Choosing the most stable members of Kou's family of iterative methods ☆

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Abstract

In this manuscript, we analyze the dynamical anomalies of a parametric family of iterative schemes designed by Kou et al. It is known that its order of convergence is three for any arbitrary value of the parameter, but it has order four (and it is optimal in the sense of Kung-Traub's conjecture) when an specific value is selected. Among all the elements of this family, one can choose this fourth-order element or any of the infinite members of third order of convergence, if only the speed of convergence is considered. However, the stability of the methods plays an important role in their reliability when they are applied on different problems. This is the reason why we analyze in this paper the dynamical behavior on quadratic polynomials of the mentioned family. The study of fixed points and their stability, joint with the critical points and their associated parameter planes, show the richness of the class and allows us to find members of it with excellent numerical properties, as well as other ones with very unstable behavior. Some test functions are analyzed for confirming the theoretical results.

Keywords: Nonlinear equation; iterative method; dynamical behavior; Fatou and Julia sets; basin of attraction; periodic orbits.

1. Introduction

Nonlinear equations $f(z) = 0$, where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a real function defined in an open interval I, are often used for modeling real problems arising in science and engineering as, for example, in the analysis of dynamical models of chemical reactors [7], preliminary orbit determination of satellites [3], in radioactive transfer [13], to simulate flow transport in a pipe [24] or even the approximation of the eigenvalues of square matrices, which is known has many applications in areas as image processing, dynamical systems, control theory, etc (see, for example, [8]).

For solving these equations, iterative schemes must be used. The best known iterative approach is Newton's method. In last decades, many researchers have proposed different iterative methods to improve Newton's scheme (see, for example, the review [21], and the references therein). These variants of Newton's method have been designed by means of different techniques, providing in the most of cases multistep schemes. Some of them come from Adomian decomposition (see [1], for example). Another procedure to develop iterative methods is the replacement of the second derivative in Chebyshev-type methods by some approximation, [23]. A common way to generate new schemes is the direct composition of known methods with a later treatment to reduce the number of functional evaluations. For example, by composing Newton's method with itself, holding the derivative "frozen" in the second step, third-order Traub's method [23] is obtained.

Recently, the weight-function procedure has been used to increase the order of convergence of known methods ([21, 22]), allowing to get optimal methods, under the point of view of Kung-Traub's conjecture [17]. These authors conjectured that an iterative methods, without memory, which uses d functional evaluations per iteration can reach, at most, order of convergence 2^{d-1}. When this bound is reached, the scheme is called optimal. Although the aim of

☆This research was supported by Ministerio de Economía y Competitividad MTM2014-52016-C2-2-P and Generalitat Valenciana PROMETEO/2016/089.

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many researches in this area is to design optimal high-order methods, it is also known that the higher the order is, the more sensitive the scheme to initial estimations will be [18]. On the other hand, recent studies on damped Newton's procedure show (see, for example [19]) that small damping parameters widen the set of initial guesses that make the method convergent, although the speed of convergence decreases.

In this paper, we present a Kou's family of iterative methods [16], whose iterative expression is

$$z_{k+1} = z_k - \left(1 - \frac{3}{4}\frac{t_k - 1}{\gamma t_k + 1 - \gamma}\right)\frac{f(z_k)}{f'(z_k)}, \quad n = 0, 1, \ldots, \tag{1}$$

where $y_k = z_k - \dfrac{2}{3}\dfrac{f(z_k)}{f'(z_k)}$, $t_k = \dfrac{f'(y_k)}{f'(z_k)}$ and γ is a free parameter.

In [16], the authors proved the following result of convergence.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently derivable function in the open interval I and let $\alpha \in I$ be a simple solution of the nonlinear equation $f(x) = 0$. We consider that x_0 is an initial approximation close enough to α. Then, the sequence $\{x_k\}_{k\geq 0}$ obtained by using Kou's family converges to α with order of convergence three for any value of parameter γ, being the error equation*

$$e_{k+1} = \frac{2}{3}(3 - 2\gamma)c_2^2 e_k^3 + O(e_k^4).$$

Moreover, for $\gamma = \frac{3}{2}$, the order of the method is four and its error equation is

$$e_{k+1} = \left(c_2^3 - c_2 c_3 + \frac{c_4}{9}\right) e_k^4 + O(e_k^5),$$

where $c_j = \frac{f^{(j)}(\alpha)}{j! f'(\alpha)}$, $j = 2, 3, \ldots$ and $e_k = z_k - \alpha$.

If we are only thinking in the speed of convergence any element of the Kou's family (except for $\gamma = \frac{3}{2}$) has a similar behavior. However, in this paper we are interested in the stability of these iterative methods. So, by using complex dynamics tools, we are going to analyze the dynamical behavior of the rational function associated to the class of iterative schemes on low degree polynomials. The qualitative information that this analysis gives us is very useful for selecting the members of the class with good stability properties and to refuse other ones with bad numerical behavior.

From the numerical point of view, the dynamical behavior of the rational function associated with an iterative method, on low degree polynomials, gives us important information about its stability and reliability. In these terms, Amat et al. in [2] described the dynamical behavior of several well-known families of iterative methods. More recently, in [4, 10, 12, 14, 19, 20] the authors analyze the qualitative behavior of different known iterative families. The most of these studies show different pathological numerical behavior, such as periodic orbits, attracting fixed points different from the solution of the problem, etc. Indeed, parameter planes associated to a family of methods allow us to understand the behavior of the different members of the family of methods, helping us in the election of a particular one.

The rest of the paper is organized as follows: in Section 2 we introduce the basic concepts on complex dynamics that we will use in all the paper, in Section 3 we obtain the fixed and critical points of the rational operator associated to Kou's family on quadratic polynomials, and the stability of the obtained fixed points is analyzed in Section 4. These stability regions also appear in the associated parameter spaces, obtained in Section 5, where we also show some dynamical planes corresponding to different elements of the family with good and bad numerical behavior. In this section, we also study the stability region of the attractive 2-periodic orbits, whose analytical expression is found, depending on the parameter. Section 6 is devoted to the numerical results, with academical examples and members of the class with different behavior, selected from the previous results. We finish the work with some remarks and conclusions.

2. Basic concepts

Under the point of view of complex dynamics, we will study the general convergence of family (1) on quadratic polynomials. It is known (see, for example [6]) that the roots of a polynomial can be transformed by an affine map with no qualitative changes on the dynamics of the family. So, we can use quadratic polynomial $p(z) = (z-a)(z-b)$. For $p(z)$, the operator of the family is the rational function:

$$T_{p,\gamma,a,b}(z) = z + \frac{(a-z)(b-z)(3a^2+3b^2+b(-15+4\gamma)z+(15-4\gamma)z^2+a(b(9-4\gamma)+(-15+4\gamma)z))}{(a+b-2z)(3a^2+3b^2+4b(-3+\gamma)z-4(-3+\gamma)z^2+a(b(6-4\gamma)+4(-3+\gamma)z))},$$

depending on parameter γ and also on the roots of the polynomial a and b.

Blanchard in [5] considered the conjugacy map $h(z) = \dfrac{z-a}{z-b}$, (a Möbius transformation) with the following properties:

$$\text{i) } h(\infty) = 1, \quad \text{ii) } h(a) = 0, \quad \text{iii) } h(b) = \infty,$$

and proved that, for quadratic polynomials, Newton's operator is conjugate to the rational map z^2. In an analogous way, operator $T_{p,\gamma,a,b}(z)$ on quadratic polynomials is conjugated to operator $O_\gamma(z)$,

$$O_\gamma(z) = \left(h \circ T_{p,\gamma,a,b} \circ h^{-1}\right)(z) = -z^3 \frac{6-4\gamma+3z}{-3-6z+4\gamma z}. \tag{2}$$

We observe that the parameters a and b have been obviated in $O_\gamma(z)$.

Now, we are going to recall some dynamical concepts of complex dynamics (see [6] for a more deep study of these concepts) that we use in this work. Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$\{z_0,\, R(z_0),\, R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

We analyze the phase plane of the map R by classifying the starting points from the asymptotic behavior of their orbits. A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of the rational function vanishes, $R'(z_0) = 0$. Moreover, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The basin of attraction of an attractor α is defined as:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \,:\, R^n(z_0) \to \alpha,\ n \to \infty\}.$$

The *immediate basin of attraction* of an attractor is the connected component of its basin of attraction that holds the attractor.

The *Fatou set* of the rational function R, $\mathcal{F}(R)$, is the set of points $z \in \hat{\mathbb{C}}$ whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the *Julia set*, $\mathcal{J}(R)$. That means that the basin of attraction of any fixed point belongs to the Fatou set and the boundaries of these basins of attraction belong to the Julia set.

The following theorem establishes a classical result of Fatou and Julia that we use in the study of parameter space associated to the family.

Theorem 2. *Let R be a rational function. The immediate basin of attraction of an attracting fixed or periodic point holds, at least, a critical point.*

By using this result, one can be sure to find all the stable behavior associated to a rational function R, by analyzing the performance of R on the set of critical points.

3. Analysis of the fixed and critical points

In the next sections, we are going to analyze, under the dynamical point of view, the stability and reliability of the members of the proposed family. Firstly, we study the fixed points of the rational function $O_\gamma(z)$ that are not related with the original roots of the polynomial $p(z)$ (called *strange fixed points*), and the *free critical points*, that is, the critical points of $O_\gamma(z)$ different from 0 and ∞, which are associated to the roots of $p(z)$.

Fixed points of $O_\gamma(z)$ are the roots of the equation $O_\gamma(z) = z$, that is, $z = 0$, $z = \infty$ and the strange fixed points

- $ex_1(\gamma) = 1$,
- $ex_2(\gamma) = \frac{1}{6}(-9 + 4\gamma - \sqrt{45 - 72\gamma + 16\gamma^2})$,
- $ex_3(\gamma) = \frac{1}{6}(-9 + 4\gamma + \sqrt{45 - 72\gamma + 16\gamma^2})$.

Some relations between the strange fixed points are described in the following result.

Lemma 1. *The number of simple strange fixed points of operator $O_\gamma(z)$ is three, except in the following cases:*

i) If $\gamma = \frac{3}{4}$, then the operator is $O_{3/4}(z) = z^3$, so there are no strange fixed points.

ii) If $\gamma = \frac{9}{4}$, then the operator is $O_{9/4}(z) = -z^3$, so there are no strange fixed points.

iii) If $\gamma = \frac{15}{4}$, there is only one strange fixed point, $ex_2 = ex_3 = 1$, as $O_{15/4}(z) = -z^3\frac{-3+z}{-1+3z}$.

iv) If $\gamma = \frac{3}{2}$, then the operator is $O_{3/2}(z) = z^4$, so there are no strange fixed points.

In order to determine the critical points, we calculate the first derivative of $O_\gamma(z)$,

$$O'_\gamma(z) = 2z^2\frac{16\gamma^2 z + 27(1+z)^2 - 6\gamma(3 + z(8+3z))}{(-3 + (-6+4\gamma)z)^2}.$$

As we have said, a classical result establishes that there is, at least, one critical point associated with each invariant Fatou component. It is clear that $z = 0$ and $z = \infty$ (related to the roots of the polynomial by means of Möbius map) are critical points and give rise to their respective Fatou components, but there exist in the family some free critical points, some of them depending on the value of the parameter.

Lemma 2. *Analyzing the equation $O'_\gamma(z) = 0$, we obtain:*

a) If $\gamma = \frac{3}{4}$, $\gamma = \frac{9}{4}$ or $\gamma = \frac{3}{2}$ there is no free critical points of operator $O_\gamma(z)$.

b) If $\gamma = 0$, then $z = -1$ is the only free critical point.

c) If $\gamma = 3$, then $z = 1$ is the only free critical point.

d) In any other case,

$$cr_1(\gamma) = \frac{27 - 24\gamma + 8\gamma^2 - 2\sqrt{-81\gamma + 171\gamma^2 - 96\gamma^3 + 16\gamma^4}}{9(-3+2\gamma)}$$

and

$$cr_2(\gamma) = \frac{27 - 24\gamma + 8\gamma^2 + 2\sqrt{-81\gamma + 171\gamma^2 - 96\gamma^3 + 16\gamma^4}}{9(-3+2\gamma)} = \frac{1}{cr_1(\gamma)},$$

are free critical points.

From the previous results, let us summarize:

- When $\gamma = 0$, $cr_1(0) = cr_2(0) = -1$ that is a pre-image of $z = 1$, that in this case is not a fixed point, and the associated operator is $O_0(z) = z^3 \dfrac{2+z}{1+2z}$.
- When $\gamma = 3$, $cr_1(3) = cr_2(3) = 1$ that is a superattractive fixed point and the associated operator is $O_3(z) = -z^3 \dfrac{-2+z}{-1+2z}$.
- At most, there is one independent free critical point. So, we will consider only $cr_1(\gamma)$.

As we will see in the following section, not only the number but also the stability of the fixed points depend on the parameter of the family. The relevance of this study yields in the fact that the existence of attracting strange fixed points can make the iterative scheme converge to a "false" solution.

4. Stability of the fixed points

As the order of convergence of the family is at least three, it is clear that the origin and ∞ (related to the roots of $p(z)$) are always superattractive fixed points, but the stability of the other fixed points gives us interesting numerical information. In the following results we show the stability of the strange fixed points.

Theorem 3. *The character of the strange fixed point $ex_1(\gamma) = 1$, $\gamma \neq \frac{9}{4}$, is as follows:*

i) If $\left|\gamma - \frac{13}{4}\right| < \frac{1}{2}$, then $ex_1(\gamma) = 1$ is an attractor and it is a superattractor when $\gamma = 3$.

ii) When $\left|\gamma - \frac{13}{4}\right| = \frac{1}{2}$, $ex_1(\gamma) = 1$ is a parabolic point.

iii) If $\left|\gamma - \frac{13}{4}\right| > \frac{1}{2}$, then $ex_1(\gamma) = 1$ is a repulsor.

Proof. It is easy to see that

$$O'_\gamma(1) = \frac{8(-3+\gamma)}{-9+4\gamma}.$$

So,

$$\left|\frac{8(-3+\gamma)}{-9+4\gamma}\right| \leq 1 \qquad \text{is equivalent to} \qquad 8|-3+\gamma| \leq |-9+4\gamma|.$$

Let us consider $\gamma = a + ib$ an arbitrary complex number. Then,

$$8^2(3^2 - 6a + a^2 + b^2) \leq 9^2 - 72a + 16a^2 + 16b^2.$$

By simplifying

$$495 - 312a + 48a^2 + 48b^2 \leq 0,$$

that is,

$$\left(a - \frac{13}{4}\right)^2 + b^2 \leq \frac{1}{4}.$$

Therefore,

$$\left|O'_\gamma(1)\right| \leq 1 \qquad \text{if and only if} \qquad \left|\gamma - \frac{13}{4}\right| \leq \frac{1}{2}.$$

■

Theorem 4. *The analysis of the stability of strange points $ex_2(\gamma)$ and $ex_3(\gamma)$ shows that:*

i) If $\left|\gamma - \frac{9}{2}\right| < \frac{3}{4}$*, then both points are attractors and they are superattractors when* $\gamma = \frac{9}{2}$.

ii) If $\left|\gamma - \frac{9}{2}\right| = \frac{3}{4}$*, then* $ex_2(\gamma)$ *and* $ex_3(\gamma)$ *are parabolic.*

iii) In any other case, both points are repulsors.

Proof. It is easy to see that

$$O'_\gamma\,(ex_i) = 6 - \frac{4\gamma}{3}, \quad i = 2, 3$$

So,

$$\left|\frac{18-4\gamma}{3}\right| \leq 1 \qquad \text{is equivalent to} \qquad |18-4\gamma| \leq 3.$$

Let us consider $\gamma = a + ib$ an arbitrary complex number. Then,

$$18^2 - 144a + 4^2a^2 + 4^2b^2 \leq 9.$$

By simplifying

$$315 - 144a + 16a^2 + 16b^2 \leq 0,$$

that is,

$$\left(a - \frac{9}{2}\right)^2 + b^2 \leq \left(\frac{3}{4}\right)^2.$$

Therefore,

$$\left|O'_\gamma\,(ex_i)\right| \leq 1 \qquad \text{if and only if} \qquad \left|\gamma - \frac{9}{2}\right| \leq \frac{3}{4}.$$

■

In Figure 1, we represent the stability regions of $ex_i(\gamma)$, $i = 1, 2, 3$, that we get from Theorem 3 and Theorem 4.

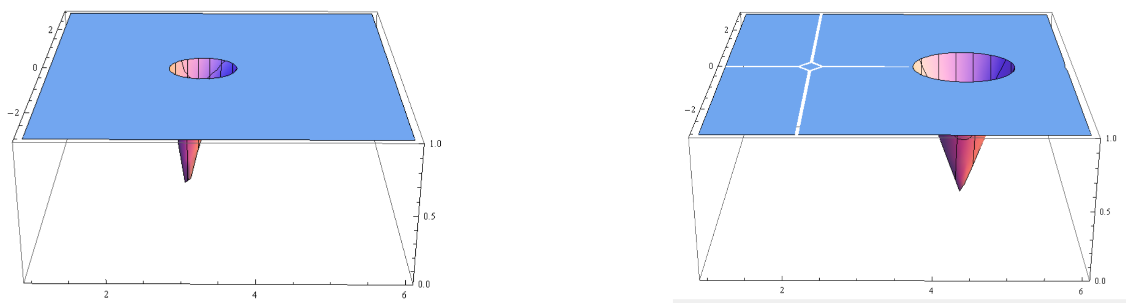


Figure 1: Stability regions of $ex_1(\gamma)$ (left) and $ex_i(\gamma)$, $i = 2, 3$ (right), respectively.

5. The parameter space

As we have seen, the dynamical behavior of operator $O_\gamma(z)$ depends on the values of parameter γ. Taking into account Theorem 2, we would like to know what happens with the free critical points, that is, do some of them belong to a basin of attraction different to those of zero and infinity? For answering this question we construct the parameter plane associated to family (1).

The parameter space associated with an independent free critical point of operator (2) is obtained by associating each point of the plane with a complex value of γ, i.e., with an element of family (1). Every value of γ belonging to the same connected component of the parameter space gives rise to subsets of schemes of family (1) with similar dynamical behavior. So, it is interesting to find regions of the parameter plane as much stable as possible, because these values of γ will give us the best members of the family in terms of numerical stability.

As $cr_1(\gamma) = \dfrac{1}{cr_2(\gamma)}$, we have at most one free independent critical point, so we can obtain different parameter planes, with complementary information. When we consider the free critical point $z = cr_1(\gamma)$ as a starting point of the iterative scheme of the family associated to each complex value of γ, we paint this point of the complex plane in red if the method converges to any of the roots (zero and infinity) and they are black in other cases. The color used is brighter when the number of iterations is lower. Then, the parameter plane P_1 is obtained; it is showed in Figure 2. The parameter plane has been generated by using the routines described in [9]. A mesh of 1000×1000 points has been used, 500 has been the maximum number of iterations involved and 10^{-3} the tolerance used as a stopping criterium.

In the following, we will focus our attention on parameter plane P_1, due to its dynamical richness.

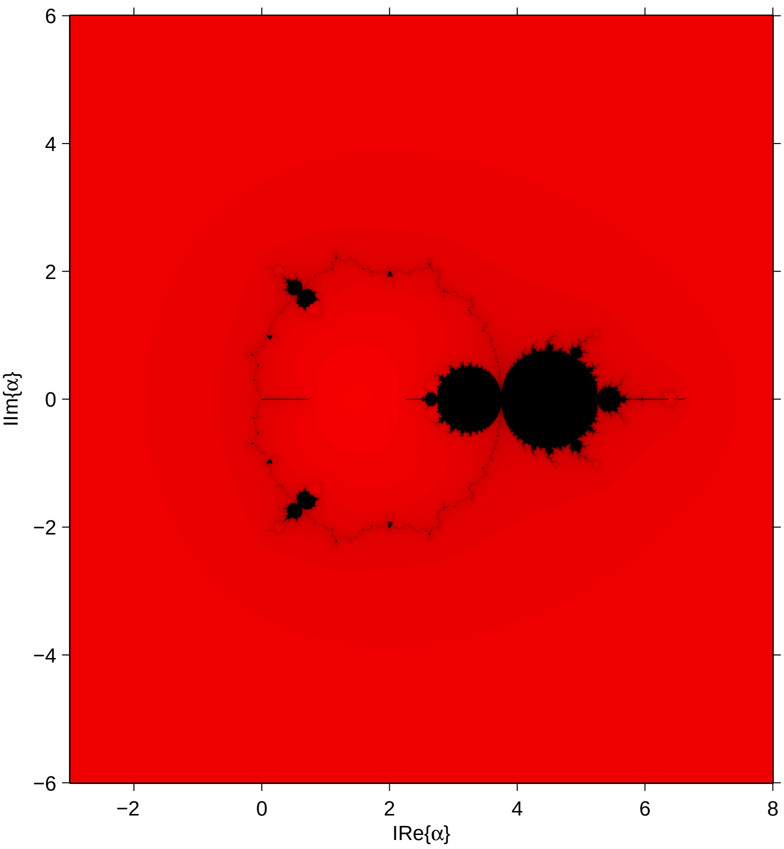


Figure 2: Parameter plane P_1 associated to $cr_i(\gamma)$, $i = 1, 2$

We can observe that the members of family (1) are, in general, very stable, the red area is very big. However, there are small black regions that inform us about different pathological behavior of some elements of the family.

Let us remark that two balls with centers $(13/4, 0)$ and $(9/2, 0)$, called D_1 and D_2, respectively. The first ball correspond to values of parameter γ for which $ex_1(\gamma)$ is attractor or superattractor (see Theorem 3). The second one corresponds to values of γ for which $ex_2(\gamma)$ and $ex_3(\gamma)$ are simultaneously attractors or superattractors (see Theorem 4). In addition, there are other black areas and bulbs that correspond to attractive orbits of different periods. In the next section, we obtain the orbits of period two.

5.1. Orbits of period two

In order to obtain the analytical expression of the elements of 2-periodic orbits, depending on γ, we calculate $O_\gamma(O_\gamma(z))$, that will be denoted by $O^2_\gamma(z)$

$$O^2_\gamma(z) = \frac{z^9(6-4\gamma+3z)^3(18-12\gamma+36z-48\gamma z+16\gamma^2 z+18z^3-12\gamma z^3+9z^4)}{(-3-6z+4\gamma z)^3(-9-18z+12\gamma z-36z^3+48\gamma z^3-16\gamma^2 z^3-18z^4+12\gamma z^4)}.$$

Figure 3: Stability regions of 2-periodic orbits $pe_i(\gamma),\ i=1,2$ (left) and $pe_i(\gamma),\ i=3,4$ (right)

The periodic points of $O_\gamma(z)$ with period two are the roots of the equation $O_\gamma^2(z)=z$, that is, the fixed points and the 2-periodic points

$$\begin{aligned} pe_{\{1,2\}}(\gamma) &= \tfrac{1}{12}(-9+4\gamma-r(\gamma)\pm\sqrt{2}\sqrt{-9+16\gamma^2+9r(\gamma)-4\gamma(12+r(\gamma))}, \\ pe_{\{3,4\}}(\gamma) &= \tfrac{1}{12}(-9+4\gamma+r(\gamma)\pm\sqrt{2}\sqrt{-9+16\gamma^2-9r(\gamma)+4\gamma(-12+r(\gamma))}, \end{aligned}$$

where $r(\gamma)=\sqrt{45-24\gamma+16\gamma^2}$.

In Figure 3, we represent the stability regions of the orbits $pe_{\{1,2\}}$ and $pe_{\{3,4\}}$. We can observe small areas where these orbits are attractive. Moreover, it can be checked that there exist several values of parameter γ that make the 2-periodic orbits superattracting, that is, satisfy ${O'_\gamma}^2(z)=0$. In Figure 4 we show all the stability regions including those of the strange fixed points and the 2-periodic orbits.

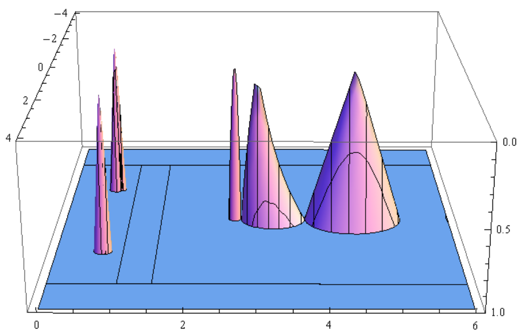


Figure 4: Stability regions of strange fixed points and 2-periodic points

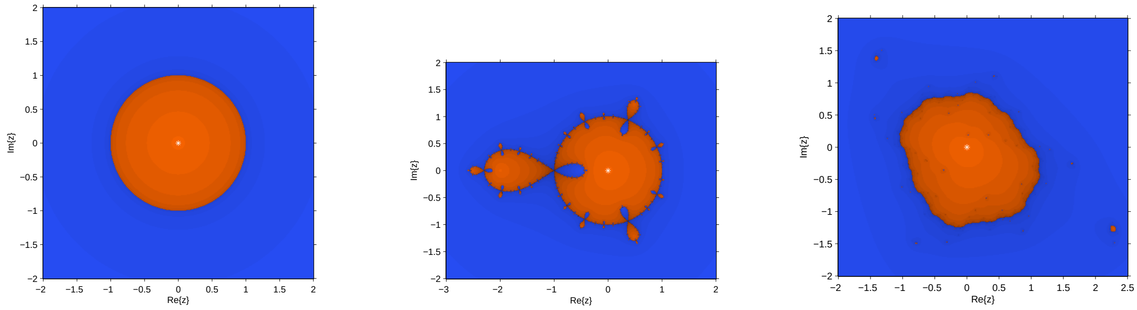
This 3D-plot allows us to identify many of the different stability regions appearing in the parameter plane P_1 (see Figure 2) as black regions.

5.2. Dynamical Planes

In this section we will show, by means of dynamical planes, the qualitative behavior of the different elements of family (1). We will select these elements by using the conclusions obtained by analyzing the parameter plane of the family and the stability analysis made on fixed and 2-periodic points.

As in case of parameter plane, these dynamical planes has been generated by using the routines appearing in [9]. The dynamical plane associated to a value of the parameter γ, that is, obtained by iterating an element of family (1), is generated by using each point of the complex plane as initial estimation (we have used a mesh of 400×400 points). We paint in blue the points whose orbit converges to infinity, in orange the points converging to zero (with a tolerance of 10^{-3}), in other colors (green, red, etc.) those points whose orbit converges to one of the strange fixed points (all fixed points appear marked as a white star in the figures) and in black if it reaches the maximum number of 40 iterations without converging to any of the fixed points.

There are some regions in parameter space P_1 whose corresponding iterative methods have good numerical behavior, in terms of stability and efficiency. They correspond to values of γ painted in red (Figure 2). In Figure 5 we



(a) $\gamma = \frac{3}{4}$ and $\gamma = \frac{3}{2}$ (b) $\gamma = 0$ (c) $\gamma = 2 - 4i$

Figure 5: Some dynamical planes with stable behavior

show different stable behavior corresponding to several values of γ selected in this red region; in particular, we use $\gamma = 0$, $\gamma = \frac{3}{4}$, $\gamma = \frac{3}{2}$ and $\gamma = 2 - 4i$.

On the other hand, unstable behavior is found when we choose values of γ in the black region of parameter plane P_1. In Figure 6, the dynamical plane of the iterative method corresponding to $\gamma = -2 \in D_2$ is presented, showing the existence of four different basins of attraction, two of them of the superattractors 0 and ∞ and the other two corresponding to the superattractors $ex_2(-2)$ and $ex_3(-2)$.

6. Numerical results

This section is devoted to verify the validity and effectiveness of our theoretical results. The members of the family not only hold the theoretical order of convergence, but also the good elements of the class for quadratic polynomials (selected by means of the complex dynamics analysis) remain being stable for more complicated functions. During these numerical experiments software Matlab R2013b have been done with double precision arithmetics. We have used as stopping criterium $|x_{k+1} - x_k| < tol$ or $|f(x_{k+1})| < tol$, being $tol = 10^{-12}$. In order to confirm the theoretical order of convergence, we use the approximated computational order of convergence ACOC introduced in [11] as

$$p \approx ACOC = \frac{\ln\left(|x_{k+1} - x_k|/|x_k - x_{k-1}|\right)}{\ln\left(|x_k - x_{k-1}|/|x_{k-1} - x_{k-2}|\right)}.$$

If ACOC is not stable along the iterative process, it is denoted by '-'. Moreover, when an scheme does not converge, it appears in Tables 2 to 4 as 'nc'.

For solving the nonlinear functions appearing in Table 1, we use some elements of Kou's family (some of them with good stability properties and others with bad ones) and also some known iterative schemes as Newton', Traub' and Homeier's ones. Let us recall that their iterative expressions are

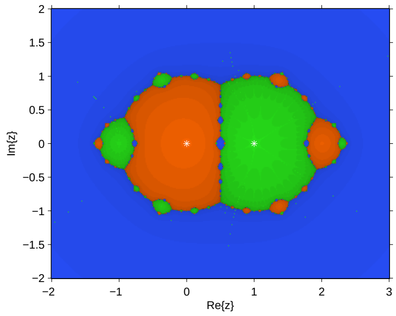
$$x_{k+1} = x_k - \frac{f(x_k) + f(y_k)}{f'(x_k)}, \tag{3}$$

$$x_{k+1} = x_k - \frac{1}{2}\left(1 + \frac{f'(x_k)}{f'(y_k)}\right)\frac{f(x_k)}{f'(x_k)}, \tag{4}$$

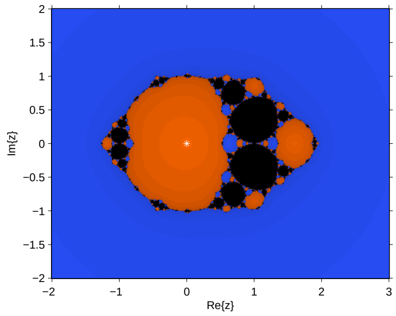
where $y_k = x_k - \frac{f(x_k)}{f'(x_k)}$ is Newton's step, corresponding to Traub' [23] and Homeier's [15] procedures, respectively.

Test functions	Zeros
$f_1(x) = \arctan(x)$	$\alpha = 0$
$f_2(x) = e^{x^2-3x}\sin x + \ln(x^2+1)$	$\alpha = 0$
$f_2(x) = e^x - 1.5 - \arctan(x)$	$\alpha_1 \approx 0.767653$, $\alpha_2 \approx -14.101270$

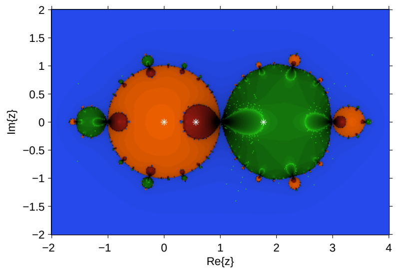
Table 1: Test functions and their zeros



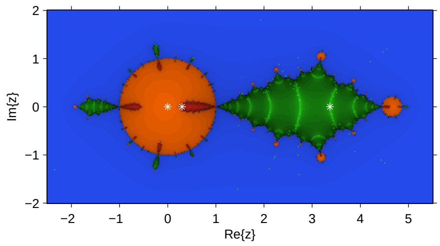
(a) $\gamma = 3,$ in D_1



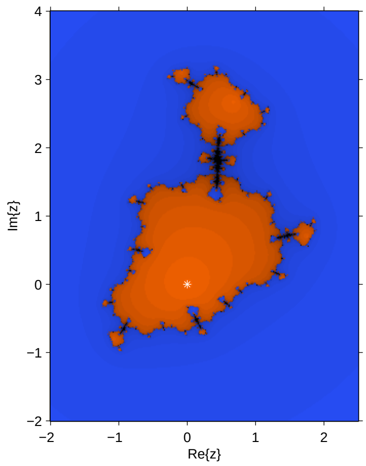
(b) $\gamma = 2.7,$ in a bulb to the left of D_1



(c) $\gamma = 4,$ in D_2



(d) $\gamma = 5,$ in D_2



(e) $\gamma = 2 + 2i,$ in a Maldelbrot type set

Figure 6: Dynamical planes with unstable behavior

Regarding Table 2, let us remark that classical methods have problems of convergence for initial estimations far from the solution, meanwhile stable members of Kou's family have a better numerical behavior.

In Tables 3 and 4, classical schemes show good results. In a similar way, those elements of Kou's class found to be stable on quadratic polynomials present as good behavior as Newton', Traub' and Homeier's methods. However, when values of γ are selected among the unstable ones, the numerical behavior is very bad.

7. Conclusions

A dynamical study on quadratic polynomials of a parametric family of iterative methods for solving nonlinear equations, constructed by Kou et al., has been presented. From the parameter plane associated to the class, it has been proved that there are many values of parameter γ, that is, elements of the family, with good stability properties and other ones with no convergence to the roots of the polynomial, and the existence of periodic orbits of period two has been showed and its analytical expression has been obtained in terms of parameter γ. Some numerical tests, with

	x_0	α	$\lvert x_{k+1}-x_k\rvert$	$\lvert f(x_{k+1})\rvert$	$Iter$	$ACOC$
Newton	1	0.0	7.96e-10	0.0	5	2.9937
	1.8	nc				
	-1.9	nc				
Traub	1	0.0	3.63e-10	0.0	4	-
	1.8	nc				
	-1.9	nc				
Homeier	1	3.31e-24	1.97e-8	3.31e-24	4	2.9951
	1.8	nc				
	-1.9	nc				
Kou $\gamma = \frac{3}{2}$	1	2.12e-22	1.65e-6	2.12e-22	3	4.6015
	1.8	0.0	5.80e-10	0.0	5	-
	-1.9	nc				
$\gamma = \frac{3}{4}$	1	0.0	2.89e-6	0.0	3	4.7770
	1.8	0.0	8.14e-6	0.0	4	-
	-1.9	0.0	1.27e-7	0.0	5	-
$\gamma = 0$	1	0.0	3.80e-7	0.0	4	-
	1.8	-1.78e-16	7.95e-4	1.78e-16	3	2.7746
	-1.9				> 1000	
$\gamma = 3$	1	-6.09e-15	0.014	6.09e-15	3	3.4617
	1.8	nc				
	-1.9	nc				
$\gamma = 5$	1	0.0	2.38e-10	0.0	4	-
	1.8	nc				
	-1.9	nc				
$\gamma = 2.7$	1	-2.62e-16	7.86e-4	2.62e-16	3	3.6158
	1.8	nc				
	-1.9	nc				

Table 2: Numerical results for $f_1(x)$

non-polynomial equations, confirm the information given by the dynamical study respect to the good or bad numerical behavior of the different elements of the family.

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	x_0	α	$\lvert x_{k+1}-x_k\rvert$	$\lvert f(x_{k+1})\rvert$	$Iter$	$ACOC$
Newton	-1.5	-6.23e-17	7.90e-9	6.23e-17	13	2.0065
	2.8	-1.06e-14	7.26e-8	1.06e-14	12	2.0128
	-3	-2.76e-23	5.26e-12	2.76e-23	21	2.0008
Traub	-1.5	-2.16e-13	3.00e-5	2.16e-13	9	3.0901
	2.8	nc				
	-3	-8.52e-20	2.20e-7	2.52e-20	15	3.0219
Homeier	-1.5	9.29e-24	3.05e-12	9.29e-24	8	3.0000
	2.8	2.38e-14	2.07e-5	2.38e-14	4	-
	-3	-2.71e-13	4.67e-5	2.71e-13	8	3.1501
Kou $\gamma=\frac{3}{2}$	-1.5	8.57e-17	5.04e-7	8.57e-17	6	-
	2.8	3.64e-17	2.87e-6	3.64e-17	5	6.8344
	-3	7.93e-15	2.57e-4	7.93e-15	8	5.8084
$\gamma=\frac{3}{4}$	-1.5	-5.03e-13	5.01e-5	5.03e-13	8	3.2028
	2.8	1.73e-19	9.16e-10	1.73e-19	7	3.0223
	-3	2.42e-22	1.56e-11	2.42e-22	13	3.0135
$\gamma=0$	-1.5	-3.32e-13	3.46e-5	3.32e-13	9	3.0961
	2.8	7.90e-19	2.33e-7	7.90e-19	7	3.0234
	-3	-1.79e-13	2.82e-5	1.79e-13	14	3.0901
$\gamma=3$	-1.5	nc				
	2.8	nc				
	-3	nc				
$\gamma=5$	-1.5				>1000	
	2.8				>1000	
	-3				>1000	
$\gamma=2.7$	-1.5	nc				
	2.8	nc				
	-3	nc				

Table 3: Numerical results for $f_2(x)$

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	x_0	α	$\|x_{k+1} - x_k\|$	$\|f(x_{k+1})\|$	$Iter$	$ACOC$
Newton	4	0.7677	1.56e-12	0.0	9	2.0001
	2	0.7677	1.34e-7	0.0	6	2.0021
	-2	-14.1013	8.80e-8	0.0	7	2.0037
Traub	4	0.7677	3.83e-7	0.0	5	2.8767
	2	0.7677	5.55e-5	0.0	4	2.7523
	-2	-14.1013	2.45e-6	0.0	5	2.9919
Homeier	4	0.7677	2.10e-8	0.0	5	3.1308
	2	0.7677	2.19e-10	0.0	4	3.0927
	-2	-14.1013	1.36e-4	0.0	4	3.1931
Kou $\gamma = \frac{3}{2}$	4	0.7677	5.18e-5	2.22e-16	4	3.3038
	2	0.7677	9.68e-5	0.0	3	3.2606
	-2	-14.1013	0.0078	4.44e-16	3	6.8668
$\gamma = \frac{3}{4}$	4	0.7677	8.49e-12	2.22e-16	6	2.9780
	2	0.7677	1.68e-6	1.11e-16	4	2.8713
	-2	-14.1013	9.77e-10	0.0	5	3.0043
$\gamma = 0$	4	0.7677	4.12e-7	1.11e-16	6	
	2	0.7677	5.64e-5	4.49e-13	4	
	-2	-14.1013	2.98e-6	2.22e-16	5	
$\gamma = 3$	4	nc				
	2	nc				
	-2	-	5.94e-13	0.0849	35	0.9723
$\gamma = 5$	4	nc				
	2				> 1000	
	-2				> 1000	
$\gamma = 2.7$	4	nc				
	2	nc				
	-2	nc				

Table 4: Numerical results for $f_3(x)$

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