# Solving the random Cauchy one-dimensional advection-diffusion equation: Numerical analysis and computing

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**Abstract**

In this paper, a random finite difference scheme to solve numerically the random Cauchy one-dimensional advection-diffusion partial differential equation is proposed and studied. Throughout our analysis both the advection and diffusion coefficients are assumed to be random variables while the deterministic initial condition is assumed to possess a discrete Fourier transform. For the sake of generality in our study, we consider that the advection and diffusion coefficients are statistical dependent random variables. Under mild conditions on the data, it is demonstrated that the proposed random numerical scheme is mean square consistent and stable. Finally, the theoretical results are illustrated by means of two numerical examples.

*Keywords:* Random Cauchy advection-diffusion equation, mean square random convergence, random finite difference scheme, random consistency, random stability

## 1. Introduction

It is well-known, from the deterministic theory, that partial differential equations (PDEs) can seldom be solved in an exact manner. This motivates the development of numerical schemes to construct reliable approximations. Deterministic finite difference methods are a class of numerical schemes which are based on replacing the partial derivatives that appear in the PDEs by their finite difference approximations. This approach leads to a system of algebraic equations that can then be solved numerically by an iterative process in order to obtain an approximate solution of the PDEs. In the deterministic scenario, the finite difference method has demonstrated to be very useful to approximate the solution of PDE [1, 2, 3]. Nevertheless, modelling real problems require to make measurements of physical variables and this entails the introduction of randomness from both error measurements and the inherent complexity of the physical phenomena under study. Starting from this initial approach, it is then natural to study finite difference

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numerical schemes for solving random/stochastic PDEs, which are mathematical representations of physical problems. It is important to highlight that the kind of randomness that is considered into the physical model formulation delineates the type of PDE. On the one hand, the consideration of uncertainty by means of a gaussian stochastic process termed white noise, the formal derivative of the Wiener process or brownian motion, leads to stochastic partial differential equations (SPDEs), usually called Itô-type SPDEs. Solving analytically these equations requires the application of a special stochastic calculus, usually referred to as Itô calculus, whose cornerstone is the Itô's lemma [4, 5]. The use of this stochastic calculus is required to handle SPDEs because the irregular behaviour of the sample trajectories of the Wiener process which are nowhere differentiable [4]. On the other hand, if uncertainty is considered through random variables (RVs) and/or stochastic processes (SPs) whose sample behaviour is milder, then one leads to random partial differential equations (RPDEs). The analysis and computing of these RPDEs are done using the so-called $L_p$-random calculus [6, 7]. This latter approach allows us the consideration of a wider kind of randomness because, apart from gaussian, other RVs like binomial, Poisson, uniform, beta, exponential, etc. can also be included in the mathematical model. Throughout this paper we will propose a random finite difference scheme (RFDS) to construct numerical approximations for the following advection-diffusion RPDE

$$U_t(x,t) + \beta\, U_x(x,t) = \alpha\, U_{xx}(x,t), \qquad t > 0, \quad -\infty < x < \infty, \tag{1}$$

with initial condition

$$U(x,0) = U_0(x). \tag{2}$$

In this random Cauchy or initial value problem (IVP), $t$ and $x$ denote the time and space variables, respectively, while $U_t$, $U_x$ and $U_{xx}$ stand for the first and the second derivatives with respect to $t$ and $x$, as usually. The coefficients $\alpha$ and $\beta$ are assumed to be positive absolutely continuous RVs, defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and satisfying certain conditions that will be specified later (see hypothesis **H2** in (23)). Henceforth, $f_{\alpha,\beta}(\alpha,\beta)$ will denote the joint probability density function (PDF) of the two-dimensional random vector $(\alpha,\beta)$. The initial condition, $U_0(x)$, is assumed to be a deterministic function such that it admits a discrete Fourier transform (see hypothesis **H4** in (36)). Keeping the standard notation, throughout this paper the solution SP will be denoted as $U(x,t)$ or $U(x,t)(\omega)$, indistinctly, when we want either to hide or emphasize its dependence on the sample parameter $\omega \in \Omega$, respectively.

**Remark 1.** Using the usual operator notation, the RPDE (1) can be written as $\mathcal{F}[U] = \mathcal{G}$, where $\mathcal{F}[U] = U_t + \beta U_x - \alpha U_{xx}$ and $\mathcal{G} \equiv 0$. We now introduce this notation because it will be used later.

The RPDE (1) arises in convection-diffusion transport problems. These problems appear in many applications in science and engineering such as in the transport of air and ground water pollutants, oil reservoir flow, in the modeling of semiconductors, and so forth [8, 9, 10]. The equation (1) is a parabolic PDE that combines the diffusion equation and the advection equation. It describes physical phenomena where particles or energy (or other physical quantities) are transferred inside a physical system due to two processes: diffusion and convection. The parameters $\alpha$ and $\beta$ are the heat diffusion coefficient and the convection velocity, respectively. The random nature of $\alpha$ and $\beta$ can be attributed because the heterogeneity and impurity of the physical medium. The solution SP, $U(x,t)$, represents species concentration for mass transfer or temperature for heat transfer [11].

In the deterministic framework the solution of the Cauchy advection-diffusion PDE (1)–(2) has been approximated using a number of finite difference schemes and related techniques [12, 13, 14]. Some of these numerical methods have been successfully extended to deal with its Itô-type SPDEs counterpart and their applications [15, 16]. In this paper, we propose a forward-time-backward/centered-space RFDS, inspired in its deterministic counterpart, to approximate the solution SP of the random Cauchy problem (1)–(2), that to the best of our knowledge has not been proposed yet. Then, we give sufficient conditions in order for the consistency and stability of the RFDS to be guaranteed in a random sense that will be specified later. Although most of the contributions have focussed on finite difference schemes for Itô-type SPDEs, some interesting studies dealing with random ordinary/partial differential equations by means of RFDSs can be found in [17, 18, 19, 20], for example.

This paper is organized as follows: In Section 2, firstly a random finite difference scheme for the Cauchy problem (1)–(2) is proposed. Secondly, the main concepts, definitions and auxiliary results that will be required throughout this paper are presented. This include the introduction of the definitions of random mean square consistency and stability, as well as several Banach spaces that will play a key role to formalize our study. In Sections 3 and 4 sufficient conditions for the mean square consistency and stability of the proposed random finite difference scheme are given and proved, respectively. In Section 5, we show two examples in order to illustrate the theoretical results established in previous sections. Section 6 summarizes the main conclusions of the paper.

## 2. Random finite difference scheme

This section is addressed to introduce the numerical scheme that will be used later to construct approximations of the random IVP (1)–(2). It is important to point out that problem (1)–(2) will be numerically solved in the fixed station sense, namely, on the domain $(x, t) \in \mathbb{R} \times [0, T]$, being $T > 0$ fixed.

With this aim, let us consider the grid points for the space variable $x$, $-\infty = x_{-\infty} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_{+\infty} = +\infty$ and for the time variable $t$, $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, $N \geq 1$ integer. Henceforth, both the space step and the time step will be assumed constant and they will be denoted by $\Delta x$ and $\Delta t$, respectively. Then, the following uniform space-time-lattice has been defined

$$x_{k+1} = x_k + \Delta x, \quad k \in \mathbb{Z}, \qquad t_{n+1} = t_n + \Delta t, \quad n = 0, 1, \ldots, N-1, \; N \geq 1,$$

where $\mathbb{Z}$ denotes the set of all integers. Let us denote by $U_k^n$ the approximation of the exact solution SP, $U(x, t)$, of the problem (1)–(2) at the lattice point $(x_k, t_n)$, i.e., $U_k^n \approx U(x_k, t_n)$ and $\mathbf{U}^n = (U_{-\infty}^n, \ldots, U_{-1}^n, U_0^n, U_1^n, \ldots, U_{+\infty}^n)$, the corresponding approximation at the $n$-time level. To formulate the random difference scheme, the following approximations for the partial derivatives will be considered

$$U_t(x_k, t_n) \approx \frac{U_k^{n+1} - U_k^n}{\Delta t}, \qquad U_x(x_k, t_n) \approx \frac{U_k^n - U_{k-1}^n}{\Delta x}, \quad U_{xx}(x_k, t_n) \approx \frac{U_{k+1}^n - 2U_k^n + U_{k-1}^n}{(\Delta x)^2}. \quad (3)$$

Substituting these approximations in the random IVP (1)–(2), one gets

$$\begin{cases} U_k^{n+1} = r(\beta\Delta x + \alpha)U_{k-1}^n + (1 - r\beta\Delta x - 2r\alpha)U_k^n + r\alpha U_{k+1}^n, \quad r = \dfrac{\Delta t}{(\Delta x)^2}, \\ U_k^0 = U_0(x_k). \end{cases} \quad (4)$$

Due to the finite differences used in (3) to approximate the corresponding partial derivatives, this RFDS is termed forward-time-backward/centered-space scheme. In the following, we will study the consistency and stability of the RFDS (4). Below, we give the definitions of consistency and stability of a RFDS. Both definitions are natural extensions of their deterministic counterparts using the $\|\cdot\|_{2,\Sigma}$-norm introduced in (14). In order to account for the accuracy of the RFDS, we shall introduce a natural definition of the order of a RFDS in terms of the $\|\cdot\|_{2,\Sigma}$-norm. With this purpose, firstly it is convenient to introduce several normed spaces that will play a key role throughout our analysis.

Firstly, the Banach space $(\mathrm{L}_p^{\mathrm{RV}}(\Omega), \|\cdot\|_{p,\mathrm{RV}})$, $p \geq 1$, of complex RVs $Y : \Omega \longrightarrow \mathbb{C}$ with finite $p$-th absolute moment with respect to the origin is finite, i.e.,

$$\|Y\|_{p,\mathrm{RV}} = \left(\mathbb{E}\left[|Y|^p\right]\right)^{1/p} < +\infty, \quad p \geq 1,$$

being $\mathbb{E}[\cdot]$ the expectation operator. For every sequence $Y_n \equiv \{Y_n : n \geq 0\}$ such that $\mathbb{E}[|Y_n|^p] < +\infty$ for each $n \geq 0$, i.e., $Y_n \in \mathrm{L}_p^{\mathrm{RV}}(\Omega)$, the convergence inferred by the $\|\cdot\|_{p,\mathrm{RV}}$-norm is usually referred to as $p$-th mean convergence, and it is defined as

$$Y_n \underset{n\to+\infty}{\overset{\|\cdot\|_{p,\mathrm{RV}}}{\longrightarrow}} Y \Longleftrightarrow \mathbb{E}\left[|Y_n - Y|^p\right] \underset{n\to+\infty}{\longrightarrow} 0.$$

Special mention deserves the Hilbert space $(\mathrm{L}_2^{\mathrm{RV}}(\Omega), \|\cdot\|_{2,\mathrm{RV}})$, corresponding to $p = 2$, which is made up for all complex RVs with finite variance. In this particular but still significant case, the norm is inferred by the inner product

$$\langle Y_1, Y_2 \rangle = \mathbb{E}[|Y_1 Y_2|], \quad Y_1, Y_2 \in \mathrm{L}_2^{\mathrm{RV}}(\Omega),$$

as

$$\|Y\|_{2,\mathrm{RV}} = +\sqrt{\langle Y, Y \rangle} = \left(\mathbb{E}\left[|Y|^2\right]\right)^{1/2} < +\infty, \quad Y \in \mathrm{L}_2^{\mathrm{RV}}(\Omega). \tag{5}$$

These RVs are called second-order RVs. As a consequence of the following classical result:

$$\text{If } Y, Z \text{ are independent RVs} \Rightarrow \mathbb{E}[YZ] = \mathbb{E}[Y]\mathbb{E}[Z],$$

provided all involved expectations exist, together with the definition of the $\|\cdot\|_{2,\mathrm{RV}}$-norm, one derives the following identity that will be used later

$$\text{if } Y, Z \in \mathrm{L}_2^{\mathrm{RV}}(\Omega) \text{ are independent} \Rightarrow \|YZ\|_{2,\mathrm{RV}} = \|Y\|_{2,\mathrm{RV}} \, \|Z\|_{2,\mathrm{RV}} \,. \tag{6}$$

In the general case that $Y$ and $Z$ are not statistically independent, but possessing moments of higher order, one can establish the following inequality [21, p.415],

$$\|YZ\|_{p,\mathrm{RV}} \leq \|Y\|_{2p,\mathrm{RV}} \, \|Z\|_{2p,\mathrm{RV}} \,, \quad p \geq 1, \quad Y, Z \in \mathrm{L}_{2p}^{\mathrm{RV}}(\Omega). \tag{7}$$

As a consequence of Liapunov's inequality,

$$\|Y\|_{r,\mathrm{RV}} = \left(\mathbb{E}\left[|Y|^r\right]\right)^{1/r} \leq \left(\mathbb{E}\left[|Y|^s\right]\right)^{1/s} = \|Y\|_{s,\mathrm{RV}} \,, \quad 1 \leq r \leq s, \tag{8}$$

one deduces

$$\mathrm{L}_s^{\mathrm{RV}}(\Omega) \subset \mathrm{L}_r^{\mathrm{RV}}(\Omega) \quad 1 \leq r \leq s, \tag{9}$$

as well as the following relationship between the convergences in these spaces

$$\text{if } Y_n \xrightarrow[n\to+\infty]{\|\cdot\|_{s,\mathrm{RV}}} Y \Longrightarrow Y_n \xrightarrow[n\to+\infty]{\|\cdot\|_{r,\mathrm{RV}}} Y, \quad 1 \le r \le s, \tag{10}$$

whenever the sequence of RVs, $\{Y_n\}$ belongs to $\mathrm{L}_s^{\mathrm{RV}}(\Omega)$, i.e., $\mathbb{E}[|Y_n|^s] < +\infty$, for every $n \ge 1$. Therefore, for RVs having finite variance the weakest $p$-th convergence corresponds to $p = 2$, namely, the mean square convergence defined in $(\mathrm{L}_2^{\mathrm{RV}}(\Omega), \|\cdot\|_{2,\mathrm{RV}})$. Mean square convergence is very important because results established in this type of stochastic convergence are also valid for another type of relevant stochastic convergences such as convergence in probability and convergence in distribution. It is important to point out that the rigorous operational manipulation of mean square convergence requires the use of the relationships (9) and (10) often. Indeed, for instance, it can be seen that the following basic operational property

$$\left.\begin{array}{l} Z \in \mathrm{L}_2^{\mathrm{RV}}(\Omega), \\ Y_n \xrightarrow[n\to+\infty]{\|\cdot\|_{2,\mathrm{RV}}} Y, \end{array}\right\} \Longrightarrow ZY_n \xrightarrow[n\to+\infty]{\|\cdot\|_{2,\mathrm{RV}}} ZY$$

does not hold in general. However, this property can be legitimated assuming further hypotheses that involve information of the Banach space $(\mathrm{L}_4^{\mathrm{RV}}(\Omega), \|\cdot\|_{4,\mathrm{RV}})$, [22]. Additionally, as it will be proved below, this basic operational property of the mean square convergence is still true when the RV $Z$ is bounded, which is just the context that will be required throughout our subsequence analysis.

**Proposition 1.** *Let $Z$ be a bounded RV in $\mathrm{L}_2^{RV}(\Omega)$, i.e., there exist constants $z_1$ and $z_2$ such that $z_1 \le Z(\omega) \le z_2$, $\omega \in \Omega$, and let us suppose that $Y_n \xrightarrow[n\to+\infty]{\|\cdot\|_{2,RV}} Y$. Then, $ZY_n \xrightarrow[n\to+\infty]{\|\cdot\|_{2,RV}} ZY$.*

**Proof.** Let us denote by $\hat{z} = \max\{|z_1|, |z_2|\} < +\infty$, then the result is straightforwardly inferred from the following sandwich-type inequality

$$0 \le \left(\|ZY_n - ZY\|_{2,\mathrm{RV}}\right)^2 = \mathbb{E}\left[|Z|^2 |Y_n - Y|^2\right] \le |\hat{z}|^2 \mathbb{E}\left[|Y_n - Y|^2\right] = |\hat{z}|^2 \left(\|Y_n - Y\|_{2,\mathrm{RV}}\right)^2 \xrightarrow[n\to+\infty]{} 0,$$

where in the last step we have used that by hypothesis $\{Y_n\}$ is mean square convergent to $Y$ as $n \to +\infty$. □

Now, we establish a crucial inequality involving $\|\cdot\|_{p,\mathrm{RV}}$-norms that will play a crucial role to study the stability of the RFDS (4). For that, let us observe that by inequality (7) with $p = 2$ one gets

$$\left\|Y^2\right\|_{2,\mathrm{RV}} \le \|Y\|_{4,\mathrm{RV}} \|Y\|_{4,\mathrm{RV}} = \left(\|Y\|_{2^2,\mathrm{RV}}\right)^2,$$

$$\left\|Y^3\right\|_{2,\mathrm{RV}} = \left\|Y^2 Y\right\|_{2,\mathrm{RV}} \le \left\|Y^2\right\|_{4,\mathrm{RV}} \|Y\|_{4,\mathrm{RV}} \le \|Y\|_{8,\mathrm{RV}} \|Y\|_{8,\mathrm{RV}} \|Y\|_{4,\mathrm{RV}} \le \|Y\|_{8,\mathrm{RV}} \|Y\|_{8,\mathrm{RV}} \|Y\|_{8,\mathrm{RV}} = \left(\|Y\|_{2^3,\mathrm{RV}}\right)^3,$$

where in the two first inequalities we have applied (7) with $p = 2$ and $p = 4$, respectively, while in the last bound Liapunov's inequality (8) has been used for the last factor, $\|Y\|_{4,\mathrm{RV}}$, with the identification $r = 4 \le 8 = s$. Reasoning recursively in the same manner it is easy to establish the following result

**Lemma 1.** *Let $Y$ be a RV such that there exist and are finite its absolute moments with respect to the origin of order $2^m$, being $m \ge 1$ integer, i.e., $\mathbb{E}\left[|Y|^{2^m}\right] < +\infty$. Then,*

$$\|Y^m\|_{2,RV} \le \left(\|Y\|_{2^m,RV}\right)^m. \tag{11}$$

Let $\mathcal{J} \subset \mathbb{R}$, we secondly introduce the Hilbert space $(\mathrm{L}_2^{\mathrm{SP}}(\mathcal{J} \times \Omega), \|\cdot\|_{2,\mathrm{SP}})$ of complex-valued SPs whose second-order moment with respect to the origin is integrable

$$\mathrm{L}_2^{\mathrm{SP}}(\mathcal{J} \times \Omega) = \left\{ Y(x) \equiv Y(x, \omega) : \mathcal{J} \times \Omega \longrightarrow \mathbb{C} : \int_{\mathcal{J}} \mathbb{E}\left[ |Y(x)|^2 \right] \mathrm{d}x < +\infty \right\} \tag{12}$$

and

$$\|Y(x)\|_{2,\mathrm{SP}} = \left( \int_{\mathcal{J}} \left( \|Y(x)\|_{2,\mathrm{RV}} \right)^2 \mathrm{d}x \right)^{1/2} = \left( \int_{\mathcal{J}} \mathbb{E}\left[ |Y(x)|^2 \right] \mathrm{d}x \right)^{1/2}.$$

Thirdly, the approximations at the $n$-time level are elements of the Banach space $(\ell_2(\Omega), \|\cdot\|_{2,\Sigma})$ being

$$\ell_2(\Omega) = \left\{ \mathbf{U} = (U_{-\infty}, \ldots, U_{-1}, U_0, U_1, \ldots, U_{+\infty}) : \sum_{k=-\infty}^{+\infty} \left( \|U_k\|_{2,\mathrm{RV}} \right)^2 < +\infty \right\} \tag{13}$$

and

$$\|\mathbf{U}\|_{2,\Sigma} = \left( \sum_{k=-\infty}^{+\infty} \left( \|U_k\|_{2,\mathrm{RV}} \right)^2 \right)^{1/2} = \left( \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[ |U_k|^2 \right] \right)^{1/2}, \tag{14}$$

where, as noticed, $\|\cdot\|_{2,\mathrm{RV}}$ corresponds to the norm defined in (5), [23].

Consistency and stability are main notions of the deterministic finite difference schemes theory that need to be translated into the random framework. Consistency means that the solution SP of the RPDE, if it is smooth enough, is an approximate solution of the RFDS. Stability can be interpreted as small errors in the initial conditions cause smalls errors in the solution. As it will shall be later, the definition of random stability permits the errors in the solution to grow, but limits them to grow not faster than exponentially. The following definitions are natural extensions of their deterministic counterparts and they are inspired in classical references like [1, 2, 3].

**Definition 1.** *The RFDS*

$$\mathbf{U}^{n+1} = Q(\mathbf{U}^n) + \Delta t\, \mathbf{G}^n, \tag{15}$$

*being*

$$\mathbf{U}^n = \left( U^n_{-\infty}, \ldots, U^n_{-1}, U^n_0, U^n_1, \ldots, U^n_{+\infty} \right),$$
$$\mathbf{G}^n = \left( G^n_{-\infty}, \ldots, G^n_{-1}, G^n_0, G^n_1, \ldots, G^n_{+\infty} \right),$$

*is said to be mean square* $\|\cdot\|_{2,\Sigma}$*-consistent with the RPDE* $\mathcal{F}[U] = \mathcal{G}$ *(see Remark 1), if the solution SP,* $U$*, of the RPDE satisfies*

$$\mathbf{U}^{n+1} = Q(\mathbf{U}^n) + \Delta t\, \mathbf{G}^n + \Delta t\, \boldsymbol{\tau}^n, \tag{16}$$

*and*

$$\|\boldsymbol{\tau}^n\|_{2,\Sigma} \underset{\Delta t \to 0}{\overset{\Delta x \to 0}{\longrightarrow}} 0, \tag{17}$$

*where the* $\|\cdot\|_{2,\Sigma}$*-norm has been introduced in* (14) *and the k-th component of* $\mathbf{U}^n$ *in* (16) *is*

$$U^n_k = U(x_k, t_n).$$

**Definition 2.** *The RFDS* (15) *is said to be mean square* $\|\cdot\|_{2,\Sigma}$*-stable if there exist positive constants* $\epsilon, \delta > 0$*, and non-negative constants* $\eta, \rho$ *such that*

$$\|\mathbf{U}^n\|_{2,\Sigma} \leq \eta \, \mathrm{e}^{\rho t} \, \|\mathbf{U}^0\|_{2,\Sigma}, \tag{18}$$

*for* $0 \leq t = n\Delta t$*,* $0 < \Delta x \leq \epsilon$*,* $0 < \Delta t \leq \delta$*.*

**Definition 3.** *In the context of Definition 1, the RFDS* (15) *is said to be of order* $(p, q)$ *if*

$$\|\boldsymbol{\tau}^n\|_{2,\Sigma} = \mathcal{O}\left((\Delta t)^p\right) + \mathcal{O}\left((\Delta x)^q\right).$$

## 3. Consistency of the random finite difference scheme

The goal of this section is to give sufficient conditions in order to guarantee the mean square $\|\cdot\|_{2,\Sigma}$-consistency of the RFDS (4) with the RPDE (1).

With this purpose let us denote, only throughout this section, $U(x_k, t_n)$ by $U_k^n$, i.e., $U_k^n$ represents the value of the exact solution SP evaluated at the lattice point $(x_k, t_n)$. According to Definition 1 and the RFDS (4), let us perform the Taylor expansion of the $k$-th component of $\mathbf{U}^{n+1} - Q(\mathbf{U}^n) - \Delta t \mathbf{G}^n$ with $\mathbf{G} \equiv \mathbf{0}$ (see (15)) taking into account that $r = \Delta t/(\Delta x)^2$,

$$\begin{aligned}
\left(\mathbf{U}^{n+1} - Q(\mathbf{U}^n)\right)_k &= U_k^{n+1} - r(\beta\Delta x + \alpha)U_{k-1}^n - (1 - r\beta\Delta x - 2r\alpha)U_k^n - r\alpha U_{k+1}^n \\
&= U_k^n + \Delta t (U_t)_k^n + \frac{1}{2}(\Delta t)^2 U_{tt}(x_k, \eta) \\
&\quad - r\beta\Delta x \left( U_k^n - \Delta x (U_x)_k^n + \frac{1}{2}(\Delta x)^2 U_{xx}(\xi_1^k, t_n) \right) \\
&\quad - r\alpha \left( U_k^n - \Delta x (U_x)_k^n + \frac{1}{2}(\Delta x)^2 (U_{xx})_k^n - \frac{1}{3!}(\Delta x)^3 U_{xxx}(\xi_2^k, t_n) \right) \\
&\quad - (1 - r\beta\Delta x - 2r\alpha)U_k^n \\
&\quad - r\alpha \left( U_k^n + \Delta x (U_x)_k^n + \frac{1}{2}(\Delta x)^2 (U_{xx})_k^n + \frac{1}{3!}(\Delta x)^3 U_{xxx}(\xi_3^k, t_n) \right) \\
&\overset{(\mathrm{I})}{=} \underbrace{(1 - r\beta\Delta x - r\alpha - 1 + r\beta\Delta x + 2r\alpha - r\alpha)}_{=0} U_k^n \\
&\quad + \Delta t \underbrace{\left( (U_t)_k^n + \beta (U_x)_k^n - \alpha (U_{xx})_k^n \right)}_{=0} \\
&\quad + \Delta t \left( \frac{1}{2}\Delta t \, U_{tt}(x_k, \eta) - \frac{1}{2}\beta\Delta x \, U_{xx}(\xi_1^k, t_n) + \frac{1}{3!}\alpha\Delta x \left( U_{xxx}(\xi_2^k, t_n) - U_{xxx}(\xi_3^k, t_n) \right) \right),
\end{aligned} \tag{19}$$

where

$$t_n < \eta < t_{n+1}, \quad x_{k-1} < \xi_1^k, \xi_2^k < x_k, \quad x_k < \xi_3^k < x_{k+1}. \tag{20}$$

Notice that in the second addend of step (I) of (19) we have used that at the lattice point $(x_k, t_n)$ the RPDE (1) holds, hence $(U_t)_k^n + \beta(U_x)_k^n - \alpha(U_{xx})_k^n = 0$. Furthermore, considering (16), from (19) one gets that the $k$-th component of $\boldsymbol{\tau}^n$ is given by

$$\tau_k^n = \frac{1}{2}\Delta t \, U_{tt}(x_k, \eta) - \frac{1}{2}\beta\Delta x \, U_{xx}(\xi_1^k, t_n) + \frac{1}{3!}\alpha\Delta x \left( U_{xxx}(\xi_2^k, t_n) - U_{xxx}(\xi_3^k, t_n) \right),$$

where $\eta$ and $\xi_i^k$, $1 \le i \le 3$, satisfy (20).

Since $U_{tt}$ and $U_{xx}$ depend on the RVs $\alpha$ and $\beta$, and using the definition of the $\|\cdot\|_{2,\mathrm{RV}}$-norm (see (5)), one gets

$$\left(\left\|\tau_k^n\right\|_{2,\mathrm{RV}}\right)^2 = \mathbb{E}\left[|\tau_k^n|^2\right] = \int_{\mathbb{R}^2} \left(\frac{1}{2}\Delta t\, U_{tt}(x_k,\eta) - \frac{1}{2}\beta\Delta x\, U_{xx}(\xi_1^k, t_n) + \frac{1}{3!}\alpha\Delta x\left(U_{xxx}(\xi_2^k, t_n) - U_{xxx}(\xi_3^k, t_n)\right)\right)^2 f_{\alpha,\beta}(\alpha,\beta)\, \mathrm{d}\alpha \mathrm{d}\beta, \tag{21}$$

where $f_{\alpha,\beta}(\alpha,\beta)$ is the joint PDF of the random vector $(\alpha,\beta)$.

Let us assume the following hypotheses **H1**–**H3**:

$$\textbf{H1}: \quad \begin{array}{c} U_{xx}(\cdot,t) = U_{xx}(\cdot,t)(\omega) \text{ and } U_{xxx}(\cdot,t) = U_{xxx}(\cdot,t)(\omega) \text{ are} \\ \text{uniformly bounded SPs for each } t \ge 0 \text{ fixed and } \forall\omega \in \Omega, \end{array} \tag{22}$$

$$\textbf{H2}: \quad \begin{array}{c} \alpha,\beta \text{ are positive bounded RVs:} \\ 0 < \alpha_1 < \alpha(\omega) < \alpha_2 \text{ and } 0 < \beta_1 < \beta(\omega) < \beta_2,\ \forall\omega \in \Omega, \end{array} \tag{23}$$

and

$$\textbf{H3}: \quad \begin{array}{c} U_{tt}(\cdot,t) = U_{tt}(\cdot,t)(\omega) \in \ell_2(\Omega) \text{ for each } t \ge 0 \text{ fixed and,} \\ U_{xx}(\cdot,t) = U_{xx}(\cdot,t)(\omega),\ U_{xxx}(\cdot,t) = U_{xxx}(\cdot,t)(\omega) \in \mathrm{L}_2^{\mathrm{SP}}(\mathbb{R}\times\Omega) \text{ for each } t \ge 0 \text{ fixed.} \end{array} \tag{24}$$

Now, bearing in mind the expression (16) involved in the definition of the mean square $\|\cdot\|_{2,\Sigma}$-consistency together with the definition of the $\|\cdot\|_{2,\Sigma}$-norm (see (14)), we deal with the following bound

$$\begin{aligned}
(\|\boldsymbol{\tau}^n\|_{2,\Sigma})^2 &= \sum_{k=-\infty}^{+\infty} \left(\|\tau_k^n\|_{2,\mathrm{RV}}\right)^2 = \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[|\tau_k^n|^2\right] \\
&\overset{(\mathrm{I})}{\le} \sum_{k=-\infty}^{+\infty} 2\,(\Delta t)^2 \int_{\mathbb{R}^2} (U_{tt}(x_k,\eta))^2 f_{\alpha,\beta}(\alpha,\beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta \\
&+ \sum_{k=-\infty}^{+\infty} 2\,(\Delta x)^2 \int_{\mathbb{R}^2} \left(\beta U_{xx}(\xi_1^k, t_n)\right)^2 f_{\alpha,\beta}(\alpha,\beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta \\
&+ \sum_{k=-\infty}^{+\infty} \frac{2}{3^2}\,(\Delta x)^2 \int_{\mathbb{R}^2} \left(\alpha U_{xxx}(\xi_2^k, t_n)\right)^2 f_{\alpha,\beta}(\alpha,\beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta \\
&+ \sum_{k=-\infty}^{+\infty} \frac{2}{3^2}\,(\Delta x)^2 \int_{\mathbb{R}^2} \left(\alpha U_{xxx}(\xi_3^k, t_n)\right)^2 f_{\alpha,\beta}(\alpha,\beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta.
\end{aligned} \tag{25}$$

Note that in step (I) we have applied the following inequality $(a+b+c+d)^2 \le 2^3(a^2+b^2+c^2+d^2)$,

$a, b, c, d \in \mathbb{R}$ to expression (21). Taking limits as $\Delta x, \Delta t \to 0$ in (25), one gets

$$\begin{aligned}\lim_{\Delta t, \Delta x \to 0} \left( \|\boldsymbol{\tau}^n\|_{2,\Sigma} \right)^2 \leq\ & 2 \left( \lim_{\Delta t \to 0} (\Delta t)^2 \right) \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}^2} (U_{tt}(x_k, \eta))^2 \, f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & + 2 \left( \lim_{\Delta x \to 0} \Delta x \right) \sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( \beta U_{xx}(\xi_1^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & + \frac{2}{3^2} \left( \lim_{\Delta x \to 0} \Delta x \right) \sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( \alpha U_{xxx}(\xi_2^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & + \frac{2}{3^2} \left( \lim_{\Delta x \to 0} \Delta x \right) \sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( \alpha U_{xxx}(\xi_3^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta. \end{aligned} \tag{26}$$

By hypothesis **H3** (see (24)), $U_{tt}(x_k, \eta) \in \ell_2(\Omega)$, hence $\sum_{k=-\infty}^{+\infty} \mathbb{E}\left[ |U_{tt}(x_k, \eta)|^2 \right] < +\infty$. Then, using the definition of the expectation in terms of the joint PDF of the random vector $(\alpha, \beta)$ and, taking into account that $U_{tt}(x_k, \eta)$ depends on $\alpha, \beta$, one gets

$$\sum_{k=-\infty}^{+\infty} \mathbb{E}\left[ |U_{tt}(x_k, \eta)|^2 \right] = \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}^2} (U_{tt}(x_k, \eta))^2 \, f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta < +\infty, \quad \forall \eta = \eta(\omega) : \ t_n < \eta(\omega) < t_{n+1}, \ \forall \omega \in \Omega.$$

In the following development we apply to the second term of the sum that appears in the right-hand side of inequality (26), firstly the hypothesis **H2** (see (23)) in step (I), and secondly, the hypothesis **H1** for $U_{xx}(x, t)$ (see (22)) in step (II), this yields

$$\begin{aligned} & \sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( \beta U_{xx}(\xi_1^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & \overset{\text{(I)}}{\leq} (\beta_2)^2 \sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( U_{xx}(\xi_1^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & \overset{\text{(II)}}{=} (\beta_2)^2 \int_{\mathbb{R}^2} \left( \lim_{\Delta x \to 0} \sum_{k=-\infty}^{+\infty} \Delta x \left( U_{xx}(\xi_1^k, t_n) \right)^2 \right) f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & = (\beta_2)^2 \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}} (U_{xx}(x, t_n))^2 \, \mathrm{d}x \right) f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ & \overset{\text{(III)}}{=} (\beta_2)^2 \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} (U_{xx}(x, t_n))^2 \, f_{\alpha,\beta}(\alpha, \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \right) \mathrm{d}x \\ & = (\beta_2)^2 \int_{\mathbb{R}} \mathbb{E}\left[ (U_{xx}(x, t_n))^2 \right] \mathrm{d}x < +\infty, \end{aligned} \tag{27}$$

where the commutation of the one-dimensional and two-dimensional integrals in the step (III) is legitimated by Fubbini's theorem because $\alpha$ and $\beta$ are bounded RVs and the two-dimensional integral exists [24]. This last assertion, that has been used to write the finiteness of the last integral, follows from hypothesis **H3** (see (24)).

It is straightforwardly to prove, following an analogous argument to the one exhibited in (27), that

$$\sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( \alpha U_{xxx}(\xi_2^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha,\beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta$$
$$\leq (\alpha_2)^2 \int_{\mathbb{R}} \mathbb{E}\left[ (U_{xxx}(x,t_n))^2 \right] \mathrm{d}x < +\infty,$$

and

$$\begin{aligned} &\sum_{k=-\infty}^{+\infty} \left( \lim_{\Delta x \to 0} \Delta x \right) \int_{\mathbb{R}^2} \left( \alpha U_{xxx}(\xi_3^k, t_n) \right)^2 f_{\alpha,\beta}(\alpha,\beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ &\leq (\alpha_2)^2 \int_{\mathbb{R}} \mathbb{E}\left[ (U_{xxx}(x,t_n))^2 \right] \mathrm{d}x < +\infty. \end{aligned} \tag{28}$$

Taking into account (27)–(28), from inequality (26) one follows

$$\lim_{\Delta t, \Delta x \to 0} \|\boldsymbol{\tau}^n\|_{2,\Sigma} = 0.$$

Summarizing, the following result has been established:

**Proposition 2.** *Under hypotheses **H1–H3** given in* (22)–(24)*, respectively, the RFDS* (4) *is mean square* $\| \cdot \|_{2,\Sigma}$*-consistent with the RPDE* (1).

**Remark 2.** Taking into account the Definition 3, then by the previous development it is clear that the order of the RFDS (4) is $(p,q) = (1,1)$.

## 4. Stability of the random finite difference scheme

This section is devoted to establish the mean square $\| \cdot \|_{2,\Sigma}$-stability of the RFDS (4) using the Von Neumann approach [1]. This method is based on the discrete Fourier transform. With this aim, we firstly need to extend the definition of this important transformation to the random context.

**Definition 4.** *Let* $\mathbf{U} \equiv \{U_k\} = (U_{-\infty}, \ldots, U_{-1}, U_0, U_1, \ldots, U_{+\infty})$ *be a sequence in the Banach space* $(\ell_2(\Omega), \|\cdot\|_{2,\Sigma})$ *introduced in* (13)–(14)*. The random discrete Fourier transform (RDFT) of* $\mathbf{U} \equiv \{U_k\}$ *is defined by*

$$\hat{\mathbf{U}}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\, k\xi} \, U_k, \qquad \mathrm{i} = +\sqrt{-1}, \quad \xi \in [0, 2\pi[. \tag{29}$$

As it shall see later, the RDFT $\hat{\mathbf{U}} : \ell_2(\Omega) \longrightarrow \mathrm{L}_2^{\mathrm{SP}}([0,2\pi[\times\Omega)$ is well-defined. Notice that $(\mathrm{L}_2^{\mathrm{SP}}([0,2\pi[\times\Omega), \|\cdot\|_{2,\mathrm{SP}})$ is just the Banach space introduced in (12) with $\mathcal{J} = [0,2\pi[$. Moreover, it can be proved by extending the deterministic techniques to the random framework that

$$U_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\, k\xi} \, \hat{\mathbf{U}}(\xi) \, \mathrm{d}\xi, \tag{30}$$

which is an inversion formula for the RDFT.

The following result shows that the norms $\|\cdot\|_{2,\mathrm{RV}}$ and $\|\cdot\|_{2,\mathrm{SP}}$ are compatible. It will be required later.

**Lemma 2.** *Let $V \in \mathrm{L}_2^{RV}(\Omega)$ and $w \equiv w(\xi) \in \mathrm{L}_2^{SP}([0, 2\pi[\times\Omega)$ such that $V$ is statistically independent of $w(\xi)$ for every $\xi \in [0, 2\pi[$. Then*

$$\|V\, w\|_{2,SP} = \|V\|_{2,RV}\, \|w\|_{2,SP}\,. \tag{31}$$

**Proof.** The result is a direct consequence of the definitions of both norms and the application of property (6) in the step (I):

$$\begin{aligned}\|V\, w\|_{2,\mathrm{SP}} &= \left(\int_0^{2\pi} \left(\|V\, w(\xi)\|_{2,\mathrm{RV}}\right)^2 \,\mathrm{d}\xi\right)^{1/2} \overset{(\mathrm{I})}{=} \left(\int_0^{2\pi} \left(\|V\|_{2,\mathrm{RV}}\, \|w(\xi)\|_{2,\mathrm{RV}}\right)^2 \,\mathrm{d}\xi\right)^{1/2} \\ &= \|V\|_{2,\mathrm{RV}} \left(\int_0^{2\pi} \left(\|w(\xi)\|_{2,\mathrm{RV}}\right)^2 \,\mathrm{d}\xi\right)^{1/2} = \|V\|_{2,\mathrm{RV}}\, \|w\|_{2,\mathrm{SP}}\,.\square\end{aligned}$$

A key result that will be used later is that the Banach spaces $(\mathrm{L}_2([0, 2\pi[\times\Omega), \|\cdot\|_{2,\mathrm{SP}})$ and $(\ell_2(\Omega), \|\cdot\|_{2,\Sigma})$ are isometric. This is a consequence of the following Parseval-type identity

$$\begin{aligned}\left(\left\|\hat{\mathbf{U}}\right\|_{2,\mathrm{SP}}\right)^2 &= \int_0^{2\pi} \left(\left\|\hat{\mathbf{U}}(\xi)\right\|_{2,\mathrm{RV}}\right)^2 \mathrm{d}\xi = \int_0^{2\pi} \mathbb{E}\left[|\hat{\mathbf{U}}(\xi)|^2\right] \mathrm{d}\xi \\ &= \int_0^{2\pi} \mathbb{E}\left[\hat{\mathbf{U}}(\xi)\overline{\hat{\mathbf{U}}(\xi)}\right] \mathrm{d}\xi = \int_0^{2\pi} \mathbb{E}\left[\left(\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_k\right)\overline{\hat{\mathbf{U}}(\xi)}\right] \mathrm{d}\xi \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[U_k \left(\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}\,k\xi}\, \overline{\hat{\mathbf{U}}(\xi)}\,\mathrm{d}\xi\right)\right] = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[U_k \left(\int_0^{2\pi} \overline{\mathrm{e}^{\mathrm{i}\,k\xi}\, \hat{\mathbf{U}}(\xi)\,\mathrm{d}\xi}\right)\right] \\ &= \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[U_k \overline{\left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\,k\xi}\, \hat{\mathbf{U}}(\xi)\,\mathrm{d}\xi\right)}\right] = \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[U_k \overline{U_k}\right] \\ &= \sum_{k=-\infty}^{+\infty} \mathbb{E}\left[|U_k|^2\right] = \sum_{k=-\infty}^{+\infty} \left(\|U_k\|_{2,\mathrm{RV}}\right)^2 = \left(\|\mathbf{U}\|_{2,\Sigma}\right)^2,\end{aligned} \tag{32}$$

or equivalently,

$$\left\|\hat{\mathbf{U}}\right\|_{2,\mathrm{SP}} = \|\mathbf{U}\|_{2,\Sigma}\,. \tag{33}$$

Observe that in (32), the basic properties of the conjugate operator for complex numbers as well as the inversion formula (30) have been used. Moreover, as a consequence of the chain of identities exhibited in (32) and the fact that if $\{U_k\} \in \ell_2(\Omega)$ (see(13)), one gets

$$\left(\left\|\hat{\mathbf{U}}\right\|_{2,\mathrm{SP}}\right)^2 = \sum_{k=-\infty}^{+\infty} \left(\|U_k\|_{2,\mathrm{RV}}\right)^2 < +\infty,$$

i.e., $\left\|\hat{\mathbf{U}}\right\|_{2,\mathrm{SP}} < +\infty$. Therefore the RDFT is well-defined in the Banach space $(\mathrm{L}_2^{\mathrm{SP}}([0, 2\pi[\times\Omega), \|\cdot\|_{2,\mathrm{SP}})$ when acting over sequences $\{U_k\}$ in the space $\ell_2(\Omega)$.

For convenience, let us rewrite the RFDS (4) in the following form

$$U_k^{n+1} = (1 - R - 2S)U_k^n + (R + S)U_{k-1}^n + S\, U_{k+1}^n, \quad \text{where} \quad R := \beta\frac{\Delta t}{\Delta x},\; S := \alpha\frac{\Delta t}{(\Delta x)^2}. \tag{34}$$

Notice that under hypothesis **H2** (see (23)) and the above definition of $R \equiv R(\omega)$ and $S \equiv S(\omega)$, $\omega \in \Omega$, both are positive bounded RVs for time step $\Delta t > 0$ and space step $\Delta x > 0$ fixed.

Let $\xi \in [0, 2\pi[$ and let us take the RDFT (29) in the RFDS (34), then one obtains

$$
\begin{aligned}
\hat{\mathbf{U}}^{n+1}(\xi) &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_k^{n+1} \\
&\overset{(\mathrm{I})}{=} \frac{1}{\sqrt{2\pi}} \left( (1 - R - 2S) \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_k^{n} + (R+S) \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_{k-1}^{n} + S \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_{k+1}^{n} \right) \\
&= (1 - R - 2S) \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_k^{n} + (R+S) \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_{k-1}^{n} + S \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\,k\xi}\, U_{k+1}^{n} \\
&= \left\{ (1 - R - 2S) + (R+S)\, \mathrm{e}^{-\mathrm{i}\,\xi} + S\, \mathrm{e}^{\mathrm{i}\,\xi} \right\} \hat{\mathbf{U}}^{n}(\xi).
\end{aligned}
\tag{35}
$$

It is important to point out that in the step (I) of (35), we have applied Proposition 1 to legitimate the commutation between the infinite sum, which is $\|\cdot\|_{2,\mathrm{RV}}$-convergent, and the bounded factors $1 - R - 2S$, $R + S$ and $S$, that depend on the bounded RVs $R$ and $S$.

If we assume that

$$
\mathbf{H4}: \quad \begin{array}{c} \text{The initial condition } U_0(x), \text{ which is assumed to be deterministic,} \\ \text{possess a discrete Fourier transform } \hat{\mathbf{U}}^0(\xi), \end{array} \tag{36}
$$

then recurrence (35) can explicitly be solved in terms of the initial term

$$
\hat{\mathbf{U}}^{n}(\xi) = G^n \hat{\mathbf{U}}^{0}(\xi), \quad \text{where} \quad G = (1 - R - 2S) + (R+S)\, \mathrm{e}^{-\mathrm{i}\,\xi} + S\, \mathrm{e}^{\mathrm{i}\,\xi}. \tag{37}
$$

As $R$ and $S$ depend on RVs $\alpha$ and $\beta$, the so-called amplification factor, $G$, also does. Now, we seek conditions in order for the random amplification factor $G \equiv G(\omega)$, $\omega \in \Omega$, has absolute value less or equal than the unit, i.e.,

$$
|G(\omega)| \leq 1, \quad \forall \omega \in \Omega. \tag{38}
$$

With this goal, let us rewrite the expression of $G$ given by (37) in the following equivalent form using the Euler's identity $\mathrm{e}^{\mathrm{i}\,x} = \cos(x) + \mathrm{i}\sin(x)$, $x \in \mathbb{R}$,

$$
\begin{aligned}
G &= 1 - 2S - R\left(1 - \mathrm{e}^{-\mathrm{i}\,\xi}\right) + S\left(\mathrm{e}^{\mathrm{i}\,\xi} + \mathrm{e}^{-\mathrm{i}\,\xi}\right) \\
&= 1 - 2S - R\,(1 - \cos(\xi) + \mathrm{i}\sin(\xi)) + 2S\cos(\xi) \\
&= 1 - 2S\,(1 - \cos(\xi)) - R\,(1 - \cos(\xi)) - \mathrm{i}\,R\sin(\xi) \\
&= 1 - (R + 2S)\,(1 - \cos(\xi)) - \mathrm{i}\,R\sin(\xi).
\end{aligned}
$$

As $|G|^2 \leq 1$ is equivalent to condition (38) and

$$
\begin{aligned}
|G|^2 &= (1 - (R+2S)(1 - \cos(\xi)))^2 + (R\sin(\xi))^2 \\
&= 1 - 2(R+2S)(1 - \cos(\xi)) + (R+2S)^2\,(1 - \cos(\xi))^2 + R^2(1 - \cos(\xi))(1 + \cos(\xi)),
\end{aligned}
$$

then condition (38) is equivalent to

$$
(R+2S)^2\,(1 - \cos(\xi))^2 + R^2(1 - \cos(\xi))(1 + \cos(\xi)) \leq 2(R+2S)(1 - \cos(\xi)). \tag{39}
$$

If $\xi = 0$, this inequality holds, while if $\xi \in ]0, 2\pi[$, $1 - \cos(\xi) > 0$, hence dividing each side of inequality (39) by this positive factor yields

$$\begin{aligned} (R+2S)^2(1-\cos(\xi)) + R^2(1+\cos(\xi)) &\le 2(R+2S), \\ \cos(\xi)\left(R^2 - (R+2S)^2\right) &\le 2(R+2S) - R^2 - (R+2S)^2. \end{aligned} \tag{40}$$

Since $S \equiv S(\omega) > 0$ for all $\omega \in \Omega$, then $R^2 - (R+2S)^2 < 0$ and from (40) one obtains

$$\cos(\xi) \ge \frac{2(R+2S) - R^2 - (R+2S)^2}{R^2 - (R+2S)^2}. \tag{41}$$

Therefore, the following condition

$$\frac{2(R+2S) - R^2 - (R+2S)^2}{R^2 - (R+2S)^2} \le -1, \tag{42}$$

guarantees that inequality (41) holds. Notice that condition (42) is equivalent to

$$2(R+2S) - R^2 - (R+2S)^2 \ge -R^2 + (R+2S)^2 \Leftrightarrow R + 2S \ge (R+2S)^2,$$

and dividing by $R + 2S$ (since $R(\omega) + 2S(\omega) > 0$ for all $\omega \in \Omega$), one concludes that the condition $|G(\omega)| \le 1$ fulfils for all $\omega \in \Omega$ if

$$1 - R(\omega) - 2S(\omega) \ge 0, \quad \forall \omega \in \Omega, \qquad R = \beta \frac{\Delta t}{\Delta x}, \; S = \alpha \frac{\Delta t}{(\Delta x)^2}. \tag{43}$$

On the other hand, it is clear that

$$\text{if } |G(\omega)| \le 1 \Rightarrow |G(\omega)|^{2^n} \le 1, \; \forall \omega \in \Omega,$$

then

$$(\|G\|_{2^n})^n = \left(\mathbb{E}\left[|G|^{2^n}\right]\right)^{n/2^n} \le 1, \quad \forall n = 1, 2, \ldots \tag{44}$$

Taking the $\|\cdot\|_{2,\mathrm{SP}}$-norm in expression (37) and, applying firstly the inequality (31) of Lemma 2 and secondly inequality (11) of Lemma 1 with the identifications, $V \equiv G^n$, $\mathbf{w} \equiv \hat{\mathbf{u}}^0(\xi)$ and $Y \equiv G$, respectively, together with (44), one obtains

$$\begin{aligned} \|\mathbf{U}^n(\xi)\|_{2,\Sigma} &\overset{\text{(I)}}{=} \left\|\hat{\mathbf{U}}^n(\xi)\right\|_{2,\mathrm{SP}} = \left\|G^n \hat{\mathbf{U}}^0(\xi)\right\|_{2,\mathrm{SP}} \overset{\text{(II)}}{=} \|G^n\|_2 \left\|\hat{\mathbf{U}}^0(\xi)\right\|_{2,\mathrm{SP}} \\ &\le (\|G\|_{2^n})^n \left\|\hat{\mathbf{U}}^0(\xi)\right\|_{2,\mathrm{SP}} \le \left\|\hat{\mathbf{U}}^0(\xi)\right\|_{2,\mathrm{SP}} \overset{\text{(III)}}{=} \left\|\mathbf{U}^0(\xi)\right\|_{2,\Sigma}. \end{aligned} \tag{45}$$

Notice that in the steps (I) and (III) we have used the identity (33) and, in the step (II) that by hypothesis the initial condition $U_0(x)$ is a deterministic function, then its RDFT $\hat{\mathbf{U}}^0(\xi)$ is statistically independent of RVs $\alpha$ and $\beta$, and hence of $G^n$ too.

The relationship (45) proves the mean square $\|\cdot\|_{2,\Sigma}$-stability of the RFDS (4) (see expression (18) with $\eta = 1$ and $\rho = 0$). However, our previous reasoning relies on condition (43) which is not completely satisfactory since it is stated in terms of RVs $R$ and $S$ rather than the input RVs $\alpha$ and $\beta$ of the RPDE (1). Therefore, it still remains to establish an explicit condition in order for the stability of the RFDS (4) can be stated in a useful manner. With this aim, let us observe that (43) writes

$$1 - \beta(\omega)\frac{\Delta t}{\Delta x} - 2\alpha(\omega)\frac{\Delta t}{(\Delta x)^2} \ge 0, \quad \forall \omega \in \Omega,$$

or

$$1 \geq \frac{\beta(\omega)\Delta x + 2\alpha(\omega)}{(\Delta x)^2}\Delta t \Leftrightarrow \Delta t \leq \frac{(\Delta x)^2}{\beta(\omega)\Delta x + 2\alpha(\omega)}, \quad \forall \omega \in \Omega.$$

Taking into account the domain of RVs $\alpha$ and $\beta$ assumed in hypothesis **H2** (see (23)), one gets

$$\beta(\omega)\Delta x + 2\alpha(\omega) \leq \beta_2 \Delta x + 2\alpha_2 \Rightarrow \frac{(\Delta x)^2}{\beta(\omega)\Delta x + 2\alpha(\omega)} \geq \frac{(\Delta x)^2}{\beta_2 \Delta x + 2\alpha_2}, \quad \forall \omega \in \Omega.$$

Summarizing the following result has been established

**Proposition 3.** *Let us consider the random IVP* (1)–(2) *where RVs $\alpha$ and $\beta$ satisfy hypothesis* ***H2*** *(see* (23)*) and the initial condition $U_0(x)$ satisfies hypothesis* ***H4*** *(see* (36)*). Then, under the following condition*

$$\Delta t \leq \frac{(\Delta x)^2}{\beta_2 \Delta x + 2\alpha_2}, \tag{46}$$

*the RFDS* (4) *is mean square $\|\cdot\|_{2,\Sigma}$-stable.*

**Remark 3.** It is very important to emphasize that the hypothesis **H2** assumed on the input data RVs $\alpha$ and $\beta$ (see (23)) to conduct our stability analysis is not very restrictive regarding applications. In fact, this assertion can be supported by the Chebyshev-Markov inequality [25]. This significant result legitimises the accurate probabilistic approximation of second-order unbounded RVs by means of the truncation of their domain. For example, this inequality guarantees that the interval $[\mu_X - 10\sigma_X, \mu_X + 10\sigma_X]$ contains the 99% of the probability of any second-order RV, say $X$, i.e. $X \in \mathrm{L}_2^{\mathrm{RV}}(\Omega)$ with mean $\mu_X$ and variance $\sigma_X^2$. This assertion holds regardless the distribution of $X$. The larger truncated interval the better probabilistic approximation, although, naturally the diameter of the above interval can be reduced if the probabilistic distribution of the RV $X$ is known. For example, if $X$ is gaussian RV, hence unbounded, $X \sim \mathrm{N}(\mu_X; \sigma_X^2)$, then the truncation of $X$ over the domain $[\mu_X - 3\sigma_X, \mu_X + 3\sigma_X]$ comprises the 99.7% of the probability of the RV $X$.

## 5. Some illustrative numerical examples

This section is addressed to illustrate the main results proved in Sections 3 and 4 by means of two examples for which reliable approximations for the expectation and the standard deviation functions of the solution SP of the random IVP (1)–(2) are constructed. Numerical approximations of these two statistical functions are computed via the RFDS (4). In order to check the accuracy of these approximations, we will compare them with the corresponding exact values. This verification is possible since input data of the IVP (1)–(2) has been devised in such a way that expressions for the expectation and the standard deviation of the solution SP are available. In the second example, we illustrate the effect of truncating adequately the input RVs in order to get accurate approximations of the mean and the standard deviation of the solution SP to the random IVP (1)–(2).

**Example 1.** *Let us consider the random Cauchy problem* (1)–(2)*. For the random coefficients $\alpha$ and $\beta$ will be assume that $\alpha$ is an exponential RV of parameter $\lambda = 1$ truncated at the interval* $[0, 6]$*, $\alpha \sim Exp_{[0,6]}(1)$, and $\beta$ is a beta RV of parameters $(a; b) = (2; 3)$, $\beta \sim Be(2; 3)$. Notice that hypothesis* ***H2*** *(see* (23)*) holds with $\alpha_2 = 6$ and $\beta_2 = 1$. Hereinafter, we will assume that $\alpha$*

*and $\beta$ are independent RVs. While for the initial condition, we take $u_0(x) = \exp(-(x/6)^2)$ which admits a DFT, [26] (see hypothesis **H4**). Likewise, we point out that it is not difficult to check that hypotheses **H1** and **H3** (see (22), (24)) hold but cumbersome, thus we will omit here the details. Then, it can easily checked that the exact solution SP of (1)–(2) is given by the SP*

$$U(x,t) = \frac{3\,\mathrm{e}^{-\frac{(x-\beta t)^2}{4(\alpha t+9)}}}{\sqrt{\alpha t+9}}.$$

*We will construct numerical approximations to the expectation and the standard deviation of the solution SP, $U(x,t)$, of the random Cauchy problem (1)–(2) on the space interval $-15 \le x \le 15$. This will done by applying the RFDS (4).*

*In order for the mean square $\|\cdot\|_{2,\Sigma}$-stability of this scheme to be guaranteed, we firstly fix the space step $\Delta x$ and taking into account that $\alpha_2 = 6$ and $\beta_2 = 1$, in accordance with condition (46) of Proposition 3, the time step $\Delta t$ must satisfy the following condition*

$$\Delta t \le \frac{(\Delta x)^2}{\Delta x + 12}. \tag{47}$$

*The numerical approximations of the expectation and the standard deviation of the solution SP $U(x,t)$ at the lattice point $(x_k, t_n)$ are computed in two steps, firstly by applying iteratively the RFDS (4) and, secondly, taking the expectation operator. The numerical results obtained by this procedure have been compared with the exact values that are computed from*

$$\mathbb{E}[U(x,t)] = \int_0^1 \int_0^6 \frac{3\,\mathrm{e}^{-\frac{(x-\beta t)^2}{4(\alpha t+9)}}}{\sqrt{\alpha t+9}} f_\alpha(\alpha) f_\beta(\beta)\,\mathrm{d}\alpha \mathrm{d}\beta \tag{48}$$

*for the mean, and*

$$\sigma[U(x,t)] = \sqrt{\int_0^1 \int_0^6 \frac{9\,\mathrm{e}^{-\frac{(x-\beta t)^2}{2(\alpha t+9)}}}{\alpha t+9} f_\alpha(\alpha) f_\beta(\beta)\,\mathrm{d}\alpha \mathrm{d}\beta - (\mathbb{E}[U(x,t)])^2} \tag{49}$$

*for the standard deviation, being*

$$f_\alpha(\alpha) = \frac{\exp(-\alpha)}{\int_0^6 \exp(-\alpha)\,\mathrm{d}\alpha},\ 0 < \alpha < 6,\ \ \textit{and}\ \ f_\beta(\beta) = 10\beta(1-\beta)^2,\ 0 < \beta < 1, \tag{50}$$

*the PDFs of the RVs $\alpha$ and $\beta$, respectively.*

*In Fig. 1, we compare, at the time instant $T = 2$ (time fixed station), the exact mean function of the solution SP calculated by (48) and the numerical approximations of the expectation obtained by means of the RFDS (4) over the spatial domain $[-15, 15]$. This comparative analysis has been carried out considering different values for the spatial step ($\Delta x$) and time step ($\Delta t$) collected in Table 1. Fixed $\Delta x$, then $\Delta t$ has been computed so that condition (47) holds. As a measure of the accuracy of the approximations, we have also included in Table 1 the mean percentage absolute error for the mean (MAPE($\mu$)) and the standard deviation (MAPE($\sigma$)) at the time fixed station $T = 2$. Specifically, if $\hat{\mu}_k$ denotes the approximation of the expectation of the solution SP to the random initial value problem (1)–(2) using the RFDS (4) at the spatial lattice point $x_k$, then*

$$MAPE(\mu) = \left( \frac{1}{2K+1} \sum_{k=-K}^{K} \left| \frac{\hat{\mu}_k - \mathbb{E}[U(x_k,2)]}{\mathbb{E}[U(x_k,2)]} \right| \right) \times 100\%, \tag{51}$$

*where* $\mathbb{E}[U(x_k, 2)]$ *is given by* (48)*, and* $K = 15/\Delta x$ *for a given value of* $\Delta x$*. The value of MAPE*($\sigma$) *has been calculated analogously. The values of both MAPEs are detailed in Table 1. Observe that these figures are in agreement with the order of the numerical method. Furthermore, the less the spatial step (and hence the time step), the less the MAPE.*

| $\Delta t$ | $\Delta x$ | MAPE($\mu$) | MAPE($\sigma$) |
|---|---|---|---|
| $1/58$ | $15/32$ | 2.27% | 2.15% |
| $2/58 = 1/29$ | $30/32 = 15/16$ | 4.70% | 4.31% |
| $4/58 = 2/29$ | $60/32 = 15/8$ | 10.17% | 8.84% |

Table 1: The two first columns collect the values of the time step ($\Delta t$) and space step ($\Delta x$) satisfying the mean square $\|\cdot\|_{2,\Sigma}$-stability condition (47) in the context of Example 1. The two last columns show the values of the mean percentage absolute error (MAPE) according to expression (51).

*In Fig. 2, we shown an analogous comparative analysis for the standard deviation at the time instant* $T = 2$*.*

*In Fig. 3 and Fig. 4 we have represented graphically the relative errors for the approximations of the expectation and standard deviation taking as spatial and time steps the figures collected in Table 1, respectively. From these graphical representations one observes that as* $\Delta x$ *is halved, the relative error is also approximately also divided by 2. This confirm the order of convergence of the random numerical scheme.*

**Example 2.** *As it has been discussed in Remark 3, the hypothesis* ***H3*** *of boundedness (see* (24)*) imposed over the input random data* $\alpha$ *and* $\beta$ *is not restrictive in practice. To justify this assertion, we now assume that the input RV* $\alpha$ *has an exponential distribution (hence* $\alpha$ *is an unbounded RV), of parameter* $\lambda = 1$ *and we keep* $\beta \sim Be(2; 3)$ *and* $u_0(x)$ *as in Example 1. For sake of clarity in the subsequent notation, henceforth this unbounded RV will be denoted by* $\hat{\alpha} \sim Exp(1)$*. Then, we have computed the exact mean and standard deviation of the solution SP using the expressions* (48) *and* (49)*, but taking* $f_{\hat{\alpha}}(\hat{\alpha}) = \exp(-\hat{\alpha})$*,* $\hat{\alpha} > 0$ *instead of the PDF* $f_\alpha(\alpha)$ *defined in* (50)*. These exact values have been compared with the ones obtained by the approximation of the unbounded RV* $\hat{\alpha} \sim Exp(\lambda = 1)$ *using the truncated (hence bounded) RV* $\alpha \sim Exp_{[0,6]}(1)$*, which contains more than* 99% *of the probability mass of* $\hat{\alpha}$*, since* $\int_0^6 f_\alpha(\alpha)\,\mathrm{d}\alpha = 0.997521$*. In Table 2, it is reported the values of the MAPE for both the mean and the standard deviation of the solution SP. From these figures we can see that the proposed RFDS* (4) *gives accurate approximations in the case that there exist unbounded input RVs. In such case, it is enough to approximate them by means of bounded RVs resulting from appropriating truncation.*

| $\Delta t$ | $\Delta x$ | MAPE($\mu$) | MAPE($\sigma$) |
|---|---|---|---|
| $1/58$ | $15/32$ | 1.87% | 2.90% |
| $2/58 = 1/29$ | $30/32 = 15/16$ | 4.24% | 4.34% |
| $4/58 = 2/29$ | $60/32 = 15/8$ | 9.61% | 7.80% |

Table 2: The two first columns collect the values of the time step ($\Delta t$) and space step ($\Delta x$) satisfying the mean square $\|\cdot\|_{2,\Sigma}$-stability condition (47). The two last columns show the values of the mean percentage absolute error (MAPE) according to expression (51) in the context of Example 2.

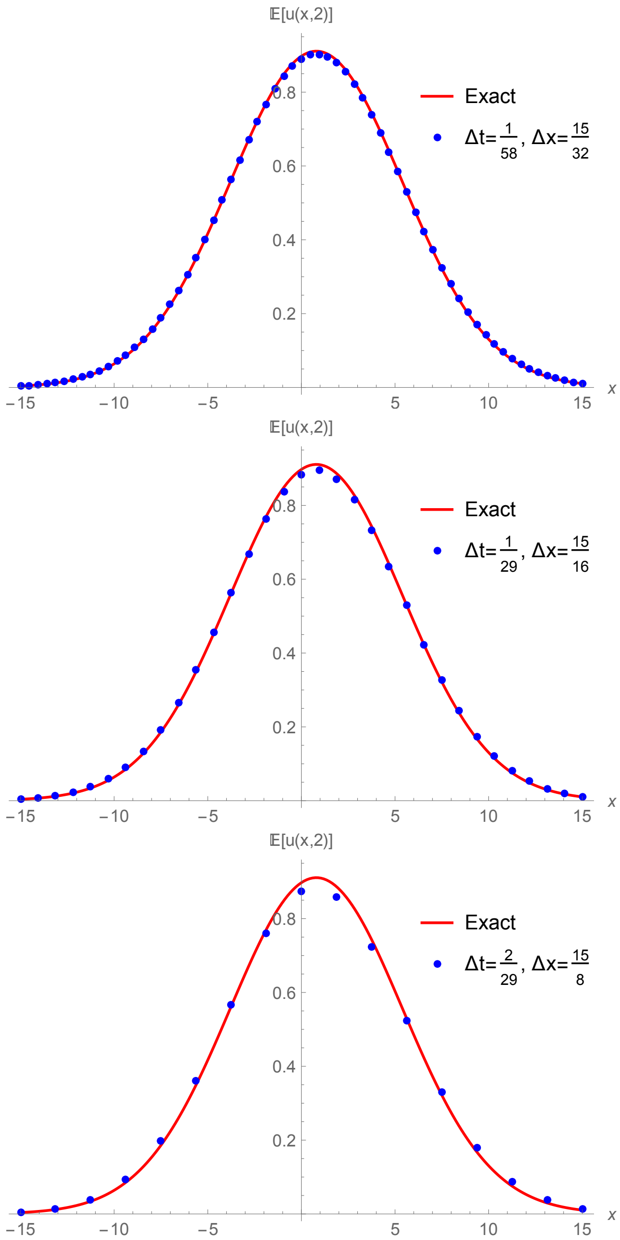


Figure 1: Comparison of the expectation of the exact solution SP and the approximations at the time instant $T = 2$ for different values of $\Delta x$ and $\Delta t$ over the spatial domain $-15 \leq x \leq 15$ in the context of Example 1.

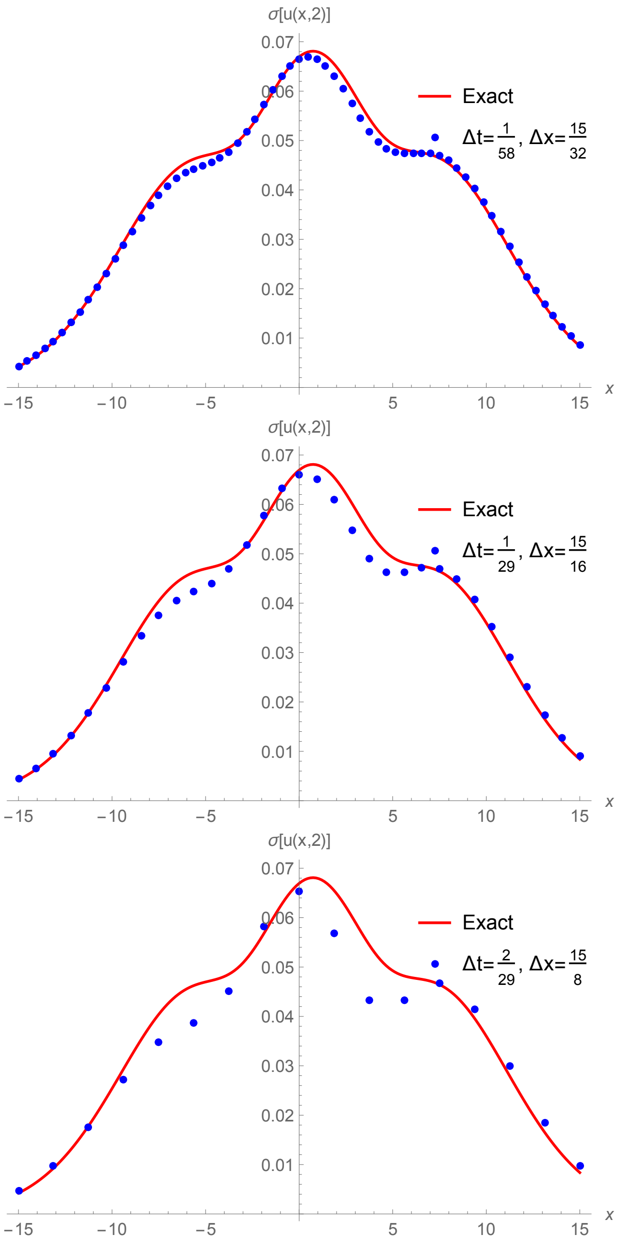


Figure 2: Comparison of the standard deviation of the exact solution SP and the approximations at the time instant $T = 2$ for different values of $\Delta x$ and $\Delta t$ over the spatial domain $-15 \leq x \leq 15$ in the context of Example 1.

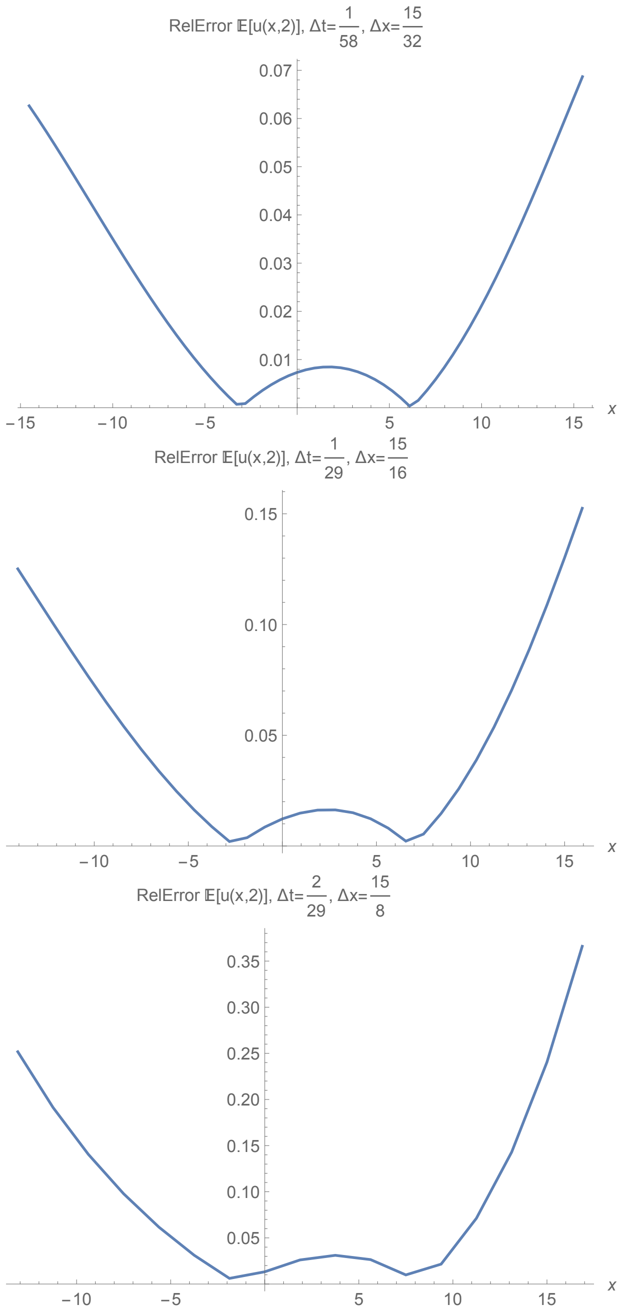


Figure 3: Relative errors at the time instant $T = 2$ for the approximations of the expectation for different values of $\Delta x$ and $\Delta t$ over the spatial domain $-15 \leq x \leq 15$ in the context of Example 1.

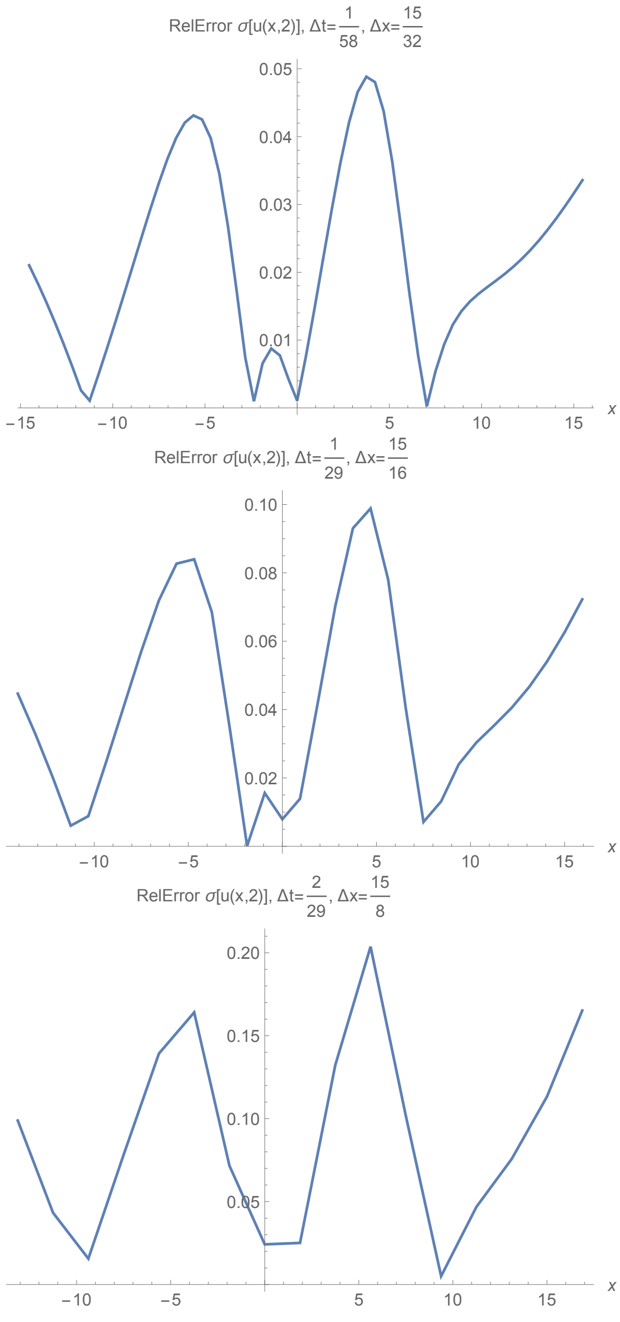


Figure 4: Relative errors at the time instant $T = 2$ for the approximations of the standard deviation for different values of $\Delta x$ and $\Delta t$ over the spatial domain $-15 \leq x \leq 15$ in the context of Example 1.

## 6. Conclusions

In this paper we have proposed a random finite difference scheme to construct reliable approximations of the one-dimensional advection-diffusion Cauchy problem with random coefficients and a deterministic initial condition. This random scheme extends the classical forward-time-backward/centered-space to the random context. We have investigated sufficient conditions on the input data (coefficients and initial condition) in order for the mean square consistency and stability of the random scheme be guaranteed. The obtained conditions are mild and they extend their deterministic counterpart in the general case that diffusion and advection coefficients are statistical dependent bounded random variables with an arbitrary joint probability density function. This latter fact is a remarkable feature regarding the present contribution since it is usual to embrace statistical independence for input random variables as well as assuming their probabilistic nature is of gaussian-type. Furthermore, it is important to point out that boundedness hypothesis on the random coefficients is not restrictive from a practical point of view since the probabilistic truncation method based on classical Chebyshev's inequality enables us to approximate unbounded RVs with a degree of accuracy previously fixed. This issue has been illustrated by means of an example where reliable numerical approximations of the mean and the standard deviation of the solution stochastic process has been computed from the proposed random numerical scheme. We have been able to check the accuracy of these approximations since we have considered a test example for which the corresponding exact values are available. In this manner, we validate the proposed method to be applied to other random one-dimensional advection-diffusion Cauchy problems whose exact solution is not available, which, of course, is the usual case in real problems. Finally, we point out that the approach considered in this paper could be carefully adapted to study another important random partial differential equations in future works.


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## Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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