Convergence of an iteration of fifth-order using weaker conditions on first order Fréchet derivative in Banach spaces

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Abstract

The convergence analysis both local under weaker Argyros-type conditions and semilocal under ω-condition is established using first order Fréchet derivative for an iteration of fifth order in Banach spaces. This avoids derivatives of higher orders which are either difficult to compute or do not exist at times. The Lipchitz and the Hölder conditions are particular cases of the ω-condition. Examples can be constructed for which the Lipchitz and Hölder conditions fail but the ω-condition holds. Recurrence relations are used for the semilocal convergence analysis. Existence and uniqueness theorems and the error bounds for the solution are provided. Different examples are solved and convergence balls for each of them are obtained. These examples include Hammerstein-type integrals to demonstrate the applicability of our approach.

Keywords: Local convergence; Semilocal convergence; Hammerstein-type integral equation; Argyros-type conditions; Fréchet derivative.

Mathematical Subject Classification 65H10, 47H99

1. Introduction

Many challenging problems in science and engineering require obtaining solutions of nonlinear equations. It is extensively studied as mathematical modeling of many real-world problems [9, 10, 27] involves differential equations, difference equations, integral equations, etc. which leads to solving these equations. Consider

$$F(x) = 0, \tag{1.1}$$

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where, $F : D \subseteq X \to Y$ be twice differentiable nonlinear operator in an open convex region $D_0 \subseteq D$ for Banach spaces X and Y. Generally, one point and multi points iterative methods along with their convergence analysis and estimation of error bounds are used for solving (1.1). Local convergence analysis [21, 23, 28, 29, 33] involves the information about the solution whereas the semilocal convergence analysis [7, 15, 5, 6, 22, 25, 26, 30, 32] uses information about the initial point. The quadratically convergent Newton's method used for solving (1.1) is given for $n = 0, 1, 2, \ldots$ by

$$x_{n+1} \quad = x_n - \Gamma_n F(x_n), \tag{1.2}$$

where, $\Gamma_n = F'(x_n)^{-1}$ and x_0 is the starting point. Kantorovich [27] established sufficient conditions for its convergence by using either majorizing sequences or recurrence relations. But recurrence relations are often used as problems in Banach spaces here are changed to simpler problems with real sequences and functions. Higher order iterative methods and their convergence analysis are also used for solving (1.1). It is worth mentioning that higher order methods require higher computational cost. Also, they involve derivatives of higher order which are either expensive to evaluate or become unbounded. Thus, they are only of theoretical interest.

Recently, higher order iterative methods without using higher order derivatives for solving (1.1) have also gained importance with the advancement of technology and development of fast computers. Thus, they are useful for applications which require quick convergence. The local convergence of a method of third order using Argyros-type conditions on F' is discussed in [17]. For a third order family of iterations, it is described in [18] for the Lipschitz condition on F'. It is given for $n \geq 0$, by

$$
\begin{aligned}
y_n &= x_n - \theta \Gamma_n F(x_n), \\
x_{n+1} &= x_n - \frac{\theta^2 + \theta - 1}{\theta^2} \Gamma_n F(x_n) - \frac{1}{\theta^2} \Gamma_n F(y_n),
\end{aligned}
\tag{1.3}
$$

where, $\theta \in \mathbb{R} - \{0\}$ and x_0 is the starting point. Its semilocal convergence analysis using same condition is given in [19]. For $\theta = 1$, its semilocal convergence is done in [31] under Hölder condition on F'. For Halley-type iterations, it is performed under the Lipschitz condition on F' in [13, 14]. For the variants of the Chebyshev-Halley iteration family, it is studied in [16] under the Hölder condition on F'. The local convergence analysis of deformed Halley method on F' under Hölder condition is developed in [24]. Likewise, a number of iterations of fourth order with convergence analysis using recurrence relations are also developed for solving (1.1) under various conditions. Hueso and Martínez [12] performed the semilocal convergence of a family of iterations of higher order under the Lipschitz condition on F''. It is given for $n \geq 0$,

by

$$
\begin{aligned}
y_n &= x_n - \theta \Gamma_n F(x_n), \\
z_n &= y_n - \Gamma_n F(y_n), \\
x_{n+1} &= z_n - \Gamma_n F(z_n),
\end{aligned}
\tag{1.4}
$$

where, $\theta \in \mathbb{R} - \{0\}$ and x_0 is the starting point. For $\theta = \pm 1$, it leads to fourth order iterative methods. Its local convergence analysis is carried out assuming the Lipschitz condition on F' in [20].

Recently, Hernández and Martínez [4] developed the semilocal convergence of (1.4) for $\theta = 1$ under the ω-condition on F'. This condition is given for a continuous and non-decreasing function $\omega : R_+ \to R_+$ such that $\omega(0) \geq 0$ by $\|F'(x) - F'(y)\| \leq \omega(\|x - y\|)$, $\forall \, x, y \in D$. This condition generalizes the Lipschitz/Hölder condition given by $\omega(x) = Nx$ and $\omega(x) = Nx^\alpha, \alpha \in (0, 1]$.

We shall describe the convergence analysis, both local and semilocal, for an iteration of fifth order discussed in [8] using weaker Argyros-type and ω-conditions on first order Fréchet derivative for solving (1.1) in Banach spaces. It is given for $n = 0, 1, 2, \ldots$, by

$$
\begin{aligned}
y_n &= x_n - \Gamma_n F(x_n), \\
z_n &= y_n - 5\Gamma_n F(y_n), \\
x_{n+1} &= z_n - \frac{1}{5}\Gamma_n[-16F(y_n) + F(z_n)],
\end{aligned}
\tag{1.5}
$$

where, x_0 is the starting point. Using recurrence relations, its convergence under Lipschitz and Hölder conditions on F' are provided in [1, 2, 3]. Clearly, the Lipschitz and Hölder conditions on F' are not satisfied for the following nonlinear Hammerstein type [11] integral equation

$$
x(r) + \sum_{i=1}^{m} \int_a^b K_i(r, s) S_i(x(s)) ds = f(r), \ r \in [a, b],
\tag{1.6}
$$

where functions f, K_i and S_i for $i = 1, 2, \ldots m$ are known, the solution x is to be determined and $-\infty < a < b < +\infty$. In order to solve (1.6), we have to solve

$$
F(x)(r) = x(r) + \sum_{i=1}^{m} \int_a^b K_i(r, s) S_i(x(s)) ds - f(r).
\tag{1.7}
$$

If $S_i'(x(s))$ is (M_i, α_i)- Hölder continuous in D, then, under max-norm, we have

$$
\|F'(x) - F'(y)\| \leq \sum_{i=1}^{m} M_i \|x - y\|^{\alpha_i}, \ M_i \geq 0, \ \alpha_i \in [0, 1], \ \forall \, x, y \in D.
\tag{1.8}
$$

For different α_i, F' neither satisfies Lipschitz nor Hölder condition but satisfies the weaker ω-condition.

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The rest of the paper is arranged as follows. In section 2, the local convergence under weaker Argyros-type condition on F' for an iteration of fifth-order in Banach spaces is described. The convergence is performed by deriving recurrence relations. The existence and uniqueness theorems and the error bounds for the solution are provided. Different numerical examples are also solved here. Section 3 is devoted to the semilocal convergence analysis under weaker ω-condition on F' for the same iteration. Here also, the recurrence relations are used. The existence and uniqueness theorems and error bounds for the solution are provided. Different numerical examples are solved. Finally, conclusions are included in section 4.

2. Local convergence analysis

Here, we shall discuss the local convergence of (1.5) for solving (1.1) under weaker Argyros-type condition on F' in Banach spaces. Let $x^* \in D$, $F(x^*) = 0$, $F'(x^*)^{-1}$ exists and F' satisfies the following Argyros-type conditions for strictly increasing functions f_0 and g_0 defined on the interval $[0, \infty)$ with values in $[0, \infty)$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq f_0(\|x - y\|), \ x, y \in D. \tag{2.1}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq g_0(\|x - x^*\|), \ x \in D. \tag{2.2}$$

where $f_0(0) = 0$, and $g_0(0) = 0$. Define a parameter ρ_0 by

$$\rho_0 = \{u \geq 0; g_0(u) < 1\}.$$

For local convergence analysis, following theorem is given.

Theorem 2.1. *Let $x^* \in D$ and $F'(x^*)^{-1}$ exists such that (2.1) and (2.2) hold. If $g_0(\rho) < 1$ then the sequence $\{x_n\}$ obtained by (1.5) is defined, conatined in $B(x^*, \rho)$ and converges to x^* for $x_0 \in B(x^*, \rho) = \{x \in X : \|x - x^*\| < \rho\} \subset D$ for some suitable radius ρ. Also, the error bounds for all $n \geq 0$ are*

$$\|y_n - x^*\| \leq h_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < \rho, \tag{2.3}$$

$$\|z_n - x^*\| \leq h_2(\|x_n - x^*\|)\|x_n - x^*\| < \rho, \tag{2.4}$$

$$\|x_{n+1} - x^*\| \leq h_3(\|x_n - x^*\|)\|x_n - x^*\| < \rho, \tag{2.5}$$

where, h_1, h_2 and h_3 are the functions to be defined.

Proof. Using $x_0 \in D$ and from (2.2), we get

$$\|I - F'(x^*)^{-1}F'(x_0)\| \leq g_0(\|x_0 - x^*\|) \leq g_0(\rho) < 1.$$

Thus, Γ_0 exists and using Banach Lemma for invertible operators, we get

$$\|\Gamma_0 F'(x^*)\| \leq \frac{1}{1 - g_0(\|x_0 - x^*\|)}. \tag{2.6}$$

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Therefore, y_0 is well defined and hence z_0 and x_1 are well defined. For $n = 0$ and using (1.5), we get

$$y_0 - x^* = -\Gamma_0 F'(x^*) \int_0^1 F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta.$$

Taking norm on both sides and using (2.1) and (2.6), we get

$$
\begin{aligned}
\|y_0 - x^*\| &\leq \frac{\int_0^1 f_0(|1 - \theta|\|x_0 - x^*\|)d\theta}{(1 - g_0(\|x_0 - x^*\|))}\|x_0 - x^*\| \\
&\leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\|,
\end{aligned}
\tag{2.7}
$$

where,

$$h_1(u) = \frac{\int_0^1 f_0(|1 - \theta|u)d\theta}{(1 - g_0(u))}.$$

Let $g_1(u) = h_1(u) - 1$. Then $g_1(0) = -1$ and $g_1(\rho_0^-) \to \infty$. Therefore, $g_1(u)$ has a minimum positive root r_1 in $[0, \rho_0)$. Thus, for $u \in [0, r_1)$, we get $0 \leq h_1(u) < 1$ and

$$\|y_0 - x^*\| \leq h_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|. \tag{2.8}$$

Again, for $n = 0$ and using (1.5), we get

$$\|z_0 - x^*\| \leq \|y_0 - x^*\| + 5\|F'(x^*)^{-1}F(y_0)\|\|\Gamma_0 F'(x^*)\|. \tag{2.9}$$

Since,

$$
\begin{aligned}
\|F'(x^*)^{-1}F(y_0)\| &\leq \left\|\int_0^1 F'(x^*)^{-1}(F'(x^* + \theta(y_0 - x^*)) - F'(x^*))d\theta + I\right\|\|(y_0 - x^*)\| \\
&\leq \left(1 + \int_0^1 g_0(|\theta|\|y_0 - x^*\|)d\theta\right)\|(y_0 - x^*)\|,
\end{aligned}
\tag{2.10}
$$

and on using (2.6) and (2.10), we get

$$
\begin{aligned}
\|z_0 - x^*\| &\leq \left(1 + 5\left(\frac{1 + \int_0^1 g_0(|\theta|\|y_0 - x^*\|)d\theta}{1 - g_0(\|(x_0 - x^*)\|)}\right)\right)\|y_0 - x^*\| \\
&\leq \left(1 + \frac{5\left(1 + \int_0^1 g_0(|\theta|h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta\right)}{1 - g_0(\|x_0 - x^*\|)}\right)h_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&\leq h_2(\|x_0 - x^*\|)\|x_0 - x^*\|.
\end{aligned}
\tag{2.11}
$$

where,

$$h_2(u) = \left(1 + \frac{5\left(1 + \int_0^1 g_0(|\theta|h_1(u)u)d\theta\right)}{1 - g_0(u)}\right)h_1(u).$$

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Let $g_2(u) = h_2(u) - 1$. Then $g_2(0) = -1$ and $g_2(\rho_0^-) \to \infty$. Therefore, $g_2(u)$ has a minimum positive root r_2 in $[0, \rho_0)$. Then for $u \in [0, r_2)$, we get $0 \le h_2(u) < 1$ and

$$\|z_0 - x^*\| \quad \le h_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|. \tag{2.12}$$

Again, from (1.5) for $n = 0$, and by using (2.6) and (2.10), we get

$$
\begin{aligned}
\|x_1 - x^*\| \quad &\le \quad \|z_0 - x^*\| + \frac{1}{5}\|\Gamma_0 F'(x^*)\|\left(16\|F'(x^*)^{-1}F(y_0)\| + \|F'(x^*)^{-1}F(z_0)\|\right)\\
&\le \quad \left(h_2(\|x_0 - x^*\|) + \frac{1}{5}\frac{1}{1 - g_0(\|x_0 - x^*\|)}\left(16(1 + \int_0^1 g_0(|\theta|h_1(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta)\right.\right.\\
&\qquad \left.\left. h_1(\|x_0 - x^*\|) + \left(1 + \int_0^1 g_0(|\theta|h_2(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta\right)h_2(\|x_0 - x^*\|)\right)\right)\|x_0 - x^*\|\\
&\le \quad h_3(\|x_0 - x^*\|)\|x_0 - x^*\|, \tag{2.13}
\end{aligned}
$$

where,

$$
h_3(u) \quad = \quad \left(h_2(u) + \frac{1}{5}\frac{1}{1 - g_0(u)}\left(16h_1(u)\left(1 + \int_0^1 g_0(|\theta|h_1(u)u)d\theta\right) + \left(1 + \int_0^1 g_0(|\theta|h_2(u)u)d\theta\right)h_2(u)\right)\right).
$$

Let $g_3(u) = h_3(u) - 1$. Then $g_3(0) = -1$ and $g_3(\rho_0^-) \to \infty$. Therefore, $g_3(u)$ has a minimum positive root r_3 in $[0, \rho_0)$. Then for $u \in [0, r_3)$, we get $0 \le h_3(u) < 1$ and

$$\|x_1 - x^*\| \quad \le h_3(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|. \tag{2.14}$$

Let $\rho = \min\{r_1, r_2, r_3\}$. Then, for $u \in [0, \rho)$, we get $0 \le h_1(u) < 1$, $0 \le h_2(u) < 1$ and $0 \le h_3(u) < 1$. Thus, theorem is proved for $n = 0$. By replacing x_0, y_0, z_0 and x_1 by x_n, y_n, z_n, x_{n+1} in the preceding way, we get the error bounds given by (2.3)-(2.5). Using the estimate $\|x_{n+1} - x^*\| < \|x_n - x^*\| < \rho$, we get $x_{n+1} \in B(x^*, \rho)$. Clearly, $h_3(u)$ is an increasing function in its domain and since, $h_3(u) < 1 \; \forall \; u \in [0, \rho)$, this gives

$$
\begin{aligned}
\|x_{n+1} - x^*\| \le h_3(u)\|x_n - x^*\| &\le h_3(u)h_3(\|x_{n-1} - x^*\|)\|x_{n-1} - x^*\|\\
&\le h_3(u)^2 h_3(\|x_{n-2} - x^*\|)\|x_{n-2} - x^*\| \le \ldots \le h_3(u)^{n+1}\|x_0 - x^*\|.
\end{aligned}
$$

Thus, $\lim_{n\to\infty} x_n = x^*$ as $\lim_{n\to\infty} h_3(u)^{n+1} = 0$. $\qquad\square$

2.1. Numerical examples

In this subsection, we have worked out a number of examples to demonstrate the applicability of our approach.

Example 2.1. *Let $X = Y = \mathbb{R}$ and define F on $D = (-1, 1)$ by $F(x) = \sin x + x^{7/4}$.*

It is obvious that $x^* = 0$. For $x, y \in D$, we get $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le \left(\|x - y\| + \frac{7}{4}\|x - y\|^{3/4}\right)$. It is easy to see that neither Lipschitz nor Hölder conditions hold on F'. Clearly, $f_0(x) = x + \frac{7}{4}x^{3/4}$ and $g_0(x) = x + \frac{7}{4}x^{3/4}$. Thus, weaker Argyros-type conditions hold on F'. By applying Theorem 2.1, $x^* \in B(0, 0.023251)$.

Example 2.2. *Let $X = Y = \mathbb{R}$ and define F on $D = (-1, 1)$ by $F(x) = e^x - 1$.*

Clearly, $x^* = 0$. For $x, y \in D$, we have $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq e\|x - y\|$. Thus, F' satisfies both Lipschitz and Hölder conditions. Clearly, $f_0(x) = ex$ and $g_0(x) = (e-1)x$. Thus, weaker Argyros-type conditions also hold. By applying Theorem 2.1, $x^* \in B(0, 0.05708)$.

Example 2.3. *Consider the nonlinear Hammerstein type integral equation given by*

$$F(x)(r) = x(r) - 5 \int_0^1 r\, s\, x(s)^{1+\alpha} ds, \ \alpha \in (0, 1), \tag{2.15}$$

with $x(r)$ in $\mathcal{C}[0, 1]$.

Clearly, $x^* = 0$. For $x, y \in D$, we have $\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq 2.5(1 + \alpha)\|x - y\|^\alpha$. Thus, Hölder condition holds on F'. Clearly, $f_0(x) = 2.5(1 + \alpha)x^\alpha$ and $g_0(x) = 2.5(1 + \alpha)x^\alpha$. Thus, weaker Argyros-type conditions also hold. Taking $\alpha = 1/2$ and by applying Theorem 2.1, $x^* \in B(0, 0.00087)$.

3. Semilocal convergence

Here, we shall discuss the semilocal convergence analysis using recurrence relations of (1.5) for solving (1.1) under weaker ω-condition on F' in Banach spaces. First of all, some preliminary results are given. Next, recurrence relations are derived. The existence, uniqueness theorems and error bounds for the solution are also provided. Finally, different examples are solved to demonstrate the efficacy of our approach.

3.1. Preliminary results

In this subsection, we shall describe some preliminary results used for the convergence of (1.5) for solving (1.1). Let $x_0 \in D$ and $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y, X)$ exists, where $\mathcal{L}(Y, X)$ denotes the set of bounded linear operators from Y to X. Let the following conditions hold.

(1) $\|\Gamma_0\| \leq \beta$,

(2) $\|\Gamma_0 F(x_0)\| \leq \eta$,

(3) $\|F'(x) - F'(y)\| \leq \omega(\|x - y\|), x, y \in D$, for a continuous non-decreasing real function $\omega(x)$, $x > 0$, $\omega(0) \geq 0$ such that, $\omega(tx) \leq t^\alpha \omega(x)$ for $t \in [0, 1]$, $x \in (0, \infty)$, $\alpha \in [0, 1]$.

Let $b_0 = \beta w(\eta)$, and define the sequence $\{b_n\}$, where,

$$b_{n+1} = b_n \phi(b_n)^{1+\alpha} \psi(b_n)^\alpha, \quad n \geq 0, \tag{3.1}$$

$$\phi(u) = \frac{1}{1 - u(1 + \psi(u))^\alpha}, \tag{3.2}$$

$$\psi(u) = \left(\frac{u}{1 + \alpha} + (u + 1)\xi(u) + \left(\frac{u}{1 + \alpha} \right) \xi(u)^{1+\alpha} \right), \tag{3.3}$$

and

$$\xi(u) = \left(\frac{4u}{5(1+\alpha)} + \frac{u}{5(1+\alpha)}\left(1 + \frac{5u}{1+\alpha}\right)^{1+\alpha}\right). \tag{3.4}$$

In order to investigate the convergence of $\{x_n\}$ generated by (1.5) to a solution of (1.1), we have to show that $\{x_n\}$ is a Cauchy sequence. For this, we have to study the properties of $\{b_n\}$. The following lemmas are proved to study the properties of the sequence $\{b_n\}$.

Lemma 3.1. *Let $\phi(u), \psi(u)$ and $\xi(u)$ be the functions given by (3.2)-(3.4) and s_0 is the minimum positive root of $h(u) = u(1 + \xi(u))^\alpha - 1$ in $(0, 1)$. Then*
(i) $\phi(u)$ is increasing and $\phi(u) > 1$ for $u \in (0, s_0)$,
(ii) $\psi(u)$ and $\xi(u)$ are the increasing functions for $u \in (0, s_0)$.

Proof. The proof is trivial. ◻

Consider the auxiliary function

$$r(u) = \left(\frac{u}{1+\alpha} + (u+1)\xi(u) + \frac{u\,\xi(u)^{1+\alpha}}{1+\alpha}\right)^\alpha - \left(\ 1 - u(1+\xi(u))^\alpha\right)^{1+\alpha}.$$

Clearly, $r(0) < 0$, $r(s_0) > 0$ and $r^{'}(u) > 0$. Thus, $r(u)$ has a real root s_1 in $(0, s_0)$.

Lemma 3.2. *Let $\phi(u)$, $\psi(u)$ and $\xi(u)$ be the functions defined by (3.2)-(3.4), respectively. If $b_0 \in (0, s_1)$, then*
(i) $\phi(b_0)^{1+\alpha}\psi(b_0)^\alpha < 1$,
(ii) $\phi(b_0)\psi(b_0) < 1$,
(iii) The sequence $\{b_n\}$ is decreasing and $b_n < s_1$ for all $n \geq 0$,
(iv) $b_n(1 + \xi(b_n))^p < 1$.

Proof. Taking $u = b_0$ in $r(u)$, we get $\phi(b_0)^{1+\alpha}\psi(b_0)^\alpha < 1$ for all $b_0 \in (0, s_1)$. Since $\phi(b_0) > 1$, this gives $(\phi(b_0)\psi(b_0))^\alpha < 1$, and hence $\phi(b_0)\psi(b_0) < 1$. To prove (iii), we use mathematical induction on (3.1). For $n = 0$, we get $b_1 = b_0\phi(b_0)^{1+\alpha}\psi(b_0)^\alpha < b_0$. Assume that $b_k < b_{k-1}$ for $k \leq n$. Since $\phi(u)$ and $\psi(u)$ are increasing functions, we get $b_{n+1} = b_n\phi(b_n)^{1+\alpha}\psi(b_n)^\alpha < b_{n-1}\phi(b_{n-1})^{1+\alpha}\psi(b_{n-1})^\alpha = b_n$. Hence, $\{b_n\}$ is a decreasing and $b_n < s_1$ for all $n \geq 0$. Since $\xi(u)$ is an increasing function, we get $b_n(1 + \xi(b_n))^\alpha < b_{n-1}(1 + \xi(b_{n-1}))^\alpha < b_0(1 + \xi(b_0))^\alpha < 1 \ \forall \ b_0 \in (0, s_1)$. This proves (iv) as $b_n(1 + \xi(b_n))^\alpha < 1$ for all $n \geq 0$ and $b_0 \in (0, s_1)$. ◻

3.2. **Recurrence relations**

In this subsection, we shall establish the recurrence relations for (1.5) under the assumptions (1),(2) and (3) described in previous subsection. The existence of Γ_0 gives the existence of y_0 and hence $\|\Gamma_0\|\omega(\|y_0 - x_0\|) \leq \beta\omega(\eta) = b_0$. Taking $n = 0$, in (1.5), we get

$$z_0 - x_0 \quad = \quad y_0 - x_0 - 5\Gamma_0 \int_0^1 \left(F^{'}(x_0 + t(y_0 - x_0)) - F^{'}(x_0)\right)(y_0 - x_0)dt.$$

Using norm, we get

$$\|z_0 - x_0\| \le \left(1 + \frac{5b_0}{1+\alpha}\right)\|y_0 - x_0\|. \tag{3.5}$$

Similarly, $\|z_0 - y_0\| \le \frac{5b_0}{1+\alpha}\|y_0 - x_0\|$. Now

$$x_1 - x_0 = -\Gamma_0 F(x_0) - \frac{9}{5}\Gamma_0 F(y_0) - \frac{1}{5}\Gamma_0 F(z_0).$$

Using Taylor's expansion of $F(y_0)$ and $F(z_0)$ about x_0 and taking norm on both sides, we get

$$\begin{aligned}
\|x_1 - x_0\| &\le \|\Gamma_0 F(x_0)\| + \frac{4}{5}\|\Gamma_0\|\int_0^1 \left\|F'(x_0 + t(y_0 - x_0)) - F'(x_0)\right\|\|y_0 - x_0\|dt \\
&+ \frac{1}{5}\|\Gamma_0\|\int_0^1 \left\|F'(x_0 + t(z_0 - x_0)) - F'(x_0)\right\|\|z_0 - x_0\|dt.
\end{aligned} \tag{3.6}$$

Substituting (3.5) in (3.6), we get

$$\begin{aligned}
\|x_1 - x_0\| &\le \left(1 + \frac{4b_0}{5(1+\alpha)} + \frac{b_0}{5(1+\alpha)}\left(1 + \frac{5b_0}{1+\alpha}\right)^{1+\alpha}\right)\|y_0 - x_0\| \\
&\le (1 + \xi(b_0))\|y_0 - x_0\|.
\end{aligned} \tag{3.7}$$

Assuming that $x_n, y_n, z_n \in D$ and $b_0 < s_0$, then, recurrence relations given below can be easily shown by mathematical induction for $n \ge 1$.

(I) $\quad \|\Gamma_n\| \le \psi(b_{n-1})\|\Gamma_{n-1}\|,$

(II) $\quad \|y_n - x_n\| \le \|\phi(b_{n-1})\psi(b_{n-1})\|y_{n-1} - x_{n-1}\|,$

(III) $\quad \|z_n - y_n\| \le \frac{5}{1+\alpha}b_n\|y_{n-1} - x_{n-1}\|,$

(IV) $\quad \|\Gamma_n\|\omega(\|y_n - x_n\|) \le b_n,$

(V) $\quad \|x_n - x_{n-1}\| \le (1 + \xi(b_{n-1}))\|y_{n-1} - x_{n-1}\|.$

Consider

$$\|I - \Gamma_0 F'(x_1)\| \le \|\Gamma_0\|\|F'(x_1) - F'(x_0)\| \le \beta\omega(\|x_1 - x_0\|).$$

Using (3.7), we get $\|I - \Gamma_0 F'(x_1)\| \le \beta(1 + \xi(b_0))^\alpha\omega(\eta) \le b_0(1 + \xi(b_0))^\alpha < 1$. From Banach Lemma, we get Γ_1 exists and

$$\|\Gamma_1\| \le \frac{1}{1 - b_0(1 + \xi(b_0))^\alpha}\|\Gamma_0\| = \phi(b_0)\|\Gamma_0\|. \tag{3.8}$$

Using Taylor's expansion of $F(x_1)$, we get

$$F(x_1) = F(y_0) + F'(y_0)(x_1 - y_0) + \int_{y_0}^{x_1}(F'(x) - F'(y_0))dx. \tag{3.9}$$

Now,

$$\|F(y_0)\| \le \frac{1}{1+\alpha}\omega(\eta)\|y_0 - x_0\|, \tag{3.10}$$

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$$\|F^{'}(y_0)(x_1 - y_0)\| \leq (b_0 + 1)\left\|\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\right\|$$
$$\leq \frac{(b_0 + 1)}{\beta}\xi(b_0)\|y_0 - x_0\|, \tag{3.11}$$

and

$$\left\|\int_{y_0}^{x_1} (F^{'}(x) - F^{'}(y_0))dx\right\| \leq \frac{1}{1+\alpha}\xi(b_0)^{1+\alpha}\omega(\|y_0 - x_0\|)\|y_0 - x_0\|. \tag{3.12}$$

Taking norm on (3.9) and using (3.10),(3.11) and (3.12), we get

$$\|F(x_1)\| \leq \frac{1}{\beta}\left(\frac{b_0}{1+\alpha} + (1+b_0)\xi(b_0) + \frac{b_0\xi(b_0)^{1+\alpha}}{1+\alpha}\right)\|y_0 - x_0\|$$
$$\leq \frac{1}{\beta}\psi(b_0)\|y_0 - x_0\|. \tag{3.13}$$

Using (3.8) and (3.13), we get

$$\|y_1 - x_1\| \leq \|\Gamma_1\|\|F(x_1)\| \leq \phi(b_0)\|\Gamma_0\|\|F(x_1)\| \leq \phi(b_0)\psi(b_0)\|y_0 - x_0\|. \tag{3.14}$$

Now,

$$\|z_1 - y_1\| \leq 5\|\Gamma_1\|F(y_1)\|\| \leq 5\|\Gamma_1\|\int_0^1 \left\|F^{'}(x_1 + t(y_1 - x_1)) - F^{'}(x_1)\right\|\|y_1 - x_1\|dt$$
$$\leq 5\phi(b_0)\|\Gamma_0\|\frac{1}{1+\alpha}\omega(\|y_1 - x_1\|)\|y_1 - x_1\|$$
$$\leq \frac{5}{1+\alpha}\beta\phi(b_0)\phi(b_0)^\alpha\psi(b_0)^\alpha\omega(\|y_0 - x_0\|)\|y_1 - x_1\|$$
$$\leq \frac{5}{1+\alpha}\beta\omega(\|y_0 - x_0\|)\phi(b_0)^{1+\alpha}\psi(b_0)^\alpha\|y_1 - x_1\|$$
$$\leq \frac{5}{1+\alpha}b_0\phi(b_0)^{1+\alpha}\psi(b_0)^\alpha\|y_1 - x_1\|.$$

This gives,

$$\|z_1 - y_1\| \leq \frac{5}{1+\alpha}b_1\|y_1 - x_1\|. \tag{3.15}$$

Using (3.8) and (3.14), we get

$$\|\Gamma_1\|\omega(\|y_1 - x_1\|) \leq \phi(b_0)\|\Gamma_0\|\phi(b_0))^\alpha\psi(b_0)^\alpha\omega(\|y_0 - x_0\|)$$
$$\leq \beta\omega(\|y_0 - x_0\|)\phi(b_0))^{1+\alpha}\psi(b_0)^\alpha$$
$$\leq b_0\phi(b_0)^{1+\alpha}\psi(b_0)^\alpha = b_1.$$

This gives,

$$\|\Gamma_1\|\omega(\|y_1 - x_1\|) \leq b_1. \tag{3.16}$$

Thus, the recurrence relations (I)-(IV) are proved for $n = 1$ by (3.8), (3.14), (3.15) and (3.16) respectively. The recurrence relation (V) is already proved in (3.7) for $n = 1$. Suppose that $x_k, y_k, z_k \in D$, and (I)-(V) holds for $n = k$. In a similar manner, we can prove all the recurrence relations (I)-(IV) by mathematical induction for $n \geq 1$.

3.3. **Convergence Theorem**

In this subsection, we shall give existence and uniqueness theorems for the convergence of the sequence generated by (1.5) to the solution x^* of (1.1). The estimation of error bounds are also provided.

Theorem 3.1. *Let $x_0 \in D$ and the conditions (1), (2) and (3) hold. Let $b_0 = \beta\omega(\eta)$ satisfies $b_0 < s_0$ where, s_0 is the smallest positive root of $u(1 + \xi(u))^\alpha - 1$, $\alpha \in [0,1]$ and $\xi(u)$ is given by (3.4). Then, starting with x_0, the sequence $\{x_n\}$ generated by (1.5) converges to x^* of (1.1) with $x_n, y_n, z_n, x^* \in \overline{B(x_0, R\eta)}$ where, $B(x_0, R\eta) = \{x \in X : \|x - x_0\| < R\} \subset D$, where $R = \frac{5}{1+\alpha}b_0 + \frac{(1+\xi(b_0))}{1-\phi(b_0)\psi(b_0)}$.*

Proof. First of all, we have to prove that y_n and $z_n \in \overline{B(x_0, R\eta)} \subset D$. Using recurrence relation (V), we get

$$
\begin{aligned}
\|x_n - x_0\| &\leq \|x_n - x_{n-1}\| + \ldots + \|x_1 - x_0\| \\
&\leq (1 + \xi(b_{n-1}))\|y_{n-1} - x_{n-1}\| + \cdots + (1 + \xi(b_0))\|y_0 - x_0\| \\
&\leq (1 + \xi(b_0))\sum_{k=0}^{n-1}(\phi(b_0)\psi(b_0))^k\|y_0 - x_0\|.
\end{aligned}
\tag{3.17}
$$

Now, using recurrence relation (II) and (3.17), we get

$$
\begin{aligned}
\|y_n - x_0\| &\leq \|y_n - x_n\| + \|x_n - x_0\| \\
&\leq \phi(b_{n-1})\psi(b_{n-1})\|y_{n-1} - x_{n-1}\| + (1 + \xi(b_0))\sum_{k=0}^{n-1}(\phi(b_0)\psi(b_0))^k\|y_0 - x_0\| \\
&\leq (\phi(b_0)\psi(b_0))^n\|y_0 - x_0\| + (1 + \xi(b_0))\sum_{k=0}^{n-1}(\phi(b_0)\psi(b_0))^k\|y_0 - x_0\| \\
&\leq (1 + \xi(b_0))\sum_{k=0}^{n}(\phi(b_0)\psi(b_0))^k\|y_0 - x_0\| \\
&\leq (1 + \xi(b_0))\frac{1 - (\phi(b_0)\psi(b_0))^{n+1}}{1 - \phi(b_0)\psi(b_0)}\|y_0 - x_0\| < R\eta.
\end{aligned}
\tag{3.18}
$$

Using recurrence relation (III) and (3.18), we get

$$
\begin{aligned}
\|z_n - x_0\| &\leq \|z_n - y_n\| + \|y_n - x_0\| \\
&\leq \frac{5}{1+\alpha}b_0(\phi(b_0)\psi(b_0))^n\|y_0 - x_0\| + (1 + \xi(b_0))^\alpha\frac{1 - (\phi(b_0)\psi(b_0))^{n+1}}{1 - \phi(b_0)\psi(b_0)}\|y_0 - x_0\| \\
&\leq \left(\frac{5}{1+\alpha}b_0 + (1 + \xi(b_0))^\alpha\frac{1 - (\phi(b_0)\psi(b_0))^{n+1}}{1 - \phi(b_0)\psi(b_0)}\right)\|y_0 - x_0\| < R\eta.
\end{aligned}
$$

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Now,

$$
\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 + \xi(b_n))\|y_n - x_n\| \\
&\leq (1 + \xi(b_n))\phi(b_{n-1})\psi(b_{n-1})\|y_{n-1} - x_{n-1}\| \\
&\leq \cdots \leq (1 + \xi(b_n))\left(\prod_{j=0}^{n-1} \phi(b_j)\psi(b_j)\right)\|y_0 - x_0\|.
\end{aligned}
$$

Now, to ensure the convergence of $\{x_n\}$, we show that $\{x_n\}$ is a Cauchy sequence. Consider

$$
\begin{aligned}
\|x_{n+k} - x_n\| &\leq \|x_{n+k} - x_{n+k-1}\| + \ldots + \|x_{n+1} - x_n\| \\
&\leq (1 + \xi(b_{n+k-1}))\left(\prod_{j=0}^{n+k-2} \phi(b_j)\psi(b_j)\right)\eta + \ldots + (1 + \xi(b_n))\left(\prod_{j=0}^{n-1} \phi(b_j)\psi(b_j)\right)\eta \\
&\leq (1 + \xi(b_0))\sum_{l=0}^{k-1}\left(\prod_{j=0}^{n+l-1} \phi(b_j)\psi(b_j)\right)\eta \\
&\leq (1 + \xi(b_0))\sum_{l=0}^{k-1}(\phi(b_0)\psi(b_0))^{l+n}\,\eta.
\end{aligned}
$$

This gives,

$$
\|x_{n+k} - x_n\| \leq (1 + \xi(b_0))\frac{1 - (\phi(b_0)\psi(b_0))^k}{1 - \phi(b_0)\psi(b_0)}(\phi(b_0)\psi(b_0))^n\eta.
$$

Hence $\{x_n\}$ is a Cauchy sequence since, $\phi(b_0)\psi(b_0) < 1$ from Lemma 3.2 and hence, it is convergent. For $n = 0$ and $k \geq 1$, we get

$$
\|x_k - x_0\| \leq (1 + \xi(b_0))\frac{1 - (\phi(b_0)\psi(b_0))^k}{1 - \phi(b_0)\psi(b_0)} < R\eta. \tag{3.19}
$$

Thus, $x_k \in \overline{B(x_0, R\eta)}$. Taking $k \to \infty$ in (3.19), we obtain $x^* \in \overline{B(x_0, R\eta)}$. Now, it is to be shown that x^* is a solution of (1.1). From (1.5), we have $y_n = x_n - \Gamma_n F(x_n)$. This gives

$$
\begin{aligned}
\|F(x_n)\| &\leq \|F'(x_n)\|\|y_n - x_n\| \\
&\leq \|F'(x_n)\|\phi(b_{n-1})\psi(b_{n-1})\|y_{n-1} - x_{n-1}\| \\
&\leq \|F'(x_n)\|(\phi(b_0)\psi(b_0))^n\|y_0 - x_0\|.
\end{aligned} \tag{3.20}
$$

Now,

$$
\begin{aligned}
\|F'(x_n)\| &\leq \|F'(x_n) - F'(x_0)\| + \|F'(x_0)\| \\
&\leq \omega(\|x_n - x_0\|) + \|F'(x_0)\| \\
&< R^\alpha \omega(\eta) + \|F'(x_0)\|.
\end{aligned}
$$

Therefore, $\|F'(x_n)\|$ is bounded. Hence from (3.20), we obtain that $\|F(x_n)\| \to 0$ as $n \to \infty$. Thus by using continuity of F in D, we get $F(x^*) = 0$. $\qquad\square$

Next theorem gives the uniqueness of x^*.

Theorem 3.2. *If r be a positive root of*

$$\frac{2\beta}{1+\alpha}\omega(R+r)\left(1-\frac{1}{2^{1+\alpha}}\right)=1, \tag{3.21}$$

then from theorem 3.1, the solution x^* is unique in $B(x_0,r)\cap D$.

Proof. In order to prove the uniqueness part, suppose there exists $s^* \in B(x_0,r)$ such that $F(s^*)=0$, $s^* \neq x^*$. Then

$$0 = F(s^*) - F(x^*) = \int_0^1 F'(x^* + \theta(s^* - x^*))d\theta(s^* - x^*) = T(s^* - x^*),$$

where $T = \int_0^1 F'(x^* + \theta(s^* - x^*))d\theta$. Consider

$$
\begin{aligned}
\|I - \Gamma_0 T\| &\leq \|\Gamma_0\| \int_0^1 \left\|\left(F'(x^* + \theta(s^* - x^*)) - F'(x_0)\right)\right\| d\theta \\
&\leq \beta \int_0^1 \omega(\|x^* + \theta(s^* - x^*) - x_0\|)d\theta \\
&\leq \beta \int_0^1 \omega(\|(1-\theta)(x^* - x_0) + \theta(s^* - x_0)\|)d\theta \\
&< \beta \int_0^{1/2} (1-\theta)^\alpha \omega(R+r)d\theta + \beta \int_{1/2}^1 \theta^\alpha \omega(R+r)dt \\
&= 2\beta\omega(R+r) \int_{1/2}^1 \theta^\alpha dt \\
&= \frac{2\beta}{1+\alpha}\omega(R+r)\left(1-\frac{1}{2^{1+\alpha}}\right) = 1.
\end{aligned}
$$

Thus, $\|I - \Gamma_0 T\| < 1$. Therefore, T^{-1} exists and hence $s^* = x^*$. $\qquad\qquad \square$

3.4. **Numerical examples**

In this subsection, examples are solved to demonstrate the applicability of our work.

Example 3.1. *Consider the following nonlinear Hammerstein type integral equation*

$$x(r) = 1 + \int_0^1 K(r,s)\left(x(s)^{7/5} + \frac{x(s)^2}{10}\right)ds, \tag{3.22}$$

where, $K(r,s)$ denotes Green's function given by

$$K(r,s) = \begin{cases} r(1-s), & r \leq s, \\ s(1-r), & s \leq r. \end{cases}$$

From (3.22), we get,

$$F(x)(r) = x(r) - 1 - \int_0^1 K(r,s)\left(x(s)^{7/5} + \frac{x(s)^2}{10}\right)ds, \qquad (3.23)$$

where $F : D \subseteq C[0,1] \to C[0,1]$. Therefore

$$F'(x)y(r) = y(r) - \int_0^1 K(r,s)\left(\frac{7}{5}x(s)^{2/5} + \frac{x(s)}{5}\right)y(s)ds.$$

This gives

$$\|F'(x) - F'(y)\| \le \frac{1}{40}\left(7\|x-y\|^{2/5} + \|x-y\|\right).$$

One can easily show the failure of Lipschitz and Hölder conditions and the success of ω-condition. Here, $\omega(x) = \frac{1}{40}\left(7x^{2/5} + x\right)$. For $x_0(r) = 1$, we obtain $\beta = 5/4$, $\eta = 11/64$ and $\alpha = 2/5$. Thus, the conditions of Lemma 1, Lemma 2 and Theorem 2.1 are satisfied. Hence the existence of x^* is guaranteed in $\overline{B(x_0, 0.30701)}$ and the uniqueness in $B(x_0, 15.14198)$.

Example 3.2. *Consider the following nonlinear Hammerstein type integral equation*

$$x(r) = 1 + \int_0^1 K(r,s)x(s)^{3/2}ds, \qquad (3.24)$$

where, $K(r,s)$ denotes the Green's function in [0,1]. Therefore,

$$F(x)(r) = x(r) - 1 - \int_0^1 K(r,s)x(s)^{3/2}ds. \qquad (3.25)$$

Clearly,

$$F'(x)y(r) = y(r) - \frac{3}{2}\int_0^1 K(r,s)x(s)^{1/2}y(s)ds.$$

This gives

$$\|F'(x) - F'(y)\| \le \frac{3}{16}\|x-y\|^{1/2}.$$

Thus, Lipschitz condition on F' fails but Hölder and ω-conditions hold. Here, $\omega(x) = \frac{3}{16}x^{1/2}$. For $x_0(r) = 1$, we obtain $\beta = \frac{16}{13}$, $\eta = \frac{2}{13}$ and $\alpha = \frac{1}{2}$. Therefore, the conditions of Lemma 3.1, Lemma 3.2 and Theorem 3.1 are satisfied. Hence x^* exists in $\overline{B(x_0, 0.23879)}$ and unique in $B(x_0, 25.03679)$. To get numerical solution, we approximate the following integral by using thee Gauss-Legendre formula given by

$$\int_0^1 \zeta(t)dt \simeq \frac{1}{2}\sum_{j=1}^m \beta_j\zeta(t_j),$$

where β_j and t_j denote the weights and nodes respectively and are to be determined. If x_i stands for $x(t_i)$, $i = 1, 2, \ldots n$, then (3.24) gives

$$x_i = 1 + \sum_{j=1}^{n} a_{ij} x_j^{3/2}, \quad i = 1, 2, \ldots m \tag{3.26}$$

where,

$$a_{ij} = \begin{cases} \frac{1}{2} \beta_j t_j (t_i - 1) & if \ \ j \leq i, \\ \frac{1}{2} \beta_j t_i (t_j - 1) & if \ \ j > i. \end{cases}$$

Thus, (3.26) gives

$$x = 1 + Bx^{3/2},$$

where, $B = (b_{ij})$, $x = (x_1, x_2, \ldots, x_n)^T$ and $x^{3/2} = (x_1^{3/2}, x_2^{3/2}, \ldots, x_n^{3/2})^T$. The Table 1 gives the value of weights and nodes for $m = 8$.

Table 1: Value of weights and nodes for m =8

j	1	2	3	4	5	6	7	8
β_j	0.1012	0.2238	0.3137	0.3627	0.3627	0.3137	0.2238	0.1012
t_j	0.0198	0.1016	0.2372	0.4083	0.5917	0.7627	0.8983	0.9801

Table 2: Solution of (3.26)

j	1	2	3	4	5	6	7	8
x_j	0.9911	0.9587	0.9192	0.8930	0.8930	0.9192	0.9587	0.9911.

Starting with x_0, we can approximate the solution by (1.5) in the following manner.

$$\begin{aligned} F'(x_n)a_n &= -F(x_n) \\ y_n &= x_n + a_n \\ F'(x_n)c_n &= -F(y_n) \\ z_n &= y_n + 5c_n \\ F'(x_n)d_n &= -F(z_n) \\ x_{n+1} &= z_n - \frac{16}{5}c_n + \frac{1}{5}d_n. \end{aligned}$$

Figure 1: **Approximate solution of** (3.26)

The numerical solution is displayed in the Table 2 for $x_0 = (1, 1, \ldots, 1)^T$ and $m = 8$. In the Figure 1, interpolating function passing through (t_i, x_i), is an approximate solution of (3.26).

4. Conclusions

Local and semilocal convergence of an iteration of fifth order is established under weaker Argyros-type and ω-conditions on F' for finding the solutions of nonlinear equations in Banach spaces. This avoids derivatives of higher order which are either expensive to compute or are unbounded. Existence and uniqueness theorems and the error bounds for the solution are provided. A number of examples including Hammerstein type nonlinear integral equation are tested to demonstrate the applicability of our approach.

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