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Improving the accessibility of Steffensen's method by decomposition of operators

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Abstract

Solving equations of the form $H(x) = 0$ is usually done by applying iterative methods. The main interest of this paper is to improve the domain of starting points for Setffensen's method. In general, the accessibility of iterative methods that use divided differences in their algorithms is reduced, since there are difficulties in the choice of starting points to guarantee the convergence of the methods. In particular, by using a decomposition of the operator H and applying a special type of iterative methods, which combine two iterative schemes in the algorithms, we can improve the accessibility of Steffensen's method. Moreover, we analyze the local convergence of the new iterative method proposed in two cases: when H is differentiable and H is non-differentiable. The dynamical properties show that the method also improves the region of accessibility of Steffensen's method for non-differentiable operators. So, we present an alternative for the non-applicability of Newton's method to non-differentiable operators that improves the accessibility of Steffensen's method. The theoretical results are illustrated with numerical experiments.

Keywords: Iterative method; Local convergence; Non-differentiable operator; Dynamics; Steffensen's method.
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1 Introduction

One of the most studied problems in numerical mathematics is the solution of nonlinear equations $H(x) = 0$, where H is a continuous nonlinear operator defined on a non-empty open convex subset Ω of a Banach space X with values in X. Iterative methods are a powerful tool for solving these equations.

It is well-known that Newton's method,

$$x_{n+1} = x_n - [H'(x_n)]^{-1} H(x_n), \quad n \geq 0; \quad x_0 \in \Omega \text{ is given}, \tag{1}$$

is one of the most used iterative methods to approximate a solution x^* of $H(x) = 0$. The quadratic convergence and the low operational cost of the method guarantee a good computational efficiency. In addition, this method has good accessibility, so that the domain of starting points of the method is large. But this method has a serious shortcoming: the derivative $H'(x)$ has to be evaluated at each iteration. This makes that the method is not applicable to equations with non-differentiable operators. It is common to approximate derivatives by divided differences for obtaining derivative free iterative methods ([13], [14]). Let us denote by $\mathcal{L}(X, X)$ the space of bounded linear operators from X to X. An operator $[x, y; D] \in \mathcal{L}(X, X)$ is called a first order divided difference for the operator $D : \Omega \subseteq X \to X$ on the points x and y $(x \neq y)$ if

$$[x, y; D](x - y) = D(x) - D(y). \tag{2}$$

Kung and Traub presented a class of multipoint iterative methods without derivatives in [11]. These methods contain Steffensen's method as a special case, where the evaluation of $H'(x)$ in each step of Newton's method is approximated by the divided difference of first order $[x, x + H(x); H]$. Stetffensen's method has been widely studied ([1, 2, 6]) and the algorithm is

$$\begin{cases} x_0 \text{ given in } \Omega, \\ x_{n+1} = x_n - [x_n, x_n + H(x_n); H]^{-1} H(x_n), \quad n \geq 0. \end{cases} \tag{3}$$

The method has quadratic convergence and the same computational efficiency as Newton's method.

Methods using divided differences in their algorithm have a drawback. As we discuss in Section 2, the accessibility of these methods they to the solution of the equation is poor, so that the domains of starting points are reduced. However, as we have already indicated, this is one of the favorable features of Newton's method (1). So, in this work, we try to improve the accessibility of Steffensen's method. For this, we use a new idea that is to perform a decomposition of the operator H that defines the equation $H(x) = 0$. So, we consider

$$H(x) = F(x) + G(x), \tag{4}$$

where $F, G : \Omega \subseteq X \to X$, and define the following Newton-Steffensen-type iterative process:

$$\begin{cases} x_0 \in \Omega \text{ is given}, \\ x_{n+1} = x_n - (F'(x_n) + [x_n, x_n + H(x_n); G])^{-1} H(x_n), \quad n \geq 0, \end{cases} \tag{5}$$

that is applied as

$$\begin{cases} x_0 \in \Omega \text{ is given,} \\ (F'(x_n) + [x_n, x_n + H(x_n); G])\alpha_n = -H(x_n) \\ x_{n+1} = x_n + \alpha_n, \quad n \geq 0. \end{cases} \tag{6}$$

The first aim of this work is to prove that (5) improves greatly the accessibility of Steffensen's method by approximating its domain of starting points to that of Newton's method and taking into account decomposition (4). Moreover, as we have already indicated above, a fundamental problem in the application of Newton's method is the fact that the operator H is non-differentiable, so that we cannot obtain H' to do the iterates. However, as we discuss in Section 3, this technique of decomposition of the operator H also plays a key role for non-differentiable operators.

The second aim of this work is to improve the approximations of solutions of equations defined from non-differentiable operators. In this case, there are two advantages of (5): first, the differentiable part of the operator is considered in the optimal situation, namely $F'(x_n)$; and second, for the non-differentiable part, iteration (3), is considered with $[x_n, x_n + H(x_n); G]$, which has quadratic convergence and the same efficiency as Newton's method. In addition, as we see, we obtain an efficient iterative process for non-differentiable problems.

On the other hand, occasionally, the study of the local convergence of derivative-free iterative processes shows a small contradiction. There are many known results of local convergence (see [3],[4],[8],[9],[15],[16] and references therein) which usually include the condition of the existence of the operator $[H'(x^*)]^{-1}$, where x^* is a solution of $H(x) = 0$, forcing the operator H to be differentiable. However, in this paper, from the technique of decomposition of operators, we obtain a local convergence result for non-differentiable operators. Moreover, from this result of local convergence, we obtain a new local convergence result for Steffensen's method and non-differentiable operators.

The paper is organized as follows. Section 2 contains the motivation of the paper. In Section 3, we establish a local convergence analysis of the new method when operators F and G are differentiable under Lipschitz conditions for both operators. Finally, the study of the non-differentiable case is done in Section 4 under w-conditions.

2 Motivation

When iterative processes defined by divided differences are applied to solve nonlinear equations, it is important to note that the region of accessibility is reduced with respect to Newton's method. In practice, we can see this with the attraction basins (the set of points in the space such that initial conditions chosen in the set dynamically evolve to a particular attractor [10], [17]) of iterative methods when they are applied to solve a complex equation $H(z) = 0$, where $H : \mathbb{C} \to \mathbb{C}$ and $z \in \mathbb{C}$. To do this, we choose for example Newton's method and method (5) for solving the complex equation $H(z) = z^3 - 1 = 0$, and show the fractal pictures that are generated to approximate the three solutions $z^* = 1$,

$z^{**} = -0.5 - 0.866025i$ and $z^{***} = -0.5 + 0.866025i$. We are interested in identifying the attraction basins of the three solutions z^*, z^{**} and z^{***} [17]. This also allows us to compare the regions of accessibility of both methods.

We take a rectangle $D \subseteq \mathbb{C}$ and iterations starting at "every" z_0 point of D. In practice, a grid of 512×512 points in D is considered and these points are chosen as starting point z_0 for the methods. The used rectangle is $[-2.5, 2.5] \times [-2.5, 2.5]$, which contains the three solutions. The numerical methods starting at a point in the rectangle can converge to some of the zeros or, eventually, diverge.

In all the cases, the tolerance 10^{-3} and a maximum of 25 iterations are used. If we have not obtained the desired tolerance with 25 iterations, do not continue and decide that the iterative method starting at z_0 does not converge to any zero.

The regions of accessibility of the two iterative methods when they are applied to approximate the solutions z^*, z^{**} and z^{***} of $H(z) = z^3 - 1 = 0$ are shown in Figures 1 and 2. The strategy taken into account is the following. A colour is assigned to each basin of attraction of a zero. The colour is made lighter or darker according to the number of iterations needed to reach the root with the fixed precision required. Finally, if the iteration does not converge, the colour black is used. For more strategies, the reader can see [17] and the references appearing there. In particular, to obtain the pictures, the cyan and magenta colours have been assigned for the attraction basins of the two zeros. We mark with black the points of the rectangle for which the corresponding iterations starting at them do not reach any root with tolerance 10^{-3} in a maximum of 25 iterations. The graphics shown here have been generated with Mathematica 5.1 [18].

If we observe the behaviour of the two methods, we see that method (5) is more demanding with respect to the starting point than Newton's method (see the black colour).

Nevertheless, the use of derivative free iterative methods is necessary when the operator H is non-differentiable. For this reason, our aim in this work is to decompose the nonlinear operator into the sum of a differentiable and a non-differentiable parts, if it is possible, since we preserve, in some way, the good accessibility of Newton's method for the differentiable part.

Then, if we use the Newton-Steffensen type method defined in (5) to solve the non-differentiable equation $H(z) = z^2 + |z| - 2 = 0$, we improve the accessibility region of Steffensen's method (3), as we can see in Figures 3 and 4, where the basins of attraction of the two solutions of this equation are drawn for the mentioned methods.

3 Differentiable operators

In this section, we prove the quadratic convergence of the new iterative method given in (5) when H is differentiable, as the Newton and Steffensen methods. To continue, we obtain a result of local convergence for the new method given in (5). Besides, we improve the accessibility of Steffensen's method from the local convergence result obtained for method (5). To finish this section, a numerical example is given to illustrate the previous results.

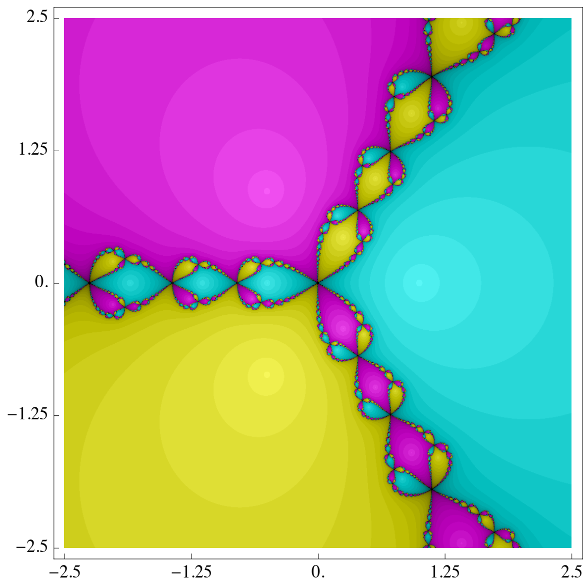

Figure 1: Attraction bassins of Newton's method for $H(z) = z^3 - 1 = 0$.

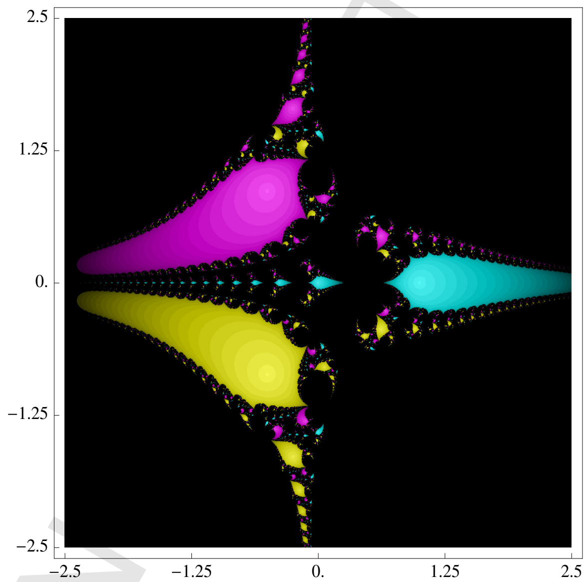

Figure 2: Attraction bassins of Steffensen's method for $H(z) = z^3 - 1 = 0$.

3.1 Local error and order of convergence

First of all, we study the local order of convergence of method (5) by assuming differentiable operators. We prove that it keeps the quadratic convergence of the Newton and the Steffensen's methods.

Theorem 1 *Let $H : \Omega \subseteq X \to X$ such that $H(x) = F(x) + G(x)$ with F and G are Fréchet differentiable operators in Ω, with $x^* \in \Omega$ such as $H(x^*) = 0$. Then, the local order of convergence of the iterative method defined in (5) is at least 2. More precisely the error equation is:*

$$e_{n+1} = (A_1 + B_1)^{-1} \left[B_2 \left(A_1 + B_1 + I \right) + A_2 \right] e_n^2 + O(e_n^3),$$

with $A_i = \dfrac{1}{i!} F^{(i)}(x^*)$ and $B_i = \dfrac{1}{i!} G^{(i)}(x^*)$, such that $A_i, B_i \in \mathscr{L}_i(X, X)$, $i = 1, 2, 3$, where $\mathscr{L}_i(X, X)$ is the space of bounded i-linear symmetric operators.

Proof: From Taylor's series of $F(x_n)$ and $G(x_n)$ around the solution x^* and $e_n = x_n - x^*$, it follows:

$$\begin{aligned} F(x_n) &= A_0 + A_1 e_n + A_2 e_n^2 + A_3 e_n^3 + O(e_n^4), \\ G(x_n) &= B_0 + B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + O(e_n^4), \end{aligned}$$

where $A_0 + B_0 = F(x^*) + G(x^*) = H(x^*) = 0$. Moreover, the derivative of $F(x_n)$ in a neighborhood of x^* takes the following form:

$$F'(x_n) = A_1 + 2A_2 e_n + 3A_3 e_n^2 + O(e_n^3). \tag{7}$$

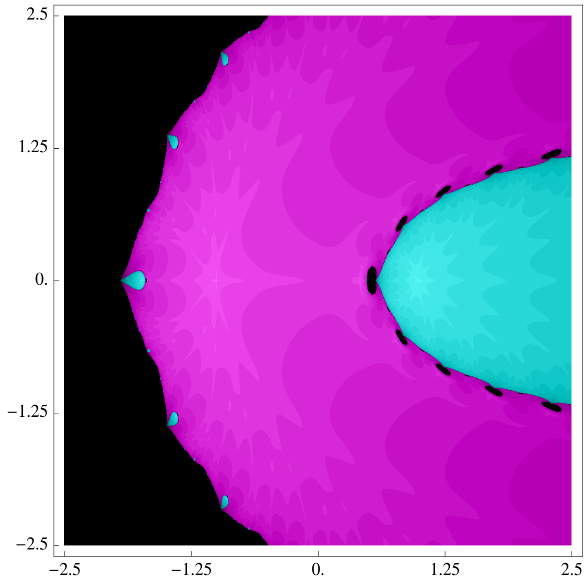

Figure 3: Attraction bassins of Steffensen's method for $H(z) = z^2 + |z| - 2 = 0$.

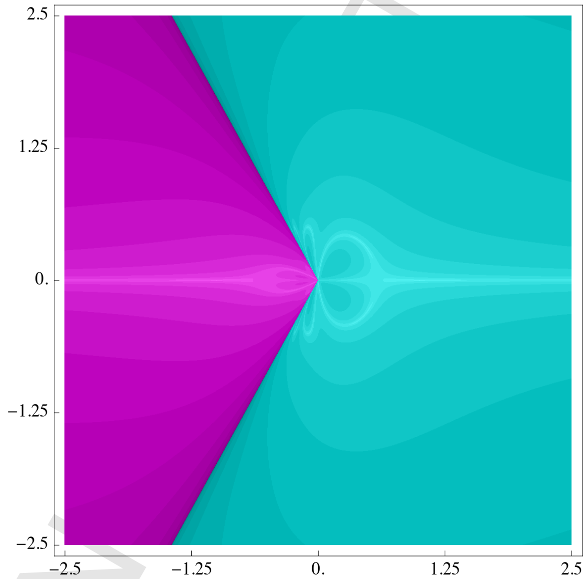

Figure 4: Attraction bassins of method (5) for $H(z) = z^2 + |z| - 2 = 0$.

Now, by using the divided difference operator mentioned in (11), we have

$$[x+h, x; G] = \int_0^1 G'(x+th)\,\mathrm{d}t$$

and, by integrating Taylor's series of $G'(x+th)$ around x, we obtain

$$[x+h, x; G] = G'(x) + \frac{1}{2}G''(x)h + \frac{1}{6}G'''(x)h^2 + O(h^3),$$

that is an operator of divided differences that approximates the Jacobian $G'(x)$ with order h.

By substituting the derivatives of G in the above expression, with $x = x_n$ and $h = H(x_n)$, we have the following error equation for the divided difference approximation:

$$\begin{aligned}[x_n, x_n + H(x_n); G] = B_1 + B_2\,(A_1 + B_1 + 2I)\,e_n + ((A_1+B_1)B_3(A_1+B_1) \\ +3B_3\,(A_1+B_1) + B_2A_2 + B_2^2 + 3B_3\big)\,e_n^2 + O(e_n^3).\end{aligned}$$

By adding this expression with that obtained for $F'(x_n)$ in (7), we have:

$$\begin{aligned}T(x_n) = F'(x_n) + [x_n, x_n + H(x_n); G] = A_1 + B_1 + (2A_2 + B_2\,(A_1 + B_1 + 2I))\,e_n + \\ + \big((A_1+B_1)B_3(A_1+B_1) + 3B_3\,(A_1+B_1) + B_2A_2 + B_2^2 + 3(A_3+B_3)\big)\,e_n^2 + O(e_n^3).\end{aligned}$$

Then, we can deduce the expression for the inverse operator $T(x_n)^{-1}$ in the way done in [5] (see equations (2) and (3)) and, by doing a step of method (5), it results the following error equation:

$$e_{n+1} = x_n - x^* - T(x_n)^{-1}\,[F(x_n) + G(x_n)] = (A_1+B_1)^{-1}\,[B_2\,(A_1 + B_1 + I) + A_2]\,e_n^2 + O(e_n^3).$$

□

3.2 Local convergence and uniqueness of solutions

In second place, we study the local convergence of method (5) for differentiable operators.

Let $H : \Omega \subseteq X \to X$ be such that $H(x) = F(x) + G(x)$, wihere F and G are Fréchet differentiable operators, that satisfy the following conditions:

(A) There exists $x^* \in \Omega$ such that $H(x^*) = 0$ and $H'(x^*)^{-1} \in \mathcal{L}(X, X)$ with $\|H'(x^*)^{-1}\| \leq \delta$. Moreover, H is a center-Lipschitz operator in x^* such that there exists a constant $L \geqslant 0$ with
$$\|H(x) - H(x^*)\| \leq L\|x - x^*\|. \tag{8}$$

(B) For $x, y \in \Omega$, there exists a constant $K_F \geqslant 0$ such that
$$\|F'(x) - F'(y)\| \leq K_F\|x - y\|. \tag{9}$$

(C) For $x, y, \in \Omega$, there exists a constant $K_G \geqslant 0$ such that
$$\|G'(x) - G'(y)\| \leq K_G\|x - y\|. \tag{10}$$

From now on, we denote $T(x) = F'(x) + [x, x + H(x); G]$.

Lemma 2 *Under conditions **(A)**, **(B)** and **(C)**, we obtain the following results:*

(i) $\|[x, y; G] - [u, v; G]\| \leq \frac{K_G}{2} \left(\|x - u\| + \|y - v\|\right)$, *for all* $x, y \in \Omega$.

(ii) If $x \in B(x^*, R) \subseteq \Omega$ *with* $R = \frac{2}{\delta(2K_F + (2+L)K_G)}$ *and* $x + H(x) \in \Omega$, *then the operator* $T(x)^{-1}$ *exists and*
$$\|T(x)^{-1}\| \leq \frac{\delta}{1 - \delta(K_F + \frac{2+L}{2}K_G)\|x - x^*\|}$$
.

Proof: We consider the following difference divided operator (see [12] and [13]):
$$[x, y; G] = \int_0^1 G'(x + t(y - x))\, \mathrm{d}t, \qquad x, y \in X. \tag{11}$$
From (10), item (i) follows easily.

Moreover, we have:
$$\begin{aligned}
\|I - H'(x^*)^{-1}T(x)\| &= \|H'(x^*)^{-1}\|\|(H'(x^*) - T(x))\| \\
&\leq \|H'(x^*)^{-1}\|\|F'(x^*) + G'(x^*) - F'(x) - [x, x + H(x), G]\| \\
&\leq \|H'(x^*)^{-1}\|\|F'(x^*) - F'(x) + [x^*, x^*, G] - [x, x + H(x), G]\| \\
&\leq \|H'(x^*)^{-1}\|(K_F\|x^* - x\| + \frac{K_G}{2}(\|x^* - x\| + \|x^* - x - H(x)\|) \\
&\leq \delta(K_F + \frac{2+L}{2}K_G)\|x - x^*\| \\
&< \delta(K_F + \frac{2+L}{2}K_G)R \\
&= 1.
\end{aligned}$$

Then, by applying the Banach lemma, it follows item (ii). □

Now, we perform the analysis of method (5).

Lemma 3 *Under the same conditions of Lemma 2, if* $g(t) = \frac{\delta(K_F+(1+L)K_G)t}{2-\delta(2K_F+(2+L)K_G)t}$ *and* $r = \frac{2}{\delta(3K_F+(3+2L)K_G)}$*, we have:*

(i) $r < R$*,* $g(t)$ *is a strictly increasing real function and* $g(r) = 1$*.*

(ii) For all starting point $x_0 \in B(x^*, r)$ *such that* $x_0 + H(x_0) \in \Omega$*, the iterate* x_1 *obtained from* (5) *verifies:*

$$\|x_1 - x^*\| \leq g(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\|.$$

Proof: Item (i) follows easily. Now, by applying (5), we get:

$$\begin{aligned}
x_1 - x^* &= x_0 - x^* - T(x_0)^{-1}\left[F(x_0) + G(x_0)\right] \\
&= x_0 - x^* - T(x_0)^{-1}(F(x_0) - F(x^*) + G(x_0) - G(x^*)) \\
&= T(x_0)^{-1}\left[T(x_0)(x_0 - x^*) - F(x_0) + F(x^*) - G(x_0) + G(x^*)\right] \\
&= T(x_0)^{-1}\left[(F'(x_0) + [x_0, x_0 + H(x_0); G])(x_0 - x^*) - F(x_0) + F(x^*) - G(x_0) + G(x^*)\right] \\
&= T(x_0)^{-1}\left[\int_{x_0}^{x^*}(F'(z) - F'(x_0))dz + ([x_0, x_0 + H(x_0); G] - [x_0, x^*; G])(x_0 - x^*)\right] \\
&= T(x_0)^{-1}\left[\int_0^1 [F'(x_0 + \tau(x^* - x_0)) - F'(x_0)]d\tau(x^* - x_0)\right] \\
&\quad + ([x_0, x_0 + H(x_0); G] - [x_0, x^*, G])(x_0 - x^*).
\end{aligned} \tag{12}$$

So, we obtain the following bound:

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|T(x_0)^{-1}\|\left[\frac{K_F}{2}\|x^* - x_0\|^2 + \frac{K_G}{2}(\|x_0 - x^*\| + \|H(x_0)\|)\|x_0 - x^*\|\right] \\
&\leq \frac{\delta}{1 - \delta(K_F + \frac{2+L}{2}K_G)\|x_0 - x^*\|}\left(\frac{K_F}{2} + \frac{K_G}{2}(1+L)\right)\|x_0 - x^*\|^2 \\
&\leq \frac{\delta(K_F + K_G(1+L))\|x_0 - x^*\|}{2 - \delta(2K_F + (2+L)K_G)\|x_0 - x^*\|}\|x_0 - x^*\| \\
&= g(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&< g(r)\|x_0 - x^*\| \qquad (13) \\
&= \|x_0 - x^*\|, \qquad (14)
\end{aligned}$$

where in the last term we have used the results obtained in item (i). □

Theorem 4 *Let $H : \Omega \subseteq X \to Y$ be such that $H(x) = F(x)+G(x)$, where F and G are Fréchet differentiable operators. Under conditions **(A)**, **(B)**, **(C)** and $B(x^*, (1+L)r) \subseteq \Omega$, if $x_0 \in B(x^*, r)$, where*

$$r = \frac{2}{\delta(3K_F + (3+2L)K_G)}, \tag{15}$$

then the sequence $\{x_n\}$ generated by method (5) is well defined, belongs to $B(x^*, r)$ and converges to x^*.

Proof: By hypothesis, we have that $\|x_0+H(x_0)-x^*\| < (1+L)r$, then $x_0+H(x_0) \in \Omega$ and $T(x_0)$ is well defined. On the other hand, $x_0 \in B(x^*, r) \subseteq B(x^*, R)$ by Lemma 3 and, then, by item (i) of Lemma 2, we obtain that there exists $T(x_0)^{-1}$. Therefore, x_1 is well defined. Now, by Lemma 3, it follows that

$$\|x_1 - x^*\| \leq g(\|x_0 - x^*\|)\|x_0 - x^*\| < g(r)\|x_0 - x^*\| = \|x_0 - x^*\| < r.$$

Then, by appliying Lemmas 2 and 3, and establishing an inductive procedure, we have for all $n \in \mathbb{N}$, the following:

$$\begin{aligned} &x_n + H(x_n) \in \Omega, \\ &x_n \in B(x^*, r) \subseteq B(x^*, R), \\ &\|x_{n+1} - x^*\| \leq g(\|x_n - x^*\|)\|x_n - x^*\| < g(r)\|x_n - x^*\| = \|x_n - x^*\| \leq r. \end{aligned}$$

That is, $\{x_n\}$ is well defined, belongs to $B(x^*, r)$ and $\{\|x_n - x^*\|\}$ is a strictly decreasing sequence of positive numbers, and then $\{x_n\}$ converges to x^*. □

Concerning to the uniqueness of solution x^*, we have the following result.

Theorem 5 *Under the conditions **(A)**, **(B)** and **(C)**, the solution x^* of the equation $H(x) = 0$, is unique in $\overline{B(x^*, S)} \cap \Omega$, where*

$$S = \frac{2}{\delta(K_F + K_G)}. \tag{16}$$

Proof. Let $y^* \in \overline{B(x^*, S)} \cap \Omega$ be such that $H(y^*) = 0$. We define the following operator:

$$P = \int_0^1 F'(x^* + t(y^* - x^*))\, dt + [y^*, x^*; G].$$

Then, using **(A)**, **(B)**, **(C)** and (16), we obtain

$$\begin{aligned} \|H'(x^*)^{-1}P - I\| &\leq \|H'(x^*)^{-1}\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x^*)\| dt \\ &\quad + \|H'(x^*)^{-1}\| \int_0^1 \|[y^*, x^*; G] - [x^*, x^*; G]\| dt \\ &\leq \|H'(x^*)^{-1}\| \left(\int_0^1 K_F t\|y^* - x^*\|\, dt + \frac{K_G}{2}\|y^* - x^*\| \right) \\ &< \delta \left(\frac{K_F + K_G}{2} \right) S = 1. \end{aligned}$$

Hence, $P^{-1} \in \mathcal{L}(X,X)$. From the identity $H(y^*) - H(x^*) = P(y^* - x^*) = 0$, we then deduce that $x^* = y^*$.

□

3.3 Particular cases

Next, we consider two particular cases that can be obtained from method (5): the Newton and Steffensen methods. We prove that the ball of convergence corresponding to Steffensen's method is the smallest one and, therefore, the accessibility of the Steffensen's method is improved from method (5).

Corollary 6 *Under conditions **(A)**, **(B)** and **(C)**, if $x_0 \in B(x^*, r_N)$, where*

$$r_N = \frac{2}{3\delta(K_F + K_G)}, \tag{17}$$

then the sequence $\{x_n\}$ generated by the Newton method (1) is well defined, belongs to $B(x^*, r_N)$ and converges to x^*. Moreover, the solution x^* of the equation $H(x) = 0$ is unique in $\overline{B(x^*, S)} \cap \Omega$, where $S = \frac{2}{\delta(K_F+K_G)}$.

Proof: In this case, the operator T is given by $T(x) = F'(x) + G'(x) = H'(x)$. Moreover,

$$\begin{aligned} \|H'(x) - H'(y))\| &\leq \|F'(x) - F'(y)\| + \|G'(x) - G'(y)\| \\ &\leq (K_F + K_G)\|x - y\| \end{aligned}$$

and, following the ideas of Lemma 2, we obtain $R_N = \frac{1}{\delta(K_F+K_G)}$. Now, by a similar reasoning to that of Lemma 3, we get:

$$g_N(t) = \frac{\delta(K_F + K_G)t}{2 - 2\delta(K_F + K_G)t} \text{ and } r_N = \frac{2}{3\delta(K_F + K_G)}.$$

So, from Theorem 4 we obtain that the sequence $\{x_n\}$ generated by Newton's method (1) is well defined, belongs to $B(x^*, r_N)$ and converges to x^*.

Finally, if we consider $P = \displaystyle\int_0^1 H'\left(x^* + t(y^* - x^*)\right) dt$, the uniqueness of solution follows easily from Theorem 5.

□

Corollary 7 *Under conditions **(A)**, **(B)**, **(C)** and $B(x^*, (1+L)r) \subseteq \Omega$, if $x_0 \in B(x^*, r_S)$, where*

$$r_S = \frac{2}{\delta(3 + 2L)(K_F + K_G)}, \tag{18}$$

then the sequence $\{x_n\}$ generated by Steffensen's method (3) is well defined, belongs to $B(x^*, r_S)$ and converges to x^*. Moreover, the solution $x^*$of the equation $H(x) = 0$ is unique in $\overline{B(x^*, S)} \cap \Omega$, where $S = \frac{2}{\delta(K_F+K_G)}$.

Proof: In this case, the operator T is given by $T(x) = [x, x+H(x); H]$ with $H = F+G$. So, as Lemma 2 (i) it is easy to follows that

$$\|[x, y; H] - [u, v; H])\| \leq \frac{(K_F + K_G)}{2}(\|x - u\| + \|y - v\|).$$

Then, by similar reasoning to that of (12), we obtain $R_S = \frac{2}{\delta(2+L)(K_F+K_G)}$. Now, following (13), we get:

$$g_S(t) = \frac{\delta(1+L)(K_F + K_G)t}{2 - \delta(2+L)(K_F + K_G)t} \text{ and } r_S = \frac{2}{\delta(3+2L)(K_F + K_G)}.$$

Then, from mathematical induction, as in Corollary 6, we obtain that Steffensen's method (3) is well defined, belongs to $B(x^*, r_S)$ and converges to x^*. The uniqueness of solution follows now easily from the Theorem 5 considering P as in Corollary 6. $\square$

Remark: From the decomposition presented for the operator $H = F + G$ and the method presented in (5), we have improved the accessibility of Steffensen's method (3), the balls of convergence satisfy that $B(x^*, r_S) \subseteq B(x^*, r)$.

3.4 Numerical example

In this section, we illustrate the application of our local convergence analysis in the differentiable case obtained in Theorem 4 with a numerical example. We justify that the new iterative method given in (5) improves the accessibility of Steffensen's method.

We consider a nonlinear integral equation of Hammerstein type, which can be used to describe applied problems in the fields of electro-magnetics, fluid dynamics, in the kinetic theory of gases and, in general, in the reformulation of boundary value problems. These equations are of the form:

$$x(s) = f(s) - \int_a^b K(s,t)\Phi(x(t))dt, \quad a \leq s \leq b, \tag{19}$$

where $x(s), f(s) \in C[a,b]$, with $-\infty < a < b < \infty$, and Φ is a polynomial function. One of the most used techniques to solve this kind of equations consists of expressing them as a nonlinear operator in a Banach space and solving the following operator equation:

$$H(x)(s) = x(s) - f(s) + \int_a^b K(s,t)\Phi(x(t))dt = 0, \tag{20}$$

where $H : D \subseteq C[a,b] \to C[a,b]$ with D a non-empty open convex subset of $C[a,b]$ with the max-norm $\|\nu\| = \max_{s\in[a,b]} |\nu(s)|$.

Specifically, we apply our theoretical results for obtaining the radius of the convergence ball of the following Hammerstein equation:

$$H(x)(s) = x(s) - \left(1 + \frac{19}{30}\lambda\right) s^2 - s + 1 + \lambda \int_0^1 s^2 t \left(3x(t)^2 - x(t)\right) dt = 0, \qquad (21)$$

where $D = B(0, w)$, $w > 0$ and $\lambda \in [0, \frac{1}{2}[$. It is easy to check that $x^*(s) = s^2 - s + 1$ is a solution of (21).

In addition,

$$H'(x)v(s) = v(s) + \lambda \int_0^1 s^2 t \,(6x(t) - 1)v(t)dt.$$

Thus, by taking norms, we have:

$$\|H'(x)\| \leq 1 + \frac{6w+1}{2}\lambda,$$

and, therefore, by the mean value theorem, we get:

$$\|H(x) - H(x^*)\| \leq (1 + \frac{6w+1}{2}\lambda)\|x - x^*\|.$$

So, the constant L of (8) is $L = 1 + \dfrac{6w+1}{2}\lambda$.

On the other hand,

$$\begin{aligned}[I - H'(x^*)]v(s) &= -\lambda s^2 \int_0^1 t\,(6x^*(t) - 1)v(t)dt \\ &= -\lambda s^2 \int_0^1 (6t^3 - 6t^2 + 5t)v(t)dt.\end{aligned}$$

Then, by using $0 \leq \lambda < 1/2$, it follows:

$$\|I - H'(x^*)\| \leq 2\lambda < 1.$$

By the Banach lemma, there exists $H'(x^*)^{-1}$ and

$$\|H'(x^*)^{-1}\| \leq \frac{1}{1 - 2\lambda}.$$

As a consequence, we choose $\delta = \dfrac{1}{1-2\lambda}$ in condition **(A).**

Next, we split the equation in two parts, so $H(x) = F(x) + G(x)$, and

$$F(x)(s) = \frac{\lambda(m-1)}{m} s^2 \int_0^1 t\,(3x(t)^2 - x(t))dt, \qquad (22)$$

$$G(x)(s) = x(s) - f(s) + \frac{\lambda}{m} s^2 \int_0^1 t\,(3x(t)^2 - x(t))dt, \qquad (23)$$

with $m \in [1, +\infty[$. The idea of considering this decomposition is due to the fact of that we obtain Steffensen's method for $m = 1$ and when m increases, we can understand that we move away from it obtaining different possibilities from method (5).

With the last decomposition, it is easy to obtain the following values for the constants appearing in **(B)** and **(C)**, ((9) and (10)), such that

$$K_F = \frac{3\lambda(m-1)}{m} \quad \text{and} \quad K_G = \frac{3\lambda}{m}.$$

Then, by applying Theorem 4, if $B(x^*, (1+L)r) \subseteq \Omega = B(0, w)$, we obtain the convergence ball $B(x^*, r)$, with

$$r = \frac{2m(1-2\lambda)}{18\lambda^2 w + 3\lambda^2 + 9\lambda m + 6\lambda}.$$

Moreover, Theorem 5 gives the radius of uniqueness, $S = \dfrac{2(1-2\lambda)}{3\lambda}$, and, by using corollaries 6 and 7, we obtain the radius of convergence balls for the Newton and Steffensen methods, that are

$$r_N = \frac{2(1-2\lambda)}{9\lambda} \quad \text{and} \quad r_S = \frac{2(1-2\lambda)}{3\lambda(5+(1+6w)\lambda)}.$$

Notice that, for applying the Theorem 4 and Corollary 7 It is necessary to check the condition: $B(x^*, (1+L)r) \subseteq B(0, w)$. But, if $\|x^*\| + (1+L)r < w$ the previous condition is verified. So, if we choose $w = 7.33$, this condition, for $\lambda = 0.1, 0.2, 0.3, 0.4$ and $m = 2, 3, 4, 5, 6, 7, 8, 9, 10$, is verified. Therefore, we can see in Table 1 the results for the radius of local convergence balls for the different values of m and λ indicated. As we can see, the accessibility of method (5) is better than that of Steffensen's method for all examples seen. Further, for bigger values of m, the radius of the convergence ball increases and the evolution of the radii is to approximate to the value of that of Newton's method.

Now, we show the application of method (5) to this example and compare the results with the Steffensen, Newton and Secant methods. For this, we discretize integral equation (21) by taking 10 subintervals in $[0, 1]$ and using the Simpson quadrature to approximate the integral involved. We solve the discretized nonlinear problem obtained from (21) by means of the last three methods and the corresponding split equation given by (22) for method (5) when $\lambda = 0.2$ and different values of m are chosen. We use program Matlab 2015b working in variable precision arithmetic with 20 digits of mantissa and iterating until the distance between consecutive iterates is less than the tolerance 10^{-15}. Table 2 shows the number of iterations, $iter$, the computational order of convergence, p, the distance between the last iterates, $incr1$, and the max-norm of $H(x)$ at the approximated solution, $incr2$, by taking as initial guess $x_0 = (1, \cdots, 1)$ in all methods and $x_1 = (0.5, \cdots, 0.5)$ for Secant's method. As we can observe, the results obtained from (5) are always better than those obtained from the Secant method. Also, they are quite similar, for different values of m, to the Steffensen and Newton methods. This confirms the competitiveness of method (5). The distance between the numerical approximation

	$m \setminus \lambda$	0.1	0.2	0.3	0.4
Steffensen	1	0.5614	0.1429	0.0480	0.0145
(5)	2	0.8533	0.2353	0.0827	0.0256
	3	1.0323	0.3000	0.1088	0.0345
	4	1.1532	0.3478	0.1293	0.0417
	5	1.2403	0.3846	0.1457	0.0476
	6	1.3061	0.4138	0.1592	0.0526
	7	1.3576	0.4375	0.1705	0.0569
	9	1.4328	0.4737	0.1882	0.0638
	10	1.4612	0.4878	0.1954	0.0667
Newton		1.7778	0.6667	0.2963	0.1111
Uniqueness		5.3333	2.0000	0.8889	0.3333

Table 1: Radius of convergence balls for different values of m and λ.

to the solution and the exact solution $x^*(s)$ in the fixed nodes for the intervals considered is $\|\mathbf{x}^* - \mathbf{x_n}\| = 3.3057\ 10^{-6}$.

	m	$iter$	p	$incr1$	$incr2$
Steffesen	1	5	1.999988	6.7944e-20	2.2472e-28
Newt-Steff	2	5	1.999989	7.027134e-25	1.304487e-28
	3	5	1.999999	7.026786e-25	1.372215e-28
	4	5	2.000003	7.026354e-25	1.292053e-28
	5	5	1.999999	7.026735e-25	1.411157e-28
	6	5	1.999996	7.027103e-25	1.414880e-28
	7	5	1.999999	7.026811e-25	1.321147e-28
	8	5	1.999999	7.026811e-25	1.321147e-28
	9	5	2.000001	7.026551e-25	1.308583e-28
	10	5	2.000003	7.026354e-25	1.342294e-28
Newton		5	2.000003	7.027572e-25	8.856434e-29
Secant		9	1.646374	4.724823e-18	2.247234e-28

Table 2: Numerical results for the differentiable problem.

4 Non-differentiable operators

In this section, we consider that the operator H is non-differentiable. Obviously, the Newton method is not applicable in this situation. However, we can approximate a solution of the equation $H(x) = 0$ with the same speed of convergence as the Newton method. Moreover, the accessibility of Steffensen's method is also improved in this situation by using method (5).

4.1 Local convergence and uniqueness of solutions.

To improve the applicability of Steffensen's method and preserve the good applicability of Newton's method in situations where the operator H is not differentiable, we consider

$$H(x) = F(x) + G(x),$$

where $F, G : \Omega \subseteq X \to X$, F is Fréchet differentiable and G is continuous and non-differentiable.

The local convergence results for method (5) require conditions on the operators F, G and the solution x^* of the equation $H(x) = 0$. Note that a local result provides what we call ball of convergence, which we denote by $B(x^*, r)$. From the value r, the ball of convergence gives information about the accessibility of the solution x^*. In this section, we analyze the local convergence of method (5). First, we consider the following conditions:

(I) H is center-ω_0-Lipschitz continuous operator in x^* such that

$$\|H(x) - H(x^*)\| \leq \omega_0(\|x - x^*\|), x \in \Omega, \tag{24}$$

where $\omega_0 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}$.

(II) F' is a ω_1-Lipschitz continuous operator such that

$$\|F'(x) - F'(y)\| \leq \omega_1(\|x - y\|), \quad x, y \in \Omega, \tag{25}$$

where $\omega_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function and we assume that there exists a continuous non-decreasing function $h : [0,1] \to \mathbb{R}_+$ such that $\omega_1(tz) \leq h(t)\omega_1(z)$, with $t \in [0,1]$ and $z \in [0, \infty)$. In addition, we denote $M = \displaystyle\int_0^1 h(t)\, dt$.

(III) We suppose that there exists $[z, w; G]$ for each pair of distinct points $z, w \in \Omega$ and the divided difference $[-,-;G]$ is a w_2-Lipschitz continuous operator such that

$$\|[x, y; G] - [u, v; G]\| \leq \omega_2(\|x - u\|, \|y - v\|); \quad x, y, u, v \in \Omega, \tag{26}$$

where $\omega_2 : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function in both arguments.

Notice that it is known that if G is a non-differentiable operator, then $\omega_2(0,0) > 0$ (see [7]). Therefore, if we consider a situation where $\omega_2(0,0) = 0$, this implies that the operator G is differentiable. Note that this is the usual condition considered by other authors: Lipschitz or Hölder continuous condition for the divided differences ($\omega_2(0,0) = 0$). However, using an auxiliary point and considering $\omega_2(0,0) > 0$, our conditions allow the operator H being not differentiable, as we see in this section.

Now, we provide a technical lemma and obtain results about the good definition of the sequence $\{x_n\}$ given by (5).

Lemma 8 *Under conditions* ***(I)***, ***(II)*** *and* ***(III)***, *we suppose that* $x_{n-1}, x_{n-1}+H(x_{n-1}) \in \Omega$, *for all* $n \geqslant 1$, *and the following conditions hold:*

(IV) *Let* x^* *be a solution of the equation* $H(x) = 0$ *and consider* $\tilde{x} \in \Omega$ *with* $\|\tilde{x}-x^*\| \leq \delta$, *so that the operator* $L^{-1} = (F'(x^*) + [x^*, \tilde{x}; G])^{-1}$ *exists and* $\|L^{-1}\| \leq \gamma$.

(V) $a_{n-1} = \gamma(\omega_1(\|x_{n-1} - x^*\|) + \omega_2(\|x_{n-1} - x^*\|, \delta + \|x_{n-1} - x^*\| + \omega_0(\|x_{n-1} - x^*\|)) < 1$, *for all* $n \geqslant 1$.

Then, x_n *is well defined and*

$$\|x_n - x^*\| \leq Q_{n-1}\|x_{n-1} - x^*\|, \quad \text{where} \quad Q_{n-1} = \frac{\tilde{a}_{n-1}}{1 - a_{n-1}} \quad \text{and}$$

$$\tilde{a}_{n-1} = \gamma(M\omega_1(\|x_{n-1} - x^*\|) + \omega_2(\|x_{n-1} - x^*\|, \omega_0(\|x_{n-1} - x^*\|))).$$

Proof: Notice that $x_{n-1}, x_{n-1}+H(x_{n-1}) \in \Omega$ and suppose $x_{n-1} \neq x_{n-1}+H(x_{n-1})$. In other case, $H(x_{n-1}) = 0$ and then $x_m = x_{n-1} = x^*$ for all $m \geqslant n-1$, so that the result follows easily. So, if $x_{n-1} \neq x_{n-1}+H(x_{n-1})$, there exists the operator $[x_{n-1}, x_{n-1} + H(x_{n-1}); G]$.

To simplify the notation, we denote $T(x) = F'(x)+[x, x + H(x); G]$. Now, taking into account (24), (25) and (26), it follows

$$\begin{aligned}
\|I - L^{-1}T(x_{n-1})\| &\leq \|I - L^{-1}(F'(x_{n-1}) + [x_{n-1}, x_{n-1} + H(x_{n-1}); G])\| \\
&\leq \|L^{-1}\|\|L - F'(x_{n-1}) - [x_{n-1}, x_{n-1} + H(x_{n-1}); G]\| \\
&\leq \|L^{-1}\| \left(\|F'(x^*) - F'(x_{n-1})\| + \|[x^*, \tilde{x}; G] - [x_{n-1}, x_{n-1} + H(x_{n-1}); G]\|\right) \\
&\leq \gamma \left(\omega_1(\|x^* - x_{n-1}\|) + \omega_2(\|x^* - x_{n-1}\|, \|\tilde{x} - x^*\| + \|x_{n-1} + H(x_{n-1}) - x^*\|)\right) \\
&\leq a_{n-1} \\
&< 1.
\end{aligned}$$

Then, by the Banach lemma on invertible operators, it follows that the operator $T(x_{n-1})^{-1}$ exists and

$$\|T(x_{n-1})^{-1}\| \leq \frac{\gamma}{1 - a_{n-1}}.$$

So, x_n is well defined. Now, from (5), it follows

$$\begin{aligned}
&x_n - x^* = x_{n-1} - T(x_{n-1})^{-1}H(x_{n-1}) - x^* = T(x_{n-1})^{-1}(T(x_{n-1})(x_{n-1} - x^*) - H(x_{n-1})) \\
&= T(x_{n-1})^{-1}((F'(x_{n-1}) + [x_{n-1}, x_{n-1} + H(x_{n-1}); G])(x_{n-1} - x^*) - F(x_{n-1}) - G(x_{n-1})) \\
&= T(x_{n-1})^{-1}\left(\int_{x_{n-1}}^{x^*} (F'(z) - F'(x_{n-1}))\, dz + [x_{n-1}, x_{n-1} + H(x_{n-1}); G](x_{n-1} - x^*) + G(x^*) - G(x_{n-1})\right) \\
&= T(x_{n-1})^{-1}\left(\int_0^1 \left(F'(x_{n-1} + t(x^* - x_{n-1})) - F'(x_{n-1})\right)(x_{n-1} - x^*)\, dt\right) \\
&+ T(x_{n-1})^{-1}([x_{n-1}, x_{n-1} + H(x_{n-1}); G] - [x^*, x_{n-1}; G])(x_{n-1} - x^*)).
\end{aligned}$$

Taking now norms in the previous expression and applying conditions (24), (25) and (26), we obtain

$$\|x_n - x^*\| \leq \|T(x_{n-1})^{-1}\| \left(\int_0^1 \omega_1 \left(\|t(x^* - x_{n-1})\| \right) dt + \omega_2(\|x_{n-1} - x^*\|, \|H(x_{n-1})\|) \right) \|x_{n-1} - x^*\|$$
$$\leq \frac{\gamma}{1 - a_{n-1}} (M\omega_1(\|x_{n-1} - x^*\|) + \omega_2(\|x_{n-1} - x^*\|, \omega_0(\|x_{n-1} - x^*\|))) \|x_{n-1} - x^*\|$$
$$= \frac{\tilde{a}_{n-1}}{1 - a_{n-1}} \|x_{n-1} - x^*\| = Q_{n-1} \|x_{n-1} - x^*\|.$$

□

To prove that the sequence $\{x_n\}$ given by method (5) converges to x^*, we are interested in the fact that $\{\|x_n - x^*\|\}$ is a strictly decreasing sequence of positive real numbers. Then, from the previous lemma, $\|x_n - x^*\| < \|x_{n-1} - x^*\|$ if $Q_{n-1} < 1$. But, $Q_{n-1} < 1$ if and only if $a_{n-1} + \tilde{a}_{n-1} < 1$. So, if we consider that r is the smallest positive real root of the equation

$$\gamma(\omega_1(t) + \omega_2(t, \delta + t + \omega_0(t))) + \gamma(M\omega_1(t) + \omega_2(t, \omega_0(t))) - 1 = 0,$$

then, for $x_{n-1} \in B(x^*, r)$ and $x_{n-1} + H(x_{n-1}) \in \Omega$, as $\|x_{n-1} + H(x_{n-1}) - x^*| \leq \|x_{n-1} - x^*\| + \omega_0(\|x_{n-1} - x^*\|) < r + \omega_0(r)$, we obtain

$$a_{n-1} < m = \gamma(\omega_1(r) + \omega_2(r, \delta + r + \omega_0(r)))) < 1,$$

since $w_1(x, y) \geqslant 0$ and $w_2(x, y) > 0$ in $\mathbb{R}_+ \times \mathbb{R}_+$. Moreover,

$$\tilde{a}_{n-1} < \tilde{m} = \gamma(M\omega_1(r) + \omega_2(r, \omega_0(r))).$$

So, as $m + \tilde{m} = 1$, we have

$$Q_{n-1} < \frac{\tilde{m}}{1 - m} = 1.$$

After that, from the previous study, we modify condition **(V)** and consider the following condition:

(V') Assume that the equation

$$\gamma \left[(1 + M)\omega_1(t) + \omega_2(t, \delta + t + \omega_0(t)) + \omega_2(t, \omega_0(t)) \right] - 1 = 0 \qquad (27)$$

has at least one positive real root and $B(x^*, r + \omega_0(r)) \subset \Omega$, where r is the smallest positive real root of (27).

In addition, we obtain the following local result.

Theorem 9 *Under conditions **(I)**, **(II)**, **(III)**, **(IV)** and **(V')**, if we choose $x_0 \in B(x^*, r)$, then the sequence $\{x_n\}$ given by method (5) is well defined, belongs to $B(x^*, r)$ and converges to a solution x^* of equation $H(x) = 0$.*

Proof: Obviously, $x_0 + H(x_0) \neq x_0$. In other case, we have $x_0 = x^*$ and then $x_n = x^*$, for all $n \geqslant 1$, so that the result is proved. From Lemma 8, as

$$a_0 = \gamma(\omega_1(\|x_0 - x^*\|) + \omega_2(\|x_0 - x^*\|, \delta + \|x_0 + H(x_0) - x^*\|) < m = \gamma(\omega_1(r) + \omega_2(r, \delta + r + \omega_0(r))) < 1,$$

we have that there exists $T(x_0)^{-1}$ and $\|T(x_0)^{-1}\| \leq \dfrac{\gamma}{1 - a_0}$. Therefore, x_1 is well defined and

$$\|x_1 - x^*\| \leq Q_0 \|x_0 - x^*\| < \|x_0 - x^*\| < r.$$

Then, $x_1 \in B(x^*, r)$. On the other hand,

$$\|x_1 + H(x_1) - x^*\| \leq \|x_1 - x^*\| + \|H(x_1)\| \leq \|x_1 - x^*\| + \omega_0(\|x_1 - x^*\|) < \|x_0 - x^*\| + \omega_0(\|x_0 - x^*\|)$$

$$< r + \omega_0(r).$$

So, $x_1 + H(x_1) \in \Omega$.

By mathematical induction on $n \geq 2$, we prove that, if $x_{n-1} \in B(x^*, r)$ and $x_{n-1} + H(x_{n-1}) \in \Omega$, with $x_{n-1} + H(x_{n-1}) \neq x_{n-1}$, then x_n is well defined, $\|x_n - x^*\| < \|x_{n-1} - x^*\|$ and $\|x_n + H(x_n) - x^*\| < r + \omega_0(r)$.

Suppose that the hypotheses are true for $n = 2, 3, \ldots, k$ and see that it is true for $n = k + 1$.

So, as $x_k \in B(x^*, r)$ and $x_k + H(x_k) \in \Omega$, from Lemma 8, there exists $T(x_k)^{-1}$. This implies that x_{k+1} is well defined and then

$$\|x_{k+1} - x^*\| \leq Q_k \|x_k - x^*\| < \|x_k - x^*\| < r.$$

On the other hand,

$$\|x_{k+1} + H(x_{k+1}) - x^*\| \leq \|x_{k+1} - x^*\| + \|H(x_{k+1})\| \leq \|x_{k+1} - x^*\| + \omega_0(\|x_{k+1} - x^*\|) < r + \omega_0(r).$$

Then, $\{x_n\} \subset B(x^*, r)$ and $\{\|x_n - x^*\|\}$ is a strictly decreasing sequence of positive real numbers. Therefore, $\{x_n\}$ converges to x^* and the proof is complete. □

Concerning to the uniqueness of the solution x^*, we have the following result.

Theorem 10 *Under conditions **(I)**, **(II)**, **(III)**, **(IV)** and **(V')**, we suppose that the equation*

$$\gamma\left(M\omega_1\left(R\right) + \omega_2\left(0, R + \delta\right)\right) = 1, \tag{28}$$

has at least one positive real root, where R is the smallest positive real root of (28). Then, the solution x^* is the unique solution of the equation $H(x) = 0$ in $\overline{B(x^*, R)} \cap \Omega$.

Proof: Let $y^* \in \overline{B(x^*, R)} \cap \Omega$ and $H(y^*) = 0$. We then define the following operator

$$P = \int_0^1 F'\left(x^* + t(y^* - x^*)\right)\, dt + [x^*, y^*; G]$$

and, using **(I)** and **(II)**, we obtain

$$\begin{aligned}
\|L^{-1}P - I\| &\leq \|L^{-1}\|\ \|P - L\| \\
&\leq \|L^{-1}\| \left[\int_0^1 \|F'\left(x^* + t(y^* - x^*)\right) - F'(x^*)\|\, dt + \|[x^*, y^*; G] - [x^*, \tilde{x}; F]\|\right] \\
&\leq \gamma \left[\int_0^1 \omega_1\left(\|t(y^* - x^*)\|\right)\, dt + \omega_2\left(0, \|y^* - \tilde{x}\|\right)\right] \\
&< \gamma\left(M\omega_1\left(R\right) + \omega_2\left(0, R + \delta\right)\right) = 1.
\end{aligned}$$

Hence, $P^{-1} \in \mathcal{L}(X, Y)$. and, by the identity $H(x^*) - H(y^*) = P(x^* - y^*) = 0$, we deduce $x^* = y^*$. □

4.2 Particular case

Taking into account that Steffensen's method (3) is a particular case of method (5), we obtain a local convergence result for this method from the last theorem.

When H is non-differentiable, to apply the Steffensen method (3), we consider $H = G$, besides $T(x) = [x, x + H(x); G]$ and $L(x) = [x^*, \tilde{x}; G]$. Then, following the previous reasoning and considering the condition:

(V')$_S$ Assume that the equation

$$\gamma\left[\omega_2(t, \delta + t + \omega_0(t)) + \omega_2(0, t + \omega_0(t))\right] - 1 = 0, \tag{29}$$

has at least one positive root and $B(x^*, r_S + \omega_0(r_S)) \subset \Omega$, where r_S is the smallest positive real root of (29),

we obtain the following result of local convergence for Steffensen's method (3).

Theorem 11 *Under conditions **(I)**, **(II)**, **(III)**, **(IV)** and **(V')**$_S$, if we take $x_0 \in B(x^*, r_S)$, then the sequence $\{x_n\}$ given by Steffensen's method (3) is well defined, remains in $B(x^*, r_S)$ and converges to a solution x^* of equation $H(x) = 0$.*

□

In the following numerical example, we see that method (5) also improves the accessibility of Steffensen's method (3) when H is non-differentiable.

4.3 Numerical example

We consider (19), where K is the Green function in $[a,b]\times[a,b]$, and then use a discretization process to transform equation (20) into a finite dimensional problem by approximating the integral by the Gauss-Legendre quadrature formula

$$\int_a^b q(t)\,dt \simeq \sum_{i=1}^{p} w_i q(t_i),$$

where the nodes t_i and the weights w_i are known.

If we denote the approximations of $x(t_i)$ and $f(t_i)$ by x_i and f_i, respectively, with $i=1,2,\ldots,p$, then equation (20) is equivalent to the following system of nonlinear equations:

$$x_i = f_i + \sum_{j=1}^{p} a_{ij}\,\Phi(x_j), \quad j=1,2,\ldots,p, \tag{30}$$

where

$$a_{ij} = w_j K(t_i,t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & j\le i,\\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & j>i.\end{cases}$$

Now, system (30) can be written as

$$\mathbb{H}(\mathbf{x}) \equiv \mathbf{x}-\mathbf{f}-A\,\mathbf{z}=0, \qquad \mathbb{H}:\Delta\subseteq\mathbb{R}^p\longrightarrow\mathbb{R}^p, \tag{31}$$

where

$$\mathbf{x}=(x_1,x_2,\ldots,x_p)^T, \qquad \mathbf{f}=(f_1,f_2,\ldots,f_p)^T, \qquad A=(a_{ij})_{i,j=1}^{p},$$
$$\mathbf{z}=(\Phi(x_1),\Phi(x_2),\ldots,\Phi(x_p))^T.$$

After that, we choose $a=0$, $b=1$, $K(s,t)$ as the Green function in $[0,1]\times[0,1]$ and $\Phi(x(t))=x(t)^3+|x(t)|$ in (19). Then, the system of nonlinear equations given in (31) is of the form

$$\mathbb{H}(\mathbf{x})=\mathbf{x}-\mathbf{f}-A\,(\mathbf{v_x}+\mathbf{w_x})=0, \qquad \mathbb{H}:\mathbb{R}^p\longrightarrow\mathbb{R}^p, \tag{32}$$

where

$$\mathbf{v_x}=(x_1^3,x_2^3,\ldots,x_p^3)^T, \qquad \mathbf{w_x}=(|x_1|,|x_2|,\ldots,|x_p|)^T.$$

It is obvious that the function $\mathbb{H}$ defined in (32) is nonlinear and non-differentiable. So, we consider $\mathbb{H}(\mathbf{x})=\mathbb{F}(\mathbf{x})+\mathbb{G}(\mathbf{x})$ where:

$$\mathbb{F}(\mathbf{x})=\mathbf{x}-\mathbf{f}-A\frac{\mathrm{m}-1}{\mathrm{m}}\mathbf{v_x} \quad\text{and}\quad \mathbb{G}(\mathbf{x})=-\mathrm{A}\left(\frac{1}{\mathrm{m}}\mathbf{v_x}+\mathbf{w_x}\right),$$

with $\mathrm{m}\in(0,+\infty)$.

As in $\mathbb{R}^p$ we can consider divided difference of first order that do not need that the function $\mathbb{G}$ is differentiable (see [13]), we use the divided difference of first order given by $[\mathbf{u},\mathbf{v};\mathbb{G}]=([\mathbf{u},\mathbf{v};\mathbb{G}]_{ij})_{i,j=1}^{p}\in\mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)$, where

$$[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij}=\frac{1}{u_j-v_j}\left(\mathbb{G}_i(u_1,\ldots,u_j,v_{j+1},\ldots,v_p)-\mathbb{G}_i(u_1,\ldots,u_{j-1},v_j,\ldots,v_p)\right),$$

$\mathbf{u} = (u_1, u_2, \dots, u_p)^T$ and $\mathbf{v} = (v_1, v_2, \dots, v_p)^T$.

Now, we consider $\mathbf{f} = \mathbf{0}$ in (32). Obviously, in this case, $\mathbf{x}^* = \mathbf{0}$ is a solution of $\mathbb{H}(\mathbf{x}) = \mathbf{0}$. Then, the system of nonlinear equations given in (32) is of the form

$$\mathbb{H}(\mathbf{x}) = \mathbf{x} - A\,\mathbf{z}, \qquad z_j = x_j^3 + |x_j|,\ j = 1, \dots, p. \tag{33}$$

Therefore,

$$\mathbb{F}'(\mathbf{x}) = I - \frac{3(\mathrm{m}-1)}{\mathrm{m}}\, A \begin{pmatrix} x_1^2 & 0 & \dots & 0 \\ 0 & x_2^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & x_p^2 \end{pmatrix}$$

and

$$[\mathbf{x}, \mathbf{y}; \mathbb{G}] = -A \cdot \ \mathrm{Diag}\left(\frac{1}{\mathrm{m}} \begin{pmatrix} x_1^2 + x_1 y_1 + y_1^2 \\ x_2^2 + x_2 y_2 + y_2^2 \\ \dots\dots\dots \\ x_p^2 + x_p y_p + y_p^2 \end{pmatrix} + \begin{pmatrix} \frac{|x_1| - |y_1|}{x_1 - y_1} \\ \frac{|x_2| - |y_2|}{x_2 - y_2} \\ \dots\dots\dots \\ \frac{|x_p| - |y_p|}{x_p - y_p} \end{pmatrix} \right)$$

Then, $[\mathbf{x}, \mathbf{y}; F] = -\left(\frac{1}{\mathrm{m}} B + C\right)$, where $B = (b_{ij})_{i,j=1}^p$ with $b_{ii} = a_{ii}(x_i^2 + x_i y_i + y_i^2)$ and $b_{ij} = 0$ if $i \neq j$, $C = (c_{ij})_{i,j=1}^p$ with $c_{ii} = a_{ii} \frac{|x_i| - |y_i|}{x_i - y_i}$ and $c_{ij} = 0$ if $i \neq j$.

If we consider $\Omega = B(0, \ell)$, then

$$\|\mathbb{F}'(\mathbf{x}) - \mathbb{F}'(\mathbf{y})\| \leq \frac{3(\mathrm{m}-1)}{\mathrm{m}}\, \|A\| \|\mathbf{x}^2 - \mathbf{y}^2\| \leq \frac{6(\mathrm{m}-1)}{\mathrm{m}}\, \ell\, \|A\| \|\mathbf{x} - \mathbf{y}\| \tag{34}$$

and

$$\|[\mathbf{x}, \mathbf{y}; \mathbb{G}] - [\mathbf{u}, \mathbf{v}; \mathbb{G}]\| \leq \frac{3}{\mathrm{m}}\, \ell\, \|A\| (\|\mathbf{x} - \mathbf{u}\| + \|\mathbf{y} - \mathbf{v}\|) + 2\, \|A\|. \tag{35}$$

Under these conditions, $x^* = 0$ is a solution of the problem and we then choose $\tilde{\mathbf{x}} \in \Omega$, so we consider $\delta = \ell$. Moreover, assuming $p = 8$, we have:

$$\|I - L\| \leq \|I - F'(0) - [0, \tilde{\mathbf{x}}; \mathbb{G}]\| \leq \|A\| \left(\frac{\|\tilde{x}\|^2}{m} + 1 \right) < \frac{1}{8} \left(\frac{\ell^2}{m} + 1 \right).$$

Then, if $\frac{1}{8} \left(\frac{\ell^2}{m} + 1 \right) < 1$, by applying the Banach lemma, L^{-1} exists and

$$\|L^{-1}\| < \frac{8}{8 - (\ell^2/m + 1)}. \tag{36}$$

Thus, we choose $\gamma = \dfrac{8m}{7m - \ell^2}$. Now, from (33), (34) and (35) we have respectively:

$$w_0(t) = t + \|A\|(t^3 + t),$$
$$w_1(t) = \frac{6(m-1)}{m} \|A\|\ \ell t,$$
$$w_2(s, t) = \|A\|(2 + \frac{3}{m} \ell(s + t)).$$

Next, we choose $\ell = 1$ and in this case equation (29) does not has a positive real root, so that condition $(V)_S$ is not satisfied and Theorem 11 cannot be applied. On the other hand, by following Theorem 9 with $h_1(t) = t$ and $M = 1/2$ for $m \geqslant 2$, the radius of the convergence ball is the smallest positive real root of the equation given in ($\mathbf{V}$'), see (27), that is reduced to

$$3t^3 + (27 + 36m)t + 16 - 12m = 0.$$

By doing similar calculations, we obtain the corresponding results for $\ell = 0.8$. In this case, Steffensen's method can be applied and results given in Table 3 show that, by applying method (5), we also improve the accessibility of Steffensen's method when H is non-differentiable. We can observe in Table 3, the value of the radius of the convergence ball for different values of m. Observe that the radius of the convergence ball increases when m does and this corresponds to the increase of the differentiable part of the equation.

Now, from equation (28), we obtain

$$R = \frac{5m - 4}{3m},$$

and, as R is always greater than 1 for $m \geqslant 2$, the solution is unique in $\Omega = B(0, 1)$. If $m = 1$, that corresponds to the Steffensen method, we obtain $R = \frac{1}{3}$ and the solution is unique in $B(0, \frac{1}{3})$.

m	$\ell = 0.8$	$\ell = 1$
1	0.0349	-
2	0.1736	0.0808
3	0.2382	0.1481
4	0.2757	0.1870
5	0.3001	0.2124
6	0.3173	0.2303
7	0.3301	0.2436
8	0.3399	0.2538
9	0.3478	0.2620
10	0.3542	0.2686

Table 3: Radius of convergence ball for different values of m.

We have already seen that method (5) improves notably the accessibility of Steffensen's method. Next, we see that method (5) keeps and even improves the approximation to the solution of Steffensen's method when it is applied to solve nonlinear and non-differentiable systems. Also, method (5) improves the Secant method, which is the usual iterative method to solve nonlinear and non-differentiable systems, which is given by

$$\begin{cases} x_{-1}, x_0 \text{ in } \Omega, \\ x_{n+1} = x_n - [x_{n-1}, x_n; H]^{-1} H(x_n), \quad n \geq 0. \end{cases} \tag{37}$$

Finally, we show the numerical application of method (5) for the non-differentiable system of nonlinear equations given in (33). Table 4 shows the results following the same notation as in Table 2, now by taking as initial guess $x_0 = (0.5, \cdots, 0.5)$ in all methods and $x_1 = (0.6, \cdots, 0.6)$ for Secant's method. As can see, the behavior of method (5) is similar to that of Steffensen's method, method (5) is better than Steffensen's method when m is higher. In all the cases, method (5) improves the Secant method, as we could expect, since this method has superlinear convergence. In this example, the computational orders of method (5) and Steffensen's method are better than those expected, while the order of convergence is very unestable for the Secant method. The distance between the numerical approximation to the solution and the exact solution $x^*(s)$ in the fixed nodes is $\|\mathbf{x}^* - \mathbf{x_n}\| = 5.605193\ 10^{-45}$.

So, all the results obtained confirm the competitiveness of the method (5).

	m	$iter$	p	$incr1$	$incr2$
Steffensen	1	6	2.976682	3.863949e-24	7.754008e-52
method (5)	2	5	3.033212	3.610817e-17	8.775674e-45
	3	5	3.033228	1.981313e-22	4.222401e-50
	4	5	3.033708	1.403275e-33	3.411129e-36
	5	5	3.034187	1.270780e-23	1.518560e-51
	6	5	3.034601	1.878968e-24	1.991741e-52
	7	5	3.034949	4.585136e-25	5.124828e-53
	8	5	3.035242	1.551433e-25	2.331296e-53
	9	5	3.035489	6.570631e-26	6.204250e-54
	10	5	3.035701	3.268590e-26	3.101501e-54
Secant		10	-	2.756964e-19	6.479021e-20

Table 4: Numerical results for problem (33).

5 Concluding remarks

As you can see in in Figures 2 and 3, Steffensen's method has a poor accessibility to the solutions of an equation. In this work, we have modified Steffensen's method and obtain method (5), that improves significantly the accessibility of Steffensen's method from decomposition of operators. This improvement can be seen experimentally in Figure 4 and theoretically from the result of local convergence given in Theorem 4. We do this study for differentiable operators and non-differentiable operators.

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