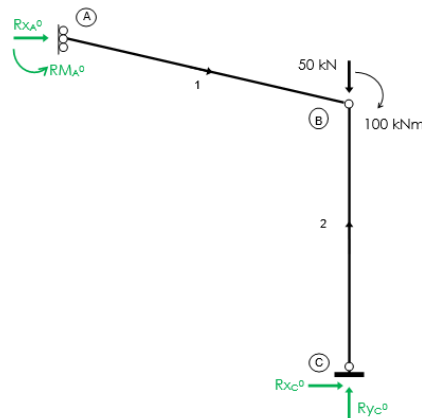

Figure 3: Equilibrium in the primary isostatic structure

Reactions:

		Internal forces functions:		
$Rx_A^0 = 0$	$RM_A^0 = 100$ kN	Member 1:	$N_1^0 = 0$	$M_1^0 = -100$
$Rx_C^0 = 0$	$Ry_C^0 = 50$ kN	Member 2:	$N_2^0 = -50$	

5. Equilibrium in the unit load structure (figure 4): we obtain the reactions, the member-end internal forces and the internal forces functions.
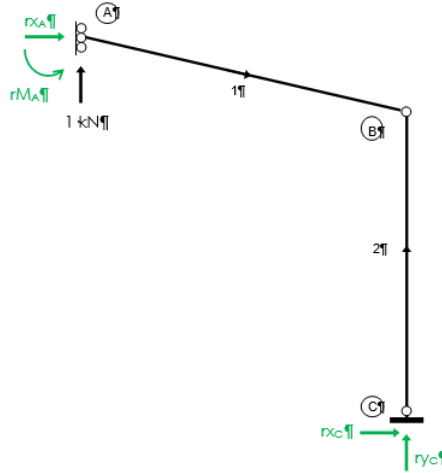

Figure 4: Equilibrium in the unit load structure

Reactions:

		Internal forces function:		
$rx_A^0 = 0$	$rM_A^0 = 4$ kN	Member 1:	$n_1 = 0.2425$	$m_1(x) = 0.97x-4$
$rx_C^0 = 0$	$ry_C^0 = -1$ kN	Member 2:	$n_2 = 1$	

6. Compatibility equation:
   There is only one compatibility equation with restores the kinematic condition in A in the vertical direction (direction and point of application of Ry_A).

$$dy_A = dy_A^0 + c_{11} \cdot Ry_A$$

7. Movement in the primary isostatic structure (dy_A^0).
   The movement will be obtained with the Principle of Virtual Work, considering a virtual unit scenario with a virtual unit vertical load applied in A (figure 5).
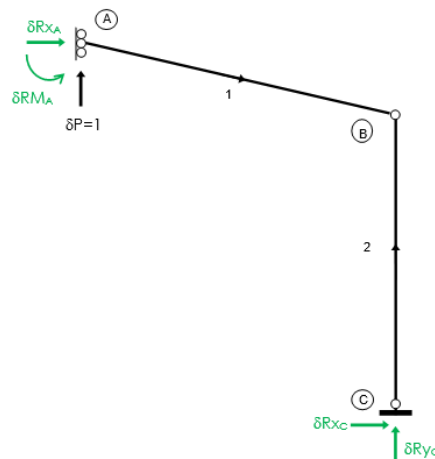

Figure 5: Virtual scenario to obtain the movements in the compatibility equation

Therefore, being this virtual scenario the same as the unit load structure, the virtual reactions and the virtual internal forces functions will be also the same.

Hence:

Virtual scenario:
Internal forces functions: $\delta N_1 = 0.2425$ $\delta M_1(x) = 0.97x - 4$ $\delta N_2 = 1$

Real structure (primary isostatic structure):
Internal forces functions: $N_1^0 = 0$ $M_1^0 = -100$ $N_2^0 = -50$

Energy balance equation of the Principle of Virtual Forces:

$$\delta W^* = \delta U^* \qquad\qquad \delta W^* = 1 \cdot dy_A^0$$

$$\delta U^* = \delta U_{b1}^* + \delta U_{a2}^* = \int_0^{\sqrt{17}} \frac{(-100) \cdot (0.97x - 4)}{17556} + \frac{(-50) \cdot 1 \cdot 4}{313320}$$

$$dy_A^0 = 4.633 \cdot 10^{-2} \text{ m}$$

8. Movement in the unit load structure (c_{11}, flexibility coefficient)
The movement of the unit load structure at the point of application and direction of Ry_A (c_{11}) will be obtained with the Principle of Virtual Work, considering the same virtual scenario as above (figure 5), that is with a virtual unit vertical load applied in A

Hence:

Virtual scenario:
Internal forces functions: $\delta N_1 = 0.2425$ $\delta M_1(x) = 0.97x - 4$ $\delta N_2 = 1$

Real structure (unit structure):
Internal forces functions: $n_1 = 0.2425$ $m_1(x) = 0.97x-4$ $n_2 = 1$

Energy balance equation of the Principle of Virtual Forces:

$$\delta W^* = \delta U^* \qquad\qquad \delta W^* = 1 \cdot c_{11}$$

$$\delta U^* = \delta U_{a1}^* + \delta U_{b1}^* + \delta U_{a2}^* = \frac{(0.2425)^2 \cdot \sqrt{17}}{1129800} + \int_0^{\sqrt{17}} \frac{(0.97x - 4)^2 \, dx}{17556} + \frac{(1)^2 \cdot 4}{313320}$$

$$C_{11} = 1.265 \cdot 10^{-3} \text{ m}$$

9. Solving the compatibility equations to get the redundant forces.

$dy_A = 0$ $dy_A^0 = 4.633 \cdot 10^{-2}$ m $C_{11} = 1.265 \cdot 10^{-3}$ m

Hence:

$$0 = 4.633 \cdot 10^{-2} + 1.265 \cdot 10^{-3} \cdot Ry_A$$

$$\boxed{Ry_A = -36.61 \text{ kN}}$$

10. Reactions and internal forces functions in the hyperstatic structure by using the equilibrium equations or by means of the principle of superposition.

Reactions:
$Rx_A = 0$ $RM_A = -46.44$ kN $Rx_C = 0$ $Ry_C = 86.61$ kN

Internal forces functions:
Member 1: $N_1 = -8.88$ $M_1(x) = 46.44 - 35.52x$
Member 2: $N_2^0 = -86.61$

The equilibrium of the hyperstatic structure is shown in figure 6.
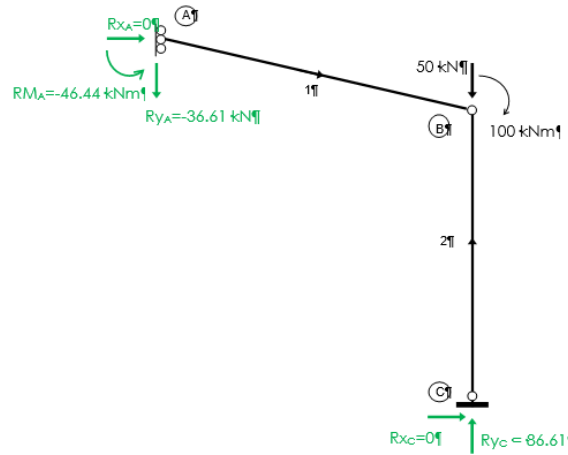

Figure 6: Equilibrium in the hyperstatic structure

5 Closing

In the document we have explained the procedure to solve statically an hyperstatic structure with the flexibility method, illustrated with a worked example.

As a practical application and self-training, we propose the student to solve statically the structure shown in figure 7
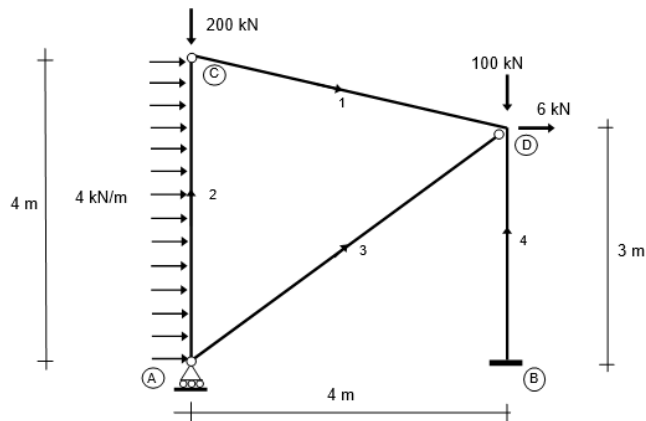

Figure 7: Self-training example

DATA: E = $2{,}1 \cdot 10^8$ kN/m^2 A = 106 cm^2 I = 11260 cm^4

(Results: DSI: 1. RM_B = 43.092 kNm; Ry_A = 198.27 kN; Rx_B = -22 kN; Ry_B = 101.73 kN. The problem has been solved taking as redundant RM_B)

6 Bibliography

6.1 Books:

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[2] Basset, L. "Aplicación del Principio de Conservación de la Energía y del Teorema de la Carga Unidad para la obtención de movimientos", Artículo Docente ETSA, 2013. Disponible en Riunet: http://hdl.handle.net/10251/30429

[3] Basset, L. "Calculation of movements in an isostatic plane-framed structure with the Principle of Virtual Forces". Artículo docente ETSA 2018. Disponible en Riunet: http://hdl.handle.net/10251/103396

6.1 Figures: Author of the figures: Luisa Basset

Figure 1: Hyperstatic structure
Figure 2: Primary isostatic structure (left) and Unit load structure (right)
Figure 3: Equilibrium in the primary isostatic structure
Figure 4: Equilibrium in the unit load structure
Figure 5: Virtual scenario to obtain the movements in the compatibility equation
Figure 6: Equilibrium in the hyperstatic structure
Figure 7: Self-training example