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Solving statically indeterminate structures with the Principle of Virtual Forces

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1 Summary of key ideas

This article explains, in an accessible and simple way, the application of the Principle of the Virtual Forces (PFV) to the calculation of statically indeterminate plane-framed structures with elastic and linear behaviour, taking redundant forces as primary unknowns. The procedure to be followed will be shown step by step, illustrated with a worked example.

2 Introduction

The principle of Virtual Work has two complementary approaches through which the structure is solved first kinematically (Principle of Virtual Displacements) or first statically (Principle of Virtual Forces), obtaining the complete structural response by means of the relationship between statics and kinematics.

We will focus in this article in the Principle of Virtual Forces which can be used to solve a statically indeterminate structure, taking as primary unknowns the redundant forces.

A set of virtual external forces (which happen to satisfy the condition of equilibrium) applied on the real structure in equilibrium and satisfying the requirement of compatibility, will induce internal virtual forces.

Considering that the real structure has undergone displacements and deformations, both an external virtual complementary work and an internal virtual complementary work (stored as a virtual complementary strain energy) will be developed.

When establishing the energy balance, an equation is generated that guarantees compliance with the conditions of compatibility of the structure.

Depending on the number of unknowns, an equal number of independent virtual forces systems must be applied, generating a system of independent equations that will allow us to obtain the redundant forces.

From the redundant forces we will calculate the reactions, the member-end internal forces and the internal forces functions.

3 Objectives

After reading this document, the student will be able to:

- Define the number of virtual forces scenarios (number of redundant forces).
- Formulate adequately the virtual forces scenarios, according to the selected redundant forces.
- Obtain the external virtual complementary work corresponding to each virtual force scenario, in the case of a plane-framed structure with elastic and linear behaviour
- Obtain the virtual complementary strain energy corresponding to each virtual force scenario, in the case of a plane-framed structure with elastic and linear behaviour

- Formulate the energy balance equations to obtain the redundant forces.

4 The Principle of Virtual Forces

4.1 Formulation

The real structure, in equilibrium and with a compatible kinematic configuration after the application of a system of external forces in a slow quasi-static loading process, is subjected to a virtual system of forces (applied instantaneously).

Characteristics of the virtual forces:

- They are arbitrary (independent of the real external forces acting on the structure)
- They satisfy the equilibrium conditions. The equilibrium in the virtual forces' scenario will be formulated with respect to the undeformed geometry, since the real structure fulfill the hypothesis of the Theory of small movements)
- They are applied instantaneously, that is with their constant final value

The virtual external forces acting through the real displacements do an instantaneous virtual work called ***external virtual complementary work*** *(*also called *virtual complementary work of the external forces)*, δW*. Internally, the virtual stresses and the real strains, induced by the real displacements, do an ***internal virtual complementary work*** (also called *virtual complementary work of the internal forces*) which is stored as ***virtual complementary strain energy*** (δU*)

The compatibility conditions of the real structure remain unmodified, hence the equation of energy balance is as follows:

$$\delta W^* = \delta U^* \qquad (1)$$

This equation represents a compatibility condition of the structure.

4.2 Virtual complementary work of the external forces (δW*)

The virtual complementary work of the external forces (δW*) or external virtual complementary work is the virtual work done by the external components of the virtual static group (forces) and the external components of the kinematic group of the real structural system (displacements). The independent variables are the forces.

The virtual complementary work of the external forces is the sum of the virtual complementary work done by all the external virtual forces working with the corresponding real displacements.

The expression of the virtual complementary work of the external forces depends on the type of load:

- Virtual complementary work done by a virtual point load δP:

  $$\delta {W^*}_{point\ load} = \pm \delta P \cdot \Delta \qquad (2)$$

  being δP a virtual point load applied at any point of the structure and Δ the real displacement of the point of application of this load in the same direction.

- Virtual complementary work done by a virtual point moment δM:

$$\delta W^*{}_{point\ moment} = \pm\delta M \cdot \theta \tag{3}$$

being δM a virtual moment applied at any point of the structure and θ the real rotation of this point in the real structure.

- Virtual complementary work done by a virtual axial load (along member x-axis) $\delta p_a(x)$:

$$\delta W^*{}_{axial\ load} = \int_0^L \pm\ \delta p_a(x) \cdot u(x) \cdot dx \tag{4}$$

being δp_a a virtual axial load (along member x-axis) applied in a member of the structure and u(x) the real axial displacement function of this member.

- Virtual complementary work done by a virtual transverse load (along member y-axis), $\delta p_n(x)$:

$$\delta W^*{}_{transverse\ load} = \int_0^L \pm\ \delta p_n(x) \cdot v(x) \cdot dx \tag{5}$$

being δp_n a virtual transverse load (along member y-axis) applied in a member of the structure and v(x) the real transverse displacement function of this member.

In all the expressions, the sign will depend on the sign of the load according to the corresponding coordinate system. In the structure shown in figure 1, the virtual point load δP and the virtual uniformly distributed transverse load δpn will work. In the case of the point load this virtual complementary work (equation 2) is positive, while the virtual complementary work of the transverse load (equation 5) is negative.

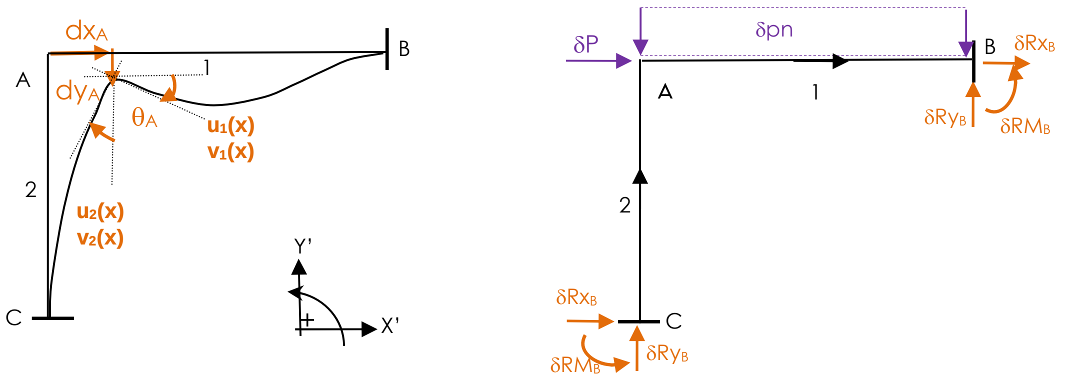


Figure 1. Displacement real scenario (left) and virtual force scenario (right)

4.3 Virtual complementary strain energy (δU)

The virtual complementary work of the internal forces (δW^*_{int}) or internal virtual complementary work is the virtual work done by the internal components of the virtual static group (stresses and internal forces) and the internal components of the kinematic group (strains and deformations) of the real structural system. This work is stored as virtual complementary strain energy (δU*). The independent variables are the stresses.

The virtual complementary strain energy is the sum of the virtual complementary strain energy of all the members of the structure which deform in the virtual scenario. In the case of plane-framed structures, with an elastic and linear behaviour and neglecting the deformation due to shear, the virtual complementary strain energy will be the sum of the virtual complementary strain energy due to axial loading and due to bending (flexure). The corresponding expressions are the following:

- Virtual complementary strain energy due to axial loading (δU^*_{ax}):

  $$\delta U^*_{ax} = \int_0^L \frac{N(x)}{EA}\,\delta N(x)\,dx \tag{7}$$

  being E the modulus of elasticity (Young's modulus), A the cross-section area, N(x) the real axial force function and δN(x) the virtual axial force function.

- Virtual complementary strain energy due to bending (δU^*_{flex}):

  $$\delta U^*_{flex} = \int_0^L \frac{M(x)}{EI}\,\delta M(x)\,dx \tag{8}$$

  being E the modulus of elasticity (Young's modulus), I the cross-section moment of inertia, M(x) the real bending moment function and δM(x) the virtual bending moment function.

4.4 Application of the PVF to the calculation of statically indeterminate structures.

With the Principle of Virtual Forces (PVF) any statically indeterminate structure can be solved, obtaining firstly all the redundant forces (primary static unknowns) which are the only unknowns in the energy balance equations. These unknowns are included in the expression of the virtual complementary strain energy (δU*), as the real internal forces functions of each member are expressed in terms of them.

Therefore, the number of virtual scenarios should be the same as the number of redundant forces. Additionally, they should be independent, fulfil the equilibrium equations and avoid adding new unknowns, in the expression of the virtual complementary work of the external forces (δW*).

Each virtual scenario will provide us with an energy balance equation (δW*=δU*) representing a compatibility equation. From the system of independent compatibility equations, the redundant forces will be obtained.

The procedure to be followed is:

1. Determining the number of primary static unknowns (n), that is the number of redundant forces, known as degree of static indeterminacy (=dsi).
2. Selection of the redundant forces.
3. REAL STRUCTURE: Fulfilment of the equilibrium conditions, obtaining the internal forces functions of all the members (N(x) and M(x)), which will be expressed in terms of the redundant forces.
4. VIRTUAL FORCES SCENARIOS: Fulfilment of the equilibrium conditions in each virtual scenario. Each virtual scenario consists of:
   - An only arbitrary virtual force or moment (we will apply a virtual unit load) in the direction and point of application of a redundant force (if it is an

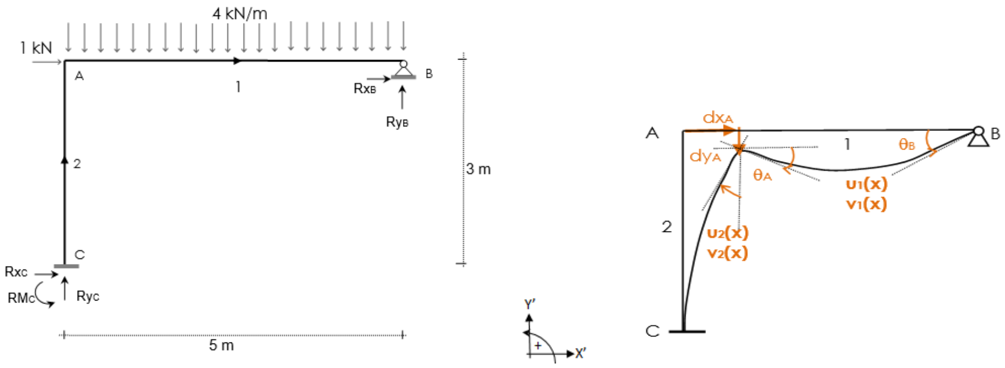
external redundant, in case of an internal redundant a pair of unit virtual forces in opposite direction would be applied)

- Virtual redundants must be defined: the redundants in the real structure are chosen as virtual redundants for all the virtual forces scenarios
- The virtual redundants are assigned a value of 0, being the virtual forces scenarios statically defined.
- In each virtual scenario, the internal forces functions of all the members ($\delta N(x)$ y $\delta M(x)$) will be obtained

5. Formulating the energy balance equation in each virtual forces scenario
6. Solving the system of equations to obtain the redundant forces
7. Solving statically the structure from the redundant forces. We obtain the real static unknowns and the internal forces functions by substituting the value of the redundant forces with their sign in the corresponding equations.

4.5 Worked example

Figure 2 shows the forces and displacements of a statically indeterminate plane-framed structure. Following the procedure established in 4.4, we will solve statically this structure, obtaining the redundant forces and then the reactions, the member end internal forces, the internal forces functions and the diagrams.



Data: member 1: IPE 200: $A = 28.5\ cm^2$ $I = 1948\ cm^4$
member 2: HEB 120: $A = 34\ cm^2$ $I = 864\ cm^4$
$E = 210000\ N/mm^2$

Figure 2. Real structure

<u>1. Number of redundant forces (degree of static indeterminacy):</u> The structure is hyperstatic to the second degree, therefore, the number of redundant forces is 2

<u>2. Selection of the redundant forces:</u> This problem will be solved taking as redundant forces the reactions in the support B (Rx_B y Ry_B). There are other possibilities such as taking as redundant forces the internal moment in member 2 j-end and the horizontal reaction in C ($M_{j2}=-M_{i1}$ y Rx_C) or the internal moment in member 2 j-end and the moment reaction in C ($M_{j2}=-M_{i1}$ y RM_C).

<u>3. Real structure</u>:
With the equilibrium equations we obtain the reactions in C, the member end internal forces and the internal forces functions of all the members (N(x) and M(x)), which will be expressed in terms of the redundant forces (figure 3).

Equilibrium in the real structure:

$\sum F_x=0$	$Rx_C + Rx_B + 1 = 0$	$Rx_C = -1 - Rx_B$	(9)
$\sum F_y=0$	$Ry_C + Ry_B = 20$	$Ry_C = 20 - Ry_B$	(10)
$\sum M_C=0$	$RM_C + Ry_B\, 5 = 53 + Rx_B\, 3$	$RM_C = 53 + Rx_B\, 3 - Ry_B\, 5$	(11)

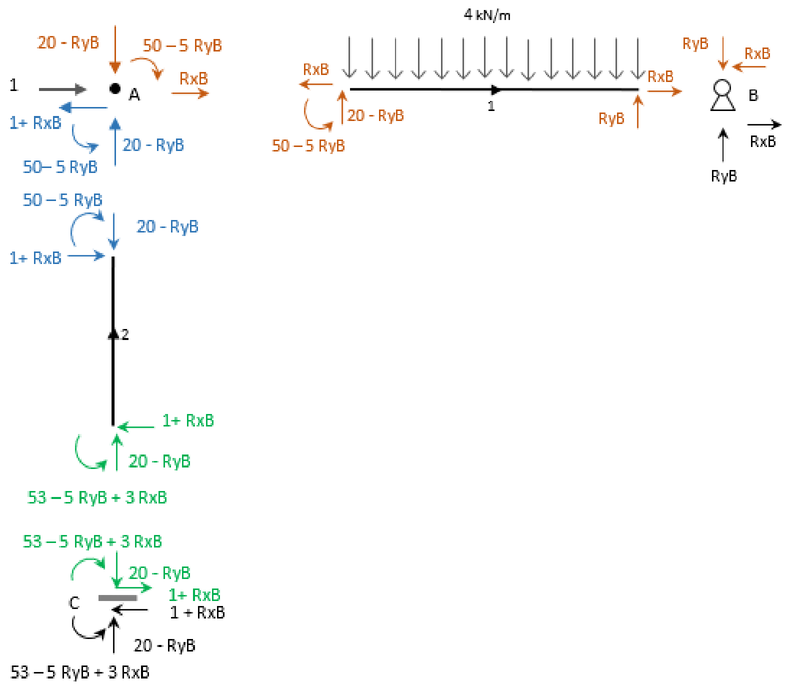


Figure 3. Equilibrium in the real structure

Internal forces functions

$N_1 = Rx_B$	$M_1(x) = -50 + 20x - 2x^2 + 5\,Ry_B - Rx_B \cdot x$	(12)(13)
$N_2 = Ry_B - 20$	$M_2(x) = -53 - 3\,Rx_B + 5\,Ry_B + Rx_B \cdot x + x$	(14)(15)

4.1. Virtual force scenario 1

In this virtual scenario a virtual positive unit load is applied in the support B, in the direction of the redundant Rx_B (figure 4). To have a static admissible configuration, we will assign a zero value to the virtual redundants which will be the same as the real redundants, δRx_B (= 0) and δRy_B (= 0).

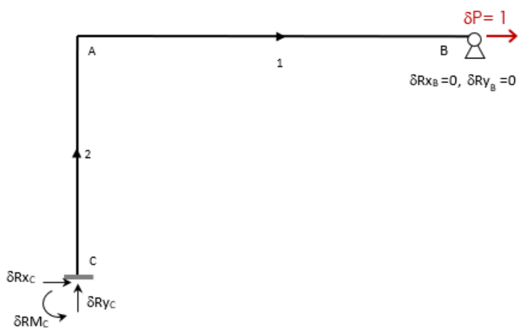


Figure 4. Virtual scenario 1

Equilibrium in the virtual scenario 1

$\sum F_x=0$	$\delta Rx_C = -1$	(16)
$\sum F_y=0$	$\delta Ry_C = 0$	(17)
$\sum M=0$	$\delta RM_C = 3$	(18)

Internal forces functions

$\delta N_1 = 1$ $\delta M_1(x) = 0$ (19)(20)

$\delta N_2 = 0$ $\delta M_2(x) = x - 3$ (21)(22)

4.2. Virtual force scenario 2

In this virtual scenario a virtual positive unit load is applied in the support B, in the direction of the redundant Ry_B (figure 5). To have a static admissible configuration, we will assign a zero value to the virtual redundants which will be the same as the real redundants, δRx_B (= 0) and δRy_B (= 0).

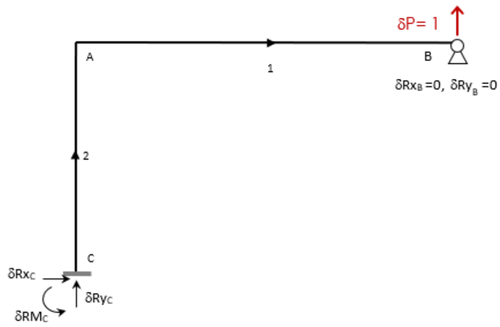


Figure 5. Virtual scenario 2

Equilibrium in the virtual scenario 2

$\sum F_x=0$ $\delta Rx_C = 0$ (23)

$\sum F_y=0$ $\delta Ry_C = -1$ (24)

$\sum M=0$ $\delta RM_C = -5$ (25)

Internal forces functions

$\delta N_1 = 0$ $\delta M_1(x) = 5 - x$ (26)(27)

$\delta N_2 = 1$ $\delta M_2(x) = 5$ (28)(29)

5.1 Energy balance equation 1 (virtual scenario 1) ($\delta W^*=\delta U^*$):

- External virtual complementary work: work done by the external virtual unit load (δP=1) and the displacement of its point of application in the real structure. This work is null because there is no horizontal displacement in B (dx_B=0) in the real structure.

$$\delta W^* = \delta P \cdot dx_B = 0 \quad \rightarrow \quad \delta W^* = 1 \cdot 0 = 0 \qquad (30)$$

- Virtual complementary stain energy: Sum of the virtual complementary strain energy due to axial loading of member 1 and the virtual complementary strain energy due to bending of member 2.

$$\delta U^* = \delta U^*_{ax1} + \delta U^*_{flex2} \qquad (31)$$

$$\delta U^*_{ax1} = \int_0^{L_1} \frac{N_1}{EA_1} \delta N_1 \, dx \quad \text{and} \quad \delta U^*_{flex2} = \int_0^{L_2} \frac{M_2(x)}{EI_2} \delta M_2(x) \, dx$$

5.2 Energy balance equation 2 (virtual scenario 2) ($\delta W^*=\delta U^*$):

- External virtual complementary work: work done by the external virtual unit load (δP=1) and the displacement of its point of application in the real structure. This work is null because there is no vertical displacement in B (dy_B=0) in the real structure.

$$\delta W^* = \delta P \cdot dy_B = 0 \quad \rightarrow \quad \delta W^* = 1 \cdot 0 = 0 \qquad (32)$$

- Virtual complementary stain energy: Sum of the virtual complementary strain energy due to bending of member 1 and the virtual complementary strain energy due to axial and bending of member 2.

$$\delta U^* = \delta U^*_{flex1} + \delta U^*_{ax2} + \delta U^*_{flex2} \qquad (33)$$

$$\delta U^*_{flex1} = \int_0^{L_1} \frac{M_1(x)}{EI_1} \delta M_1(x)\, dx$$

$$\delta U^*_{ax2} = \int_0^{L_2} \frac{N_2}{EA_2} \delta N_2\, dx \quad \text{and} \quad \delta U^*_{flex2} = \int_0^{L_2} \frac{M_2(x)}{EI_2} \delta M_2\, dx$$

<u>6.</u> Solving the system of equations to obtain the redundant forces
The two energy balance equations (comparability equations) are formulated, after solving the integrals. The only unknowns are the redundant forces

(30) = (31) $\quad 0 = 0.1289 + 4.9687 \cdot 10^{-3}\, Rx_B - 1.2401 \cdot 10^{-2}\, Ry_B$ (34)

(32) = (33) $\quad 0 = -0.5022 - 1.2401 \cdot 10^{-2}\, Rx_B + 5.1525 \cdot 10^{-2}\, Ry_B$ (35)

$Rx_B = -4.07$ kN $\qquad Ry_B = +8.77$ kN

<u>7. Solving statically the structure from the redundant forces.</u>
Replacing the value of Rx_B and Ry_B in (9), (10), (11), (12), (13), (14) y (15) we obtain all the reactions and the internal forces functions.

Rx_C= 3.07 kN $\qquad Ry_C$= 11.23 kN $\qquad RM_C$= -3.06 kNm

$N_1 = -4.07 \qquad V_1(x) = 11.23 - 4x \qquad M_1(x) = -6.15 + 11.23x - 2x^2$

$N_2 = -11.23 \qquad V_2 = -3.07 \qquad M_2(x) = 3.06 - 3.07x$

Figure 6 shows the equilibrium and the diagrams in the real structure

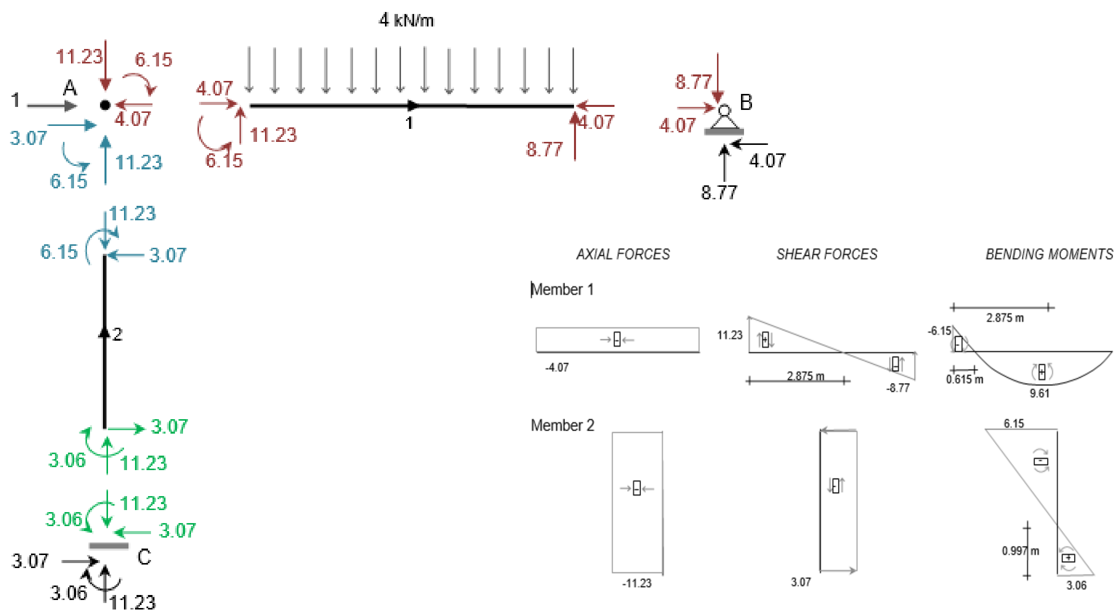


Figure 6. Equilibrium and diagrams in the real structure

5 Closing

In this document we have solved statically an hyperstatic structure with the Principle of Virtual Forces.

As a practical application and self-training, apart from solving the problem in 4.5 taking the other 2 possible set of redundant forces (which of course will lead to the same result), we propose to obtain the reactions of the structure shown in figure 7. (To simplify the problem, neglect the axial and shear deformation, thus considering that there will only be virtual strain energy due to bending).

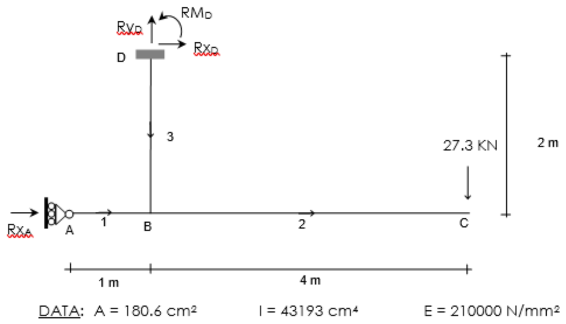


Figure 7. Self-training example

(Results. The structure is hyperstatic to the first degree, therefore we will need only a virtual force scenario.

Rx_A = 140.54 kN, Rx_D = -140.54 kN, Ry_D =27.3 kN, RM_D = -171.88 kNm)

6 Bibliography

6.1 Books:

[1] Abdilla E. "Fundamentos energéticos de la Teoría de Estructuras. Segunda parte-Aplicaciones. Volumen 1". Editorial UPV, ref.: 2003.718, 2003

[2] Basset-Salom L. (2013). Aplicación del Principio de las Fuerzas Virtuales a la resolución estática de estructuras hiperestáticas Colección Artículos docentes ETSA. Disponible en: http://hdl.handle.net/10251/30428

6.2 Figures: Author of the figures: Luisa Basset