Intersection and point-to-line solutions for geodesics on the ellipsoid

Sergio Baselga[1,*] and José Carlos Martínez-Llario[1]

[1]*Cartographic Engineering, Geodesy and Photogrammetry Department, Universitat Politècnica de València, Camino de Vera s/n, 46022 València, Spain*
[*]*Corresponding author: serbamo@cgf.upv.es*

Abstract

The paper presents two algorithms for the computation of intersection of geodesics and minimum distance from a point to a geodesic on the ellipsoid, respectively. They are based on the iterative use of direct and inverse problems of geodesy by means of their implementations with machine-precision accuracy in GeographicLib. The algorithms yield the same results as those obtained by Karney's approach based on the use of auxiliary ellipsoidal gnomonic projections with the advantage on our side that the algorithms are not limited to distances below 10000 km. This results in our algorithm being the only general solution for the problem of minimum distance from a point to a geodesic on the ellipsoid.

Keywords

Geodesic line; Intersection; Ellipsoid; GeographicLib

1 Introduction

A geodesic on the ellipsoid is a curve with null geodetic curvature (i.e. null curvature on the tangent space). Given two points *A* and *B* on the surface on the ellipsoid, the geodesic *AB* provides the line of minimum length that connects *A* and *B*. Over the last centuries, geodesists have devised algorithms of increasing accuracy for solving the so-called direct and inverse problems of geodesy, that is, obtaining the coordinates of a point B located on a geodesic starting at point A with a given azimuth α_A and distance s (direct problem) and obtaining the distance s and azimuths at both ends α_A and α_B given the coordinates of two points A and B on the ellipsoid (inverse problem). Among the most widespread methods today we find the algorithms by Vincenty (1975), suitable for small programmable calculators, Sjöberg (2006, 2009) and Karney (2011a, 2013). The latter are implemented in the GeographicLib suite (Karney 2017), a library of geodetic functions developed in several programming languages that permits to take advantage of the high precision of standard computers today. State-of-the-art open source software for spatial analysis use an implementation of GeographicLib (e.g. PostGIS Development Group 2017).

The question of intersection of geodesics has received not so much attention as the direct and inverse problems of geodesy. Sjöberg (2002, 2008, 2009) deals with the problem directly on the ellipsoid by using numerical integration and Karney (2011a, 2011b, 2013) ingeniously solves the question by using a particular map projection (ellipsoidal gnomonic projection) in which geodesics are very nearly straight and angles are preserved at the origin. Even less attention has received the problem of minimum distance from a point to a geodesic on the ellipsoid, for which the only available procedure we find is the one based on Karney's approach using the ellipsoidal gnomonic projection (Karney 2011a, 2013), which is implemented in Karney (2011b) and JaVaWa (2017). In this paper we present new algorithms for solving these two problems. Without delving into the difficulties in dealing with the underlying systems of ordinary differential equations, their solution methods and the corresponding analyses of numerical stabilities, we resort to prior accurate implementations of the direct and inverse problems of geodesy (we recommend and will use their two implementations in GeographicLib) in order to build two easy-to-implement robust algorithms. As we will show, they yield the same results as Karney and Sjöberg's methods for intersection of geodesics and Karney's auxiliary projection method for minimum point-to-line distance except for the case of very long distances where only our methods succeed.

2 Intersection of geodesics on the ellipsoid

We want to derive an algorithm for computing the intersection of geodesics on the ellipsoid that is based on existing implementations of the direct and inverse problems of geodesy. We have in mind, in particular, the corresponding functions in GeographicLib, which are optimized to deliver accuracy close to machine precision. Beyond ensuring no loss of precision, we want our algorithm to be fast enough to be used not only for a single intersection but also as a possible tool in the future for more complex problems requiring multiple intersections. We describe next an iterative algorithm for the intersection of geodesics with fast convergence based on prior accurate implementations of direct and inverse problems of geodesy.

Given points *A*, *B*, *C* and *D* on the ellipsoid we will compute the intersection of geodesics *AB* and *CD*. The problem is depicted in Fig. 1, where the intersection point is denoted by *X*.

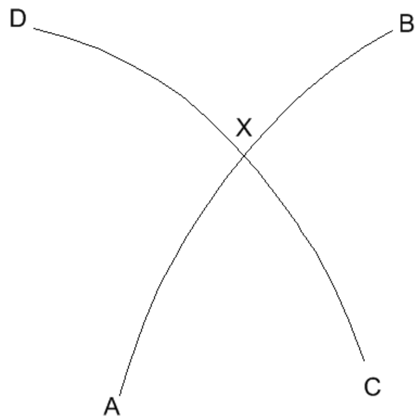

Fig. 1 Intersection of geodesics *AB* and *CD*

A geodesic is mathematically described by a set of differential equations which are embedded in the corresponding implementations of the direct and inverse problems of geodesy. By using the implementation of the direct problem, for instance, we are able to move along the geodesic *AB* and obtain one point at a given distance from *A*. The idea of the current method is to find an expression for the distance from *A* to *X*, s_{AX}, that is correct for small figures and approximate enough for larger ones to yield a new point on the geodesic that is closer to the intersection. This new point will subsequently be used as base for a new iteration.

For the computation of the approximate expression of the distance between a point, say *A*, and the intersection *X* of geodesics on the ellipsoid let us assume that *A*, *C* and *X* form a spherical triangle (where we will use a suitable value for the radius of the sphere R as e.g. the semimajor axis of the ellipsoid). Now we resort to a rarely used formula of spherical trigonometry known as cotangent four-part formula, whose derivation can be found in Todhunter (1886), Sects. 43-44, that sets the following relation for a spherical triangle of angles A, X, C (as it is common we will denote both the vertex and the angle at the vertex by the same uppercase letter) and corresponding opposite sides a, x, c

$$\cos x \cos A = \cot c \sin x - \cot C \sin A \tag{1}$$

By transposition we can write

$$\tan c = \frac{\sin x}{\cos x \cos A + \cot C \sin A} \tag{2}$$

and obtain the approximate expression for s_{AX} that we are looking for

$$s_{AX} = R \arctan\left(\frac{\sin(s_{AC}/R)}{\cos(s_{AC}/R)\cos A + \cot C \sin A} \right) \tag{3}$$

Similarly we obtain

$$s_{CX} = R \arctan\left(\frac{\sin(s_{AC}/R)}{\cos(s_{AC}/R)\cos C + \cot A \sin C} \right) \tag{4}$$

These expressions have no significant singularities apart from the trivial case when both A and C are zero, i.e. both geodesics are coincident, and the entire geodesics are then the intersection. We also assume that the implementing software is able to deal with arctangents of arbitrarily large values yielding $\pm\pi/2$.

Now Eqs. (3) and (4) do not solve the problem unless distances are sufficiently small. They provide, however, approximate values for the correct distances so that an iterative process can be started as follows.

After initial determination of azimuths α_{AB} and α_{CD} by means of the inverse geodetic problem with points A, B and points C, D (and considering that $\alpha_{AX} = \alpha_{AB}$ and $\alpha_{CX} = \alpha_{CD}$) we apply the direct geodetic problem at C with azimuth α_{CX} and the distance s_{CX} obtained from Eq. (4), as well as the direct geodetic problem at A with azimuth α_{AX}

and the distance s_{AX} obtained from Eq. (3). As shown in Fig. 2a) we obtain in the geodesic *AB* a point A_2 which is much closer to the intersection than *A*. Similarly we obtain a point C_2 in the geodesic *CD* which is much closer to the intersection than *C*. These points will act as new base points for the next iteration (i.e. *A* is replaced by A_2 and *C* by C_2). The new distances bring the new points, A_2 and C_2, closer to the intersection (Fig. 2b). Note that azimuths change in every iteration since the azimuth of a geodesic changes along its different base points, i.e. the values α_{AB} and α_{CD} are different in Fig. 2b) than in Fig. 2a). Iterations can be stopped when the coordinate differences between A_2 and C_2 are below the required level of accuracy. In general the method converges fast and ensures that the intersection point *X* belongs to both geodesics (the successive application of inverse and direct problems ensures not to divert from the original geodesics).

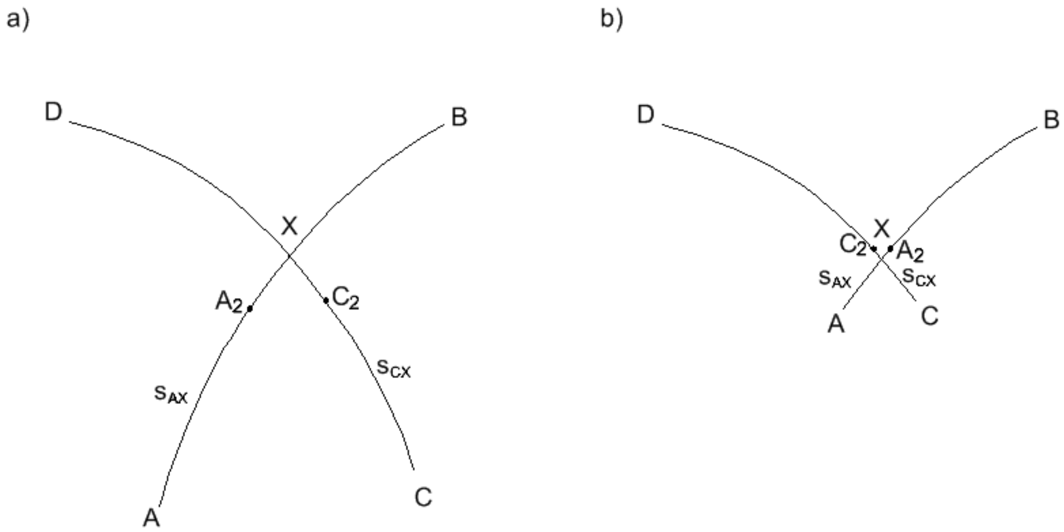

Fig. 2 Iterative procedure for the computation of the intersection point

Several notes of caution are worth mentioning here. First, the distances resulting from Eq. (3) and Eq. (4) can be negative (as it would be the case of the next iteration in Fig. 2b). Direction reversal with a positive distance shall be understood then, either by the particular implementation of the direct geodetic problem (as the corresponding function of GeographicLib does) or by inspection and modification before application of the corresponding function. This provides support also to the case where one or both lines need to be prolonged to meet the intersection. Second, apart from the intersection point *X* shown in Fig. 1 the two geodesics intersect also in an almost-antipodal point. Our method naturally yields the intersection point that is closer to *A* and C (assuming the arctangent function implemented returns values in the interval -π/2, π/2), but it is easy to obtain the nearly-antipodal point provided we take only for the initial iteration the formulas

$$s_{AX} = R\left[\arctan\left(\frac{\sin(s_{AC}/R)}{\cos(s_{AC}/R)\cos A + \cot C \sin A}\right) + \pi\right] \tag{5}$$

$$s_{CX} = R\left[\arctan\left(\frac{\sin(s_{AC}/R)}{\cos(s_{AC}/R)\cos C + \cot A \sin C}\right) + \pi\right] \tag{6}$$

and then we continue the course of iterations with Eqs. (3) and (4).

2.1 Practical examples

We start with an example where the maximum distance to the intersection point (here s_{XB}) is relatively small, 74 km. The geodetic coordinates of the points defining geodesics *AB* and *CD* are φ_A = 52º, λ_A = 5º, φ_B = 51.4º, λ_B = 6º, φ_C = 51.5º, λ_C = 4.5º, φ_D = 52º and λ_D = 5.5º. The course of iterations is shown in Table 1. As it can be seen our method quickly converges to coordinates for A_2 and C_2 that differ less than 0.00001 arc seconds, which constitutes our finishing criterion.

Table 1 Intersection of geodesics, example 1. Successive distances for the application of the direct problem of geodesy and corresponding coordinates of points on geodesics from *A* (point A_2) and *C* (point C_2).

Iteration No.	s_{AX} (m)	s_{CX} (m)	φ_{A2}	λ_{A2}	φ_{C2}	λ_{C2}
1	21637.0270	64700.0631	51º51'56.3972"	5º13'38.8427"	51º51'56.3311"	5º13'38.7157"
2	0.0762	3.1832	51º51'56.3954"	5º13'38.8456"	51º51'56.3954"	5º13'38.8456"
3	0.0000	0.0000	51º51'56.3954"	5º13'38.8456"	51º51'56.3954"	5º13'38.8456"

We can check the correctness of the final value (φ_X = 51º 51' 56.3954", λ_X = 5º 13' 38.8456") by inspection of azimuth equalities: $\alpha_{AX} = \alpha_{AB}$ (in this case 133º 36' 13.4578") and $\alpha_{BX} = \alpha_{BA}$ (in this case 314º 23' 18.7197") needed since the intersection point *X* belongs to the first geodesic, and $\alpha_{CX} = \alpha_{CD}$ (in this case 50º 45' 17.6875") and $\alpha_{DX} = \alpha_{DC}$ (in this case 231º 32' 24.8830") since the intersection point *X* belongs to the second geodesic. All of them are dutifully fulfilled. We can also see that, as expected, in every geodetic line the azimuths from the intersection point to both ends are equal (if the line had to be prolonged) or differ exactly in 180º. In this example α_{XA} = 133º

46' 58.1263" and α_{XB} = 313º 46' 58.1263", as well as α_{XC} = 51º 19' 32.4304" and α_{XD} = 231º 19' 32.4304".

We obtain exactly the same coordinates for the intersection by Karney's method whereas we find some discrepancies (1" in latitude, 2" in longitude) after the application of Sjöberg's (they are probably due to the numerical evaluation of the integral involved and not to the method itself).

Now we compute another example which appears in Karney (2011b) involving much longer distances (the distances to the intersection point will result in values from 1100 to 5600 km). We take φ_A = 42º, λ_A = 29º, φ_B = 39º, λ_B = -77º, φ_C = 6º, λ_C = 0º, φ_D = 64º and λ_D = -22º as coordinates defining the geodesics *AB* and *CD* and we have the course of iterations displayed in Table 2.

Table 2 Intersection of geodesics, example 2. Successive distances for the application of the direct problem of geodesy and corresponding coordinates of points on geodesics from *A* (point A_2) and *C* (point C_2).

Iteration No.	s_{AX} (m)	s_{CX} (m)	φ_{A2}	λ_{A2}	φ_{C2}	λ_{C2}
1	3402464.8393	4589822.8334	54º40'00.1134"	-13º45'42.2728"	46º24'36.9467"	-10º27'32.0442"
2	52037.0641	957583.2555	54º43'01.3449"	-14º33'50.5265"	54º37'33.5290"	-14º30'34.4773"
3	-11.6217	10723.2692	54º43'01.3066"	-14º33'49.8807"	54º43'01.3061"	-14º33'49.8804"
4	0.0000	0.0152	54º43'01.3066"	-14º33'49.8807"	54º43'01.3066"	-14º33'49.8807"
5	0.0000	0.0000	54º43'01.3066"	-14º33'49.8807"	54º43'01.3066"	-14º33'49.8807"

We can check that the final value (φ_X = 54º 43' 1.3066", λ_X = -14º33'49.8807") is correct by inspection of azimuth equalities $\alpha_{AX} = \alpha_{AB}$ = 309º 18' 22.4891", $\alpha_{BX} = \alpha_{BA}$ = 47º 44' 07.2214", $\alpha_{CX} = \alpha_{CD}$ = 349º 1' 47.4802", $\alpha_{DX} = \alpha_{DC}$ = 154º 29' 46.5383". We can also see that, as expected, α_{XA} = 275º 51' 17.7681" and α_{XB} = 95º 51' 17.7681" differ in 180º, as well as α_{XC} = 340º 55' 2.9801" and α_{XD} = 160º 55' 2.9801". Although the computation by Karney's method assumes that geodesics are straight lines in the ellipsoidal gnomonic projection centered in *X* while in fact they are only "very nearly straight" (Karney 2013) we do not find any differences between our final results and those by Karney's method. By contrast, this problem does not find a reasonable result by means of Sjöberg's method. It is due to the known inability of the method to follow the geodesic beyond its vertices (Sjöberg and Shirazian 2012), since in the present example the geodesic arc *AB* crosses the point of maximum latitude (φ = 54º 55' 43"), which is in fact quite close to the intersection point.

Finally, we compute a third example that is also intractable by Karney's method based on the gnomonic projection. Since this projection cannot represent more than one hemisphere and it has to be centered in the intersection point, the intersection problems for which at least one of the distances from the intersection point to the initial points *A*, *B*, *C*, or *D* are longer than some 10000 km are impossible to solve by Karney's method. We take for instance φ_A = 35º, λ_A = -92º, φ_B = 40º, λ_B = 52º, φ_C = -8º, λ_C = 20, φ_D = 49º and λ_D = -95º, for which the distances to the intersection point will eventually result in values from 1100 to 11300 km). Table 3 shows the course of iterations.

Table 3 Intersection of geodesics, example 3. Successive distances for the application of the direct problem of geodesy and corresponding coordinates of points on geodesics from *A* (point A_2) and *C* (point C_2).

Iteration No.	s_{AX} (m)	s_{CX} (m)	φ_{A2}	λ_{A2}	φ_{C2}	λ_{C2}
1	1604865.7945	4348568.2893	47º31'26.0523"	-82º22'38.4586"	21º52'41.8516"	-5º53'11.1551"
2	389269.4887	4565612.7810	50º24'28.6348"	-79º21'46.4166"	46º28'27.8630"	-46º38'35.2232"
3	9744.5640	2273004.8317	50º28'44.6977"	-79º16'58.1459"	50º30'44.8473"	-77º01'21.0483"
4	2.0208	160366.4992	50º28'44.7508"	-79º16'58.0861"	50º28'44.8137"	-79º16'55.5158"
5	0.0000	50.7146	50º28'44.7508"	-79º16'58.0861"	50º28'44.7508"	-79º16'58.0861"
6	0.0000	0.0000	50º28'44.7508"	-79º16'58.0861"	50º28'44.7508"	-79º16'58.0861"

We obtain the result φ_X = 50º 28' 44.7508", λ_X = -79º 16' 58.0861" and azimuth equalities $\alpha_{AX} = \alpha_{AB}$ = 27º 0' 50.0492", $\alpha_{BX} = \alpha_{BA}$ = 330º 57' 3.3917", $\alpha_{CX} = \alpha_{CD}$ = 319º 57' 19.8704", $\alpha_{DX} = \alpha_{DC}$ = 75º 46' 33.9244". We can also see that, as expected, α_{XA} = 35º 44' 35.7125" and α_{XB} = 215º 44' 35.7125" differ in 180º, as well as α_{XC} = 267º 48' 9.5053" and α_{XD} = 87º 48' 9.5053". In this case, Sjöberg's method is unable to find the correct solution since the intersection is beyond the vertex of geodesic *CD* when computed from *C*. We can however use Sjöberg's method and attain a result with rather small discrepancies with respect to ours (0.1" both in latitude and longitude) and 0.5" of maximum discrepancy in the azimuth check if we interchange points *C* and *D*, or, in other words, if we compute the intersection point from *D* not *C* (so that the vertex of the geodesic is not reached before the intersection). At any rate, Sjöberg's method demonstrates to be not always applicable, in contrast to ours.

3 Minimum point-to-line distance on the ellipsoid

The problem now is to find the minimum distance from a point *P* to a geodesic *AB* on the ellipsoid (Fig. 3). We can devise an algorithm based on the iterative use of the implementations of the direct and inverse problems of geodesy plus the necessary condition of intersection at right angles (Patrikalakis *et al.* 2009).

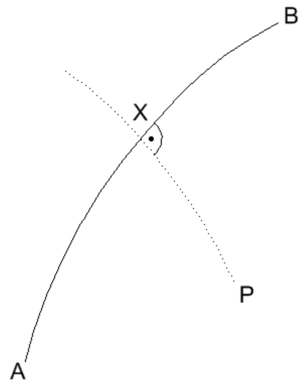

Fig. 3 Minimum point-to-line distance

Again the idea is to use approximate expressions for the distances from *A* to *X*, s_{AX}, and *P* to *X*, s_{PX}, that are correct for small figures and approximate enough for larger ones to yield a new point on the geodesic *AB* that is closer to the intersection. This new point will subsequently be used as base for a new iteration.

For the computation of the approximate expressions for distances let us assume that points *A*, *P* and *X* form a spherical triangle, one that has a right angle *X*, where we will use a suitable value for the radius of the sphere *R* (e.g. the semimajor axis of the ellipsoid).

The sine rule for angles *A*, *X* (90º) and corresponding opposite sides *a*, *x* reads

$$\sin a = \sin x \sin A \tag{7}$$

$$s_{PX} = R \arcsin\left[\sin\left(s_{AP} / R\right) \sin A\right] \tag{8}$$

The use of one of Napier's analogies for the spherical triangle of angles *A*, *P*, *X* (90º) and corresponding opposite sides *a*, *p*, *x* permits us to write

$$\tan\frac{p}{2} = \frac{\sin\frac{90^{\circ}+A}{2}}{\sin\frac{90^{\circ}-A}{2}} \tan\frac{x-a}{2} \tag{9}$$

$$s_{AX} = 2R\arctan\left(\frac{\sin\frac{90^\circ+A}{2}}{\sin\frac{90^\circ-A}{2}}\tan\frac{s_{AP}-s_{PX}}{2R}\right) \tag{10}$$

Eqs. (8) and (10) are not approximate enough unless the distances are sufficiently small, so an iterative process must be started as follows.

Using the implementation of the inverse problem of geodesy we determine the distance s_{AP} as well as azimuths α_{AP} and α_{AB}. Angle A is the difference of these azimuths. By means of Eq. (8) we obtain an approximate value for the distance s_{PX} only to be introduced in Eq. (10) so that an approximate value for s_{AX} can be obtained. Now we use the implementation of the direct problem of geodesy to compute a point A_2 along the geodesic *AB* located at distance s_{AX} from *A*, which will act as base point for the next iteration (i.e. *A* is replaced by A_2), just the same as we did in the first method (Fig. 2). The difference now is that we do not know the azimuth of the geodesic *XP* so we cannot iterate *P*. After several iterations the differences between successive coordinates of A_2 are below the required level of accuracy (or, equivalently, the distance s_{AX} can be regarded zero for our level of accuracy) and the iterations can be finished.

The method converges fast to the solution of minimum distance from point *P* to geodesic *AB* – note that we were not interested in the other possible larger value of the arcsin function in Eq. (8) – supports line prolonging and has no significant singularities (it yields the correct solution even if the point *P* is in the geodesic *AB*).

Next we apply the algorithm to some examples and compare the results with those that can be obtained by Karney (2011b) by means of use of the auxiliary gnomonic projection.

3.1 Practical examples

Let us start with an example where the point-to-line distance is relatively small, some 24 km. The geodetic coordinates of the points defining the geodesic *AB* and point *P* are φ_A = 52º, λ_A = 5º, φ_B = 51.4º, λ_B = 6º, φ_P = 52º and λ_P = 5.5º. The course of iterations is shown in Table 4. As it can be seen, our method converges to the final value in just three iterations (our finishing condition is that successive coordinates for A_2 differ less than 0.00001 arc seconds).

Table 4 Point to geodesic, example 1. Successive coordinates of point A_2 and corresponding distances s_{AX}.

Iteration No.	φ_{A2}	λ_{A2}	s_{AX} (m)
1	51º50'45.9212"	5º15'37.5426"	24784.2886
2	51º50'45.9212"	5º15'37.5426"	-0.0002
3	51º50'45.9212"	5º15'37.5426"	0.0000

We ensure that the solution φ_X = 51º 50' 45.9212", λ_X = 5º 15' 37.5426" is correct by checking right-angle conditions (in this case α_{XA} = 313º 48' 31.4766", α_{XP} = 43º 48' 31.4766" and α_{XB} = 133º 48' 31.4766") along with the condition that X is on both geodesics (which was indeed part of our implementation of intersection of geodesics: α_{AX} = 133º 36' 13.4578" and α_{AB} = 133º 36' 13.4578", as well as α_{BX} = 314º 23' 18.7191" and α_{BA} = 314º 23' 18.7191"). The results are the same following Karney's approach.

Our second example is taken from Karney (2011b) and involves a point-to-line distance much longer (around 1000 km). With φ_A = 42º, λ_A = 29º, φ_B = 39º, λ_B = -77º, φ_P = 64º and λ_P = -22º we have the course of iterations shown in Table 5.

Table 5 Point to geodesic, example 2. Successive coordinates of point A_2 and corresponding distances s_{AX}.

Iteration No.	φ_{A2}	λ_{A2}	s_{AX} (m)
1	54º55'42.7066"	-21º56'18.1328"	3928857.7554
2	54º55'42.7134"	-21º56'14.2477"	-69.1851
3	54º55'42.7134"	-21º56'14.2477"	0.0000

Again it is checked that the solution φ_X = 54º 55' 42.7134", λ_X = -21º 56' 14.2477" fulfills right-angle conditions (α_{XA} = 89º 49' 31.0947", α_{XP} = 359º 49' 31.0943" and α_{XB} = 89º 49' 31.0947"), belongs to geodesic AB (α_{AX} = 309º 18' 22.4891" and α_{AB} = 309º 18' 22.4891", as well as α_{BX} = 47º 44' 7.2214" and α_{BA} = 47º 44' 7.2214") and is exactly coincident with the one in Karney (2011b).

Finally, we compute an example that is intractable by Karney's method based on the gnomonic projection due to its limitation to one hemisphere (i.e. requires that all distances from the projection center, point X, be below 10000 km). With φ_A = 42º, λ_A

= 29º, φ_B = -35º, λ_B = -70º, φ_P = 64º and λ_P = -22º we have the course of iterations shown in Table 6.

Table 6 Point to geodesic, example 3. Successive coordinates of point A_2 and corresponding distances s_{AX}.

Iteration No.	φ_{A2}	λ_{A2}	s_{AX} (m)
1	37º58'39.9186"	18º20'53.7147"	1012443.9063
2	37º58'41.2237"	18º20'56.6280"	-81.6969
3	37º58'41.2237"	18º20'56.6280"	0.0000

The solution φ_X = 37º 58' 41.2236", λ_X = 18º 20' 56.6279" verifies right-angle conditions (α_{XA} = 60º 29' 33.8570", α_{XP} = 330º 29' 33.8570" and α_{XB} = 240º 29' 33.8570") and belongs to geodesic *AB* (α_{AX} = 247º 21' 11.9791" and α_{AB} = 247º 21' 11.9791", as well as α_{BX} = 56º 53' 14.0821" and α_{BA} = 56º 53' 14.0821"). As mentioned before, the resulting distance $s_{XB} \approx$ 12200 km is longer than 10000 km, what makes the problem intractable by Karney's method and, as far as we know, by any other method that has been published so far.

4 Conclusions

We have presented two algorithms for solving the problems of intersection of geodesics and minimum distance from a point to a geodesic on the ellipsoid, respectively. They can be applied to all range of distances: from local-scale problems (a few km) to global-scale ones (even more than 10000 km). Being iterative in nature, and the second algorithm also resorting to the implementation of the first, their computational cost is dependent on the distances involved. They represent a competitive implementation with the currently standard approach by Karney based on the use of an auxiliary ellipsoidal gnomonic projection, with the advantage on our side that Karney's method is inapplicable for distances longer that 10000 km. This results in our algorithm being the only possibility for accurately solving the general problem of minimum distance from a point to a geodesic on the ellipsoid.

Acknowledgments

The authors are sincerely thankful to the editor and two anonymous reviewers for their detailed comments and suggestions that helped to considerably improve the methods developed in the manuscript as well as its readability.

References

JaVaWa, 2017. *JAVAWA GPS Tools*. (online at http://www.javawa.nl/coordcalc_en.html. Accessed 23 January 2017).

Karney C.F.F., 2011a. *Geodesics on an ellipsoid of revolution*. Tech rep, SRI International. (online at http://arxiv.org/abs/1102.1215v1 Accessed 15 December 2016).

Karney C.F.F., 2011b. *Intersection between two geodesic lines*. (online at https://sourceforge.net/p/geographiclib/discussion/1026621/thread/21aaff9f. Accessed 15 February 2017).

Karney C.F.F., 2013. Algorithms for geodesics. *J. Geod.,* 87(1), 43–55.

Karney C.F.F., 2017. *GeographicLib*, version 1.47. (online at http://geographiclib.sourceforge.net)

Patrikalakis N.M., Maekawa T., Cho W., 2009. *Shape interrogation for computer aided design and manufacturing.* (online at http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/mathe.html. Accessed 18 April 2017).

PostGIS Development Group, 2017. *PostGIS 2.3.3dev Manual*. (online at http://postgis.net/docs/index.html. Accessed 23 February 2017).

Sjöberg L.E., 2002. Intersections on the sphere and ellipsoid. *J. Geod.*, 76, 115–120.

Sjöberg L.E., 2006. Direct and indirect geodetic problems on the ellipsoid. *ZfV*, 131, 35–39.

Sjöberg L.E., 2008. Geodetic intersection on the ellipsoid . *J. Geod.*, 82(9), 565–567.

Sjöberg L.E., 2009. New solutions to classical geodetic problems on the ellipsoid. In: Sideris M.G. (Ed.) *Observing our changing earth*, International Association of Geodesy Symposia 133, 781–784, Springer-Verlag Berlin Heidelberg.

Sjöberg L.E. and Shirazian M, 2012. Solving the direct and inverse geodetic problems on the ellipsoid by numerical integration. *J. Surv. Eng.*, 138(1), 9-16.

Todhunter I., 1886. *Spherical trigonometry*, 5th Ed., MacMillan and Co. (available at http://www.gutenberg.org/ebooks/19770. Accessed 2 May 2017)

Vincenty T., 1975. Direct and inverse solutions of geodesics on the ellipsoid with application of nested equations. *Surv. Rev*., 23(176), 88–93 [addendum: Surv Rev 23(180), 294 (1976)].